Title: Homotopy types and geometries below Spec(Z)

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Abstract: This talk is based on joint work with Yuri Manin. The idea of a â€œgeometry over the ï¬•eld with one element F1â€• arises in connection with the study of properties of zeta functions of varieties deï¬•ned over Z. Several diï¬€erent versions of F1 geometry (geometry below Spec(Z)) have been proposed over the years (by Tits, Manin, Deninger, Kapranovâ€“Smirnov, etc.) including the use of homotopy theoretic methods and â€œbrave new algebraâ€• of ring spectra (ToÂ¨enâ€“VaquiÂ´e). We present a version of F1 geometry that connects the homotopy theoretic viewpoint, using Zakharevichâ€™s approach to the construction of spectra via assembler categories, and a point of view based on the Bostâ€“Connes quantum statistical mechanical system, and we discuss its relevance in the context of counting problems, zeta-functions and generalised scissors congruences.

Homotopy types and geometries below $\mathrm{Spec}(\mathbb{Z})$

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joint work with Yuri Manin

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This talk is based on:

- Yu.I. Manin, M. Marcolli, *Homotopy types and geometries below* $\mathrm{Spec}(\mathbb{Z})$, arXiv:1806.10801.

Other relevant references:

- Yu.I. Manin, M. Marcolli, *Moduli operad over* $\mathbb{F}_1$, in "Absolute arithmetic and $\mathbb{F}_1$-geometry", pp. 331–361, Eur. Math. Soc., 2016.
- C. Connes, C. Consani, M. Marcolli, *Fun with* $\mathbb{F}_1$, Journal of Number Theory, 129 (2009) 1532–1561
- M. Marcolli, *Cyclotomy and endomotives*, p-Adic Numbers Ultrametric Anal. Appl. 1 (2009), no. 3, 217–263
- Yu.I. Manin, *Cyclotomy and analytic geometry over* $\mathbb{F}_1$, in "Quanta of Maths", pp. 385–408, Clay Math. Proc. 11, Amer. Math. Soc., 2010.
- I. Zakharevich, *The K-theory of assemblers*, Adv. Math. 304 (2017), 1176–1218

What is the "field with one element"?

Finite geometries $(q = p^k,\ p \text{ prime})$

$$\#\mathbb{P}^{n-1}(\mathbb{F}_q) = \frac{\#(\mathbb{A}^n(\mathbb{F}_q) \smallsetminus \{0\})}{\#\mathbb{G}_m(\mathbb{F}_q)} = \frac{q^n - 1}{q - 1} = [n]_q$$

$$\#\mathrm{Gr}(n,j)(\mathbb{F}_q) = \#\{\mathbb{P}^j(\mathbb{F}_q) \subset \mathbb{P}^n(\mathbb{F}_q)\}$$

$$= \frac{[n]_q!}{[j]_q![n-j]_q!} = \binom{n}{j}_q$$

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad [0]_q! = 1$$

The origin of $\mathbb{F}_1$-geometry: Jacques Tits observed if take $q = 1$

$$\mathbb{P}^{n-1}(\mathbb{F}_1) := \text{ finite set of cardinality } n$$

$$\mathrm{Gr}(n,j)(\mathbb{F}_1) := \text{ set of subsets of cardinality } j$$

Is there an algebraic geometry over $\mathbb{F}_1$?

Extensions $\mathbb{F}_{1^n}$ (Kapranov-Smirnov)
Monoid $\{0\} \cup \mu_n$ (n-th roots of unity)
- Vector space over $\mathbb{F}_{1^n}$: pointed set (V, v) with free action of μ_n on $V \smallsetminus \{v\}$
- Linear maps: permutations compatible with the action

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[t, t^{-1}]/(t^n - 1)$$

Counting of points: for geometries X over $\mathbb{Z}$, reductions mod p

$$N_q(X) = \#X(\mathbb{F}_q), \qquad q = p^r$$

Polynomially countable if $N_q(X) = P_X(q)$ polynomial in q.
Counting of "points over the field with one element and its extensions"

$$P_X(m+1) = \#X(\mathbb{F}_{1^m})$$

Zeta functions (Manin) as variety over $\mathbb{F}_1$:

$$Z(\mathbb{T}^n, s) = \frac{s-n}{2\pi} \quad \text{with } \mathbb{T}^n = \mathbb{G}_m^n$$

$\Rightarrow$ $\mathbb{F}_1$-geometry and infinite primes (Arakelov)

$$\Gamma_{\mathbb{C}}(s)^{-1} = ((2\pi)^{-s}\Gamma(s))^{-1} = \prod_{n\geq 0} \frac{s+n}{2\pi}$$

like a "dual" infinite projective space $\oplus_{n=0}^{\infty}\mathbb{T}^{-n}$ over $\mathbb{F}_1$

General approaches: Descent data for rings from $\mathbb{Z}$ to $\mathbb{F}_1$:

- by cyclotomic points (Soulé)
- by Λ-ring structure (Borger)
- by sphere spectrum (Toën–Vaquié)

Different approaches to $\mathbb{F}_1$-geometry: Soulé, Haran, Deitmar, Dourov, Toën–Vaquié, Connes–Consani, López-Peña–Lorscheid, Borger, ...

Comparative view:

- J. López-Peña, O. Lorscheid, *Mapping $\mathbb{F}_1$-Land*, in "Noncommutative geometry, Arithmetic, and Related Topics", pp. 241–265, Johns Hopkins Univ. Press, 2011.

