Title: The Duistermaatâ€“Heckman distribution for the based loop group

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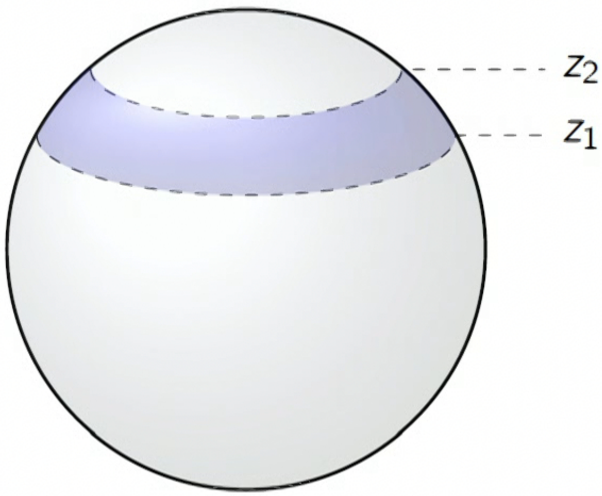
Abstract: The based loop group is an inï¬•nite-dimensional manifold equipped with a Hamiltonian action of a ï¬•nite dimensional torus. This was studied by Atiyah and Pressley. We investigate the Duistermaatâ€“Heckman distribution using the theory of hyperfunctions. In applications involving Hamiltonian actions on inï¬•nite-dimensional manifolds, this theory is necessary to accommodate the existence of the inï¬•nite order diï¬€erential operators which aries from the isotropy representation on the tangent spaces to ï¬•xed points. (Joint work with James Mracek)

Main goals for today:

- What is localization?
- What is a hyperfunction?

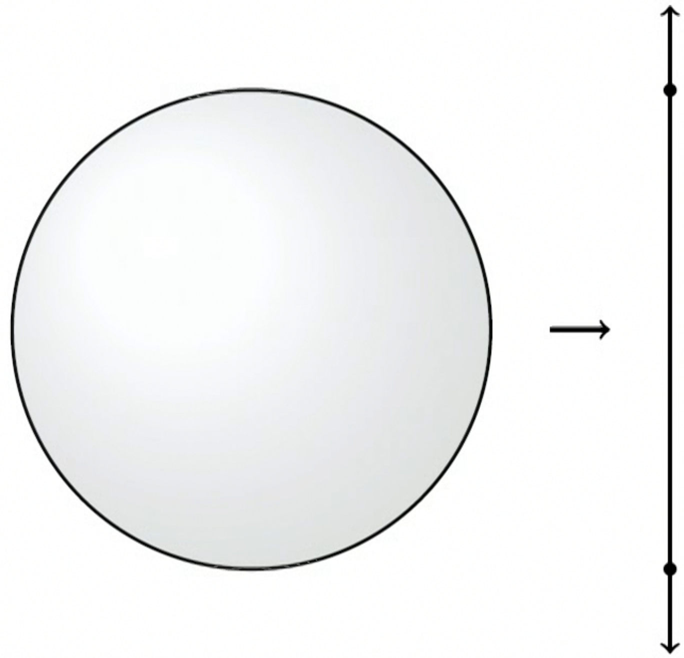
Archimedes' Hat-Box theorem

The surface area on a unit 2-sphere enclosed by two latitudes is proportional only to the difference in height between the latitudes.

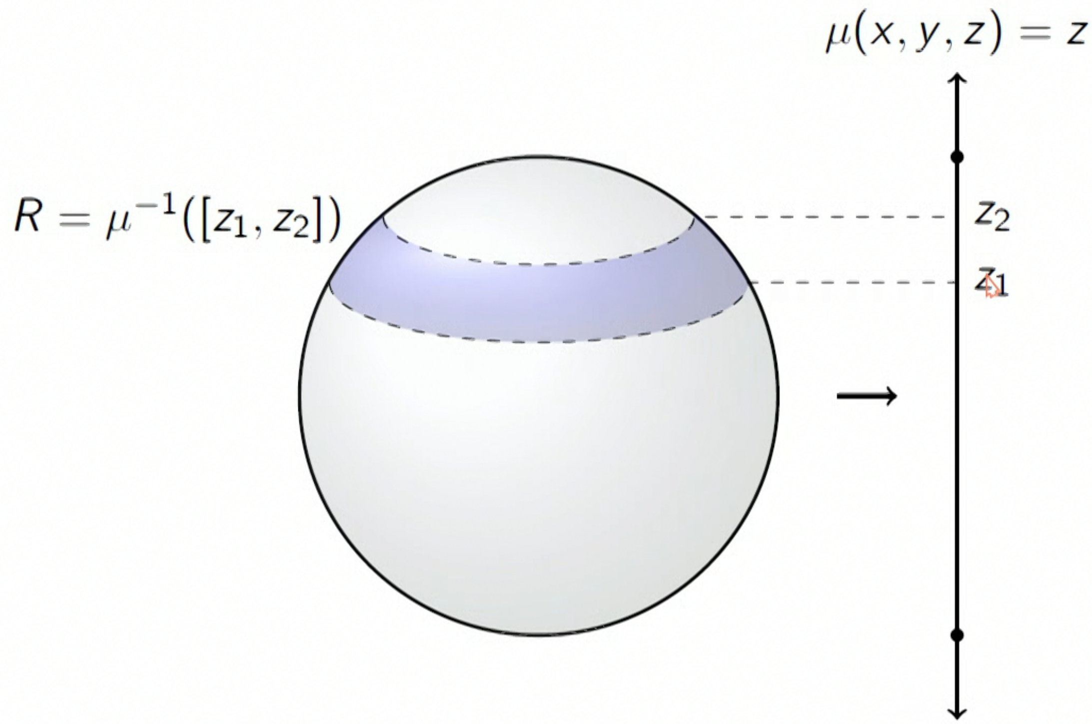


$$\omega = dz \wedge d\theta$$

$$\begin{aligned}\text{Area}(R) &= \int_R \omega \\ &= \int_0^{2\pi} \int_{z_1}^{z_2} dz \wedge d\theta \\ &= 2\pi(z_1 - z_2)\end{aligned}$$



$\mu(x, y, z) = z$



Define a measure on $\mathbb{R}$:

$$\mu_*(\omega)(S) = \int_{\mu^{-1}(S)} \omega$$

Related to Lebesgue measure; How?

$$\frac{d\mu_*(\omega)}{dm} = \lim_{\epsilon\to 0} \frac{\mu_*(\omega)(B_\epsilon(x))}{m(B_\epsilon(x))}$$
$$= \chi_{\mu(S^2)}(x)$$

Fourier transform of $\mu_*(\omega)$?

$$\int_{\mathbb{R}} e^{i\xi z} d\mu_*(\omega) = \int_{\mathbb{R}} e^{i\xi z} \frac{d\mu_*(\omega)}{dm} dm(z)$$
$$= \int_{-1}^{1} e^{i\xi z}\, dz$$
$$= \frac{e^{i\xi}}{\xi} + \frac{e^{-i\xi}}{(-\xi)}$$

Geometry?

$$\mu : S^2 \to \mathbb{R}$$
$$(x, y, z) \mapsto z$$

$$\omega(\partial_\theta, -) = -dz = -d\mu$$

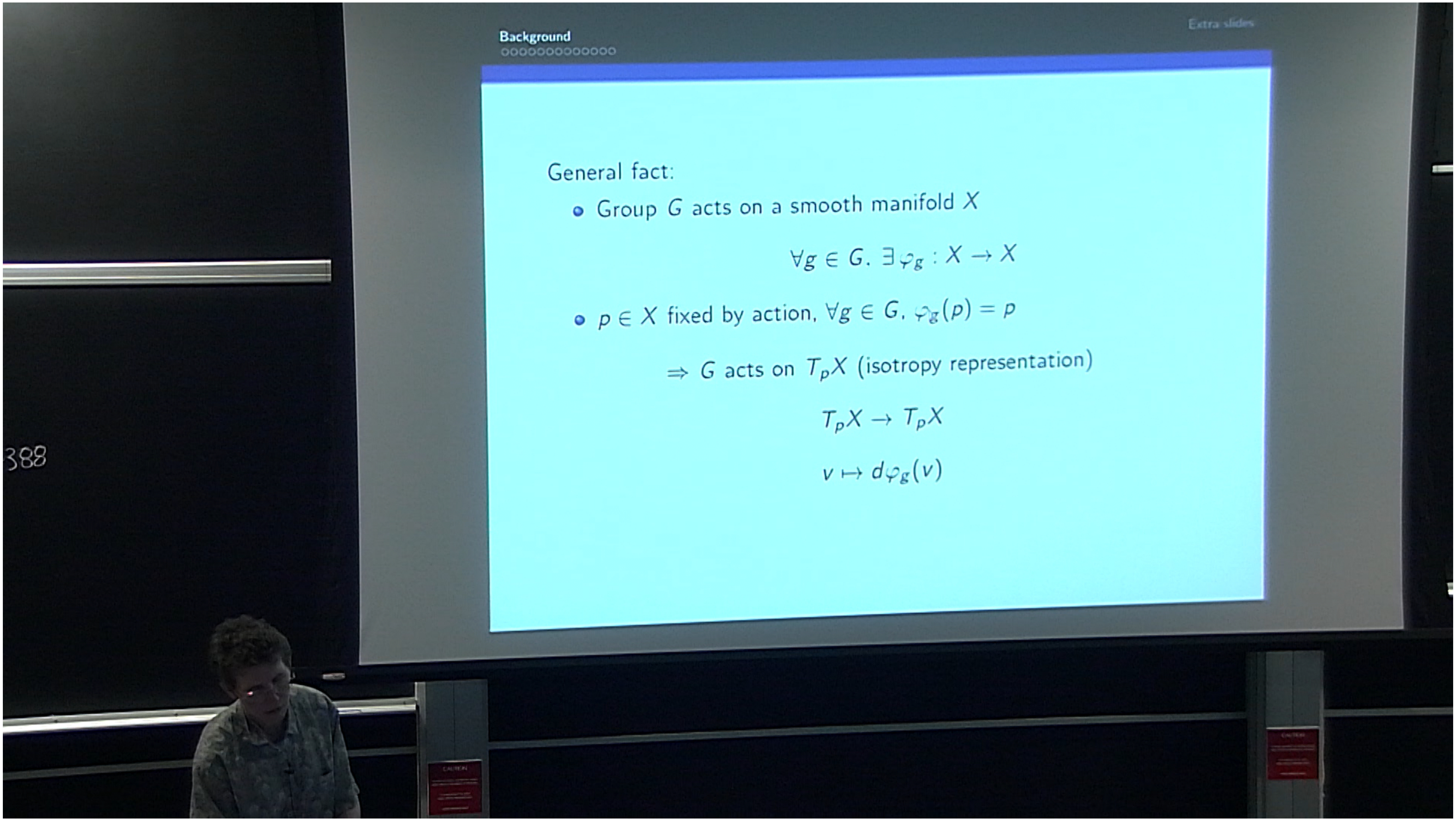
- Vector field ∂_θ generates rotations around the z-axis
- Get a circle action on S^2.

