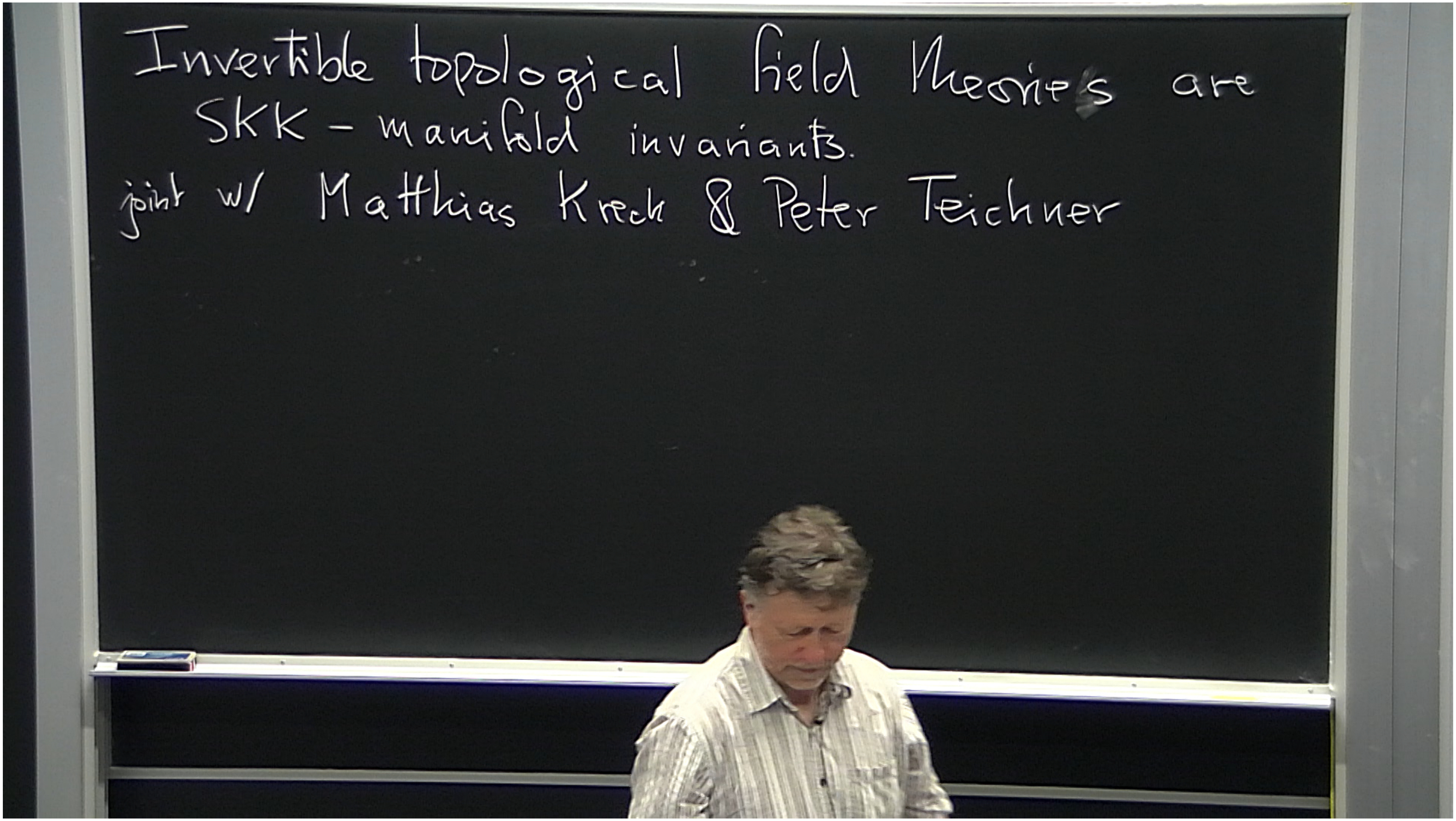
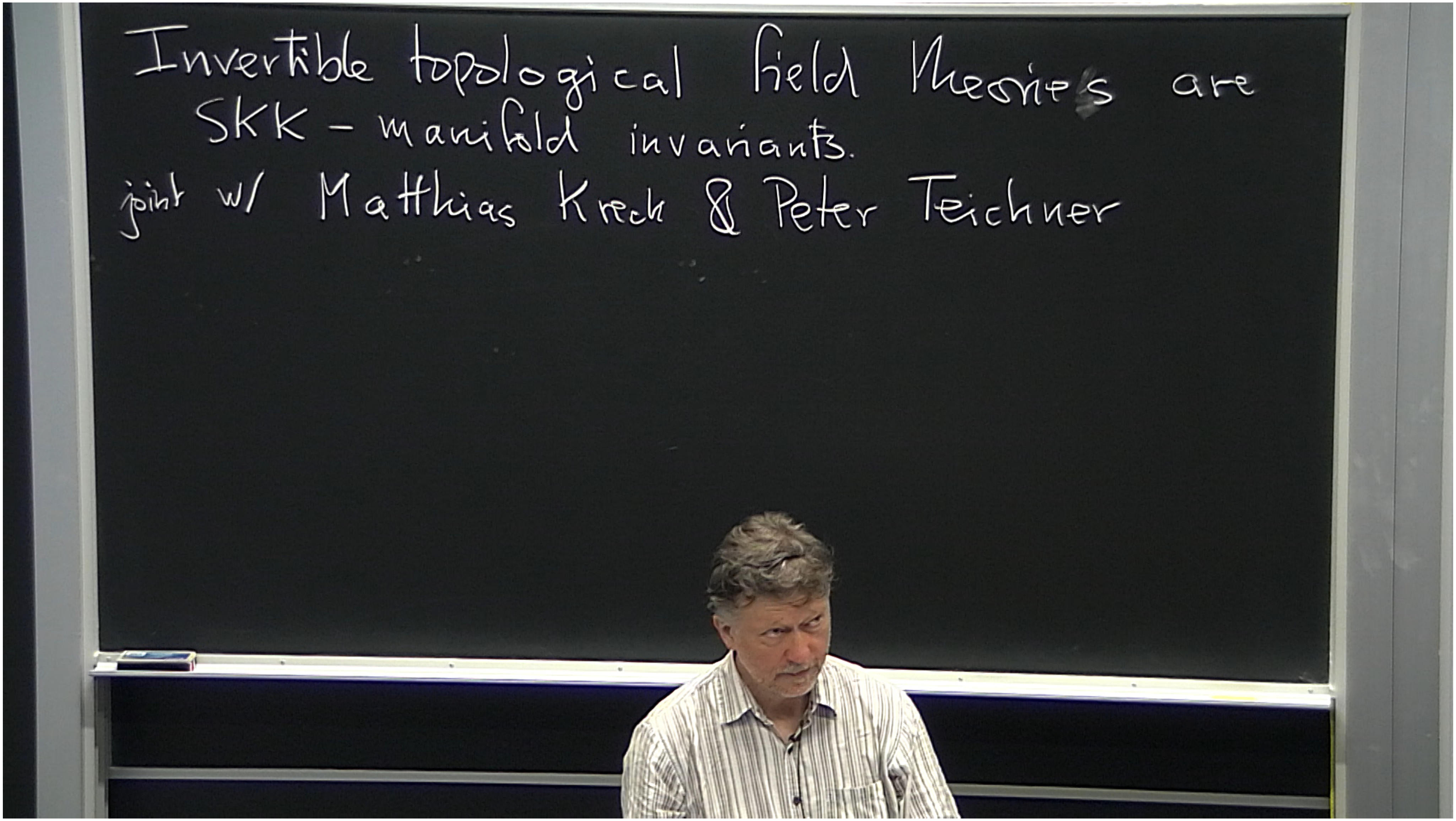
Title: Invertible topological ï¬•eld theories are SKK manifold invariants

Date: Aug 17, 2018 09:00 AM

URL: http://pirsa.org/18080048

Abstract: Topological ï¬•eld theories in the sense of Atiyahâ€"Segal are symmetric monoidal functors from a bordism category to the category of complex (super) vector spaces. A ï¬•eld theory E of dimension d associates vector spaces to closed (d-1)-manifolds and linear maps to manifolds of dimension d. It turns out that if E is invertible, i.e., if the vector spaces associated to (d-1)-manifolds have dimension one, then the complex number E(M) that E associates to a closed d-manifold M, is an SKK manifold invariant. Here these letters stand for schneiden=cut, kleben=glue and kontrolliert=controlled, meaning that E(M) does not change when modifying the manifold by cutting and gluing along hypersurfaces in a controlled way. The main result of this joint work with Matthias Kreck and Peter Teichner is that the map described above gives a bijection between topological ï¬•eld theories and SKK manifold invariants.