Focus here:

- combining the *integral Bost–Connes* viewpoint on $\mathbb{F}_1$ and Kapranov–Smirnov extensions $\mathbb{F}_{1^m}$ with the *spectra* viewpoint of Toën–Vaquié and the *torifications* point of view of López-Peña–Lorscheid
- a proposal for a *dynamical* approach to $\mathbb{F}_1$-geometry

Background: the integral Bost–Connes viewpoint on $\mathbb{F}_1$

- C. Connes, C. Consani, M. Marcolli, *Fun with* $\mathbb{F}_1$, Journal of Number Theory, 129 (2009) 1532–1561
- C. Soulé, *Les variétés sur le corps à un élément*, Mosc. Math. J. Vol.4 (2004) N.1, 217–244.

Soulé's "gadgets" and varieties over $\mathbb{F}_1$:

• $X : \mathcal{R} \to Sets$ covariant functor, rings (fin gen flat) to sets
• key idea: $R = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \Rightarrow X(R)$ cyclotomic points
• $\mathcal{A}_X$ complex algebra and evaluation maps: points $x \in X(R)$, $\sigma : R \to \mathbb{C} \Rightarrow e_{x,\sigma} : \mathcal{A}_X \to \mathbb{C}$ algebra homomorphism $e_{f(y),\sigma} = e_{y,\sigma\circ f}$ for $f : R' \to R$ ring homom
• Affine varieties $V_{\mathbb{Z}} \Rightarrow$ gadget $X = G(V_{\mathbb{Z}})$ with $X(R) = \mathrm{Hom}(O(V), R)$ and $\mathcal{A}_X = O(V) \otimes \mathbb{C}$
• Soulé's "gadgets" and varieties over $\mathbb{F}_1$: gadget with $X(R)$ finite; variety $X_{\mathbb{Z}}$ and morphism of gadgets $X \to G(X_{\mathbb{Z}})$ such that all $X \to G(V_{\mathbb{Z}})$ come from $X_{\mathbb{Z}} \to V_{\mathbb{Z}}$

The Bost–Connes quantum statistical mechanical system

Algebra $\mathcal{A}_{\mathbb{Q},BC} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ generators and relations

$$\mu_n \mu_m = \mu_{nm}$$
$$\mu_n \mu_m^* = \mu_m^* \mu_n \quad \text{when } (n,m)=1$$
$$\mu_n^* \mu_n = 1$$

$$e(r+s) = e(r)e(s), \quad e(0) = 1$$

$$\rho_n(e(r)) = \mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$

- Time evolution

$$\sigma_t(e(r)) = e(r), \quad \sigma_t(\mu_n) = n^{it} \mu_n$$

Partition function $\mathrm{Tr}(e^{-\beta H}) = \zeta(\beta)$ Riemann zeta

Properties of the Bost–Connes system

- Low temperature equilibrium states ($\beta = 1/T > 1$)

$$\varphi_{\beta,\rho}(a) = \frac{\mathrm{Tr}(\pi_\rho(a) e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}, \quad \rho \in \hat{\mathbb{Z}}^*$$

polylog function at roots of unity, normalized by $\zeta(\beta)$; at high temperature unique equilibrium (Euler totient function)

- Zero temperature: evaluations $\varphi_{\infty,\rho}(e(r)) = \zeta_r$

$$\varphi_{\infty,\rho}(a) = \langle \epsilon_1, \pi_\rho(a)\epsilon_1 \rangle$$

arithmetic subalgebra $a \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$, Galois symmetries $\gamma \in \hat{\mathbb{Z}}^*$

$$\varphi_{\infty,\rho}(\gamma a) = \theta_\gamma(\varphi_{\infty,\rho}(a))$$

$$\theta : \hat{\mathbb{Z}}^* \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

class field theory isomorphism

Integer model of the Bost–Connes algebra

$\mathcal{A}_{\mathbb{Z},BC}$ generated by $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and μ_n^*, $\tilde{\mu}_n$

$$\begin{aligned} \tilde{\mu}_n \tilde{\mu}_m &= \tilde{\mu}_{nm} \\ \mu_n^* \mu_m^* &= \mu_{nm}^* \\ \mu_n^* \tilde{\mu}_n &= n \\ \tilde{\mu}_n \mu_m^* &= \mu_m^* \tilde{\mu}_n \quad (n,m)=1. \end{aligned}$$

$$\mu_n^* x = \sigma_n(x)\mu_n^* \quad \text{and} \quad x\tilde{\mu}_n = \tilde{\mu}_n \sigma_n(x)$$

where $\sigma_n(e(r)) = e(nr)$ for $r \in \mathbb{Q}/\mathbb{Z}$

Note: $\rho_n(x) = \mu_n x \mu_n^*$ ring homomorphism but not $\tilde{\rho}_n(x) = \tilde{\mu}_n x \mu_n^*$ (correspondences "crossed product" $\mathcal{A}_{\mathbb{Z},BC} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\tilde{\rho}} \mathbb{N}$)

Integer Bost–Connes algebra and Soulé $\mathbb{F}_1$-geometry

Roots of unity as varieties over $\mathbb{F}_1$*:*

$$\underline{\mu}^{(k)}(R) = \{x \in R \,|\, x^k = 1\} = \mathrm{Hom}_{\mathbb{Z}}(A_k, R)$$

