Abstract: The based loop group is an infinite-dimensional manifold equipped with a Hamiltonian action of a finite dimensional torus. This was studied by Atiyah and Pressley. We investigate the Duistermaat–Heckman distribution using the theory of hyperfunctions. In applications involving Hamiltonian actions on infinite-dimensional manifolds, this theory is necessary to accommodate the existence of the infinite order differential operators which arise from the isotropy representation on the tangent spaces to fixed points. (Joint work with James Mracek)
Main goals for today:
- What is localization?
- What is a hyperfunction?
Archimedes’ Hat-Box theorem

The surface area on a unit 2-sphere enclosed by two latitudes is proportional only to the difference in height between the latitudes.

\[ \omega = dz \wedge d\theta \]

\[
\text{Area}(R) = \int_R \omega \\
= \int_0^{2\pi} \int_{z_1}^{z_2} dz \wedge d\theta \\
= 2\pi(z_1 - z_2)
\]
\[ \mu(x, y, z) = z \]
Define a measure on $\mathbb{R}$:

$$\mu^*(\omega)(S) = \int_{\mu^{-1}(S)} \omega$$
Related to Lebesgue measure; How?

\[
\frac{d\mu_*(\omega)}{dm} = \lim_{\varepsilon \to 0} \frac{\mu_*(\omega)(B_\varepsilon(x))}{m(B_\varepsilon(x))} = \chi_\mu(s²)(x)
\]

Fourier transform of \(\mu_*(\omega)\)?

\[
\int_{\mathbb{R}} e^{i\xi z} d\mu_*(\omega) = \int_{\mathbb{R}} e^{i\xi z} \frac{d\mu_*(\omega)}{dm} dm(z)
\]

\[
= \int_{-1}^{1} e^{i\xi z} dz = \frac{e^{i\xi}}{\xi} + \frac{e^{-i\xi}}{(-\xi)}
\]
Geometry?

\[ \mu : S^2 \rightarrow \mathbb{R} \]

\[(x, y, z) \mapsto z\]

\[\omega(\partial_\theta, -) = -dz = -d\mu\]

- Vector field \( \partial_\theta \) generates rotations around the z-axis
- Get a circle action on \( S^2 \).
General fact:

- Group $G$ acts on a smooth manifold $X$

\[ \forall g \in G, \exists \varphi_g : X \to X \]

- $p \in X$ fixed by action, $\forall g \in G$, $\varphi_g(p) = p$
General fact:

- Group $G$ acts on a smooth manifold $X$

$$\forall g \in G, \exists \varphi_g : X \rightarrow X$$

- $p \in X$ fixed by action, $\forall g \in G, \varphi_g(p) = p$

$$\Rightarrow G \text{ acts on } T_pX \text{ (isotropy representation)}$$

$$T_pX \rightarrow T_pX$$

$$v \mapsto d\varphi_g(v)$$
Back to our example...
Localization: Oscillatory integral on $X$ = Sum of terms involving local geometry near fixed points
Theorem [Duistermaat-Heckman]

Let \((X, \omega, T, \mu : X \to \mathfrak{t}^*)\) be a Hamiltonian action of a compact, rank \(r\) torus on a compact, finite dimensional manifold \(X\).

- The critical values of \(\mu\) separate the moment map image into chambers where \(\mu_*(\omega^n/n!)\) has a piecewise polynomial density function for the Lebesgue measure on \(\mathfrak{t}^*\).
- The inverse Fourier transform of \(\mu_*(\omega^n/n!)\) has an exact expression coming from the method of stationary phase:

\[
\int_X e^{i\mu(p)(\xi)} \omega^n / n! = \frac{1}{(2\pi i)^r} \sum_{q \in \mathcal{F}} \frac{e^{i\mu(q)(\xi)}}{e_q^T(\xi)}
\]

(1)

where \(\xi \in \mathfrak{t}\) is such that \(e_q^T(\xi) \neq 0\) for all \(q \in \mathcal{F}\).
Dunstermeier-Heckman 1983 ? localization

\[ \leftarrow \text{localization in equivariant cohomology} \]

\[ (\text{Berline-Vergne} \ 1983, \ \text{for equivariant cohomology}) \]

\[ \text{Atiyah-Bott} \ 1983, \ \text{for any equivariant class} \]
Definition

- \( \Omega \subseteq \mathbb{R} \)
- Choose \( U \subseteq \mathbb{C} \) such that \( U \cap \mathbb{R} = \Omega \).