Invertible topological field theories are
SKK - manifold invariants.
joint w/ Matthias Kreck & Peter Teichner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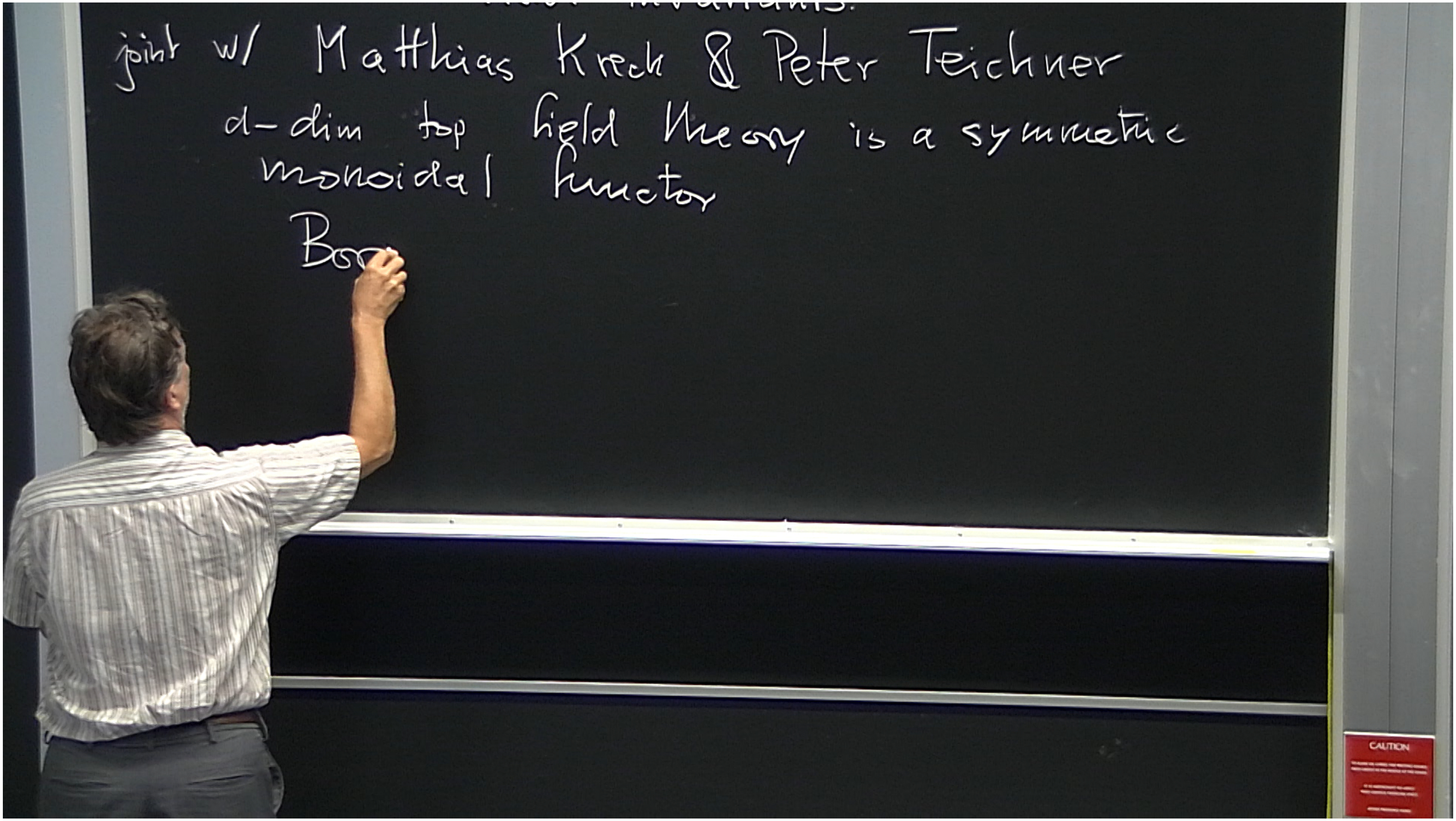
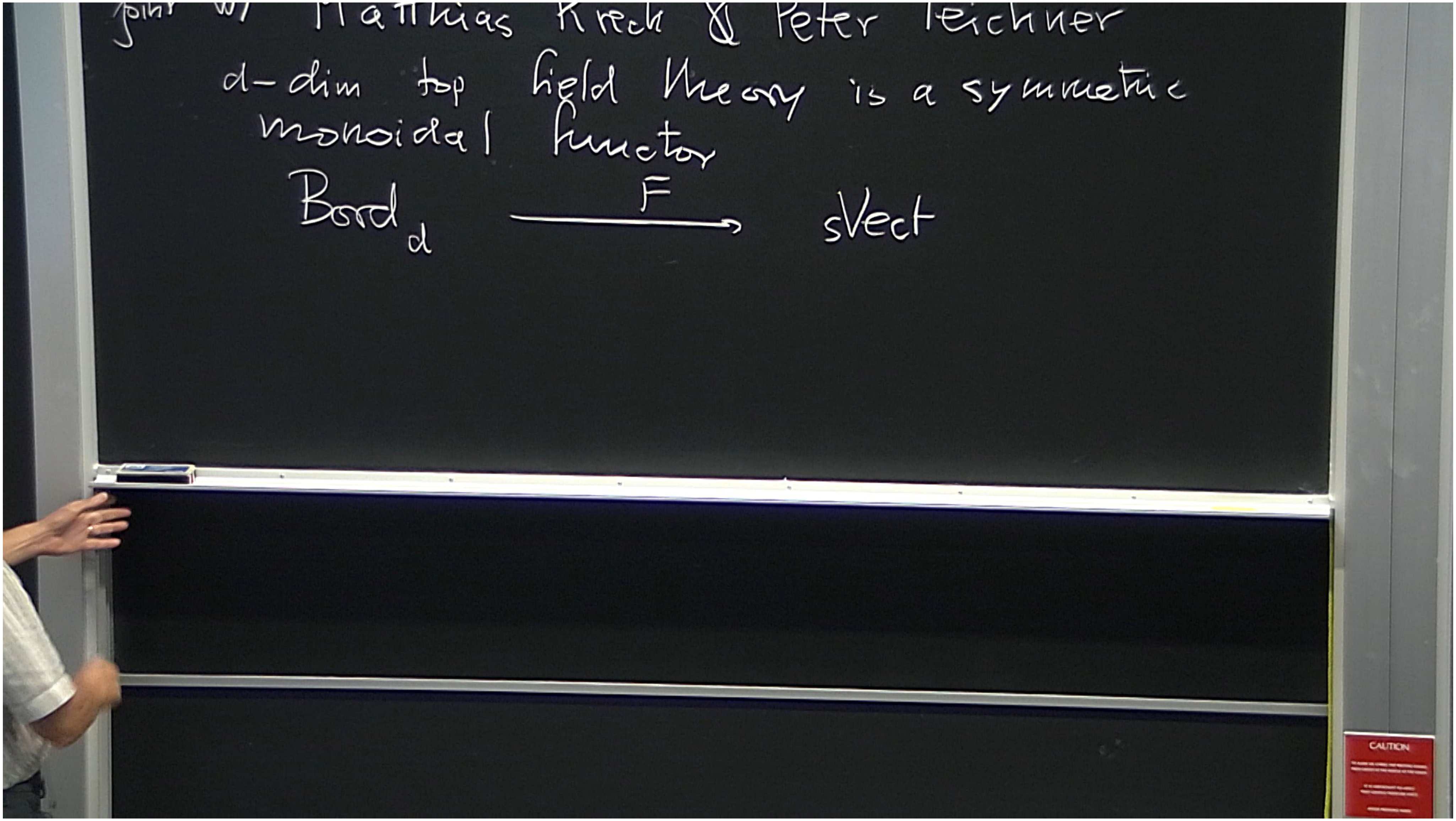