$A_k = \mathbb{Z}[t, t^{-1}]/(t^k - 1)$

- Inductive system $\mathbb{G}_m$ over $\mathbb{F}_1$:

$$\underline{\mu}^{(n)}(R) \subset \underline{\mu}^{(m)}(R), \quad n|m, \quad A_m \twoheadrightarrow A_n$$

$$\mathcal{A}_X = C(S^1)$$

- Projective system (BC): $\xi_{m,n} : X_n \twoheadrightarrow X_m$

$$\xi_{m,n} : \underline{\mu}^{(n)}(R) \twoheadrightarrow \underline{\mu}^{(m)}(R), \quad n|m$$

$$\underline{\mu}^{\infty}(R) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], R)$$

$\Rightarrow$ projective system of affine varieties over $\mathbb{F}_1$

$$\xi_{m,n} : \mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} \to \mathbb{F}_{1^m} \otimes_{\mathbb{F}_1} \mathbb{Z}$$

$$\mathcal{A}_X = \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$$

• Affine varieties $\mu^{(n)}$ over $\mathbb{F}_1$ defined by gadgets $G(\mathrm{Spec}(\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]))$; projective system

• Endomorphisms σ_n (of varieties over $\mathbb{Z}$, of gadgets, of $\mathbb{F}_1$-varieties)

• Extensions $\mathbb{F}_{1^n}$: free actions of roots of 1 (Kapranov–Smirnov)

$$\zeta \mapsto \zeta^n, \ n \in \mathbb{N} \quad \text{and} \quad \zeta \mapsto \zeta^\alpha \leftrightarrow e(\alpha(r)), \ \alpha \in \hat{\mathbb{Z}}$$

Frobenius action on $\mathbb{F}_{1^\infty}$

• in reductions mod p of integral Bost–Connes endomotive $\Rightarrow$ this action becomes the Frobenius

Conclusion from this earlier work: integral Bost–Connes algebra as a model for $\mathbb{F}_1$-geometry, namely the tower of extensions $\mathbb{F}_{1^n}$ plus a good notion of Frobenius

Current work: lift this integral Bost–Connes viewpoint to the level of Grothendieck rings of varieties and further to the homotopy level of spectra

Examples of additive invariants:

- Topological Euler characteristic
- *Couting points over finite fields*
- Gillet–Soulé motivic $\chi_{mot}(X)$:

$$\chi_{mot} : K_0(\mathcal{V})[\mathbb{L}^{-1}] \to K_0(\mathcal{M}), \quad \chi_{mot}(X) = [(X, id, 0)]$$

for X smooth projective; complex $\chi_{mot}(X) = W^{\cdot}(X)$

Note: counting points over finite fields factors through the Grothendieck ring, geometrize expected behavior of the counting function as a condition on $[X] \in K_0(\mathcal{V}_{\mathbb{Z}})$.

Grothendieck rings and the integral Bost–Connes algebra

- G-equivariant Grothendieck ring $K_0^G(\mathcal{V})$:
  - variety X with good G action: each G-orbit contained in an affine open subset
  - $K_0^G(\mathcal{V})$ generated by isomorphism classes of varieties with good G-action, inclusion-exclusion relation $[X] = [Y] + [X \smallsetminus Y]$ with G-equivariant embeddings $Y \hookrightarrow X$ and $X \smallsetminus Y \hookrightarrow X$
  - diagonal G-action on the product

- case of group of all roots of unity

$$G = \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ varieties with action of $\hat{\mathbb{Z}}$ factoring through good action of finite quotient $\mathbb{Z}/n\mathbb{Z}$

Lifting the integral Bost–Connes algebra

• Euler characteristic ring homomorphism

$$\chi^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{C}}) \to K_0^{\hat{\mathbb{Z}}}(\mathbb{C}),$$

to Grothendieck ring of finite dimensional representations: character group $\mathrm{Hom}(\hat{\mathbb{Z}}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$ so

$$K_0^{\hat{\mathbb{Z}}}(\mathbb{C}) = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

• endomorphisms $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with $\sigma_n(e(r)) = e(nr)$ lift to endomorphisms $\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{C}}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{C}})$ with commuting

$$\begin{array}{ccc} K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{C}}) & \xrightarrow{\chi^{\hat{\mathbb{Z}}}} & \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \downarrow{\sigma_n} & & \downarrow{\sigma_n} \\ K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{C}}) & \xrightarrow{\chi^{\hat{\mathbb{Z}}}} & \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{array}$$

Semigroup action of $\mathbb{N}$ on $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{C}})$.

• X variety with good $\hat{\mathbb{Z}}$-action $\alpha : \hat{\mathbb{Z}} \times X \to X$ factoring through $\mathbb{Z}/N\mathbb{Z}$ and action

$$\sigma_n : (X, \alpha) \mapsto (X, \alpha_n = \alpha \circ \sigma_n)$$

• maps $\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$ lift to morphisms (of abelian groups):

$$\mathbb{V}_n = [Z_n, \gamma_n] \in K_0^{\hat{\mathbb{Z}}}(\mathcal{V}) \qquad Z_n = \mathrm{Spec}(\mathbb{Q}^n)$$

$\gamma_n : \hat{\mathbb{Z}} \times Z_n \to Z_n$ cyclic permutations

$$\Phi_n(\alpha) : \hat{\mathbb{Z}} \times X \times Z_n \to X \times Z_n$$

$$\Phi_n(\alpha)(\zeta, x, a_i) = \begin{cases} (x, \gamma_n(\zeta, a_i)) & i = 1, \ldots, n-1 \\ (\alpha(\zeta, x), \gamma_n(\zeta, a_n)) & i = n. \end{cases}$$

$$\tilde{\rho}_n[X, \alpha] := [X \times Z_n, \Phi_n(\alpha)]$$

Geometric version of Verschiebung map ("inverse" of Frobenius, raising to power)