General fact:

- Group G acts on a smooth manifold X

$$\forall g \in G,\ \exists\, \varphi_g : X \to X$$

- $p \in X$ fixed by action, $\forall g \in G,\ \varphi_g(p) = p$


Background
Extra slides
General fact:
Group G acts on a smooth manifold X
$\forall g \in G, \exists \varphi_g : X \to X$
$p \in X$ fixed by action, $\forall g \in G, \varphi_g(p) = p$
$\Rightarrow G$ acts on T_pX (isotropy representation)
$T_pX \to T_pX$
$v \mapsto d\varphi_g(v)$
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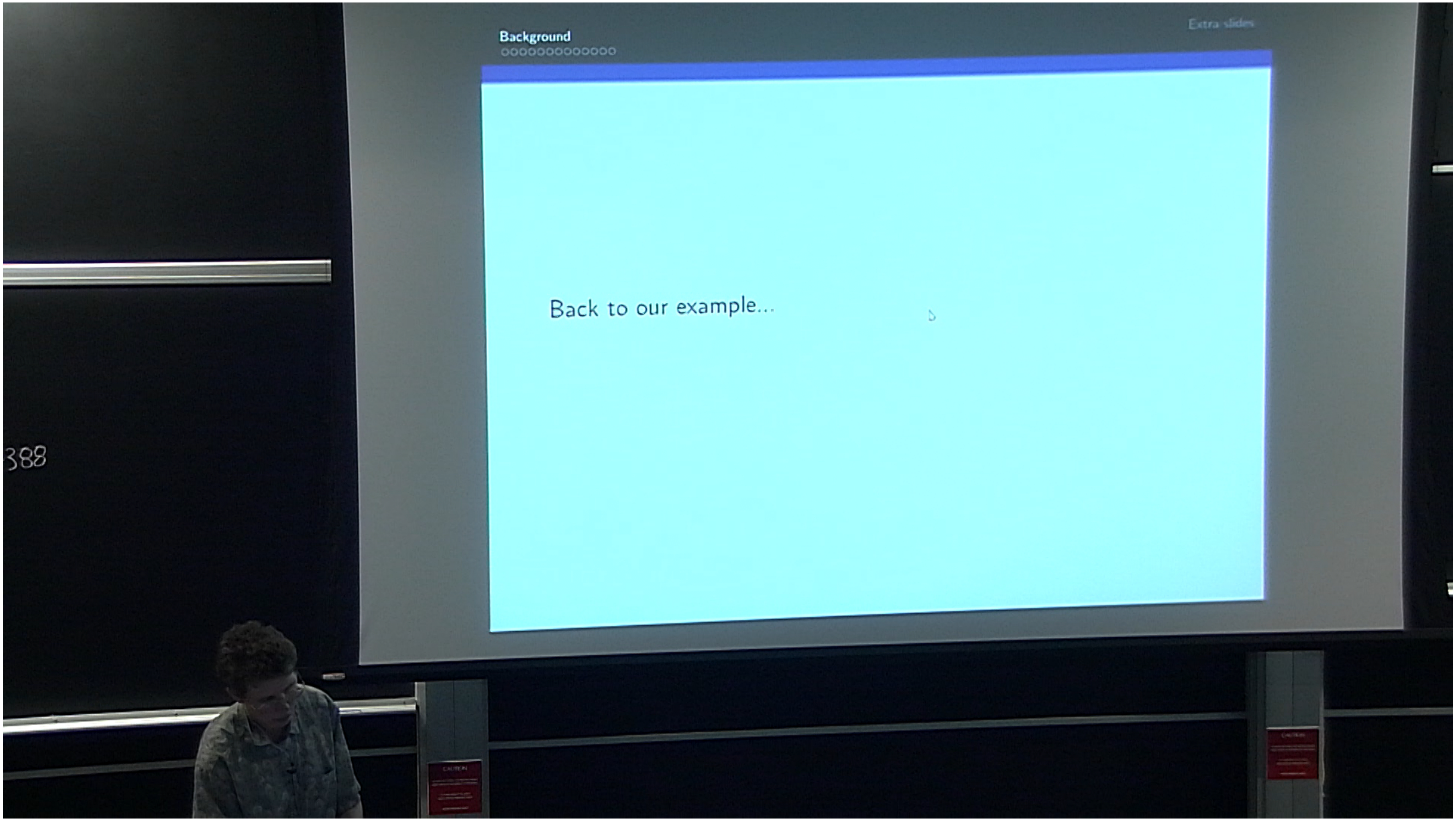
Background
Extra slides
Back to our example...
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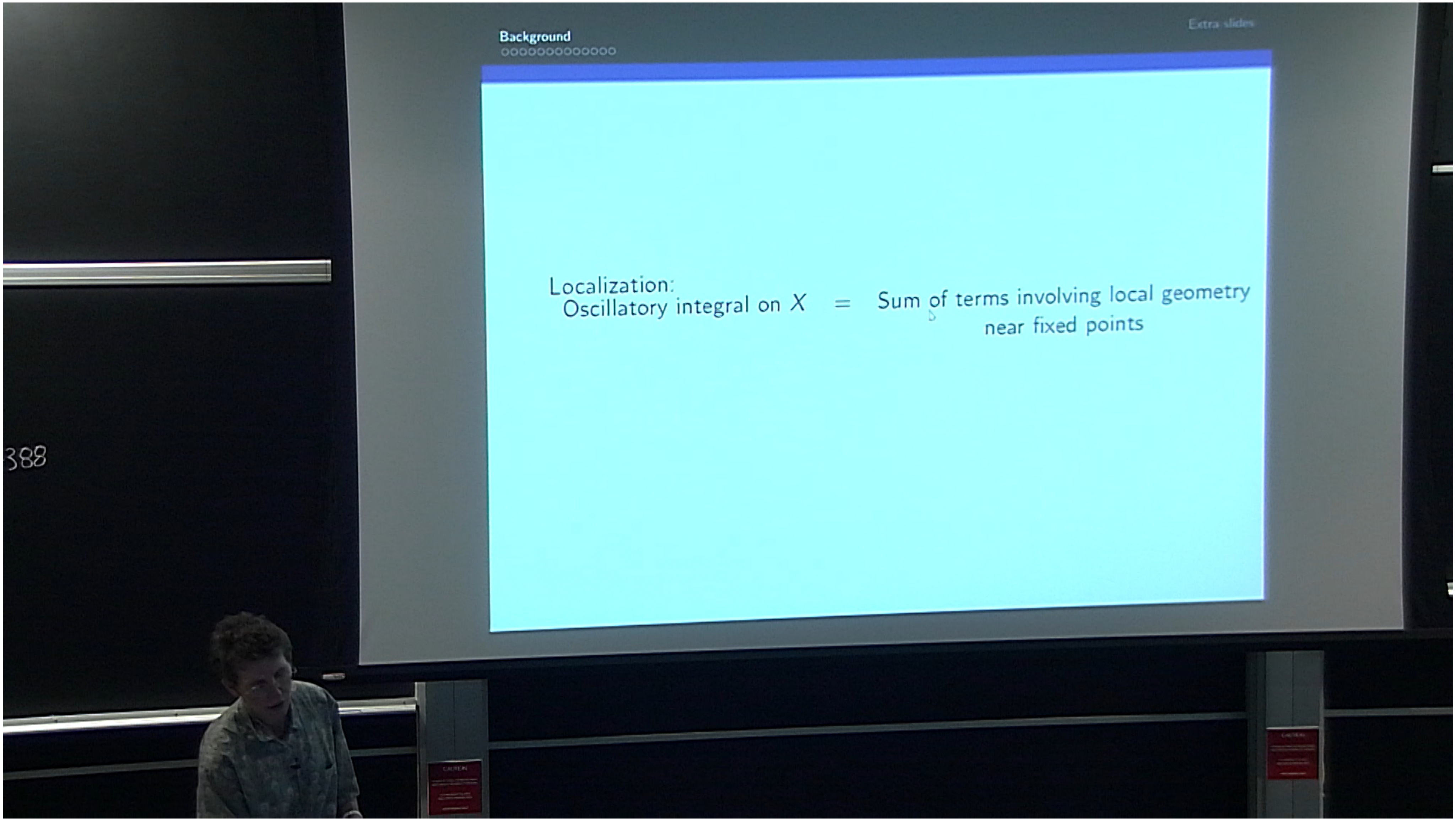
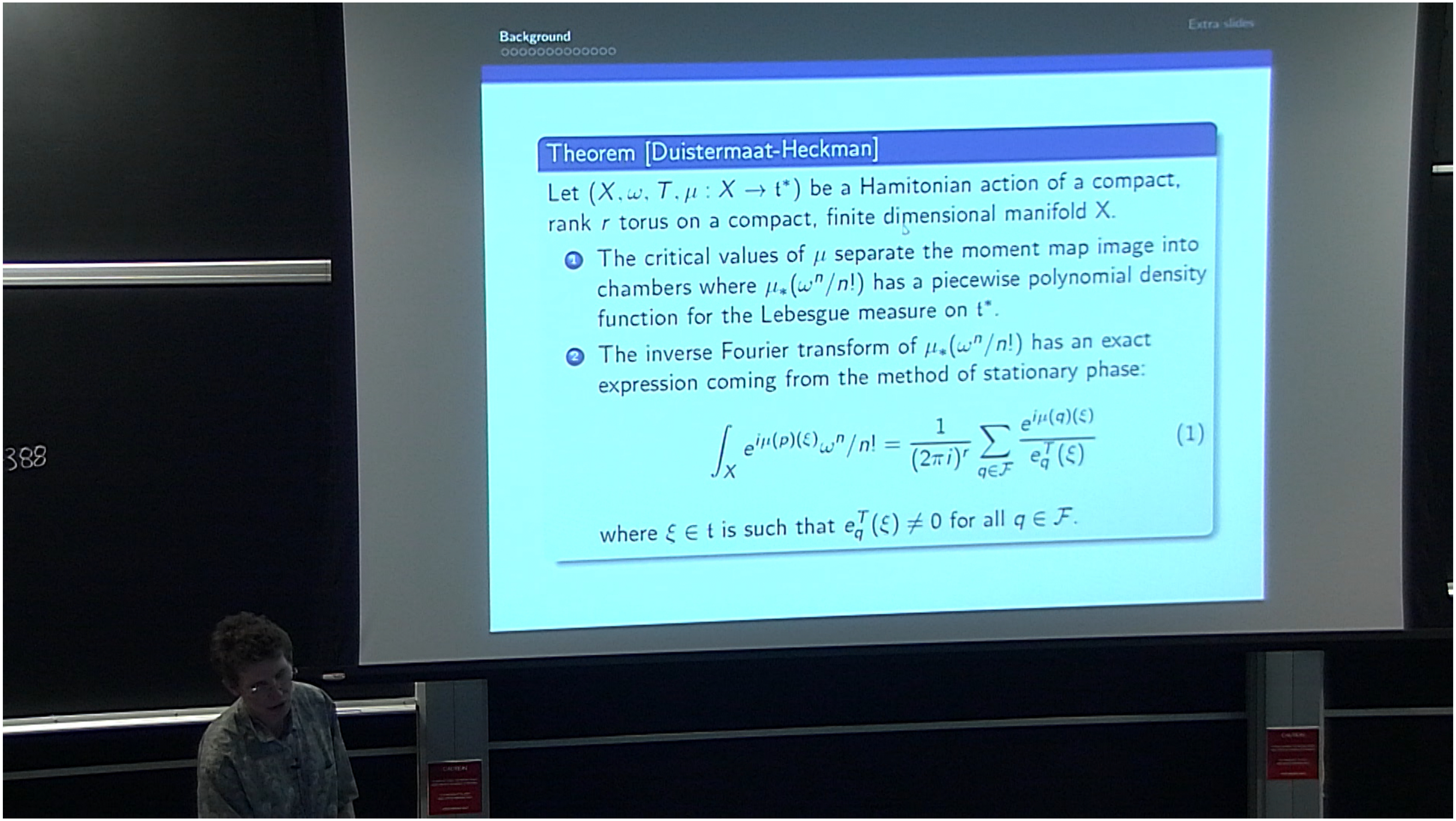

