Title: Symplectic duality and geometric Langlands

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Abstract: In this talk I would like to briefly sketch how one can use the tools of derived symplectic geometry and holomorphically twisted gauge theories to derive a relationship between symplectic duality and local Langlands. Our starting point will be an observation due to Gaiotto-Witten that a 3d N=4 theory with a G-flavor symmetry is a boundary condition for 4d N=4 SYM with gauge group G.

By examining the relationship between boundary observables and bulk lines we will be able to derive constructions originally due to Braverman, Finkelberg, Nakajima. By examine the relationship between boundary lines and bulk surface operators one can derive new connections to local geometric Langlands.

This is based on joint work with Philsang Yoo, Tudor Dimofte, and Davide Gaiotto

S(ymplectic) Duality

joint with Philsang Yoo

[Davide Gaiotto
Tudor Dimofte
Kevin Costello]

Thanks to Natalie Paquette for the snazzy new title!!

Symplectic duality [BLPW] $\subseteq$ 3d $N=4$ mirror symmetry [IS]

[BDGH, BFN]

T 3d $N=4$ theory

Too complicated. Twist!

$\mathbb{Z}_B$-graded

T_A — T_{HT} — T_B

topological

Holomorphic topological

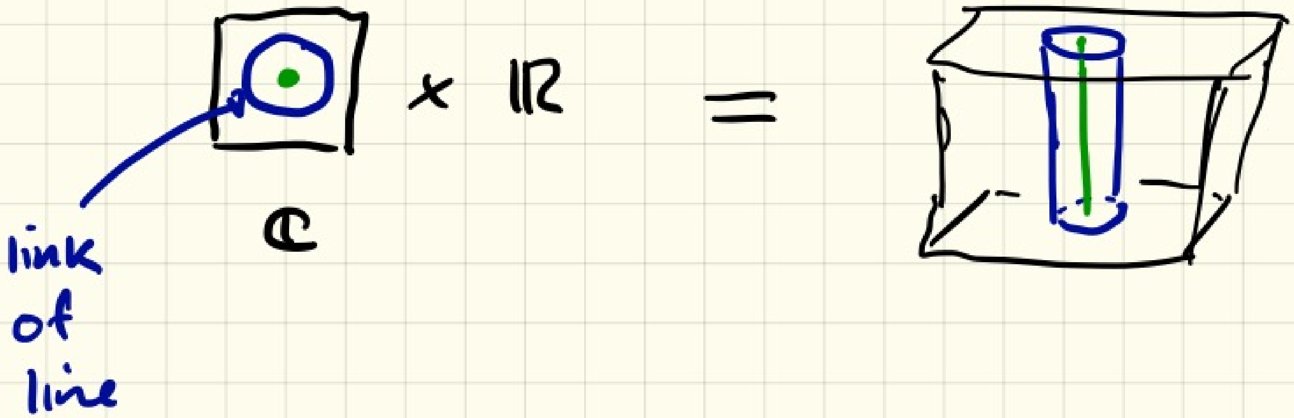
$[\Sigma \times M']$

topological

$\mathbb{Z}_A$-graded

$\mathbb{Z}/2\mathbb{Z}$ grading can be lifted to distinct $\mathbb{Z}_A$, $\mathbb{Z}_B$ gradings

In this talk we will consider the categories of line operators in the HT, A, B twists.



Line operator $\Longleftrightarrow$ boundary condition on link S^1

Since our S^1 is in a holomorphic direction we should use $D^* = \mathrm{Spec}\ \mathbb{C}((t))$ instead.

If $\mathcal{T}$ is of cotangent type ($T^*[2/0]X$)
and we put it on D^+ we get

$\mathbb{Z}_B / \mathbb{Z}_A$

HT] $\mathrm{Maps}(D^+_{Dol}, T^*[2/0]X)$

$\mathbb{Z}_A$ $T^*[1]\,\mathrm{Maps}(D^+, X)_{Dol}$

$\mathbb{Z}_B$ $T^*[1]\,\mathrm{Maps}(D^+_{Dol}, X)$

A] $T^*[1]\,\mathrm{Maps}(D^+, X)_{dR}$

B] $T^*[1]\,\mathrm{Map}(D^+_{dR}, X)$

semiclassically.

more generally can try to use any 2/0-shifted symplectic stack.