A hyperfunction on \( \Omega \subseteq \mathbb{R} \) is an element of the vector space:

\[
f \in \frac{\mathcal{O}(U \setminus \Omega)}{\mathcal{O}(U)}
\]
- $G$ Lie group
- $\mathfrak{g}$ Lie algebra
- $R$ roots.
- $T \subseteq G$ maximal torus, $t$ its Lie algebra

\[ \Omega G = \{ \gamma : S^1 \to G \mid \gamma \in C^\infty(S^1, G), \gamma(1) = e \} \]

\[ T_\gamma \Omega G \simeq \Omega \mathfrak{g} \]
\( \Omega G \) is an infinite-dimensional Kähler manifold
- \( \omega_e(X, Y) = \int_{\Omega}^{2\pi} \langle X, Y' \rangle \, d\theta \)

\( T \times S^1 \) acts on \( \Omega G \)
- \( t \in T, \gamma \in \Omega G \) then \( (t \cdot \gamma)(\theta) = t\gamma(\theta)t^{-1} \)
- \( \psi \in S^1, \gamma \in \Omega G \) then \( (\psi \cdot \gamma)(\theta) = \gamma(\theta + \psi)\gamma(\psi)^{-1} \)

**Proposition [Atiyah-Pressley]**
- \( T \times S^1 \) action on \( \Omega G \) is Hamiltonian.
- Moment map \( \mu : \Omega G \to t \oplus \mathbb{R} \)
Theorem [Atiyah-Pressley]

- $\text{Fix}_{T \times S^1}(\Omega G) = \text{Hom}(S^1, T)$
- The image of $\mu : \Omega G \to t \oplus \mathbb{R}$ is the convex hull of the image of the fixed point set.
Classifying the singular values of $\mu$

$x \in \Omega G$ is critical for $\mu$ if and only if $\exists T' \subseteq T \times S^1, x \in \Omega G^{T'}$

$T_s = \{ \exp(\beta s) \in T \times S^1 \mid s \in \mathbb{R} \}$

- $\Omega G^{T_s}$ non-trivial?
- $\Omega G^{T_s} = \Omega G^{T_{s'}}$?
- $\Omega G^{T_s} =$?
Dunstermeier-Heckman 1983

\[ \text{localization in equivariant cohomology} \]

(Cervera-Vergne 1983; for equiv.

Atiyah-Bott 1983, for any

Hyperfunctions, Kato, Kawai

Hyperfunctionals, and others

extension of symplectic volume

equivariant class
Our goal is to understand how this theorem generalizes to the case where $X$ is infinite dimensional. Two sub-problems:

1. Classify the singular values of $\mu$
2. Make sense of the localization formula
Classifying the singular values of $\mu$

$x \in \Omega G$ is critical for $\mu$ if and only if $\exists T' \subseteq T \times S^1$, $x \in \Omega G^{T'}$

$$T_\beta = \overline{\{\exp(\beta s) \in T \times S^1 \mid s \in \mathbb{R}\}}$$

- $\Omega G^{T_\beta}$ non-trivial?
- $\Omega G^{T_\beta} = \Omega G^{T_{\beta'}},$?
- $\Omega G^{T_\beta} =$?
Classifying the singular values of $\mu$

$x \in \Omega G$ is critical for $\mu$ if and only if $\exists \ T' \subseteq T \times S^1, \ x \in \Omega G^{T'}$

\[
T_\beta = \{\exp(\beta s) \in T \times S^1 \mid s \in \mathbb{R}\}
\]

- $\Omega G^{T_\beta}$ non-trivial?
- $\Omega G^{T_\beta} = \Omega G^{T_{\beta'}}$?
- $\Omega G^{T_\beta} =$?

Theorem [J.-M.]

(a) For any cocharacter $\beta = (\lambda, m) \in X_*(T \times S^1) \subseteq \text{Lie}(T \times S^1)$, there exists $L_\beta, T \subseteq L_\beta \subseteq G$, such that $\gamma \in \Omega G^{T_\beta}$ if and only if $(\gamma(\theta), \theta)$ is a one parameter subgroup of $L_\beta \ltimes_{\beta} S^1$.

(b) Let $\beta = (\lambda, m)$ and $\beta' = (\lambda', m')$ generate rank one subgroups $T_\beta, T_{\beta'} \subseteq T \times S^1$, and let $L_\beta, L_{\beta'}$ be the Levi subgroups from (a). $\Omega G^{T_\beta} = \Omega G^{T_{\beta'}}$ if and only if $\lambda/m - \lambda'/m' \in \text{Lie}(Z(L_\beta))$.