Background
Extra slides
Localization:
Oscillatory integral on X = Sum of terms involving local geometry near fixed points


Background
Extra slides
Theorem [Duistermaat-Heckman]
Let $(X, \omega, T, \mu : X \to \mathfrak{t}^*)$ be a Hamitonian action of a compact, rank r torus on a compact, finite dimensional manifold X.
1. The critical values of μ separate the moment map image into chambers where $\mu_*(\omega^n/n!)$ has a piecewise polynomial density function for the Lebesgue measure on $\mathfrak{t}^*$.
2. The inverse Fourier transform of $\mu_*(\omega^n/n!)$ has an exact expression coming from the method of stationary phase:
$$\int_X e^{i\mu(p)(\xi)}\omega^n/n! = \frac{1}{(2\pi i)^r}\sum_{q\in\mathcal{F}}\frac{e^{i\mu(q)(\xi)}}{e_q^T(\xi)} \quad (1)$$
where $\xi \in \mathfrak{t}$ is such that $e_q^T(\xi) \neq 0$ for all $q \in \mathcal{F}$.
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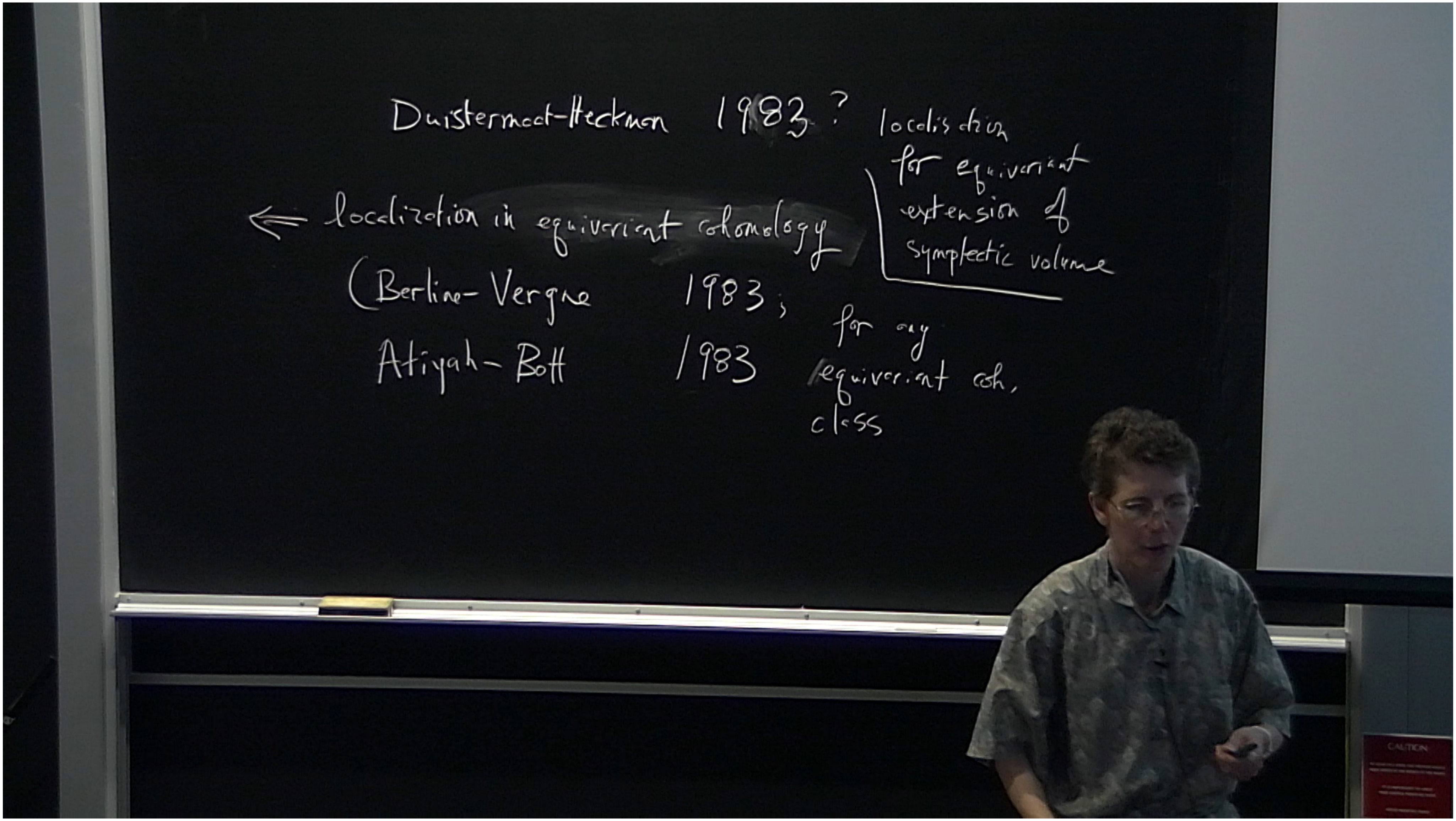
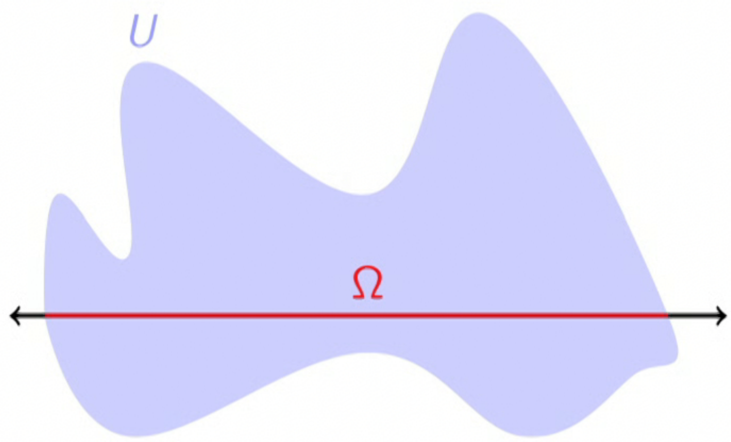

Duistermaat-Heckman 1983 ?
localisation for equivariant extension of symplectic volume
⇐ localization in equivariant cohomology
(Berline-Vergne 1983;
Atiyah-Bott 1983
for any equivariant coh. class

Definition

- $\Omega \subseteq \mathbb{R}$
- Choose $U \subseteq \mathbb{C}$ such that $U \cap \mathbb{R} = \Omega$.

A hyperfunction on $\Omega \subseteq \mathbb{R}$ is an element of the vector space:

$$f \in \frac{\mathcal{O}(U \backslash \Omega)}{\mathcal{O}(U)}$$



Background on the based loop group

- G Lie group
- $\mathfrak{g}$ Lie algebra
- R roots.
- $T \subseteq G$ maximal torus, $\mathfrak{t}$ its Lie algebra

$$\Omega G = \{\gamma : S^1 \to G \,|\, \gamma \in C^\infty(S^1, G), \gamma(1) = e\}$$

$$T_\gamma \Omega G \simeq \Omega \mathfrak{g}$$

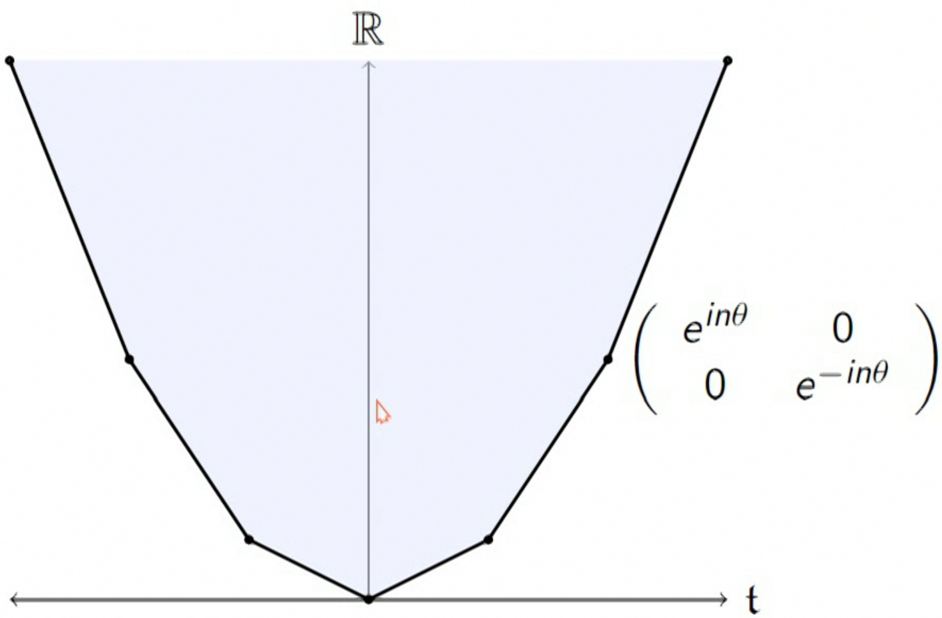
- ΩG is an infinite-dimensional Kähler manifold
  - $\omega_e(X, Y) = \int_0^{2\pi} \langle X, Y' \rangle \, d\theta$
- $T \times S^1$ acts on ΩG
  - $t \in T$, $\gamma \in \Omega G$ then $(t \cdot \gamma)(\theta) = t\gamma(\theta)t^{-1}$
  - $\psi \in S^1$, $\gamma \in \Omega G$ then $(\psi \cdot \gamma)(\theta) = \gamma(\theta + \psi)\gamma(\psi)^{-1}$