"Lagrangian"

Need additional data to quantize

Doesn't always exist.

There is a burgeoning theory of geometric quantization for shifted symplectic stacks:

$$n\text{-shifted} \rightsquigarrow n\text{-category}$$

Applying it gives our categories of line ops

HT] $\mathbb{Z}_A$ $QC(\mathrm{Maps}(D_+, X)_{Dol})$

$\mathbb{Z}_B$ $QC(\mathrm{Maps}(D^{+}_{Dol}, X))$

$\mathbb{Z}/2\mathbb{Z}$ graded Fourier transform should relate choice of polarizations

A] $QC(\mathrm{Maps}(D^+, X)_{dR}) = D\text{-mod}(\mathrm{Maps}(D^+, X))$

B] $QC(\mathrm{Maps}(D^+_{dR}, X))$

3d Mirror Symmetry

If $\mathcal{T}$ and $\mathcal{T}^!$ are 3d mirror we have

$$
\begin{array}{ccccc}
 & & \mathcal{T} & & \\
 & & \wr & & \\
\mathcal{T}_B & \sim & \mathcal{T}_{HT} & \sim & \mathcal{T}_A \\
\cong & & \cong & & \cong \\
\mathcal{T}^!_A & \sim & \mathcal{T}^!_{HT} & \sim & \mathcal{T}^!_B \\
 & & \wr & & \\
 & & \mathcal{T}^! & &
\end{array}
$$

These isomorphisms of theories induce isomorphisms between categories of line operators.

Ex

$$\mathcal{T} = \begin{tikzcd} \boxed{4} \arrow[d, no head] \\ \textcircled{2} \end{tikzcd} = T^*[2/0]\underbrace{\left[\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^4)/GL(2)\right]}_{X}$$

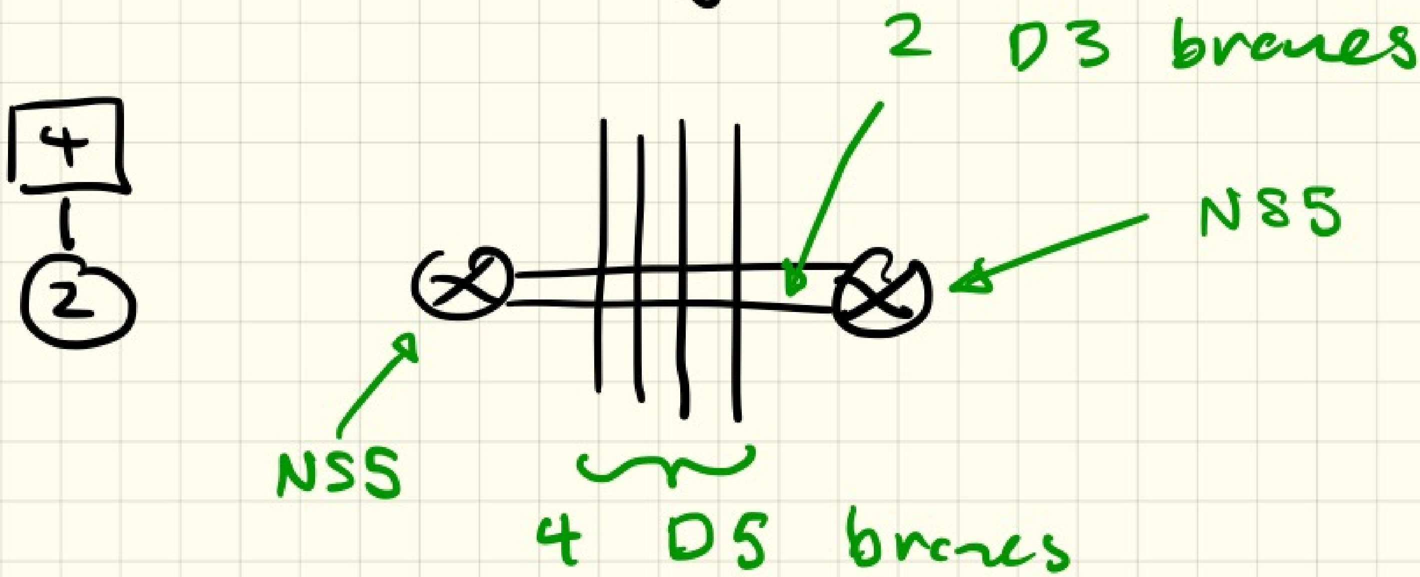
$$\mathcal{T}^! = \begin{tikzcd} & \boxed{2} \arrow[d, no head] & \\ \textcircled{1} \arrow[r, no head] & \textcircled{2} \arrow[r, no head] & \textcircled{1} \end{tikzcd} = T^*[2/0]\, X^!$$