• X variety with good $\hat{\mathbb{Z}}$-action $\alpha : \hat{\mathbb{Z}} \times X \to X$ factoring through $\mathbb{Z}/N\mathbb{Z}$ and action

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Geometric version of Verschiebung map ("inverse" of Frobenius, raising to power)

• noncommutative version: $\mathfrak{K}_0^{\hat{\mathbb{Z}}}(\mathcal{V}_\mathbb{Q})$ non-commutative ring generated by $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_\mathbb{Q})$ and $\tilde{\mu}_n$, μ_n^* for $n \in \mathbb{N}$ satisfying the Bost–Connes relations

• Euler characteristic $\chi^{\hat{\mathbb{Z}}}$ extends to ring homomorphism $\chi : \mathfrak{K}_0^{\hat{\mathbb{Z}}}(\mathcal{V}_\mathbb{Q}) \to \mathcal{A}_\mathbb{Z}$ to the integral Bost–Connes algebra

• after tensoring with $\mathbb{Q}$ semigroup crossed product rings

$$\chi : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_\mathbb{Q})_\mathbb{Q} \rtimes \mathbb{N} \to \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$$

with $\mathfrak{K}_0^{\hat{\mathbb{Z}}}(\mathcal{V}_\mathbb{Q}) \otimes_\mathbb{Z} \mathbb{Q} = K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_\mathbb{Q})_\mathbb{Q} \rtimes \mathbb{N}$ and $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_\mathbb{Q})_\mathbb{Q} = K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_\mathbb{Q}) \otimes_\mathbb{Z} \mathbb{Q}$, and $\mathcal{A}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} = \mathcal{A}_\mathbb{Q} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ rational Bost–Connes algebra

• noncommutative version: $\mathfrak{K}_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{Q}})$ non-commutative ring generated by $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{Q}})$ and $\tilde{\mu}_n$, μ_n^* for $n \in \mathbb{N}$ satisfying the Bost–Connes relations

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Spectra

- symmetric spectrum:
  - pointed spaces (pointed simplicial sets) $X = \{X_n\}_{n\geq 0}$ with action of symmetric group S_n on X_n
  - structure maps: based maps $\varphi_n^X : S^1 \wedge X_n \to X_{n+1}$
  - $S_n \times S_m$-equivariant composition

$$S^m \wedge X_n \stackrel{\varphi_n^X}{\to} S^{m-1} \wedge X_{n+1} \to \cdots S^1 \wedge X_{n+m-1} \stackrel{\varphi_{n+m-1}^X}{\to} X_{n+m}$$

  - morphisms S_n-equivariant based maps $f_n : X_n \to Y_n$ with $f_{n+1} \circ \varphi_n^X = \varphi_n^Y \circ (f_n \wedge Id_{S^1})$
  - *ring spectrum* with $S_n \times S_m$-equivariant multiplication maps $M_{n,m} : X_n \wedge X_m \to X_{n+m}$ and unit maps $\iota_0 : S^0 \to X_0$ and $\iota_1 : S^1 \to X_1$ with associativity and unit diagrams

Spectra and homotopy

- $\gamma_n^X : X_n \to \Omega X_{n+1}$ be the maps induced by the adjoints of the structure maps
- homotopy groups of spectra $\pi_k(X) = \varinjlim_n \pi_{k+n}(X_n)$ over maps induced by γ_n^X
- n–connected spectrum if $\pi_k(X) = 0 \ \forall k \leq n$
connective if -1-connected
- *sphere spectrum*: $\mathbb{S}$ with $\mathbb{S}_n = S^n \simeq S^1 \wedge \cdots \wedge S^1$ and $\varphi_n^{\mathbb{S}} : S^1 \wedge S^n \to S^{n+1}$, stable homotopy groups of spheres
- generalized homology theories (Steenrod axioms: homotopy invariance, exactness, excision, additivity, but not dimension axioms on homology of point)

Assemblers: from Grothendieck rings to spectra

- Inna Zakharevich, *The K-theory of assemblers*, Adv. Math. 304 (2017), 1176–1218

• category $\mathcal{C}$ with initial object $\emptyset$: two morphisms $f : Y \to X$ and $g : W \to X$ are disjoint if pullback $Y \times_X W$ exists and equal to $\emptyset$. A collection $\{f_i : X_i \to X\}_{i \in I}$ in $\mathcal{C}$ is disjoint if f_i and f_j are disjoint for all $i \neq j \in I$

• *sieve* in a category is a full subcategory closed under precomposition by morphisms in $\mathcal{C}$; Grothendieck topology: assignment of sieves $\mathcal{J}(X)$ in the over category $\mathcal{C}/X$ with pullback compatibility

• $\mathcal{C}$ with Grothendieck topology: collection of morphisms $\{f_i : X_i \to X\}_{i \in I}$ in $\mathcal{C}$ is a covering family if full subcategory of $\mathcal{C}/X$ containing all morphisms of $\mathcal{C}$ factoring through the f_i, is sieve $\mathcal{J}(X)$

• assembler category: $\mathcal{C}$ small category with Grothendieck topology, initial object $\emptyset$, where all morphisms are monomorphisms, and any two finite disjoint covering families of X in $\mathcal{C}$ have a common refinement also finite disjoint covering family

• associate to an assembler $\mathcal{C}$ a category $\mathcal{W}(\mathcal{C})$: objects $\{A_i\}_{i\in I}$ collections noninitial objects A_i of $\mathcal{C}$ indexed by finite set I, morphisms $f : \{A_i\}_{i\in I} \to \{B_j\}_{j\in J}$ map of finite sets $f : I \to J$ and morphisms $f_i : A_i \to B_{f(i)}$ with $\{f_i : A_i \to B_j \, : \, i \in f^{-1}(j)\}$ disjoint covering family $\forall j \in J$.

