Title: Elliptic quantum groups and their ï¬•nite-dimensional representations

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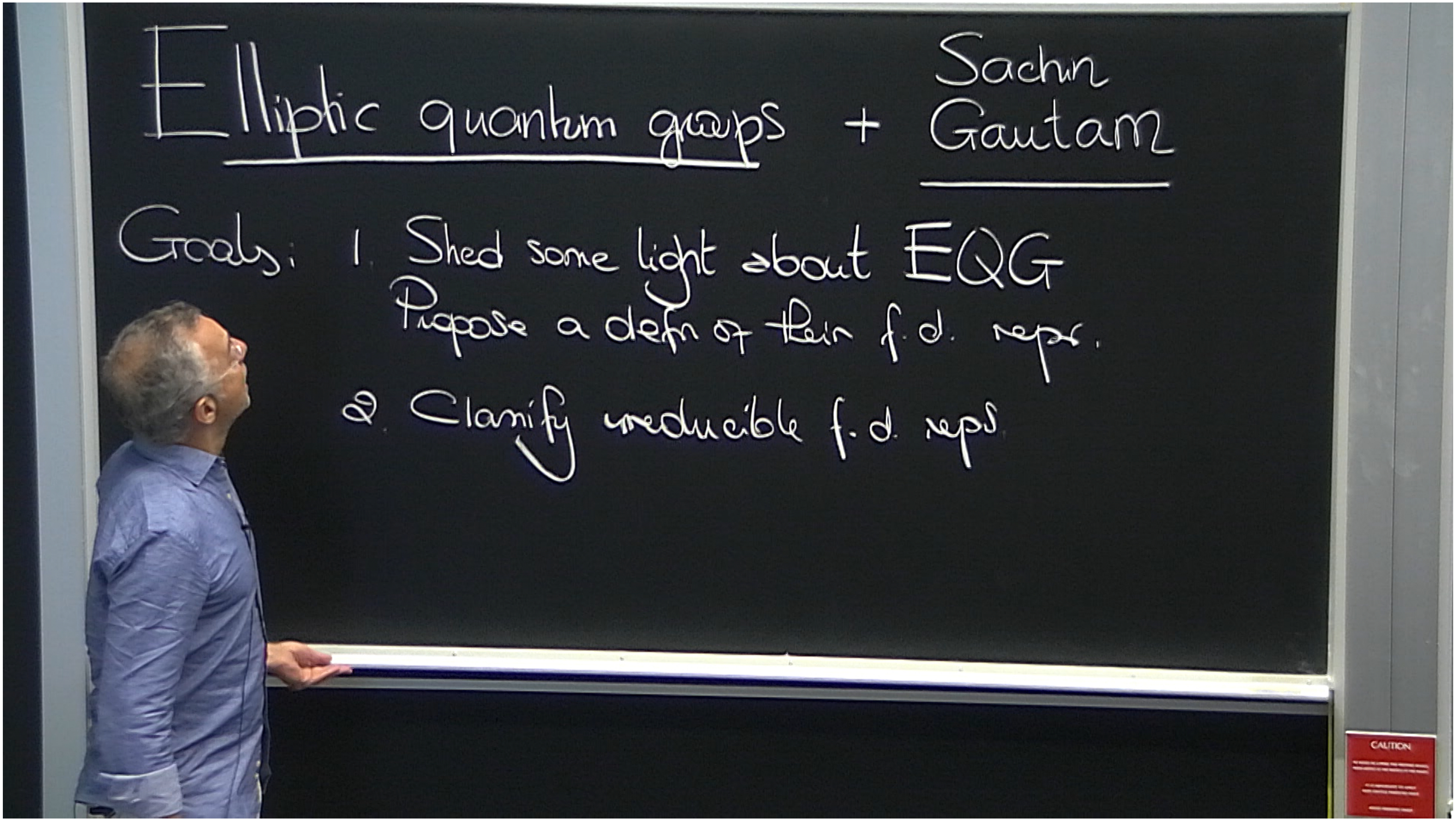
Abstract: I will describe joint work with Sachin Gautam where we give a deï¬•nition of the category of ï¬•nite-dimensional representations of an elliptic quantum group which is intrinsic, uniform for all Lie types, and valid for numerical values of the deformation and elliptic parameters. We also classify simple objects in this category in terms of elliptic Drinfeld polynomials. This classiï¬•cation is new even for sl(2), as is our deï¬•nition outside of type A.


Elliptic quantum groups + Sachin Gautam
Goals: 1. Shed some light about EQG
Propose a defn of their f.d. reps.
2. Classify irreducible f.d. reps
CAUTION

quantum groups $\neq$ groups

elliptic quantum groups $\neq$ quantum groups

EQG : Felder '94

QYBE

$$R_{12}(u_{12})R_{13}(u_{13})R_{23}(u_{23}) = R_{23}(u_{23})R_{13}(u_{13})R_{12}(u_{12})$$

- $R : \mathbb{C} \xrightarrow[\text{mero}]{} GL(V \otimes V)$, V f.d. $/\mathbb{C}$
- $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$
- $u_{ij} = u_i - u_j$

Answer 1

(Fadeev–Reshetikhin–Takhtajan)

$$R \rightsquigarrow \underset{+\ \text{fiber functor}}{\otimes\ \text{cat}\ \mathrm{Rep}(R) \xrightarrow{f} \mathrm{Vec}} \rightsquigarrow H_R := \mathrm{End}(f)$$

Hopf alg.

Flavour of Rep(R)