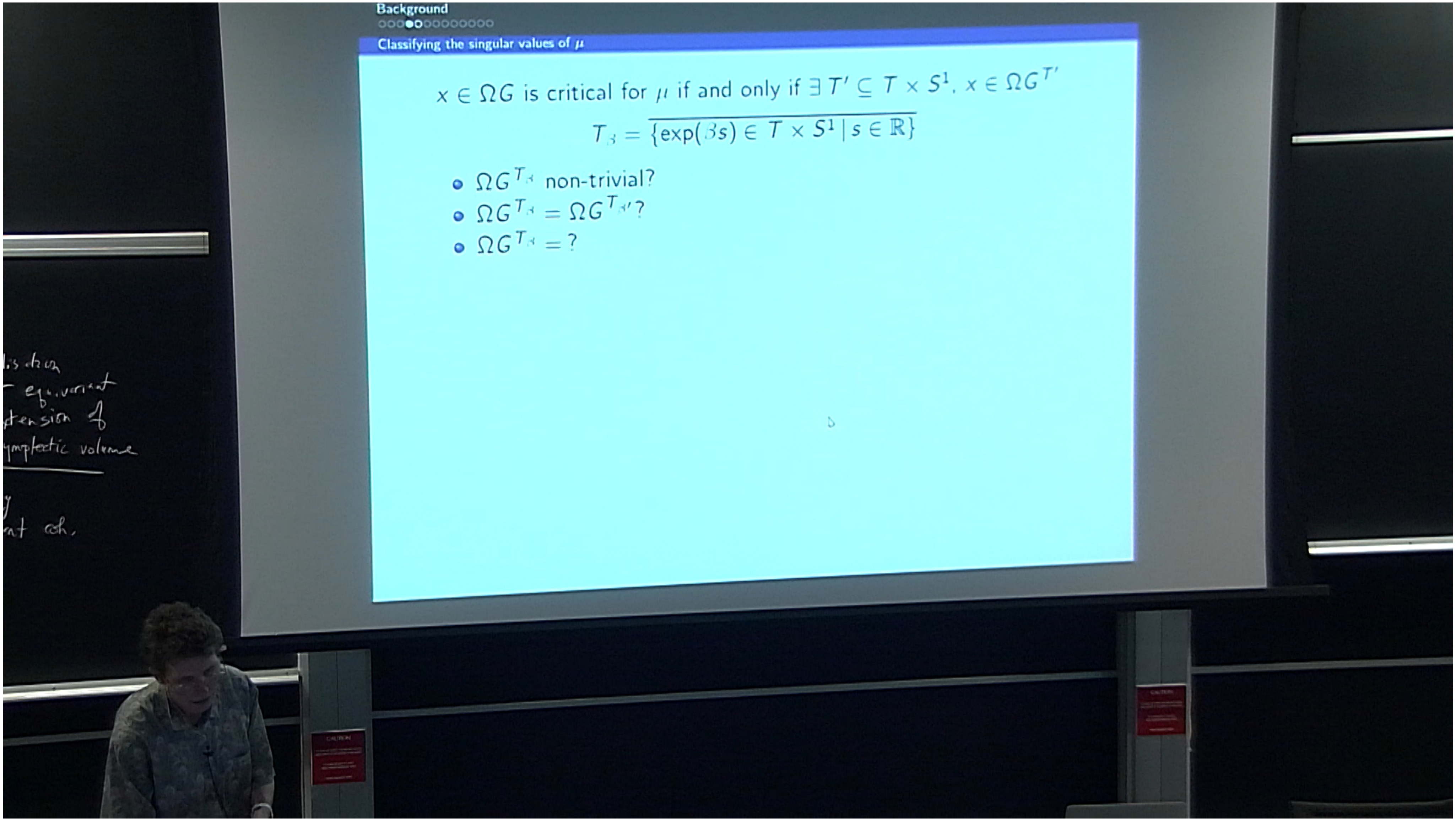
Proposition [Atiyah-Pressley]

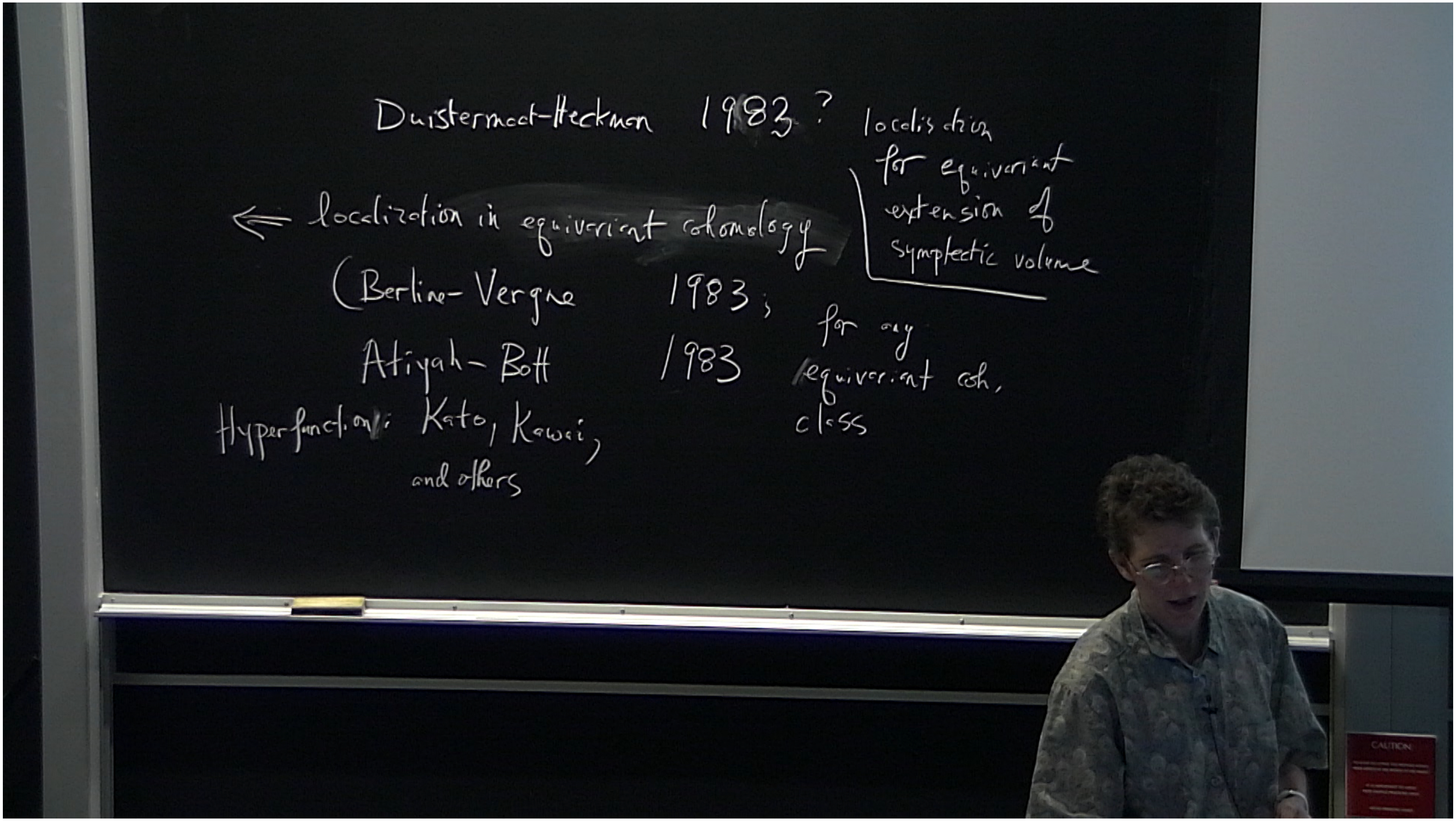
- $T \times S^1$ action on ΩG is Hamiltonian.
- Moment map $\mu : \Omega G \to \mathfrak{t} \oplus \mathbb{R}$

Theorem [Atiyah-Pressley]

- $\mathrm{Fix}_{T\times S^1}(\Omega G) = \mathrm{Hom}(S^1, T)$
- The image of $\mu : \Omega G \to \mathfrak{t} \oplus \mathbb{R}$ is the convex hull of the image of the fixed point set.




Background
Classifying the singular values of μ
$x \in \Omega G$ is critical for μ if and only if $\exists\, T' \subseteq T \times S^1$, $x \in \Omega G^{T'}$
$T_\beta = \overline{\{\exp(\beta s) \in T \times S^1 \mid s \in \mathbb{R}\}}$
ΩG^{T_β} non-trivial?
$\Omega G^{T_\beta} = \Omega G^{T_{\beta'}}$?
$\Omega G^{T_\beta} = ?$


Duistermaat-Heckman 1983 ?
localisation for equivariant extension of symplectic volume
⟸ localization in equivariant cohomology
(Berline-Vergne 1983;
Atiyah-Bott 1983
for any equivariant coh. class
Hyperfunctions: Kato, Kawai, and others

Our goal is to understand how this theorem generalizes to the case where X is infinite dimensional. Two sub-problems:

1. Classify the singular values of μ
2. Make sense of the localization formula

Classifying the singular values of μ

$x \in \Omega G$ is critical for μ if and only if $\exists\, T' \subseteq T \times S^1$, $x \in \Omega G^{T'}$

$$T_\beta = \overline{\{\exp(\beta s) \in T \times S^1 \mid s \in \mathbb{R}\}}$$

- ΩG^{T_β} non-trivial?
- $\Omega G^{T_\beta} = \Omega G^{T_{\beta'}}$?
- $\Omega G^{T_\beta} = ?$

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- $\Omega G^{T_\beta} = ?$

Theorem [J.-M.]

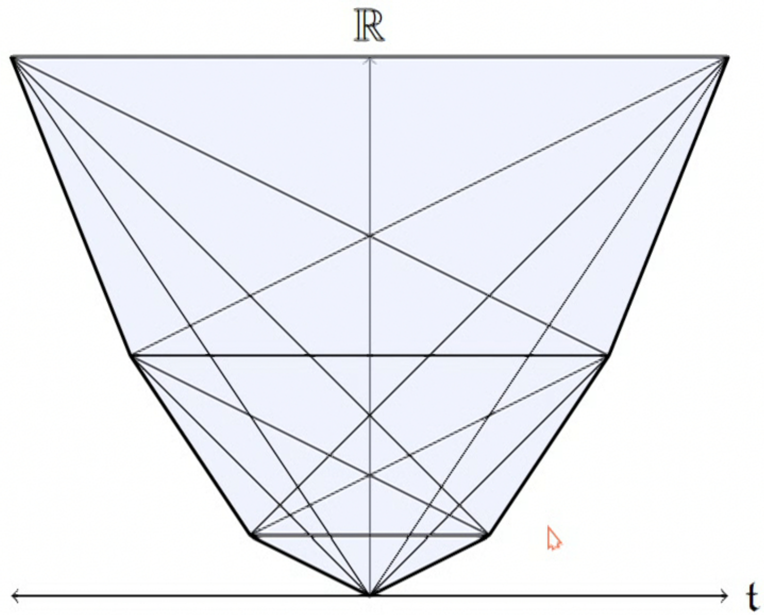
(a) For any cocharacter $\beta = (\lambda, m) \in X_*(T \times S^1) \subseteq \mathrm{Lie}(T \times S^1)$, there exists L_β, $T \subseteq L_\beta \subseteq G$, such that $\gamma \in \Omega G^{T_\beta}$ if and only if $(\gamma(\theta), \theta)$ is a one parameter subgroup of $L_\beta \rtimes_\beta S^1$.

(b) Let $\beta = (\lambda, m)$ and $\beta' = (\lambda', m')$ generate rank one subgroups T_β, $T_{\beta'} \subseteq T \times S^1$, and let $L_\beta, L_{\beta'}$ be the Levi subgroups from (a). $\Omega G^{T_\beta} = \Omega G^{T_{\beta'}}$ if and only if $\lambda/m - \lambda'/m' \in \mathrm{Lie}(Z(L_\beta))$

(c) Every connected component of the fixed point set of T_β is a translate of an adjoint orbit in $\mathrm{Lie}(L_\beta) \subseteq \mathfrak{g}$.