• coproduct of assemblers $\bigvee_{x\in X} \mathcal{C}_x$ with X finite set: objects initial object $\emptyset$ and all non-initial objects of assemblers $\mathcal{C}_x$, morphisms of non-initial objects induced by those of $\mathcal{C}_x$

• finite pointed set (X, x_0) and assembler $\mathcal{C}$: assembler $X \wedge \mathcal{C} := \bigvee_{x\in X\smallsetminus\{x_0\}} \mathcal{C}$. Then assignment $X \mapsto \mathcal{N}\mathcal{W}(X \wedge \mathcal{C})$ with $\mathcal{N}$ nerve

intuition: this category keeps track of abstract scissor-congruence relations that will be realized in the Grothendieck ring

Integral Bost–Connes algebra and spectra

• Zakharevich constructed an assembler $\mathcal{C}_{\mathcal{V}}$ of algebraic varieties and pairs of embeddings of closed subvarieties and their complement so that the associated spectrum $K(\mathcal{C}_{\mathcal{V}})$ has zeroth homotopy the Grothendieck ring of varieties (ring structure from monoidal structure): $\pi_0 K(\mathcal{C}_{\mathcal{V}}) = K_0(\mathcal{V})$

• here we consider a variant of this construction with an assembler $\mathcal{C}_{\mathcal{V}}^{\hat{\mathbb{Z}}}$ so that

$$\pi_0 K(\mathcal{C}_{\mathcal{V}}^{\hat{\mathbb{Z}}}) = K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{Q}})$$

and lift the integral Bost–Connes algebra to endofunctors of $\mathcal{C}_{\mathcal{V}}^{\hat{\mathbb{Z}}}$ that induce the Bost–Connes endomorphisms on $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{Q}})$

the assembler $\mathcal{C}_{\mathcal{V}}^{\hat{\mathbb{Z}}}$:

- Objects: varieties X with good $\hat{\mathbb{Z}}$-action (factoring through some finite $\mathbb{Z}/N\mathbb{Z}$)
- Grothendieck topology generated by covering families

$$\{Y \hookrightarrow X, X \smallsetminus Y \hookrightarrow X\}$$

$\hat{Z}$-equivariant embeddings, Y closed subvariety

Lift of Bost–Connes algebra:

- $\sigma_n(X, \alpha) = (X, \alpha \circ \sigma_n)$ and $\tilde{\rho}_n(X, \alpha) = (X \times Z_n, \Phi_n(\alpha))$
- endofunctors of $\mathcal{C}_{\mathcal{V}}^{\hat{\mathbb{Z}}}$
- σ_n comptabile with product

$$\sigma_n(X, \alpha) \times \sigma_n(X', \alpha') = (X \times X', (\alpha \times \alpha') \circ \Delta \circ \sigma_n) = \sigma_n((X, \alpha) \times (X', \alpha'))$$

- $\tilde{\rho}_n$ not compatible with products (group homomorphisms of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{Q}})$)

Lifting the equivariant Euler characteristic

• $\chi^G : K_0^G(\mathcal{V}) \to A(G) \to R(G)$ with $A(G)$ Burnside ring (finite group G) Grothendieck ring of G-sets

• for $\hat{\mathbb{Z}}$ completion $\hat{A}(\hat{\mathbb{Z}}) = \varprojlim A(\mathbb{Z}/n\mathbb{Z})$ Grothendieck ring of almost-finite $\hat{\mathbb{Z}}$-sets

• there is an assembler $\mathcal{AF}^G$ of almost-finite-G-sets (Zakharevich, 2018) and equivariant Euler characteristic $\chi^G : K_0^G(\mathcal{V}) \to A(G)$ lifts to morphism of assemblers

$$\chi^G : \mathcal{C}^G \to \mathcal{AF}^G$$

Variant: Kontsevich-Tschinkel Burnside ring

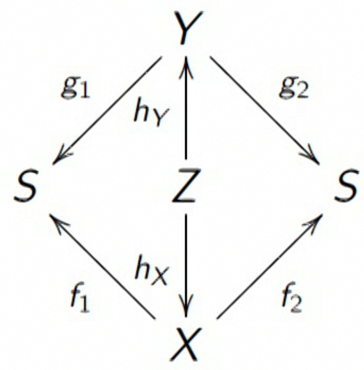
A refined version of the Grothendieck ring of varieties based on birational equivalence

- Assembler $\mathcal{C}^{\hat{\mathbb{Z}}}_{\mathrm{Burn}}$ with objects equivalence classes of smooth varieties with good $\hat{\mathbb{Z}}$-action morphisms $\hat{\mathbb{Z}}$-equivariant open embeddings $U \hookrightarrow X$
- $\pi_0 K(\mathcal{C}^{\hat{\mathbb{Z}}}_{\mathrm{Burn}})$ generated by objects of $\mathcal{C}^{\hat{\mathbb{Z}}}_{\mathrm{Burn}}$ with scissor-congruence relations $[X] = [U]$ for any dense open embedding $U \hookrightarrow X$

Epimorphic assemblers with sinks (Zakharevich)