(c) Every connected component of the fixed point set of $T_\beta$ is a translate of an adjoint orbit in $\text{Lie}(L_\beta) \subseteq g$. 
Classifying the singular values of $\mu$.

Chamber structure on $\mu(\Omega SU(2))$.
$$\int_{\Omega G} e^{i\mu(p)(x)} \omega^n / n! = \frac{1}{(2\pi i)^r} \sum_{q_{\in F}} \frac{e^{i\mu(q)(x)}}{e^T_q(x)}$$

Problems:
- Not rigorous, but related to Feynman path integral
- $e^T_q(x)$ is an infinite product $\Rightarrow$ convergence issues
- Behaviour of $1/e^T_q(x)$ as a distribution is not evident
Isotropy representation: \( T \times S^1 \cong T_\gamma \Omega G \)
Isotropy representation: $T \times S^1 \cap T_\gamma \Omega G$

**Theorem [J.-M.]**

- Suppose $\gamma(\theta) \in \Omega G^{T \times S^1}$
- Then $T \times S^1$ action on $T_\gamma \Omega G$ decomposes into irreducible subrepresentations:

$$
T_\gamma \Omega G \simeq \Omega g \simeq \bigoplus_{k=1}^\infty \left( \bigoplus_{n \in \mathbb{R}} V_{n,k} \oplus \bigoplus_{i=1}^\infty V_{i,k} \right)
$$

with an explicitly determined weight basis
Isotropy representation: $T \times S^1 \bowtie T_\gamma \Omega G$

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- **Suppose** $\gamma(\theta) \in \Omega G^{T \times S^1}$
- **Then** $T \times S^1$ action on $T_\gamma \Omega G$ decomposes into irreducible subrepresentations:

$$T_\gamma \Omega G \simeq \Omega g_s \simeq \bigoplus_{k=1}^{\infty} \left( \bigoplus_{\alpha \in R} V_{\alpha, k} \bigoplus_{i=1}^{n} V_{i, k} \right)$$

with an explicitly determined weight basis
Regularized equivariant Euler class of the normal bundle to $\gamma$:

$$e_{\gamma}^{T \times S^1}(z_1, z_2) = \prod_{k=1}^{\infty} \left( \prod_{\alpha \in R} \frac{\lambda_{\alpha}^k(z_1, z_2)}{kz_2} \right)$$
Regularized equivariant Euler class of the normal bundle to $\gamma$:

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**Proposition [J.-M.]**

- **Suppose** $\gamma_n \in \Omega SU(2)^{T \times S^1}$
- **Then** regularized equivariant Euler class is:

$$e_n^{T \times S^1}(z_1, z_2) = \frac{\sin (2\pi z_1/z_2)}{2\pi(n + z_1/z_2)}$$

(2)

$$\eta = \text{Fourier transform} \left( 1/e_n^{T \times S^1} \right)$$
Definition

A **hyperfunction** on $\Omega \subseteq \mathbb{R}^n$ is an element:

$$\sum_{i=1}^{n} b_{\gamma_i}(F_i(z)) \in \bigoplus_{\gamma \in \Gamma} \mathcal{O}(\Omega \times i\gamma 0)/ \sim$$

If $\gamma_1, \gamma_2, \gamma_3$ open convex cones, $\gamma_3 \subseteq \gamma_1 \cap \gamma_2$ and $F_i \in \mathcal{O}(\Omega \times i\gamma_i)$, then

$$F_1(z) + F_2(z) \sim F_3(z) \iff (F_1(z) + F_2(z))|_{\gamma_3} = F_3(z)$$
Local model:

- Suppose $T$ has a Hamiltonian action on a complex vector space with weights $\lambda_i$.
- **Weights**: $W = \{\lambda_i : t \rightarrow \mathbb{R}\}_{i \in I}$.
  - **Half space**: $H_\lambda = \{y \in t^* \mid \lambda(y) < 0\}$,
  - **Hyperfunction**: $f_\lambda(x) = b_{H_\lambda} \left( \frac{1}{\lambda(z)} \right)$
Hyperfunction version of $1/e^T(x)$?
Hyperfunction version of $1/e^T(x)$?

Theorem [J.-M.]