1) (W, L) $\quad W$ fd$/\mathbb{C}$, $L : \mathbb{C} \to GL(V \otimes W)$

s.t. $R_{12}(\ )L_{13}(\ )L_{23}(\ ) = L_{23}(\ )L_{13}(\ )R_{12}(\ )$

$V \otimes W \otimes W$

Ex: a) $(W,L) = (V,R)$ "vector" repr

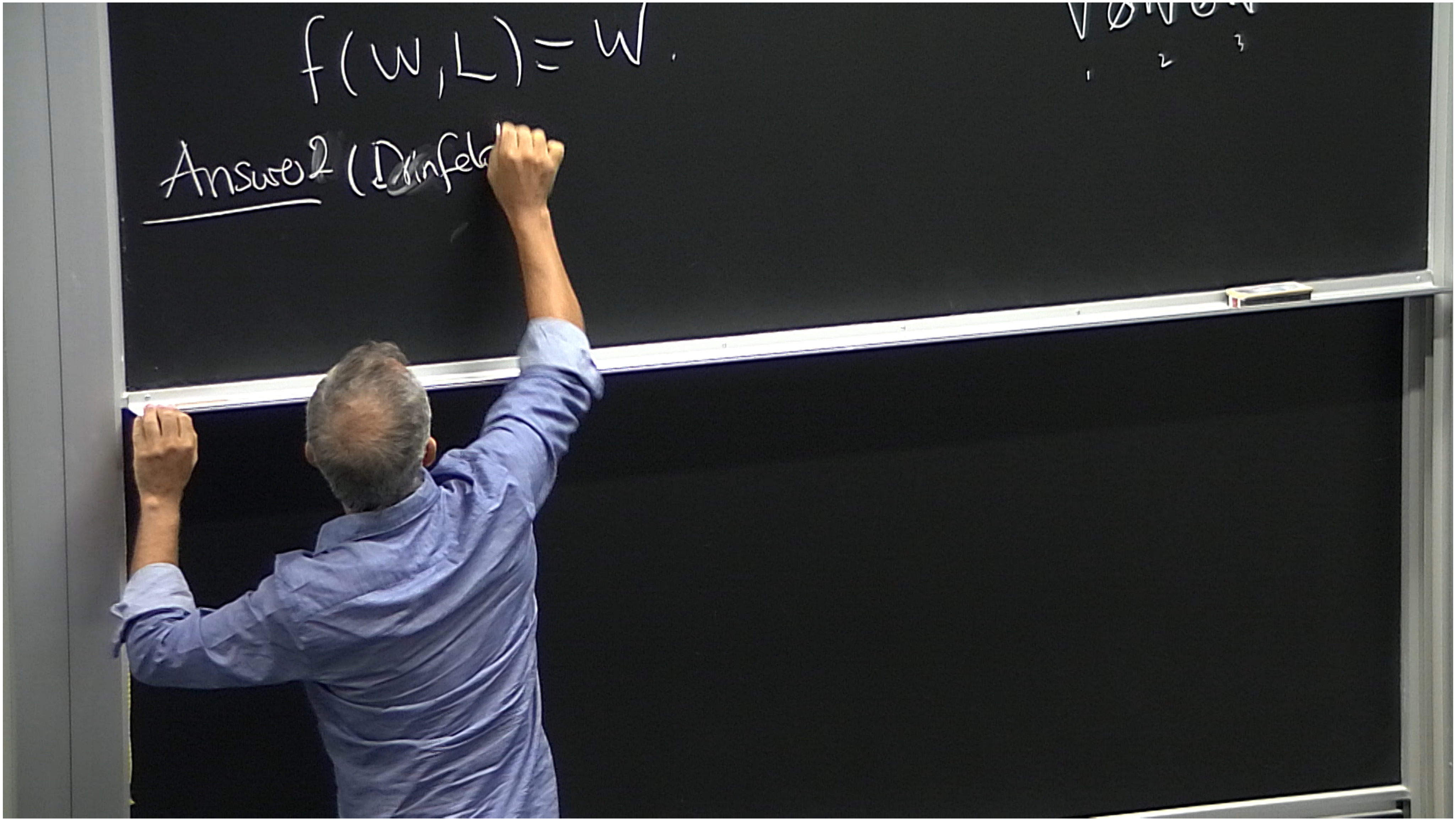
b) $(W,L) = (\mathbb{C}, \mathrm{id}_{V\otimes W})$ "trivial repr"