Classifying the singular values of μ

Chamber structure on $\mu(\Omega SU(2))$

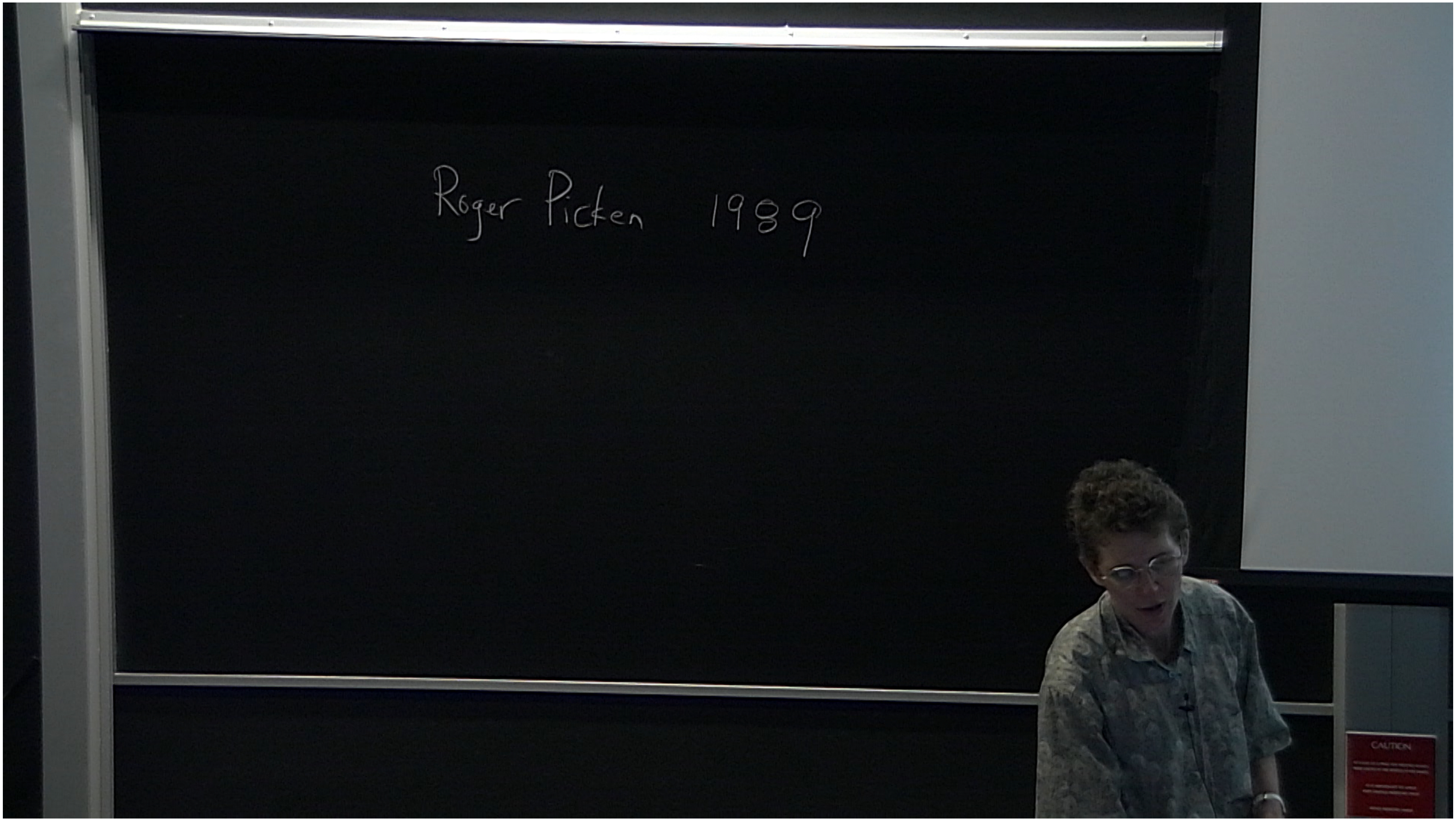


Localization formula

$$\int_{\Omega G} e^{i\mu(p)(x)}\omega^n/n! = \frac{1}{(2\pi i)^r}\sum_{q\in\mathcal{F}}\frac{e^{i\mu(q)(x)}}{e_q^T(x)}$$

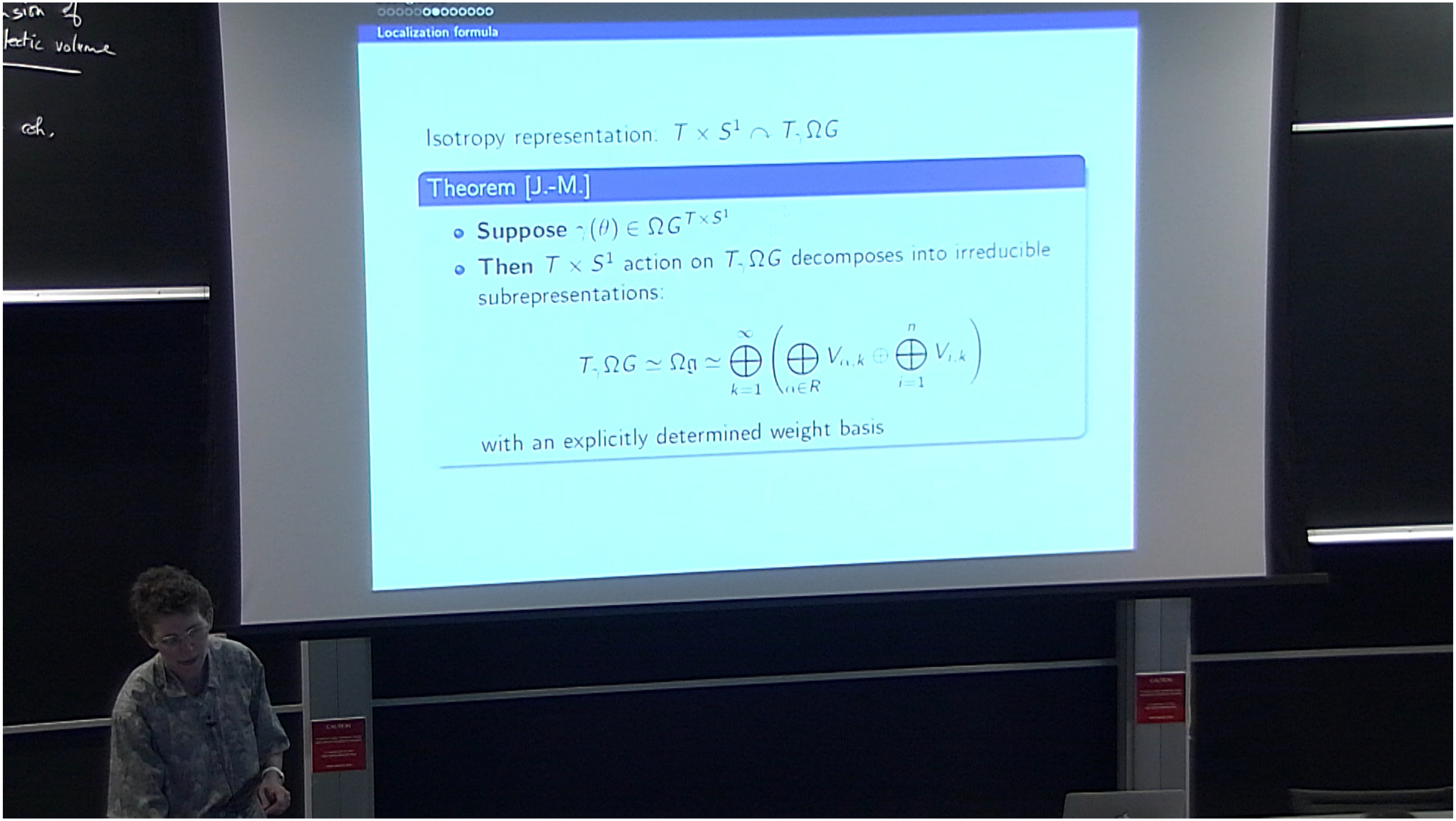
Problems:

- $\int_{\Omega G}$ not rigorous, but related to Feynman path integral
- $e_q^T(x)$ is an infinite product $\Rightarrow$ convergence issues
- Behaviour of $1/e_q^T(x)$ as a distribution is not evident


Roger Picken 1989

Localization formula

Isotropy representation: $T \times S^1 \curvearrowright T_\gamma \Omega G$


Localization formula
Isotropy representation: $T \times S^1 \curvearrowright T_\gamma \Omega G$
Theorem [J.-M.]
Suppose $\gamma(\theta) \in \Omega G^{T \times S^1}$
Then $T \times S^1$ action on $T_\gamma \Omega G$ decomposes into irreducible subrepresentations:
$T_\gamma \Omega G \simeq \Omega \mathfrak{g} \simeq \bigoplus_{k=1}^{\infty} \left(\bigoplus_{\alpha \in R} V_{\alpha,k} \oplus \bigoplus_{i=1}^{n} V_{i,k} \right)$
with an explicitly determined weight basis

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Localization formula

Regularized equivariant Euler class of the normal bundle to γ:

$$e_{\gamma}^{T\times S^1}(z_1, z_2) = \prod_{k=1}^{\infty}\left(\prod_{\alpha\in R}\frac{\lambda_{\alpha}^{k}(z_1, z_2)}{kz_2}\right)$$

Localization formula

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Proposition [J.-M.]

- **Suppose** $\gamma_n \in \Omega SU(2)^{T\times S^1}$
- **Then** regularized equivariant Euler class is:

$$e_{n}^{T\times S^1}(z_1,z_2)=\frac{\sin\left(2\pi z_1/z_2\right)}{2\pi(n+z_1/z_2)} \tag{2}$$

$$\eta = \text{Fourier transform}\left(1/e_n^{T\times S^1}\right)$$

Localization formula

Definition

A **hyperfunction** on $\Omega \subseteq \mathbb{R}^n$ is an element:

$$\sum_{i=1}^{n} b_{\gamma_i}(F_i(z)) \in \bigoplus_{\gamma \in \Gamma} \mathcal{O}(\Omega \times i\gamma 0)/ \sim$$

If $\gamma_1, \gamma_2, \gamma_3$ open convex cones, $\gamma_3 \subseteq \gamma_1 \cap \gamma_2$ and $F_i \in \mathcal{O}(\Omega \times i\gamma_i)$, then

$$F_1(z) + F_2(z) \sim F_3(z) \Leftrightarrow (F_1(z) + F_2(z))|_{\gamma_3} = F_3(z)$$

Localization formula

Local model:

- Suppose T has a Hamiltonian action on a complex vector space with weights λ_i.
- **Weights**: $W = \{\lambda_i : \mathfrak{t} \to \mathbb{R}\}_{i \in I}$.
  - **Half space**: $H_\lambda = \{y \in \mathfrak{t} \mid \lambda(y) < 0\}$,
  - **Hyperfunction**: $f_\lambda(x) = b_{H_\lambda}\left(\frac{1}{\lambda(z)}\right)$

Localization formula

Hyperfunction version of $1/e^T(x)$?