$$QC(\mathrm{Maps}(D^*_{dR}, X)) = D\text{-mod}(\mathrm{Maps}(D^*, X^!))$$

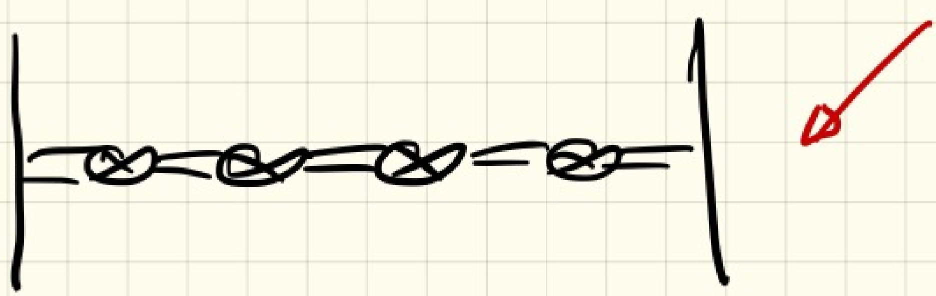
$$D\text{-mod}(\mathrm{Maps}(D^*, X)) = QC(\mathrm{Maps}(D^*_{dR}, X^!))$$

Q: How does one prove a theorem like this???

A: Turn your quiver into branes!
(in IIB string theory)



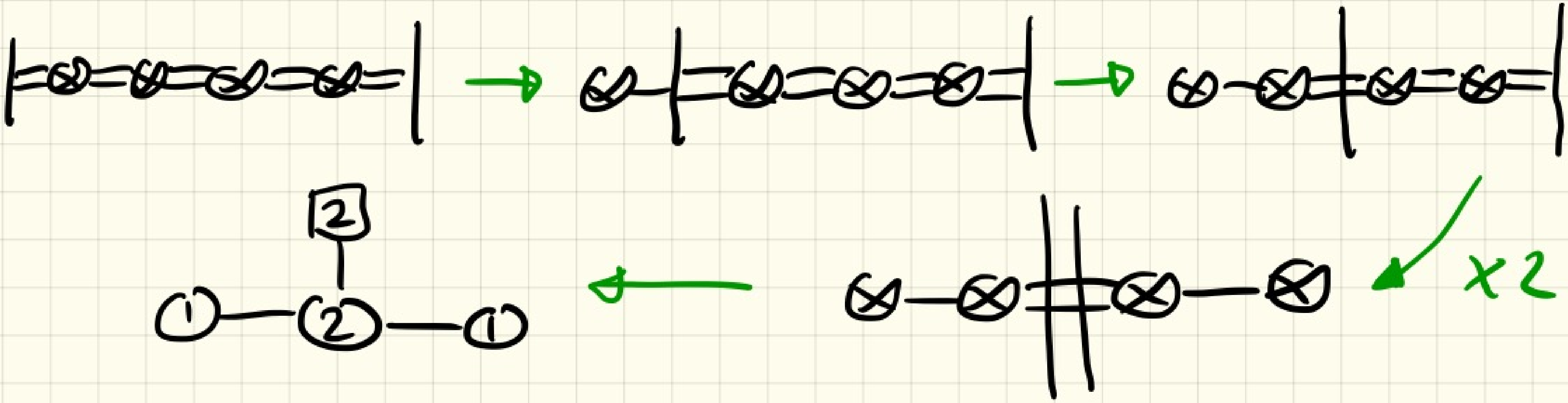
Then apply S-duality (NS5 $\leftrightarrow$ D5)
to get mirror



Not quiver. Not even cotangent.