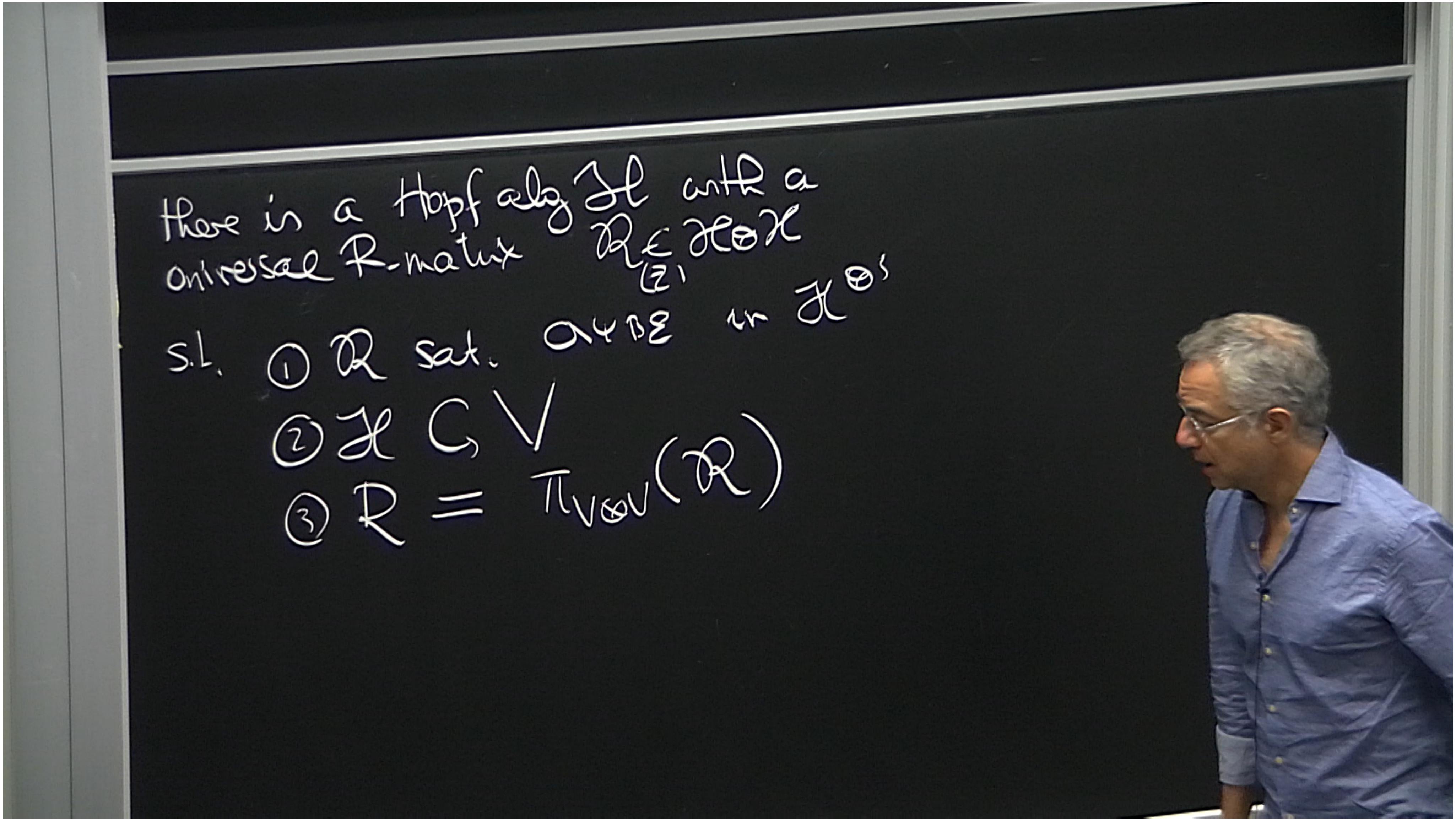
$\vdots$

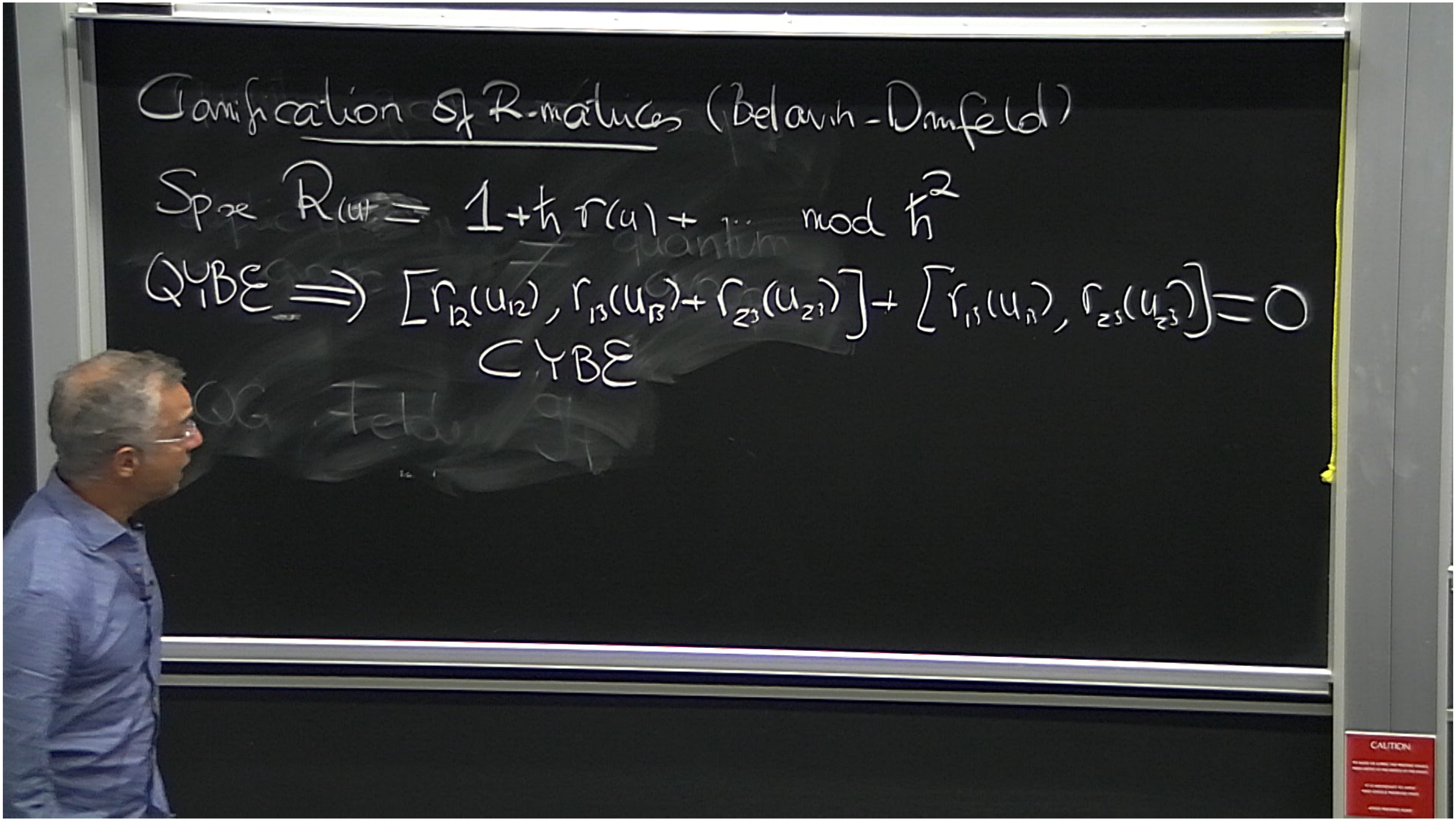
$$(W,L)\otimes(W',L') := (W\otimes W',\ L_{12}(u)\ L'_{13}(u))$$