Hyperfunction version of $1/e^T(x)$?

Theorem [J.-M.]

- $\gamma = \bigcap_{\lambda \in W} H_\lambda$.
- **Suppose** $\mu : V \to \mathfrak{t}^*$ proper
- **Then** $\dfrac{1}{e^T(x)} = \prod_{\lambda \in W} f_\lambda(x) = b_\gamma \left(\prod_{\lambda \in W} \dfrac{1}{\lambda(z)} \right)$ is well defined.

Localization formula

- Duistermaat-Heckman *hyperfunction* η:

$$\eta(\xi) = \mathscr{F}\left(\frac{1}{(2\pi i)^r}\sum_{q\in\mathcal{F}}\frac{e^{i\mu(q)(x)}}{e_q^T(x)}\right)$$

  $\mathscr{F}$ is the hyperfunction Fourier transform.

- Key point: In computing $\mathscr{F}$, deform contour off the real axis (where poles are located)

Analytic requirement:

$$\frac{e^{i\mu(q)(x)}}{e_q^T(x)}$$

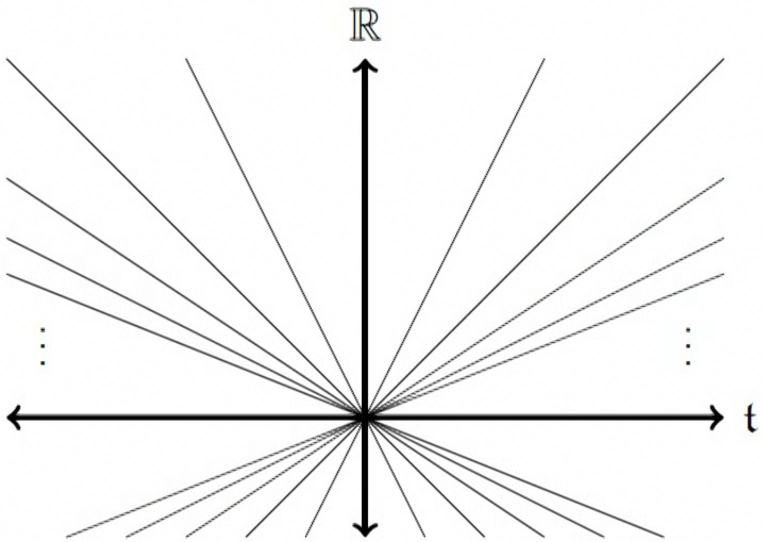
must be a *slowly increasing hyperfunction*

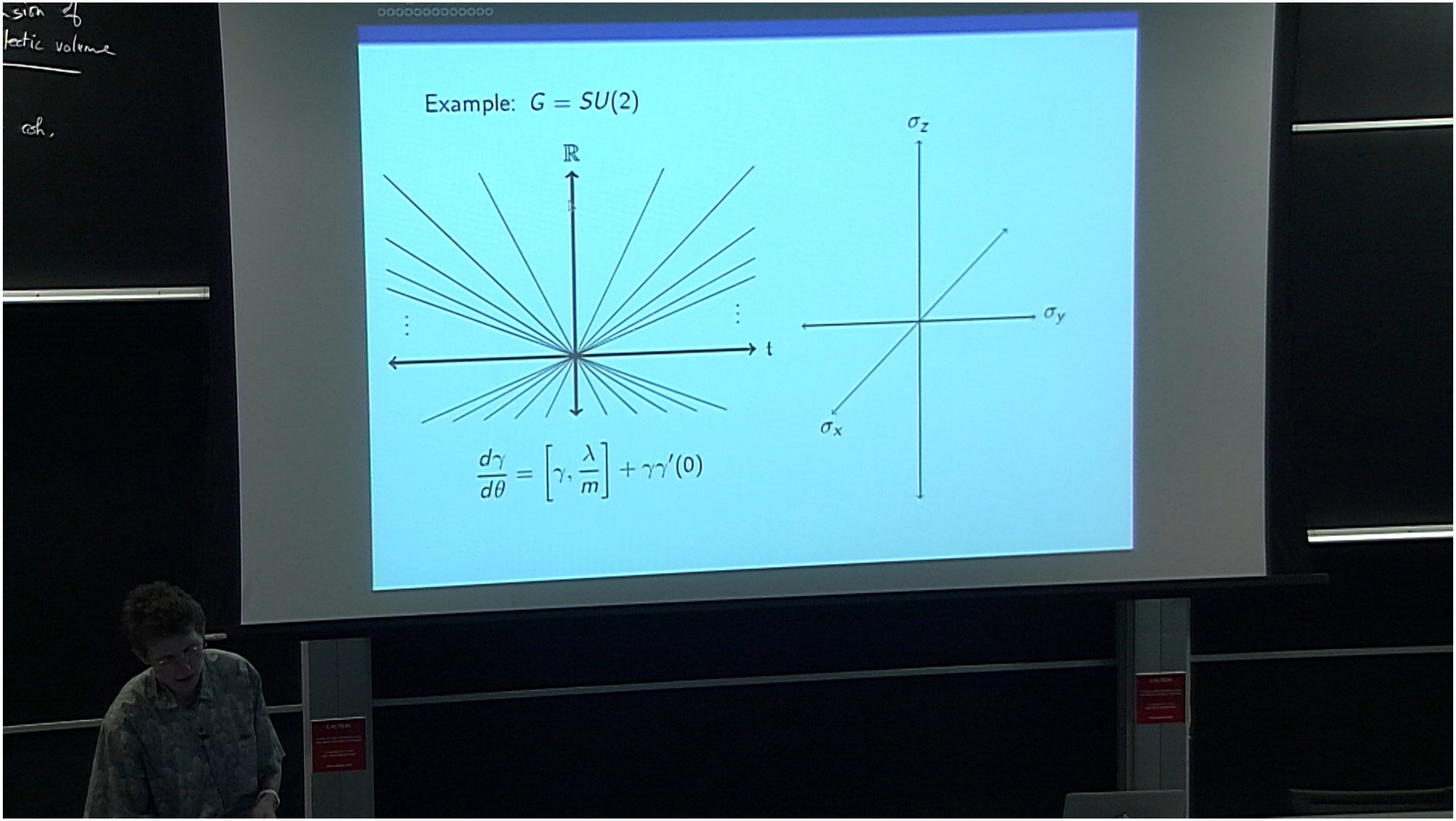
Theorem [J.-M.]

For every n, $1/e_n^T(x)$ is slowly increasing.

$$\eta(\xi_1,\xi_2) = \frac{1}{(2\pi i)^2}\sum_{n\in\mathbb{Z}} \mathscr{F}\left[b_{\gamma_n}\left(e^{-inz_1 - in^2 z_2/2}\frac{2\pi(n+z_1/z_2)}{\sin(2\pi z_1/z_2)}\right)\right]$$

Example: $G = SU(2)$

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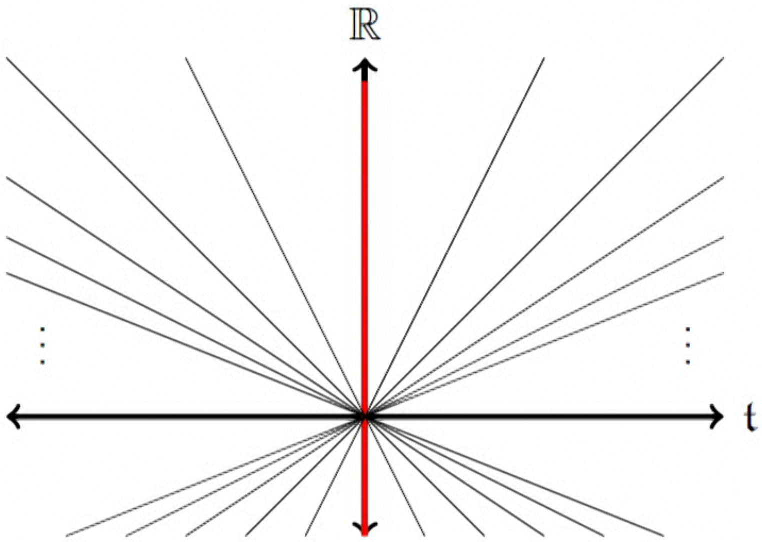

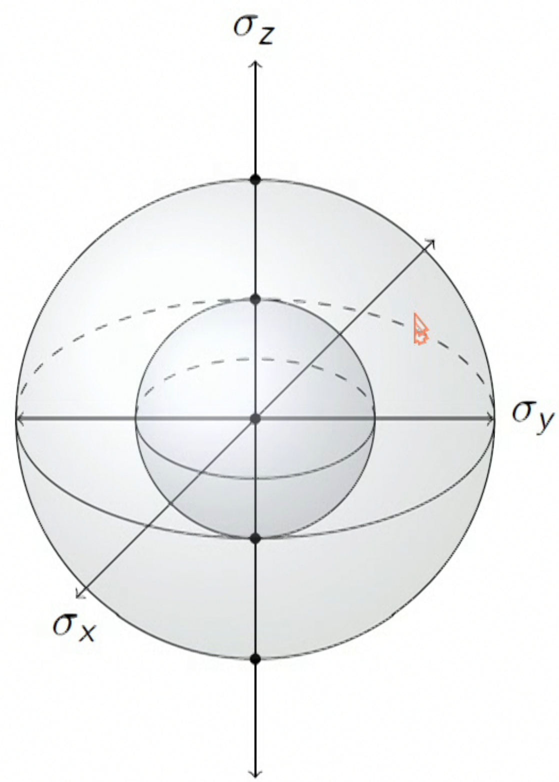

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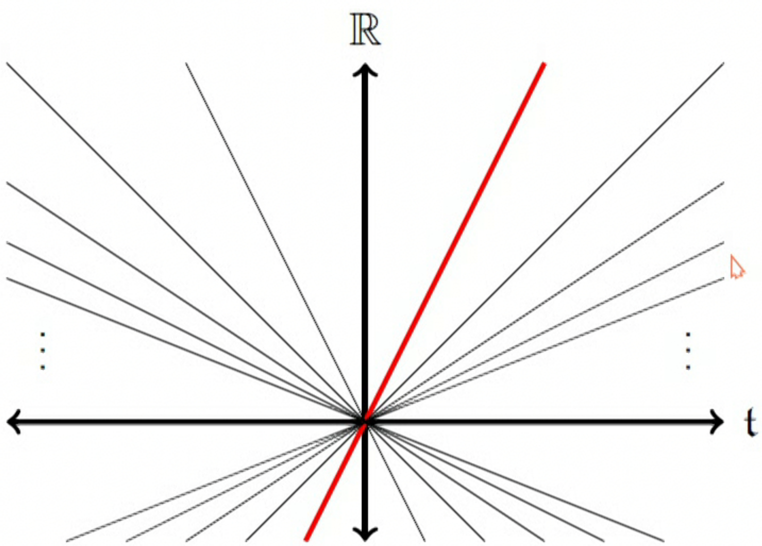