- for a group G assembler $\mathbb{S}_G$ with objects $\emptyset$ and $\star$, a non-invertible morphism $\emptyset \to \star$ and invertible morphisms $\mathrm{Aut}(\star) = G$, spectrum $K(\mathbb{S}_G) = \Sigma^\infty_+ BG$
- assembler $\mathcal{C}$ with a sink object S such that $\mathrm{Hom}(X,S) \neq \emptyset$ for all other objects $X \in \mathcal{C}$
  - $f : X \to Y$ in $\mathcal{C}$ epimorphism (X non-initial)
  - single morphism set $\{f : X \to Y\}$ is covering family
  - for $X, Y \neq \emptyset$ no two morphisms $X \to Z$ and $Y \to Z$ are disjoint

• group $G_{\mathcal{C}}$ associated to epimorphic assembler with sink: elements equivalence classes of pairs of morphisms $f_1, f_2 : X \to S$ from non-initial object to sink with equivalence existence of object Z and maps $h_X : Z \to X$ and $h_Y : Z \to Y$



Composition by completing a pullback-type diagrams (unique up to equivalence)

• morphism of assemblers $\mathcal{C} \to \mathbb{S}_G$ equivalence on K-theory

• The assembler $\mathcal{C}^{\hat{\mathbb{Z}}}_{\rm Burn}$ of the Kontsevich-Tschinkel Burnside ring is a coproduct of epimorphic assemblers with sinks

$$\mathcal{C}^{\hat{\mathbb{Z}}}_{\rm Burn} = \bigvee_{[X,\alpha]} \mathcal{C}^{\hat{\mathbb{Z}}}_{\rm Burn}(X,\alpha)$$

$$K(\mathcal{C}^{\hat{\mathbb{Z}}}_{\rm Burn}(X,\alpha)) \simeq \Sigma^\infty_+ B{\rm Aut}^{\hat{\mathbb{Z}}}(\mathbb{K}(X,\alpha))$$

with ${\rm Aut}^{\hat{\mathbb{Z}}}(\mathbb{K}(X,\alpha))$ group of $\hat{\mathbb{Z}}$-equivariant birational automorphisms of X with good $\hat{\mathbb{Z}}$-action α

• relation between Kontsevich–Tschinkel Burnside ring $\mathrm{Burn}^{\hat{\mathbb{Z}}}(\mathbb{K})$ and equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})_{\mathbb{K}})$ at the level of spectra:

$$\mathcal{C}^{\hat{\mathbb{Z}}}_{\mathrm{Burn},n} := \bigvee_{[X,\alpha]\in B_n^{\hat{\mathbb{Z}}}} \mathcal{C}^{\hat{\mathbb{Z}}}_{\mathrm{Burn}}(X,\alpha)$$

$$K(\mathcal{C}^{\hat{\mathbb{Z}}}_{\mathrm{Burn},n}) \simeq \bigvee_{[X,\alpha]\in B_n^{\hat{\mathbb{Z}}}} \Sigma^\infty_+ B\mathrm{Aut}^{\hat{\mathbb{Z}}}(\mathbb{K}(X,\alpha))$$

$$\simeq \mathrm{hocofib}(K(\mathcal{C}_{\mathbb{K}}^{\hat{\mathbb{Z}},(n-1)}) \to K(\mathcal{C}_{\mathbb{K}}^{\hat{\mathbb{Z}},(n)}))$$

with $B_n^{\hat{\mathbb{Z}}}$ birational ($\hat{\mathbb{Z}}$-equivariant) isomorphism classes of varieties of dimension n with good $\hat{\mathbb{Z}}$-action and $\mathcal{C}_{\mathbb{K}}^{\hat{\mathbb{Z}},(\ell)}$ full sub-assembler varieties dimension $\leq n$

Dynamical view of $\mathbb{F}_1$-geometry

A new possible viewpoint on $\mathbb{F}_1$-geometry: general ideas

- want a natural replacement for $(\mathrm{Fr}_q, H^*_{\mathrm{et}}(V))$ Frobenius action on étale cohomology
- want an $\mathbb{F}_1$-structure to be related to roots of unity (counting of roots of unity as $\mathbb{F}_{1^m}$-points replacing counting of $\mathbb{F}_q$-points)

A topological setting: Morse–Smale diffeomorphisms

- M compact smooth manifold, $f : M \to M$ diffeomorphism
- $x \in M$ non-wandering if $\forall U \ni x \; \exists n > 0$ with $U \cap f^n(U) \neq \emptyset$
- f is *Morse-Smale* if finite number of non-wandering points and *structurally stable* (small deformation of f is isotopic to f)
- induced map f_* on homology $H_*(M, \mathbb{Z})$ is *quasi-unipotent*: eigenvalues are roots of unity

• consider varieties X endowed with an endomorphism $f : X \to X$ such that the induced action f_* on homology $H_*(X, \mathbb{Z})$ is quasi-unipotent

• $H_*(X, \mathbb{Z})$ as $\mathbb{Z}[T, T^{-1}]$-module with T acting as f_*

• Grothendieck ring $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ of pairs (X, f) with f_* unipotent, addition disjoint union, Cartesian product

• spectrum (in operator sense) of f_* acting on homology gives a ring homomorphism

$$\sigma : K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \sigma(X, f) = \sum_{\lambda \in \Sigma(f_*)} m_\lambda \, \lambda$$