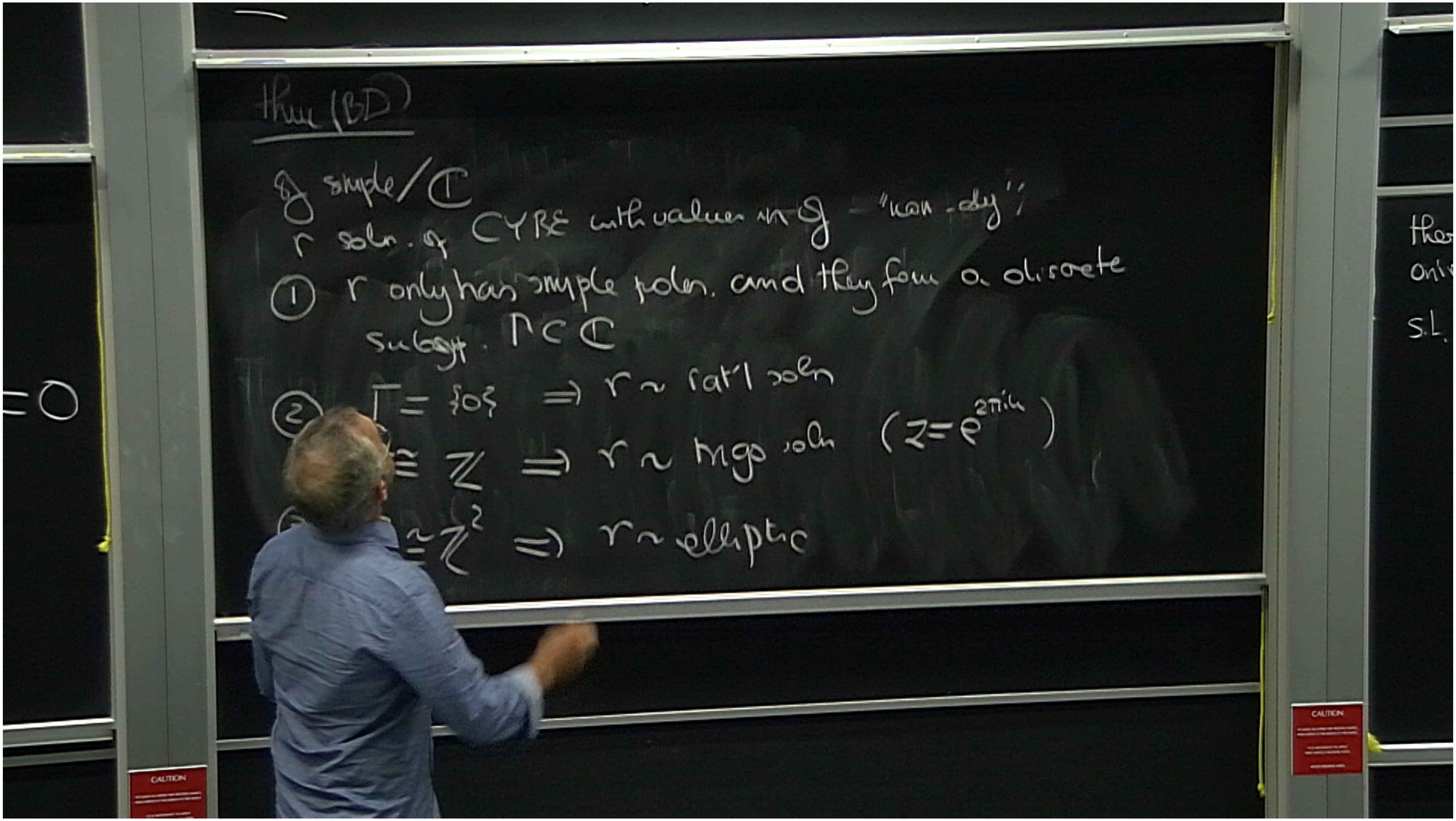
$$\circlearrowright$$

$$\underset{1}{V}\otimes\underset{2}{W}\otimes\underset{3}{W'}$$

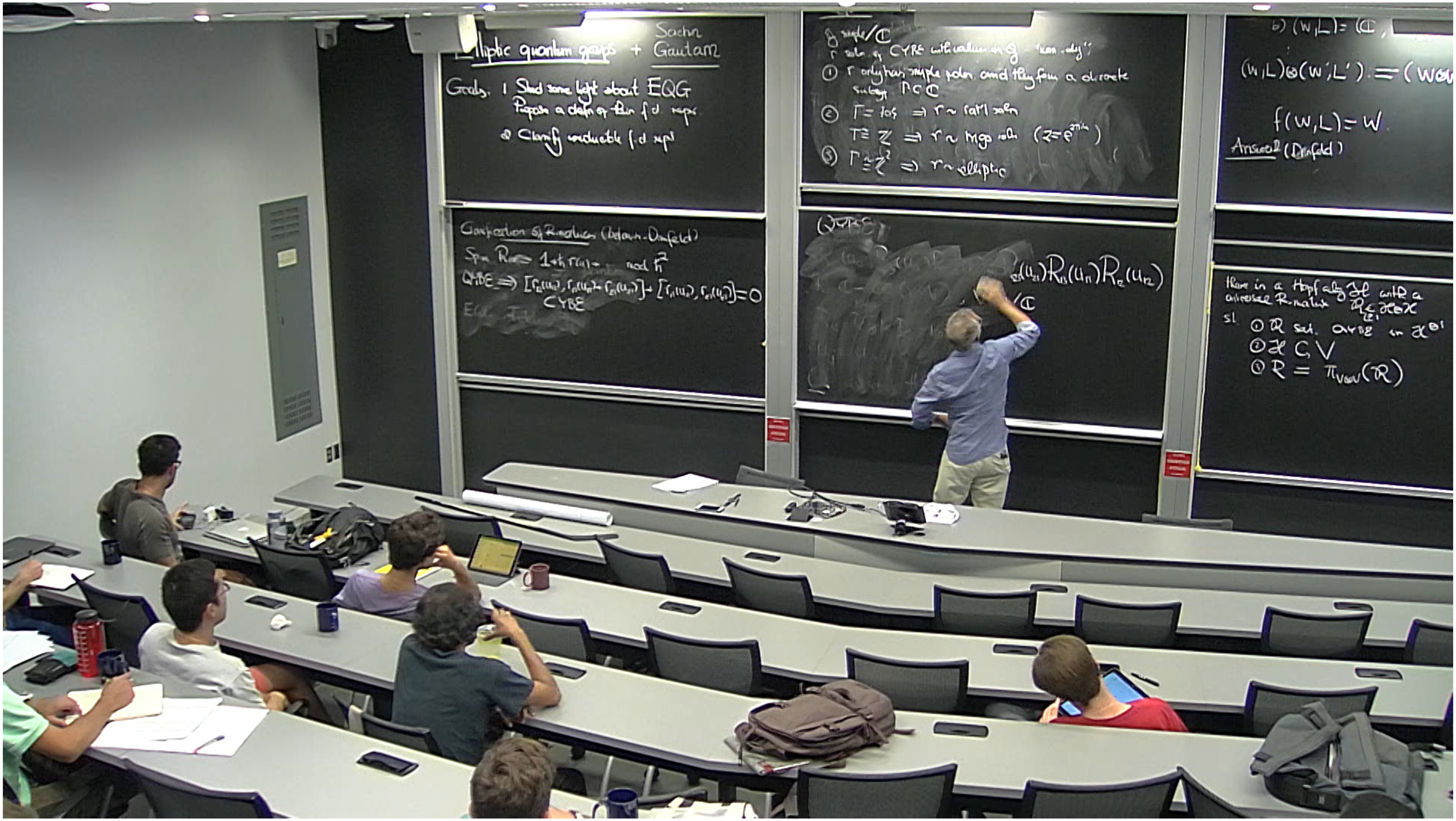

f(W,L)=W.
Answer 2 (Drinfel

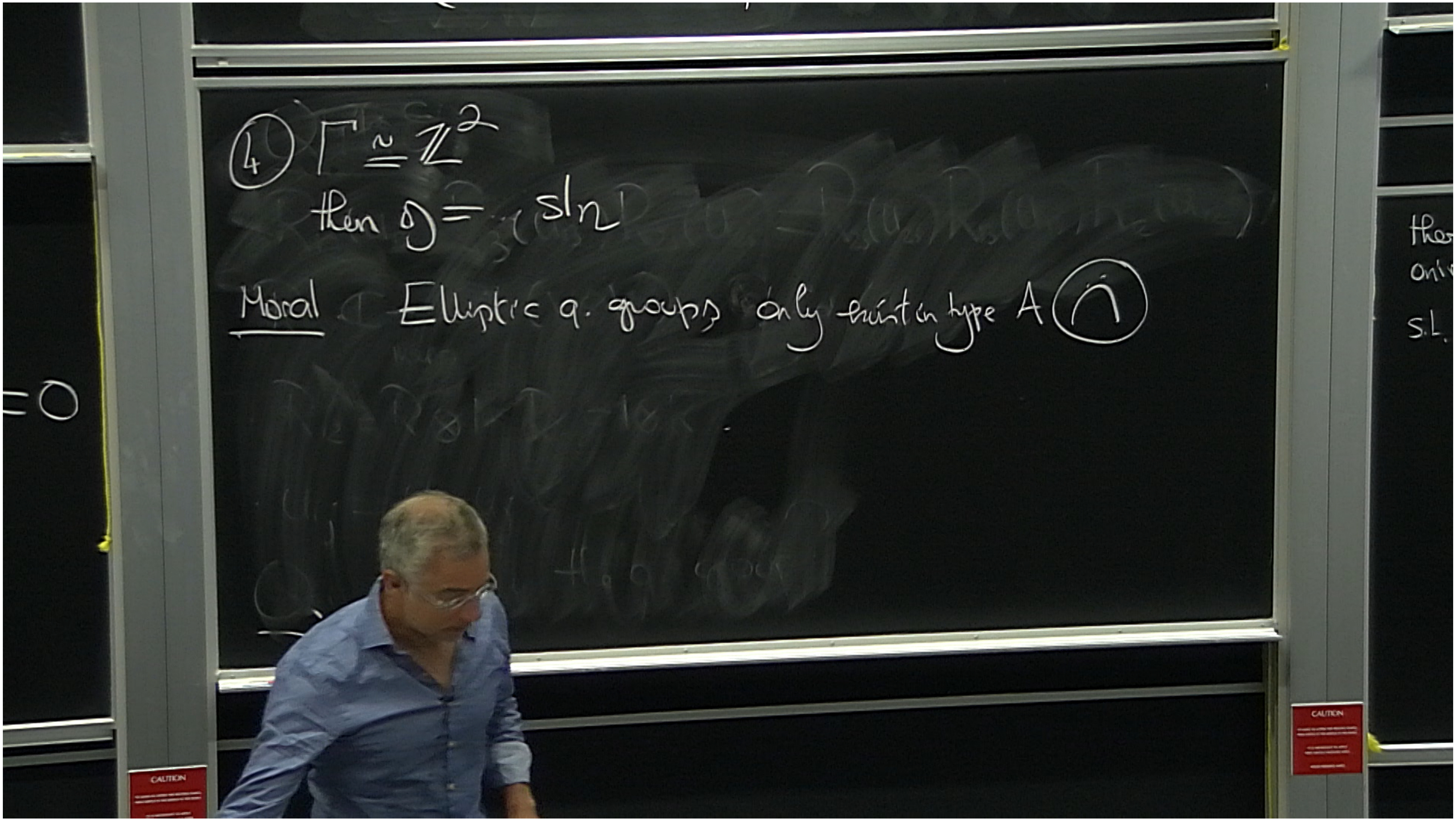

there is a Hopf alg $\mathcal{H}$ with a
universal R-matrix $\mathcal{R} \in \mathcal{H}\otimes\mathcal{H}$
s.t. ① $\mathcal{R}$ sat. QYBE in $\mathcal{H}^{\otimes 3}$
② $\mathcal{H} \curvearrowright V$
③ $R = \pi_{V\otimes V}(\mathcal{R})$


Classification of R-matrices (Belavin-Drinfeld)
Suppose $R(u) = 1 + \hbar r(u) + \dots \mod \hbar^2$
QYBE $\Longrightarrow [r_{12}(u_{12}), r_{13}(u_{13}) + r_{23}(u_{23})] + [r_{13}(u_{13}), r_{23}(u_{23})] = 0$
CYBE



thm (BD)
g simple/ℂ
r soln. of CYBE with values in g - "non-deg";
① r only has simple poles and they form a discrete subgp. Γ ⊂ ℂ
② Γ = {0} ⇒ r ~ rat'l soln
≅ ℤ ⇒ r ~ trig soln (z = e^2πiu)
≅ ℤ^2 ⇒ r ~ elliptic