- $\gamma = \bigcap_{\lambda \in \mathcal{W}} H_\lambda$.
- **Suppose** $\mu : V \to t^*$ proper
- **Then** $\frac{1}{e^T(x)} = \prod_{\lambda \in \mathcal{W}} f_\lambda(x) = b_\gamma \left( \prod_{\lambda \in \mathcal{W}} \frac{1}{\lambda(z)} \right)$ is well defined.
Duistermaat-Heckman *hyperfunction* \( \eta \):

\[
\eta(\xi) = \mathcal{F} \left( \frac{1}{(2\pi i)^r} \sum_{q \in \mathcal{F}} e^{i\mu(q)(x)} \frac{e^T_q(x)}{e_q(x)} \right)
\]

\( \mathcal{F} \) is the hyperfunction Fourier transform.

Key point: In computing \( \mathcal{F} \), deform contour off the real axis (where poles are located).
Analytic requirement:

\[ \frac{e^{i\mu(q)}(x)}{e^{T}(x)} \]

must be a slowly increasing hyperfunction

**Theorem [J.-M.]**

For every \( n \), \( 1/e_n^T(x) \) is slowly increasing.

\[ \eta(\xi_1, \xi_2) = \frac{1}{(2\pi i)^2} \sum_{n \in \mathbb{Z}} \mathcal{F} \left[ b_{\gamma_n} \left( e^{-inz_1-in^2z_2/2} \frac{2\pi(n+z_1/z_2)}{\sin(2\pi z_1/z_2)} \right) \right] \]
Example: $G = SU(2)$
Example: \( G = SU(2) \)

\[
\frac{d \gamma}{d \theta} = \left[ \gamma, \frac{\lambda}{m} \right] + \gamma \gamma'(0)
\]
Example: $G = SU(2)$

\[ \frac{d\gamma}{d\theta} = \left[ \gamma, \frac{\lambda}{m} \right] + \gamma\gamma'(0) \]
Example: $G = SU(2)$

\[
\frac{d\gamma}{d\theta} = \left[ \gamma, \frac{\lambda}{m} \right] + \gamma\gamma'(0)
\]
The weight of $T \times S^1$ on $V_{\alpha,k}$ is:

$$\lambda^k_{\alpha} : \text{Lie} (T \times S^1)_\mathbb{C} \rightarrow \mathbb{C}$$

$$\lambda^k_{\alpha}(x_1, x_2) = \alpha (x_1 + \eta x_2) + kx_2$$

A basis of weight vectors for $V_{\alpha,k}$ is:

$$X^{(1)}_{\alpha,k} = i \sigma_y^\alpha \cos(k\theta) \pm i \sigma_x^\alpha \sin(k\theta)$$

$$X^{(2)}_{\alpha,k} = i \sigma_x^\alpha \cos(k\theta) \mp i \sigma_y^\alpha \sin(k\theta)$$

$\pm$ is taken depending on whether $\alpha$ is a positive or negative root, respectively. The weight of $T \times S^1$ on $V_{i,k}$ is:

$$\lambda^k_i : \text{Lie} (T \times S^1)_\mathbb{C} \rightarrow \mathbb{C}$$

$$\lambda^k_i(x_1, x_2) = kx_2$$

A basis of weight vectors for $V_{i,k}$ is given by:

$$X^{(1)}_{i,k} = i \sigma_i^\alpha \cos(k\theta)$$

$$X^{(1)}_{i,k} = i \sigma_z^\alpha \sin(k\theta)$$
Proof sketch (a)

- Set \( L_\beta = Z_G(\Lambda(2\pi / m)) \) (\( \Lambda(\theta) = \exp(\lambda \theta) \)). We get a map:

\[
\varphi_\beta : S^1 \rightarrow \text{Aut } L_\beta
\]

\[
\varphi_\beta(\psi) \cdot x = \Lambda \left( \frac{\psi}{m} \right)^{-1} x \Lambda \left( \frac{\psi}{m} \right)
\]

From which we build \( L_\beta \rtimes_\beta S^1 \).

- One parameter subgroups of \( L_\beta \rtimes_\beta S^1 \) are obtained by conjugating one parameter subgroups of \( T \times S^1 \).

- Write down a formula for the conjugate using the group multiplication law of the semidirect product, then check by hand that it has the desired properties.
Roger Picken 1989
Relation to Feynman-Kac formula
for standard model
WZW model?
Roger Picken 1989
Relation to Feynman-Kac formula
for standard model/
WZW model?
Any modular properties?