Lift of integral Bost–Connes algebra

- maps $\sigma_n(X,f)=(X,f^n)$ and $\tilde{\rho}_n(X,f)=(X\times Z_n,\Phi_n(f))$ with Z_n zero-dimensional with $\#Z_n(\mathbb{C})=n$ and $\Phi_n(f)(x,a_i)=(x,a_{i+1})$ for $i=1,\dots,n-1$ and $\Phi_n(f)(x,a_n)=(f(x),a_1)$
- induced map $\Phi_n(f)_*:H_*(X\times Z_n,\mathbb{Z})\to H_*(X\times Z_n,\mathbb{Z})$ Verschiebung $V(f_*):H_*(X,\mathbb{Z})^{\oplus n}\to H_*(X,\mathbb{Z})^{\oplus n}$

$$V(f_*)=\begin{pmatrix} 0 & 0 & \dots & \dots & f_* \\ 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots 1 & 0 \end{pmatrix}.$$

- Bost–Connes relations (Verschiebung and Frobenius $V_n(F_n(a)b)=aV_n(b)$)

$$\sigma_n\circ\tilde{\rho}_n(X,f)=(X\times Z_n,\Phi_n(f)^n)=(X\times Z_n,f\times 1)=(X,f)^{\oplus n}$$

$$\tilde{\rho}_n\circ\sigma_n(X,f)=\tilde{\rho}_n(X,f^n)=(X\times Z_n,\Phi_n(f^n))=(X,f)\times(Z_n,\gamma),$$

• under the "spectrum Euler characteristic" map $\sigma : K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ commutative diagrams

$$\begin{array}{ccc} K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) & \xrightarrow{\sigma} & \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \downarrow \sigma_n & & \downarrow \sigma_n \\ K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) & \xrightarrow{\sigma} & \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{array} \qquad \begin{array}{ccc} K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) & \xrightarrow{\sigma} & \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \\ \tilde{\rho}_n \uparrow & & \uparrow \tilde{\rho}_n \\ K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) & \xrightarrow{\sigma} & \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \end{array}$$

• the σ_n are ring homomorphism and the $\tilde{\rho}_n$ are group homomorphisms.

• noncommutative version $\mathfrak{K}_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ generated by $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ and $\tilde{\mu}_n$ and μ_n^* with Bost–Connes relations: semigroup crossed product

$$\mathfrak{K}_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})_{\mathbb{Q}} \rtimes \mathbb{N}$$

with $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})_{\mathbb{Q}} = K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ with algebra homomorphism

$$\sigma : \mathfrak{K}_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$$

to Bost–Connes algebra

Assemblers and spectra in the dynamical setting

- assembler for $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$: category $\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}$ with objects the pairs (X,f) with f_* quasi-unipotent and morphisms $\phi : (Y, f|_Y) \hookrightarrow (X,f)$ embeddings $Y \hookrightarrow X$ of components preserved by f
- this gives $\pi_0 K(\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}) = K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$
- σ_n and $\tilde{\rho}_n$ on $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ lift to endofunctors of $\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}$, with σ_n compatible with monoidal structure
- ...but here spectrum $K(\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}})$ not so interesting topologically: decompositions into connected components (so that splitting of homology $H_*(X,\mathbb{Z}) = H_*(X_1,\mathbb{Z}) \oplus H_*(X_2,\mathbb{Z})$ with $\sigma(X,f) = \sigma(X_1,f_1) + \sigma(X_2,f_2)$)

Integral Bost–Connes and Λ-rings

- James Borger, *Lambda-rings and the field with one element*, arXiv:0906.3146

• Λ-ring structure Grothendieck: characteristic classes, Riemann–Roch

• Λ-ring structure as "descent data" for a ring from $\mathbb{Z}$ to $\mathbb{F}_1$

• Torsion free R with action of semigroup $\mathbb{N}$ by endomorphisms lifting Frobenius

$$s_p(x) - x^p \in pR, \quad \forall x \in R$$

Morphisms: $f \circ s_k = s_k \circ f$

• $\mathbb{Q}$-algebra $A \Rightarrow \Lambda$-ring
iff action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{N}$ on $\mathcal{X} = \mathrm{Hom}(A, \bar{\mathbb{Q}})$ factors through an action of $\hat{\mathbb{Z}}$

•Characteristic p versions of the Bost–Connes (Connes-Consani-Marcolli)

$$\mathbb{Q}/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}/\mathbb{Z})^{(p)}$$

denom = power of p; denom = prime to p

$$\mathbb{K}[\mathbb{Q}_p/\mathbb{Z}_p] \rtimes p^{\mathbb{Z}^+}$$

endomorphisms σ_n for $n = p^\ell$, $\ell \in \mathbb{Z}^+$
$\varphi_{\mathbb{F}_p}(x) = x^p$ Frobenius of $\mathbb{K}$ char p

$$(\sigma_{p^\ell} \otimes \varphi_{\mathbb{F}_p}^\ell)(f) = f^{p^\ell} \quad f \in \mathbb{K}[\mathbb{Q}/\mathbb{Z}]$$

$$(\sigma_{p^\ell} \otimes \varphi_{\mathbb{F}_p}^\ell)(e(r) \otimes x) = e(p^\ell r) \otimes x^{p^\ell} = (e(r) \otimes x)^{p^\ell}$$

$\Rightarrow$ BC endomorphisms restrict to Frobenius on mod p reductions:
σ_{p^ℓ} Frobenius correspondence on pro-variety $\mu^\infty \otimes_{\mathbb{Z}} \mathbb{K}$

- Embeddings of Λ-rings (Borger–de Smit)

Every torsion free finite rank Λ-ring embeds in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ with action of $\mathbb{N}$ compatible with $S_{n,diag} \subset M_n(\mathbb{Z})^+$