Elliptic quantum groups + Sachin Gautam
Goals: 1 Shed some light about EQG
Propose a defn of their f.d. reps
2 Classify irreducible f.d. reps
Quantization of R-matrices (Belavin-Drinfeld)
CYBE
Q CYBE
g simple/ℂ
① r only has simple poles and they form a discrete subgroup Γ ⊂ ℂ
② Γ = {0} ⇒ r ~ rational soln
Γ ≅ ℤ ⇒ r ~ trigo soln
③ Γ ≅ ℤ² ⇒ r ~ elliptic
(W,L) ⊗ (W',L')
f(W,L) = W
Answer 2 (Drinfeld)
there is a Hopf alg with a universal R-matrix
① R sat. QYBE
② C V
③ R = π_{V⊗V}(R)


4 $\Gamma \cong \mathbb{Z}^2$
then $\mathfrak{g} = \mathfrak{sl}_2$
Moral Elliptic q. groups only exist in type A

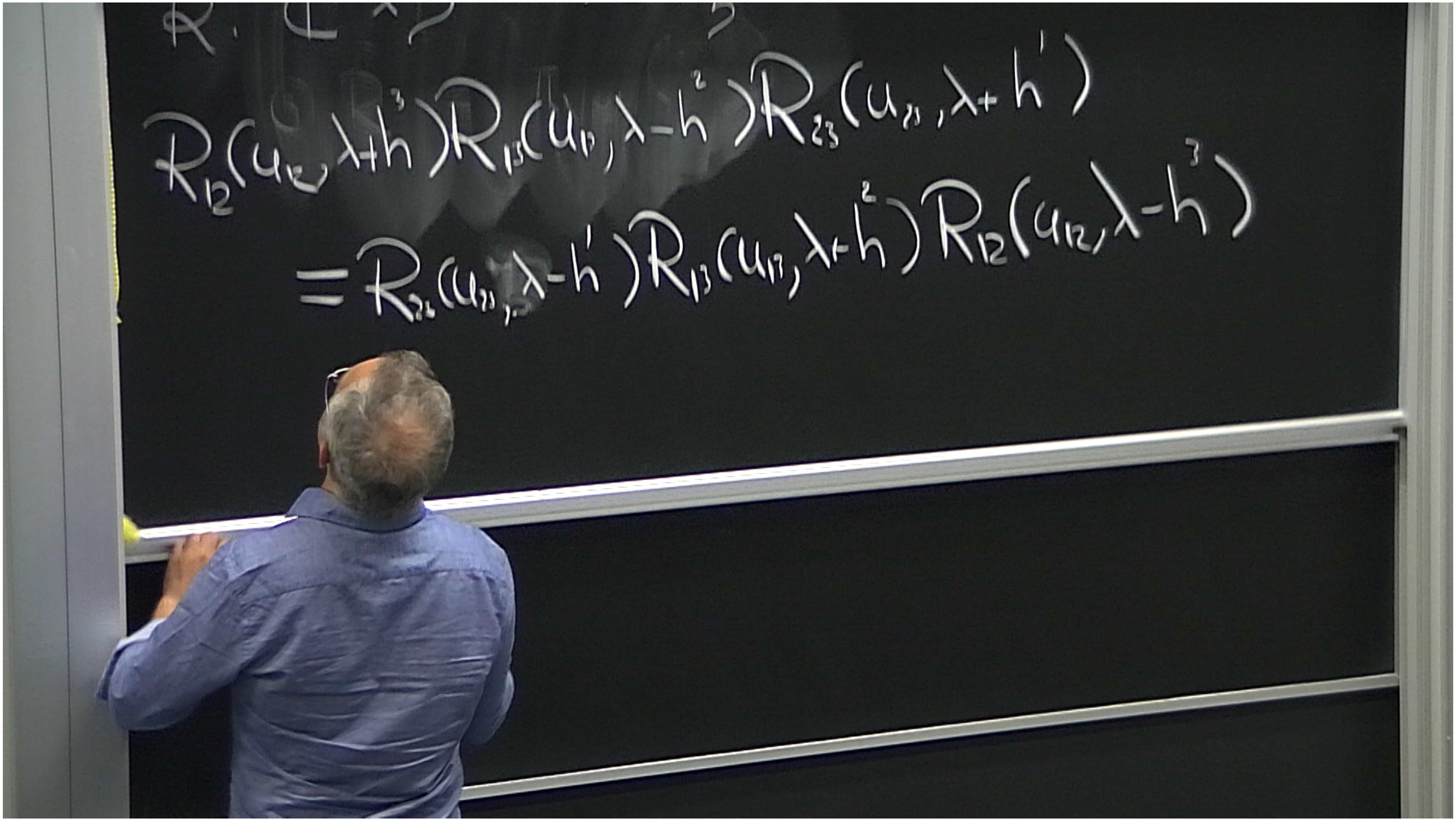
Felder ('94)

dynamical QYBE: V f.d./$\mathbb{C}$ $\rightsquigarrow$ $V = \bigoplus_{\mu \in \mathfrak{h}^*} V(\mu)$, $\mathfrak{h}$ abelian Lie alg.

$$R : \mathbb{C} \times \mathfrak{h}^* \longrightarrow GL_{\mathfrak{h}}(V \otimes V)$$

$$R_{12}(u_{12}, \lambda \quad) R_{13}(u_{13}, \lambda \quad) R_{23}(u_{23}, \lambda \quad)$$

$$= R_{23}(u_{23}, \lambda \quad) R_{13}(u_{13}, \lambda \quad) R_{12}(u_{12}, \lambda \quad)$$