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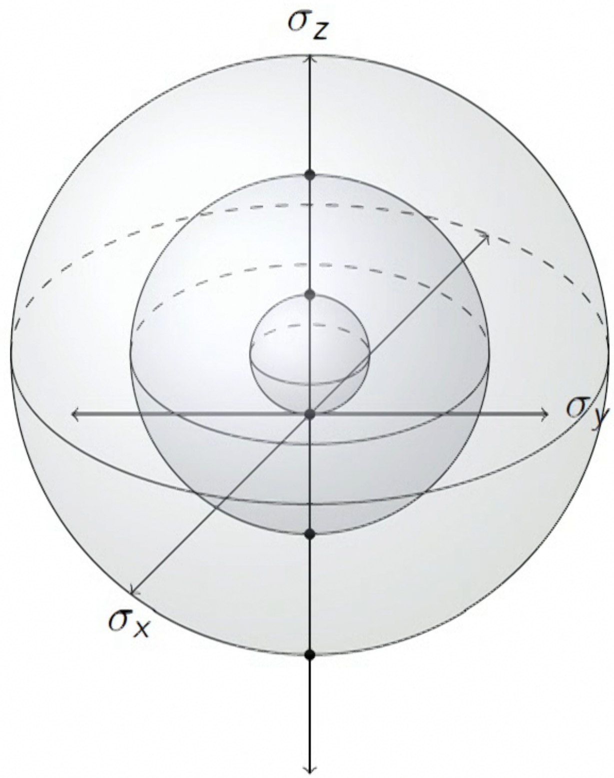
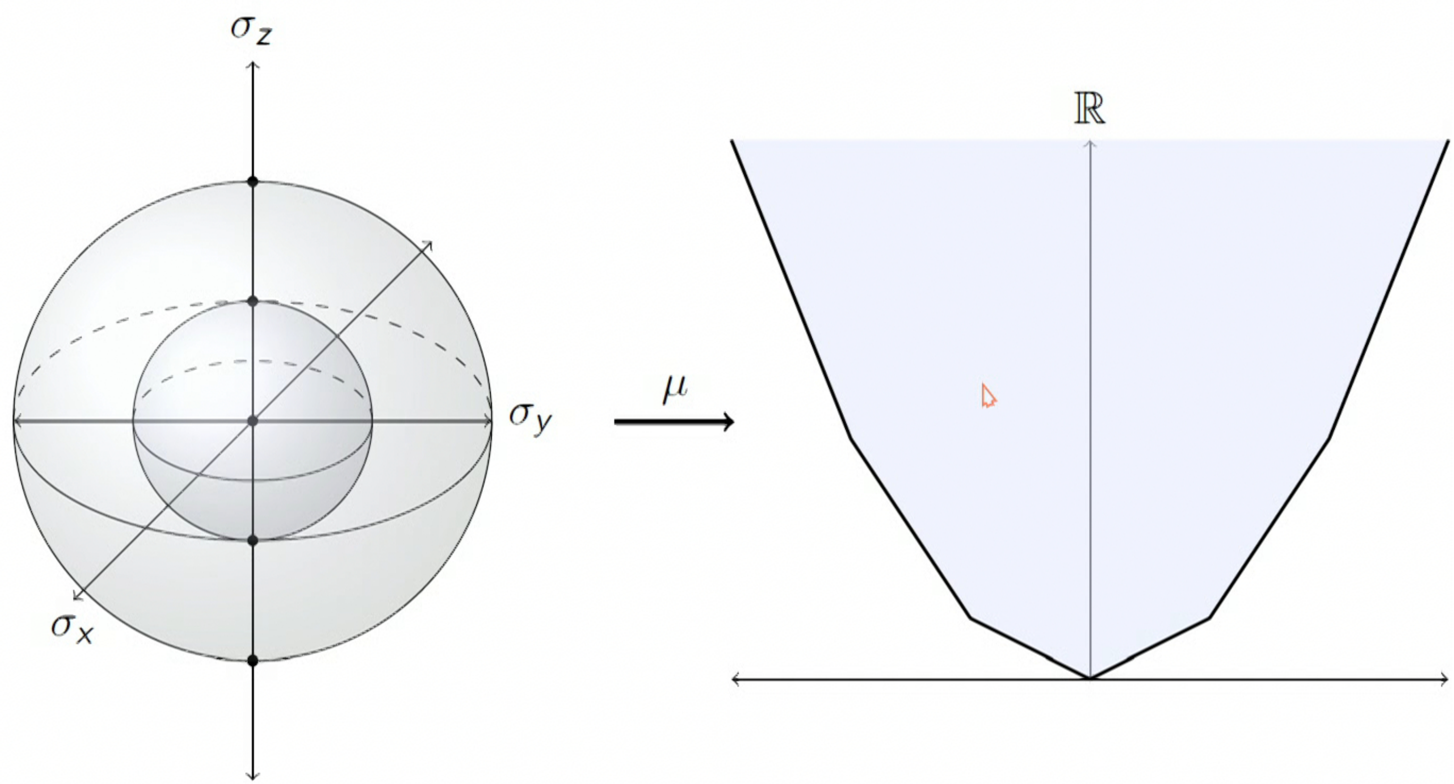
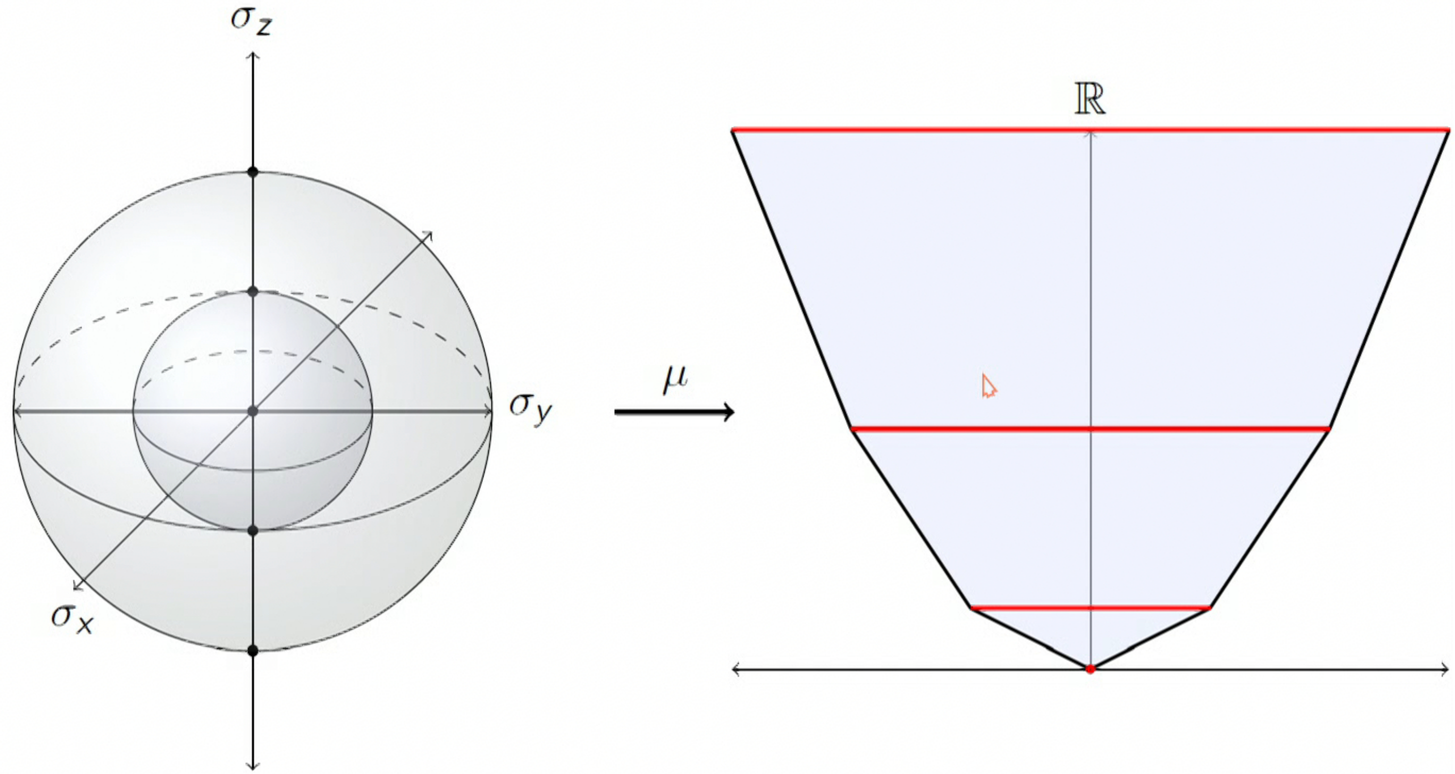



σ_z
σ_y
σ_x
μ
$\mathbb{R}$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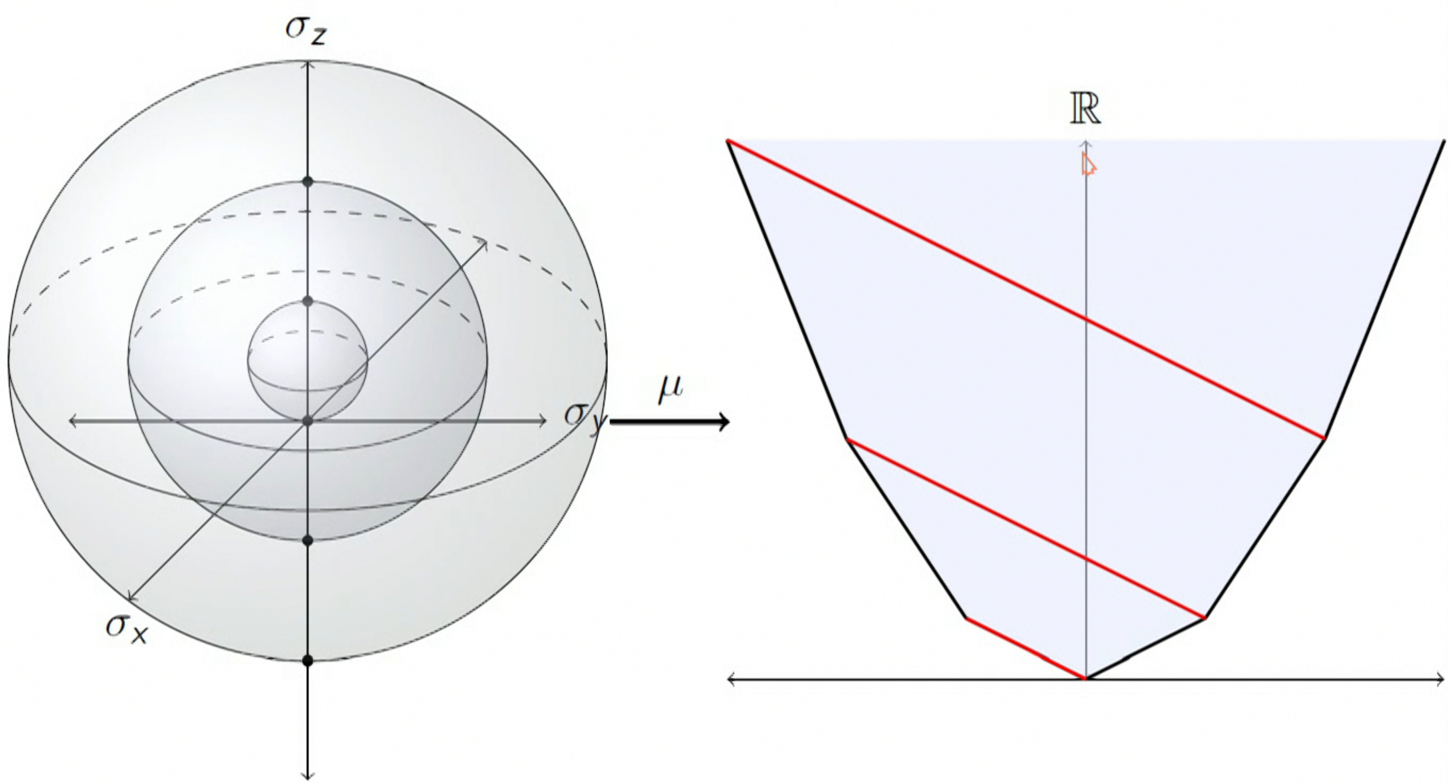
σ_z
σ_y
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$\mathbb{R}$