Is twisted cotangent so can make sense of it directly. But I won't.

Instead use Hanany-Witten moves to rewrite mirror as quiver:



Q: How does one make sense of this mathematically??

A: Use local Langlands!

n

n D3 branes carry

4d $N=4$ Gl_n - S.Y.M.

Has HT, A, B twists. A slight extension of results of Elliott - Yoo shows that on D^* G - S.Y.M. gives:

A] $T^*[2]\,\mathrm{Bun}_G(D^*)_{dR}$

$$\Big\downarrow$$

HT] $T^*[2]\,\mathrm{Bun}_G(D^*)_{Dol}$ $\sim$ $T^*[2]\,\mathrm{Higgs}_G(D^*)$

$\mathbb{Z}_A$ $\qquad$ $\mathbb{Z}_B$

$$\Big\uparrow$$

B] $T^*[2]\,\mathrm{Flat}_G(D^*)$

After geometric quantization get:

$$\mathrm{Shv}_{G,A} := \mathrm{ShvCat}/\mathrm{Bun}_G(D^\times)_{dR}$$

$$\simeq D(G_K)\text{-mod}$$

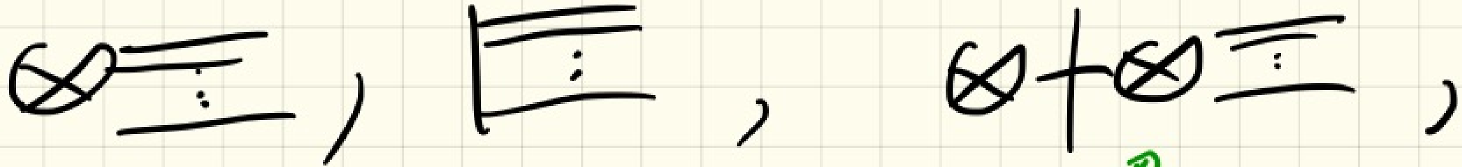
$$\mathrm{Shv}_{G,HT} := \mathrm{ShvCat}/\mathrm{Higgs}_G(D^\times)$$

$$\mathrm{Shv}_{G,B} := \mathrm{ShvCat}/\mathrm{Flat}_G(D^\times)$$

$$\cup\!\!\mid$$

$$QC(\mathrm{Flat}_G(D^\times))\text{-mod}$$

The link of a surface in 4d is an S^1 $(D^\times)$. Should think of these as surface defects.



, etc

$GL_n \circlearrowright \boxed{n} - (1) - \boxed{1}$

are boundary conditions for 4d $N=4$ GL_n - S.Y.M.

More generally, $G \circlearrowright^0 T_{3d}$, $N=4$ gives a b.c. $B[T_{3d}]$ for 4d $N=4$ G - S.Y.M with gauge group G.

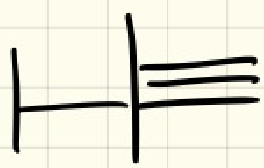
A b.c. B is Lagrangian (in the physics sense) if it can be given by Lagrangian (in the math sense)

$$L_B \longrightarrow T^*[3/1]BG$$

Ex $G \circlearrowright T^*[2/0]X \longrightarrow \mathfrak{g}^*[2/0]$ Hamiltonian

$$\Longleftrightarrow$$

$$\underbrace{\left(T^*[2/0]X\right)/G}_{T^*_{X/G}[3/1]BG} \longrightarrow \mathfrak{g}^*[2/0]/G = T^*[3/1]BG$$

, , etc are all Lagrangian.