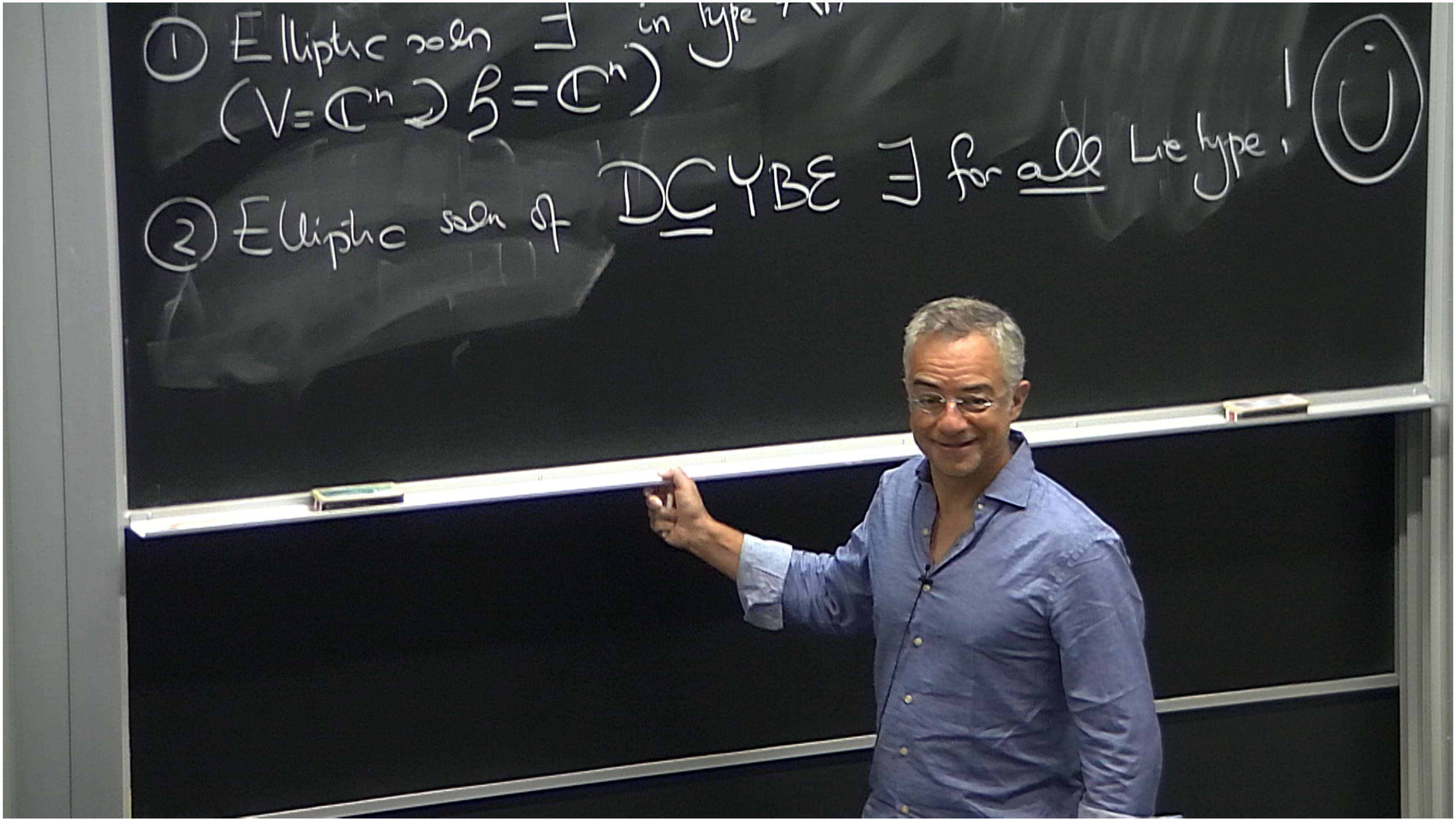

$R_{12}(u_{12}, \lambda + h^3) R_{13}(u_{13}, \lambda - h^2) R_{23}(u_{23}, \lambda + h^1)$
$= R_{23}(u_{23}, \lambda - h^1) R_{13}(u_{13}, \lambda + h^2) R_{12}(u_{12}, \lambda - h^3)$

$$R_{12}(u,\lambda+h^{3})\, v_1\otimes v_2\otimes \underset{\substack{\cap \\ V_3[\mu]}}{v_3} = \left(R(u,\lambda+\mu)\, v_1\otimes v_2\right)\otimes v_3$$

<u>Moreover</u>

① Elliptic soln $\exists$ in type A_n
($V=\mathbb{C}^n \supseteq \mathfrak{h}=\mathbb{C}^n$)

② Elliptic soln of <u>DC</u>YBE


① Elliptic soln ∃ in type
(V=ℂ^n ⊇ 𝔥=ℂ^n)
② Elliptic soln of DCYBE ∃ for all Lie type!

Moreover

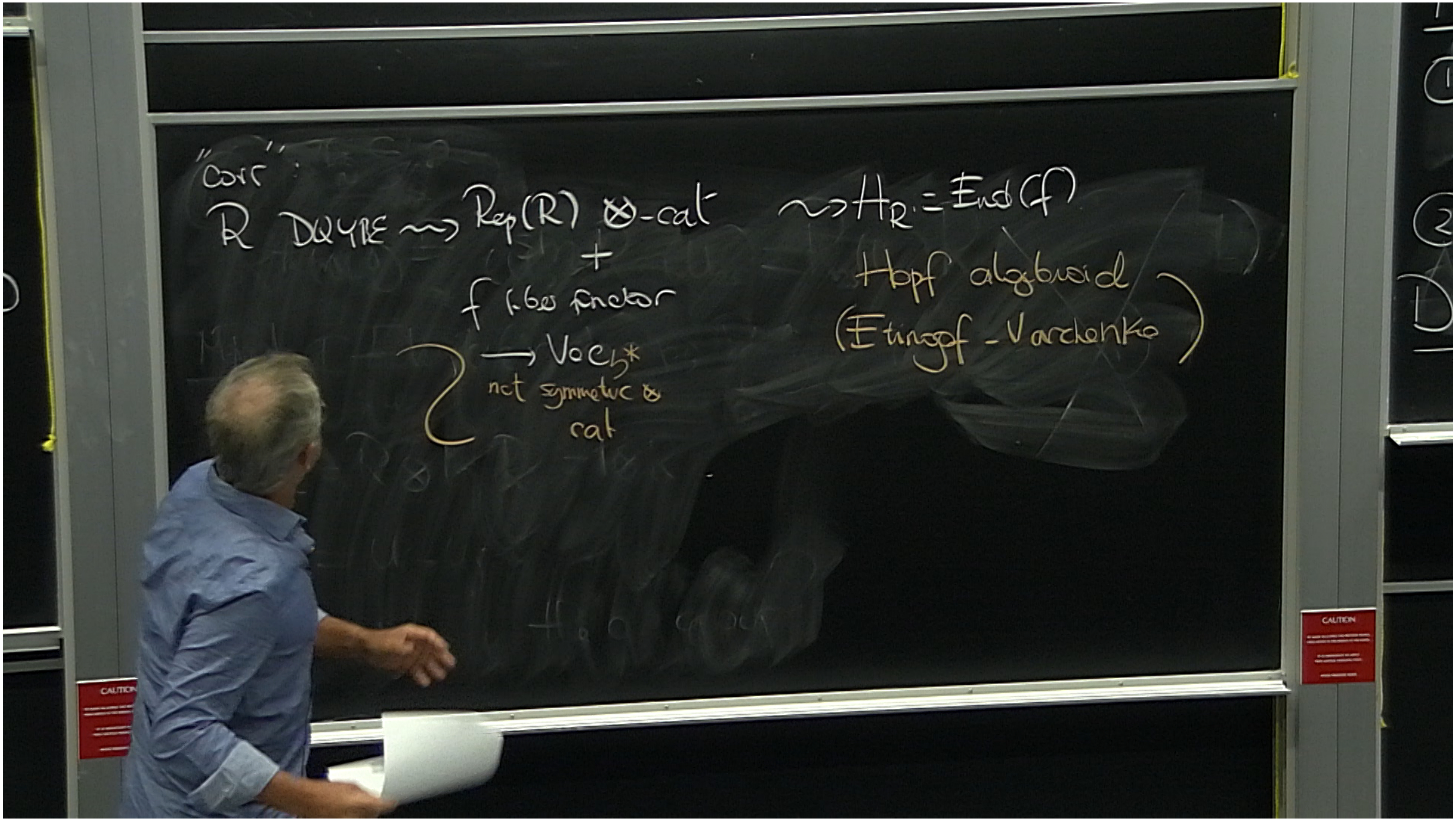
① Elliptic soln $\exists$ in type A_n

$(V = \mathbb{C}^n \supseteq \mathfrak{h} = \mathbb{C}^n)$

② Elliptic soln of DCYBE $\exists$ for all Lie type!

Defn 1.0 (Felder)

EQG are the q. groups corr. to elliptic soln. of the DQYBE.


"corr":
R DQYBE ⇝ Rep(R) ⊗-cat ⇝ H_R = End(f)
+
f fiber functor
→ Vec_h*
not symmetric ⊗
cat
Hopf algebroid
(Etingof - Varchenko)

Etingof - Schiffmann ('98)

Jimbo - Konno - Odake - Shiraishi ('97)

Elliptic soln. of the DQYBE $\exists\ \forall\ \mathfrak{g}$

& V f.d. repr. of the $U_q L\mathfrak{g}$ - q. loop algebra.

$$J^{21}(\lambda)^{-1}_{V(z)\otimes V} \cdot R_{VV}(z) \cdot J_{V(z)\otimes V}(\lambda) \quad \leftarrow \text{fusion op.}$$

R_{VV} ↑ R-matrix of $U_q(L\mathfrak{g})$

$$R_{V(z),V}(\mathcal{R}) \qquad J_{V(z)\otimes V}(\lambda) = \Pi_{V(z)\otimes V}(J(\lambda))$$