The weight of $T \times S^1$ on $V_{\alpha,k}$ is:

$$\lambda_\alpha^k : \mathrm{Lie}(T \times S^1)_{\mathbb{C}} \to \mathbb{C}$$

$$\lambda_\alpha^k(x_1, x_2) = \alpha(x_1 + \eta x_2) + kx_2$$

A basis of weight vectors for $V_{\alpha,k}$ is:

$$X_{\alpha,k}^{(1)} = i\sigma_y^\alpha \cos(k\theta) \pm i\sigma_x^\alpha \sin(k\theta)$$

$$X_{\alpha,k}^{(2)} = i\sigma_x^\alpha \cos(k\theta) \mp i\sigma_y^\alpha \sin(k\theta)$$

$\pm$ is taken depending on whether α is a positive or negative root, respectively. The weight of $T \times S^1$ on $V_{i,k}$ is:

$$\lambda_i^k : \mathrm{Lie}(T \times S^1)_{\mathbb{C}} \to \mathbb{C}$$

$$\lambda_i^k(x_1, x_2) = kx_2$$

A basis of weight vectors for $V_{i,k}$ is given by:

$$X_{i,k}^{(1)} = i\sigma_z^{\alpha_i} \cos(k\theta)$$

$$X_{i,k}^{(1)} = i\sigma_z^{\alpha_i} \sin(k\theta)$$

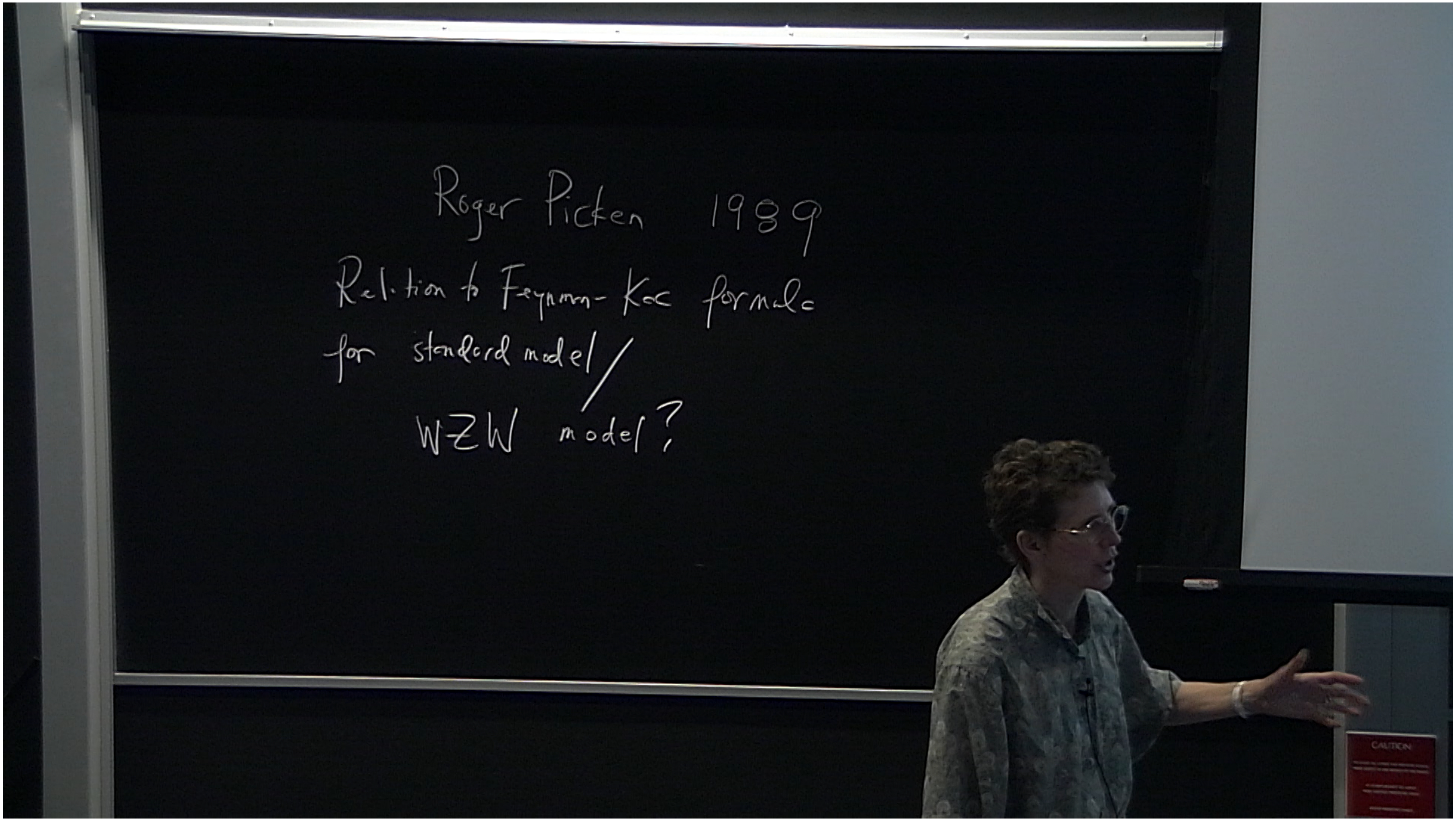
Proof sketch (a)

- Set $L_\beta = Z_G(\Lambda(2\pi/m))$ $(\Lambda(\theta) = \exp(\lambda\theta))$. We get a map:

$$\varphi_\beta : S^1 \to \mathrm{Aut}\, L_\beta$$

$$\varphi_\beta(\psi) \cdot x = \Lambda\left(\frac{\psi}{m}\right)^{-1} x \Lambda\left(\frac{\psi}{m}\right)$$

  From which we build $L_\beta \rtimes_\beta S^1$.
- One parameter subgroups of $L_\beta \rtimes_\beta S^1$ are obtained by conjugating one parameter subgroups of $T \times S^1$.
- Write down a formula for the conjugate using the group multiplication law of the semidirect product, then check by hand that is has the desired properties.


Roger Picken 1989
Relation to Feynman-Kac formula
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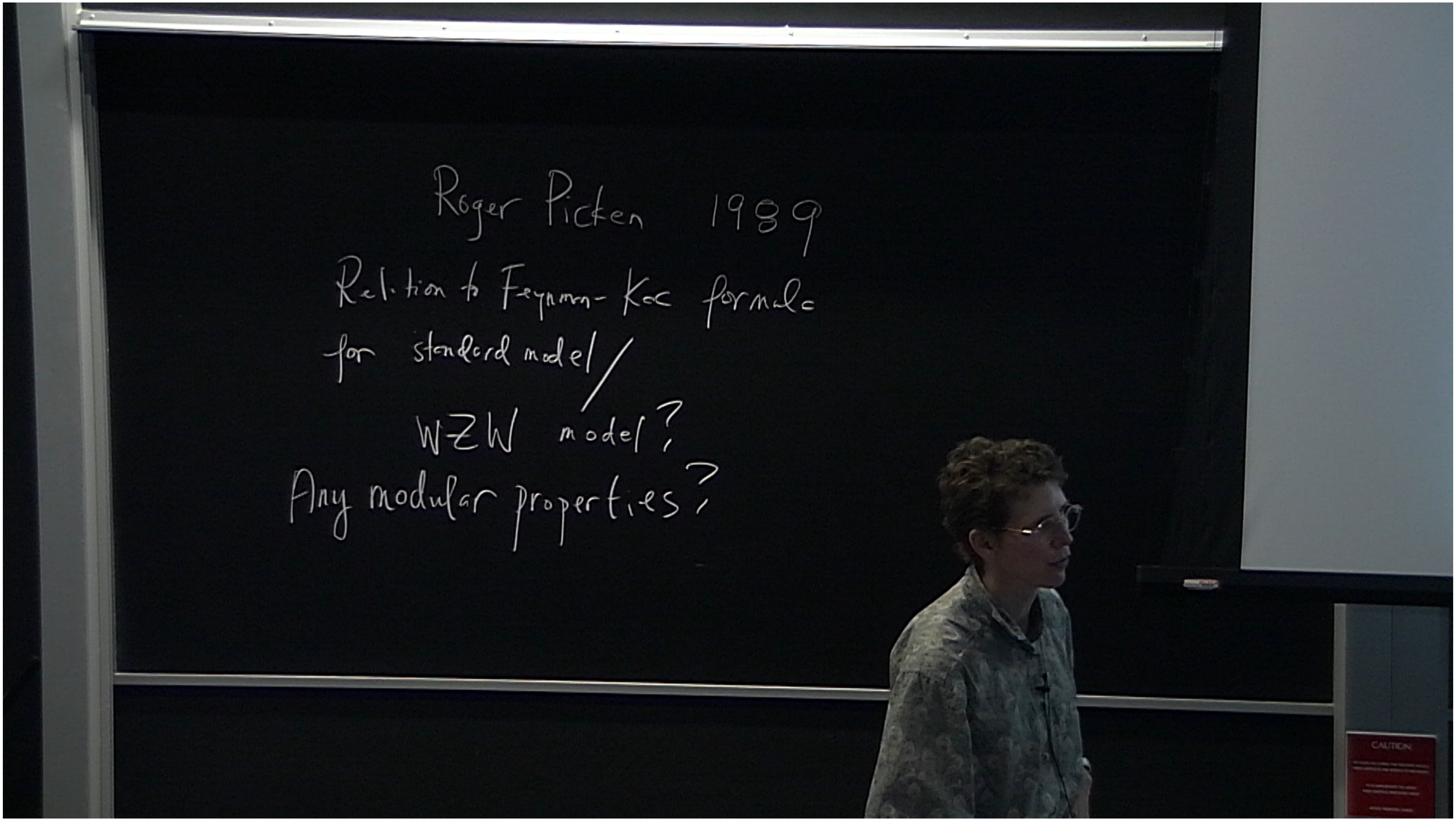
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Any modular properties?