canonical

twisted canonical

Use AKSZ to get A, B, HT Lagrangians

B]

$$\overbrace{\mathrm{Maps}(D^+_{dR}, \mathcal{L}_B)}^{\mathcal{L}_{B,B}} \longrightarrow T^*[2]\,\mathrm{Flat}_G(D^+)$$

$$\left[T^*_{\mathrm{Maps}(D^+_{dR}, X/G)}[2]\,\mathrm{Loc}_G(D^+) \right]$$

HT]

$$\overbrace{\mathrm{Maps}(D^+_{Dol}, \mathcal{L}_B)}^{\mathcal{L}_{B,HT}} \longrightarrow T^*[2]\,\mathrm{Higgs}_G(D^*)$$

$$\Big\updownarrow \ \mathbb{Z}/2\mathbb{Z}$$

difficult to write down $\longrightarrow T^*[2]\,\mathrm{Bun}_G(D^*)_{Dol}$

$$\left[T^*_{\mathrm{Maps}(D^+, X/G)_{Dol}}[2]\,\mathrm{Bun}_G(D^*)_{Dol} \right]$$

A]

$$\mathcal{L}_{B,A} \longrightarrow T^*[2]\, \mathrm{Bun}_G(D^\times)_{dR}$$

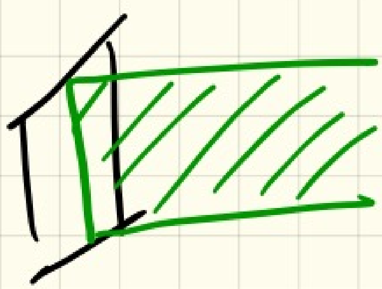
deform from HT

$$\left[\; T^*_{\mathrm{Maps}(D^\times, V(G))_{dR}}[2]\, \mathrm{Bun}_G(D^\times)_{dR} \;\right]$$

can apply geometric quantization to get sheafy version of 3d line operators

$$B[Y]_{\_} \in \mathrm{Shv}_{G,\_}$$

<u>Moral</u> There is a relation between surfaces in bulk and lines on ∂



<u>Note</u> Geometric quantization in the shifted setting is recursive.

$$L \xrightarrow{i} T^*[2]X \longrightarrow X$$

To compute the stalk of $GQ(L) \in \mathrm{ShvCat}/X$ above $p \in X$:

1) $T^*_p[2]X \hookrightarrow T^*[2]X$ is Lagrangian

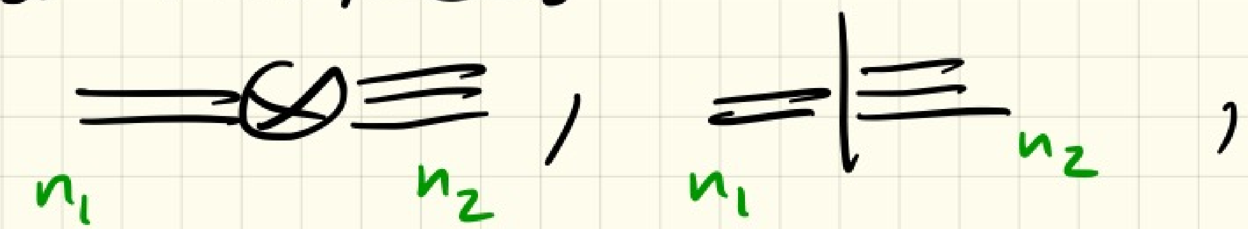
2) $T_p[2]X \cap L$ is 1-shifted symplectic

Thus we expect

$$GQ(L)_p = GQ(\underbrace{T^*_p[2]X \cap L}_{1\text{-category}})$$

Safronov – Can actually assemble these into a sheaf.

For interfaces



, etc

get Lagrangian correspondences

$$T^*[3]BG_1 \longleftarrow L \longrightarrow T^*[3]BG_2$$

and, after AKSZ and quantization, functors

$$\mathrm{ShvCat}_{G_1,-} \xrightarrow{L_-} \mathrm{ShvCat}_{G_2,-}$$

Consider Lagrangian correspondences

$$\begin{tikzcd}[column sep=small]
 & & L_1 \circ L_2 \arrow[dl] \arrow[dr] \arrow[d, phantom, "L"] & & \\
 & L_1 \arrow[dl] \arrow[dr] & \; & L_2 \arrow[dl] \arrow[dr] & \\
T^*[2]X & & T^*[2]Y & & T^*[2]Z
\end{tikzcd}$$