$\mathfrak{g}$ ss/$\mathbb{C}$

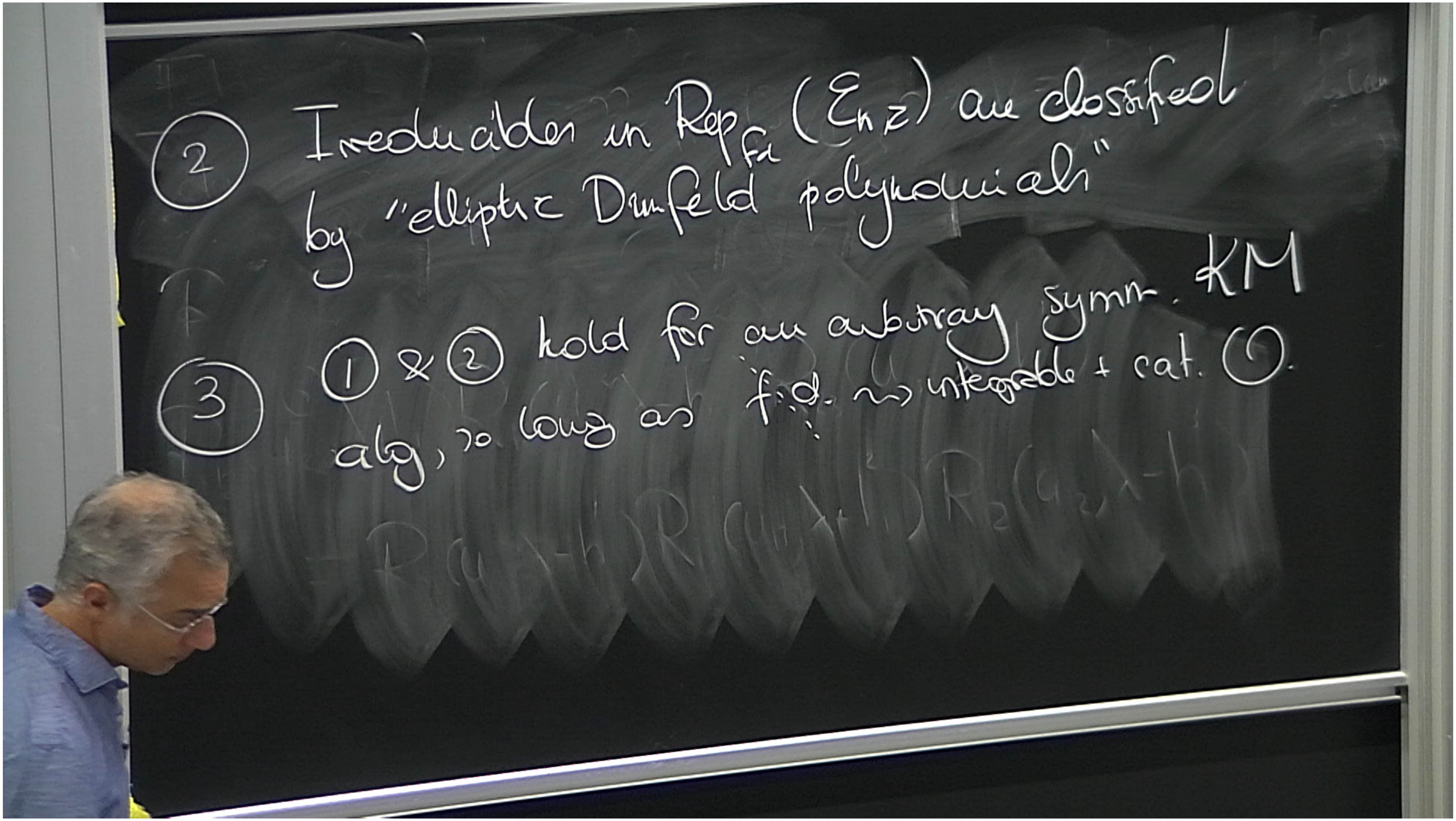
(1) $\exists$ a definition of $\mathrm{Rep}_{fd}(\mathcal{E}_{\hbar,\tau}\mathfrak{g})$ which is

× intrinsic (no $U_q L\mathfrak{g}$ f.ex. ...)

× Uniform for all Lie types.

× valid for numerical values of $\hbar \in \mathbb{C}$ s.t. $\hbar$ has ∞-order in $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$.

$\mathrm{Im}\,\tau > 0$


② Irreducibles in Rep$_{fd}$($\mathcal{E}_{n,z}$) are classified
by "elliptic Drinfeld polynomials"
KM
③ ① & ② hold for an arbitrary symm. KM
alg, so long as f.d. ⇝ integrable + cat. $\mathcal{O}$.

Remarks

1. in our Defn 3.0

2. Extend the classification of irrep. f.d. rep. of Yangian $Y_\hbar\mathfrak{g}$ & q-loop alg $U_qL\mathfrak{g}$

(Drinfeld) $Y_\hbar\mathfrak{g}$ irreps $\longleftrightarrow$ $(S\mathbb{C})^I$

(Chari-Pressley) $U_qL\mathfrak{g}$ " $\longleftrightarrow$ $(S\mathbb{C}^\times)^I$

(GTL) $E_{\hbar,\tau}\mathfrak{g}$ " $\longleftrightarrow$ $(SE_\tau)^I$

Defn of Rep($E_{\tau,2}\, sl_2$)

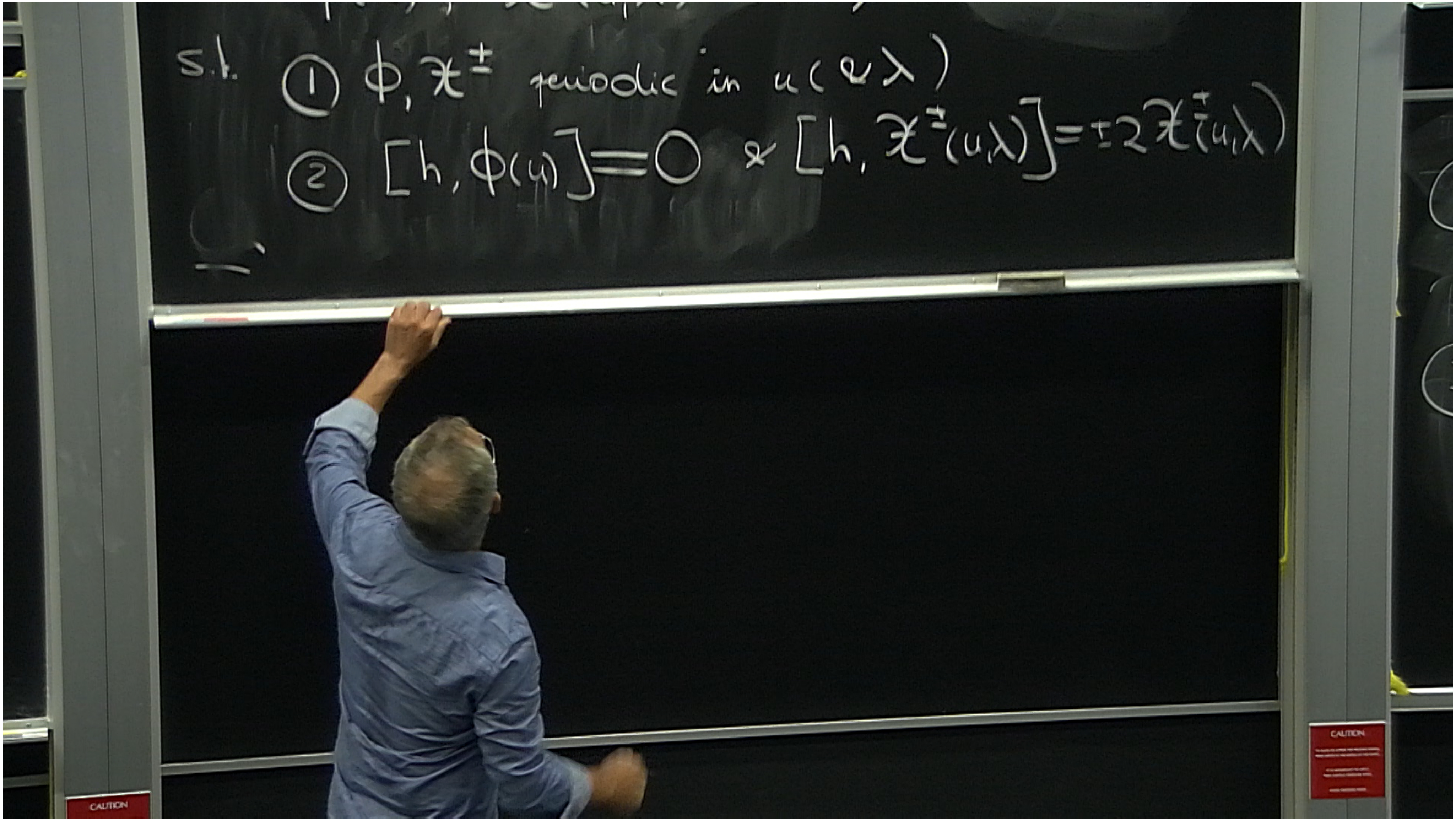
A f.d. rep. is .. f.d. v. space $V=\bigoplus_{\mu\in\mathfrak{h}^*} V[\mu]$, $\mathfrak{h}=\mathbb{C}h$