- multivariable Bost–Connes endomotives as universal Λ-rings

Question: associate to data (X, f) with f_* quasi-unipotent a Λ-ring as an $\mathbb{F}_1$-structure in the sense of Borger

- idea: take minimal subcategory $\mathcal{C}$ of $\mathbb{Z}[T, T^{-1}]$–modules, containing $H_*(X, \mathbb{Z})$ closed with respect to direct sums, tensor products, and exterior powers, then produce Grothendieck λ–ring structure on $K_0(\mathcal{C})$ using exterior powers to define the λ-structure

Torifications and $\mathbb{F}_1$-geometry

approach to $\mathbb{F}_1$-geometry via torifications (López-Peña–Lorscheid)

Idea: *geometrize* the expected behavior of the "counting of points" function

Levels of torified structures:

- Torification of the class in the Grothendieck ring
- Geometric torification
- Affine torification
- Regular torification

Also have a further *weaker* form of geometric torification: *constructible torifications*

- Grothendieck class torification

The class $[X] \in K_0(\mathcal{V}_{\mathbb{Z}})$ in the Grothendieck ring satisfies

$$[X] = \sum_k a_k \mathbb{T}^k$$

with $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$, and with coefficients $a_k \geq 0$.

- If $[X] \in \mathbb{Z}[\mathbb{L}]$ Tate motive, then polynomially countable $N_q(X) = P_X(q)$, then

$$\#X(\mathbb{F}_1) = P_X(1) = \lim_{q \to 1} N_q(X) = a_0$$

points over $\mathbb{F}_1$ and points over $\mathbb{F}_{1^m}$:

$$\#X(\mathbb{F}_{1^m}) = P_X(m+1) = \sum_k a_k \, m^k$$

where $N_q(\mathbb{T}) = N_q(\mathbb{A}^1 \smallsetminus \{pt\}) = q - 1$

Example: $\mathbb{F}_{1^m}$-points of $\mathbb{P}^1$

- Class $[\mathbb{P}^1] = 1 + \mathbb{L} = 2 + \mathbb{T}$ and counting $N_q(\mathbb{P}^1) = 1 + q$
- $\mathbb{F}_1$-points: $\#\mathbb{P}^1(\mathbb{F}_1) = 2$, say $\mathbb{P}^1 = \mathbb{G}_m \cup \{0, \infty\}$
- $\mathbb{F}_{1^m}$-points: $\#\mathbb{P}^1(\mathbb{F}_{1^m}) = m + 2$, given by $\{0, \infty\}$ *and* m-th roots of unity in $\mathbb{G}_m$

This suggest a more *geometric* notion of torification, as a decomposition of the variety, not only of the Grothendieck class.

- Geometric torification (López-Peña–Lorscheid)

Morphism of schemes $e_X : T \to X$ from (finite) disjoint union of tori $T = \coprod_{j\in I} T_i$, $T_j = \mathbb{G}_m^{d_j}$, with restriction of e_X to each torus an immersion inducing bijection of k–points, $e_X(k) : T(k) \to X(k)$, for every field k

- Affine torification

$\exists$ affine covering $\{U_\alpha\}$ of X compatible with e_X: $\forall\ U_\alpha$, $\exists$ subfamily $\{\, T_j \,|\, j \in I_\alpha\}$ of torification such that restriction $e_X|_{\cup_{j\in I_\alpha} T_j}$ torification of U_α

- Regular torification

Closure of tori T_j is union of other tori of the torification

Examples: $\mathbb{P}^n = \mathbb{A}^n \cup \cdots \cup \mathbb{A}^1 \cup \mathbb{A}^0$ affinely torified; Grassmanians $\mathrm{Gr}(n,j)$ torified (cell decomposition), but not affine; smooth toric varieties regular torification (torus orbits)

Torified varieties, Grothendieck rings, spectra

• Grothendieck rings $K_0(\mathcal{T})^s$, $K_0(\mathcal{T})^o$, and $K_0(\mathcal{T})^w$

• classes $[X]_s$, $[X]_o$, $[X]_w$ of torified varieties up to strong, ordinary, or weak isomorphisms of torified varieties

• scissor-congruence relations $[X]_a = [Y]_a + [X \smallsetminus Y]_a$ for $a = s, o, w$ if $Y \hookrightarrow X$ and $X \smallsetminus Y \hookrightarrow X$ are strong, ordinary, or weak morphisms of torified varieties

• assembler category $\mathcal{C}^a_{\mathcal{T}}$ for $a = s, o, w$ objects torified varieties X, morphisms locally closed embeddings that are strong, ordinary, or weak morphisms of torified varieties.

• Grothendieck topology generated by covering families

$$\{Y \hookrightarrow X, X \smallsetminus Y \hookrightarrow X\}$$

with both embeddings strong, ordinary, or weak morphisms

• assembler with spectrum $K(\mathcal{C}^a_{\mathcal{T}})$ with $\pi_0 K(\mathcal{C}^a_{\mathcal{T}}) = K^a_0(\mathcal{T})$.

Further aspects (ongoing)

- dynamical zeta functions for data (X, f) with f_* quasi-unipotent
- dynamical zeta function on torified varieties
- positive characteristic case and dynamical zeta functions (rationality questions)
- zeta functions and counting of $\mathbb{F}_1$-points
- zeta functions lift to assembler and spectra
- role of exponentiable motivic measures and Witt rings
- view of λ-ring structures through exponentiable motivic measures

... to be continued

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