Need pre-quantum data and relative polarizations

<u>Conj</u>

$$\begin{tikzcd}
\mathrm{ShvCat}/_X \arrow[r, "(L_1)"] \arrow[rr, bend right, "(L_1 \circ L_2)"'] & \mathrm{ShvCat}/_Y \arrow[r, "(L_2)"] \arrow[d, phantom, "\circlearrowright"] & \mathrm{ShvCat}/_Z \\
 & \; &
\end{tikzcd}$$

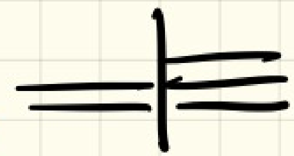
<u>Safronov</u> Holds for twisted conormals.

$\Longrightarrow$

<u>Cor</u> Hanany-Witten relations hold.

S-duality

D5



A

B

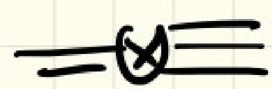
$$Shv_{GL_2,A} \xrightarrow{(D5)_A} Shv_{GL_3,A} \qquad Shv_{GL_2,B} \xrightarrow{(D5)_B} Shv_{GL_3,B}$$

Local Langlands

$$Shv_{GL_2,B} \xrightarrow{(NS5)_B} Shv_{GL_2,B} \qquad Shv_{GL_2,A} \xrightarrow{(NS5)_A} Shv_{GL_3,A}$$

B

A



NS5

Def $G \circlearrowright \mathcal{T}_{3d}$ and $G^L \circlearrowright \mathcal{T}^\vee_{3d}$ are S-dual if

depends on action!!!

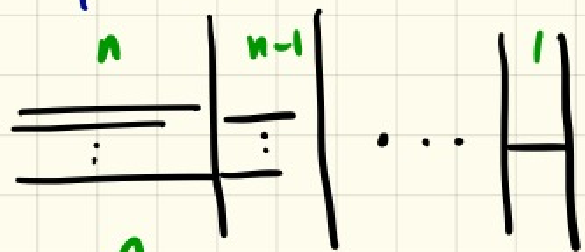
$$\begin{array}{ccc} Shv_{G,A} & \cong & Shv_{G^L,B} \\ \Downarrow & & \Downarrow \\ \mathcal{B}[\mathcal{T}]_{,A} & = & \mathcal{B}[\mathcal{T}^\vee]_{,B} \end{array}$$

$$\begin{array}{ccc} Shv_{G,B} & \cong & Shv_{G^L,A} \\ \Downarrow & & \Downarrow \\ \mathcal{B}[\mathcal{T}]_{,B} & = & \mathcal{B}[\mathcal{T}^\vee]_{,A} \end{array}$$

Can write symplectic duality in terms of S-duality and kernel $T[G]$

Ex $GL_n \circlearrowright \mathcal{T}$ and $GL_n \circlearrowright \mathcal{T}^\vee$ are S-dual

Forgets symmetry

$\mathcal{T} = B[\mathcal{T}]$ 

Dirichlet $D[GL_n]$

S-duality

S-dual of $\mathcal{T}$ with trivial symmetry

$\mathcal{T}^! = B[\mathcal{T}^\vee]$ 

$$B\big[T[GL_n]\big] := B\big[\,\boxed{n} - (n-1) - (n-2) - \cdots - (1)\,\big]$$

For all groups $D[G]$ is Lagrangian but in general $T[G^L] \circlearrowright G^L$ is not.

Braverman has some ideas about this

Horizons:

0) S-dual of Lagrangian is not always Lagrangian.

1) Same techniques apply for global curve
[Ginzburg - Rozenblyum A-twist]

2) Same techniques apply for $\mathbb{B} = D \cup_{D^*} D$ get relation between local ops in 3d and lines in 4d
[BFN - Ring objects A-twist]

3) Analogous relation between local ops [quantum cohomology] in 2d $N=(2,2)$ theories and lines in 3d $N=4$. Teleman, Arkhipov - Kapranov

4) Interfaces between 3d $N=4$ theories can be handled just like in 4d.

5) Can combine talk + 3) + 4) into A, B, HT $(\infty, 3)$ - functors from a suitable category of iterated Lagrangian correspondences.