and is endowed with

$$\phi(u),\ \mathcal{X}^\pm(u,\lambda) : \mathbb{C}\times\mathfrak{h}^*\longrightarrow \mathrm{End}(V)$$

s.t. (1) $\phi, \mathcal{X}^\pm$ periodic in u (& λ)

(2) $[h,\phi(u)]=0$


s.t. ① φ, X± periodic in u (& λ)
② [h, φ(u)] = 0 & [h, X±(u,λ)] = ±2X±(u,λ)
CAUTION

③ $[\phi(u), \phi(v)] = 0 \quad \forall u, v.$

④ $\phi(u)\, x^{\pm}(v,\lambda)\, \phi(u)^{-1}$

$$= \frac{\Theta(u-v\pm\hbar)}{\Theta(u-v\mp\hbar)}\, x^{\pm}(v, \lambda\pm 2\hbar)$$

$$\pm \frac{\Theta(2\hbar)}{\Theta(\lambda)} \frac{\Theta(u-v-\lambda\mp\hbar)}{\Theta(u-v\mp\hbar)}\, x^{\pm}(u\mp\hbar, \lambda\pm 2\hbar)$$

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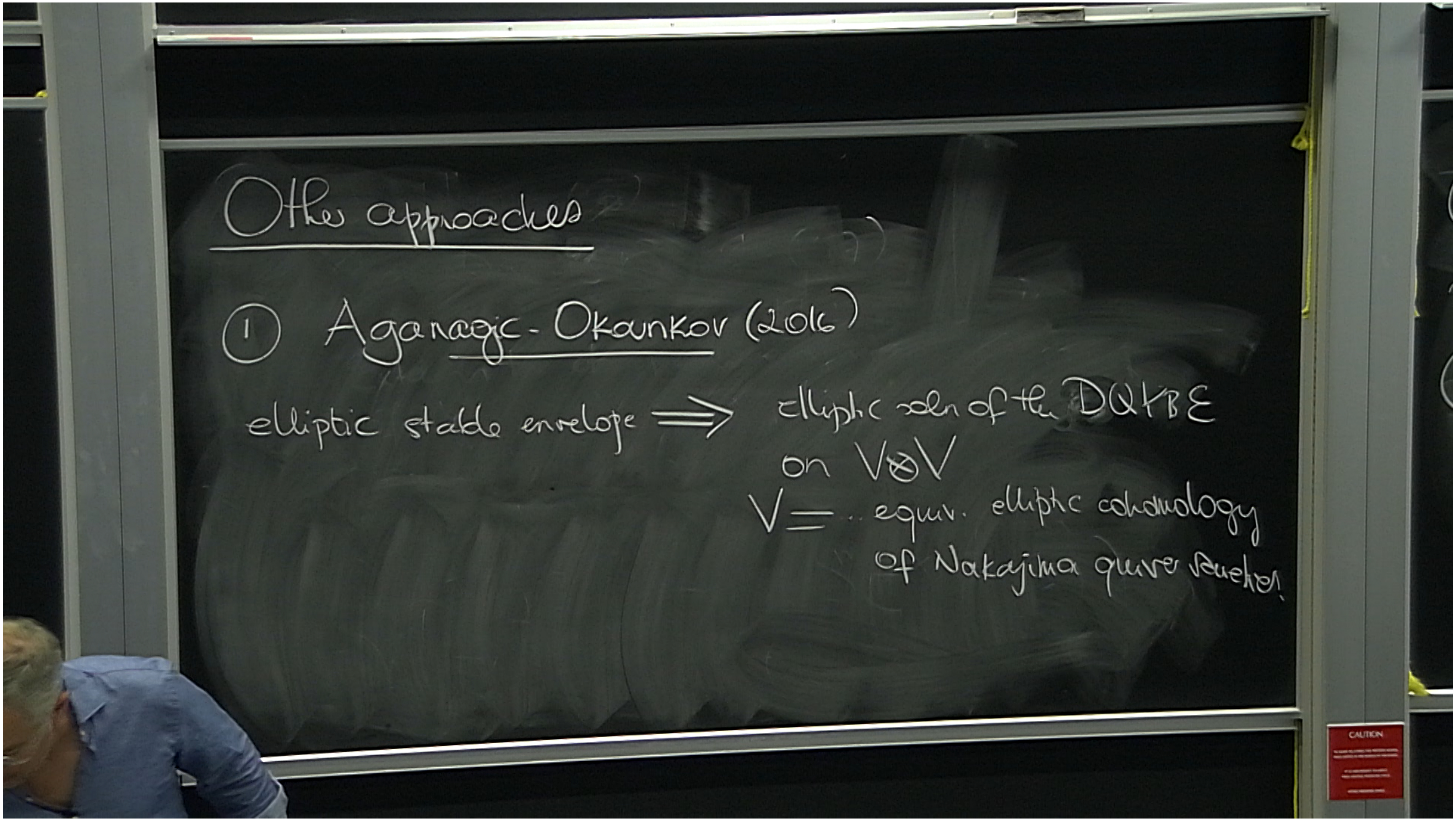
5. Reln b/w $\mathcal{X}^{\pm}(u,\lambda)$ & $\mathcal{X}^{\pm}(v,\lambda)$

6. ——— $\mathcal{X}^{+}(u,\lambda)$ & $\mathcal{X}^{-}(v,\lambda)$


Other approaches
① Aganagic - Okounkov (2016)
elliptic stable envelope ⟹ elliptic soln of the DQYBE
on V⊗V
V = .. equiv. elliptic cohomology
of Nakajima quiver varieties

(b) Yang-Zhao (arXiv: 17 8 ...)

Q : sym. quivers

(1) $\mathcal{E}ll_G(\Pi) \curvearrowright \mathcal{E}ll_G(\mathcal{M})$

coho. Hall alg

(2) contains operators which satisfy the comm. relns. (1) – (6)

③ Sheafified version of elliptic q. group $\mathcal{E}_{\tau,z}(\mathfrak{g})$

is a bialg. object in a meso $\otimes$-cat (braided)

For $\mathfrak{g} = \mathfrak{sl}_2$:

$$\left(\coprod_{n\geq 0} \mathrm{Coh}(S^n E_\tau),\ \underset{\text{shuffle}}{\otimes},\ \frac{\theta(\ldots)}{\theta(\ldots)},\ \text{abelian braiding} \right)$$