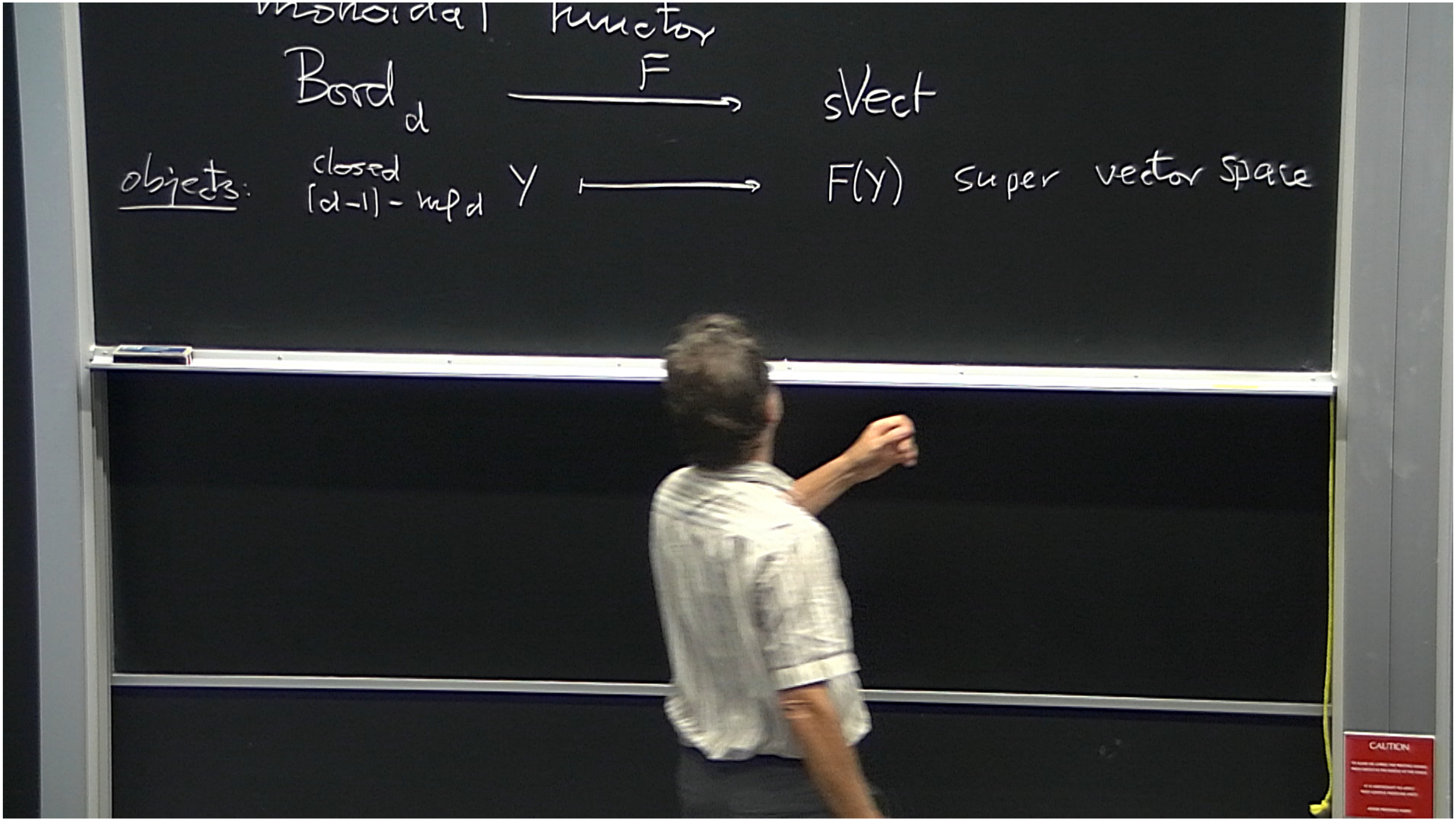
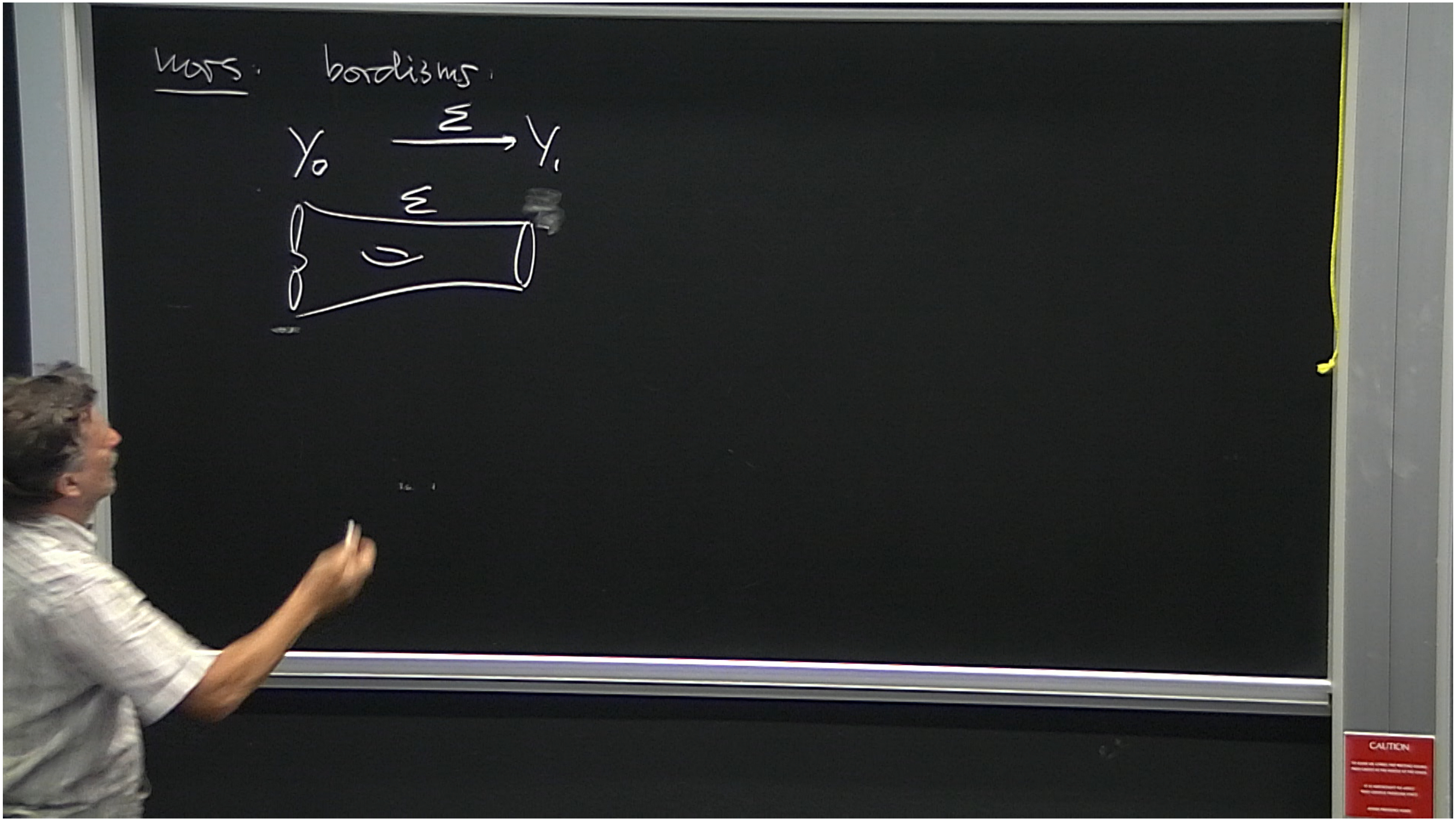
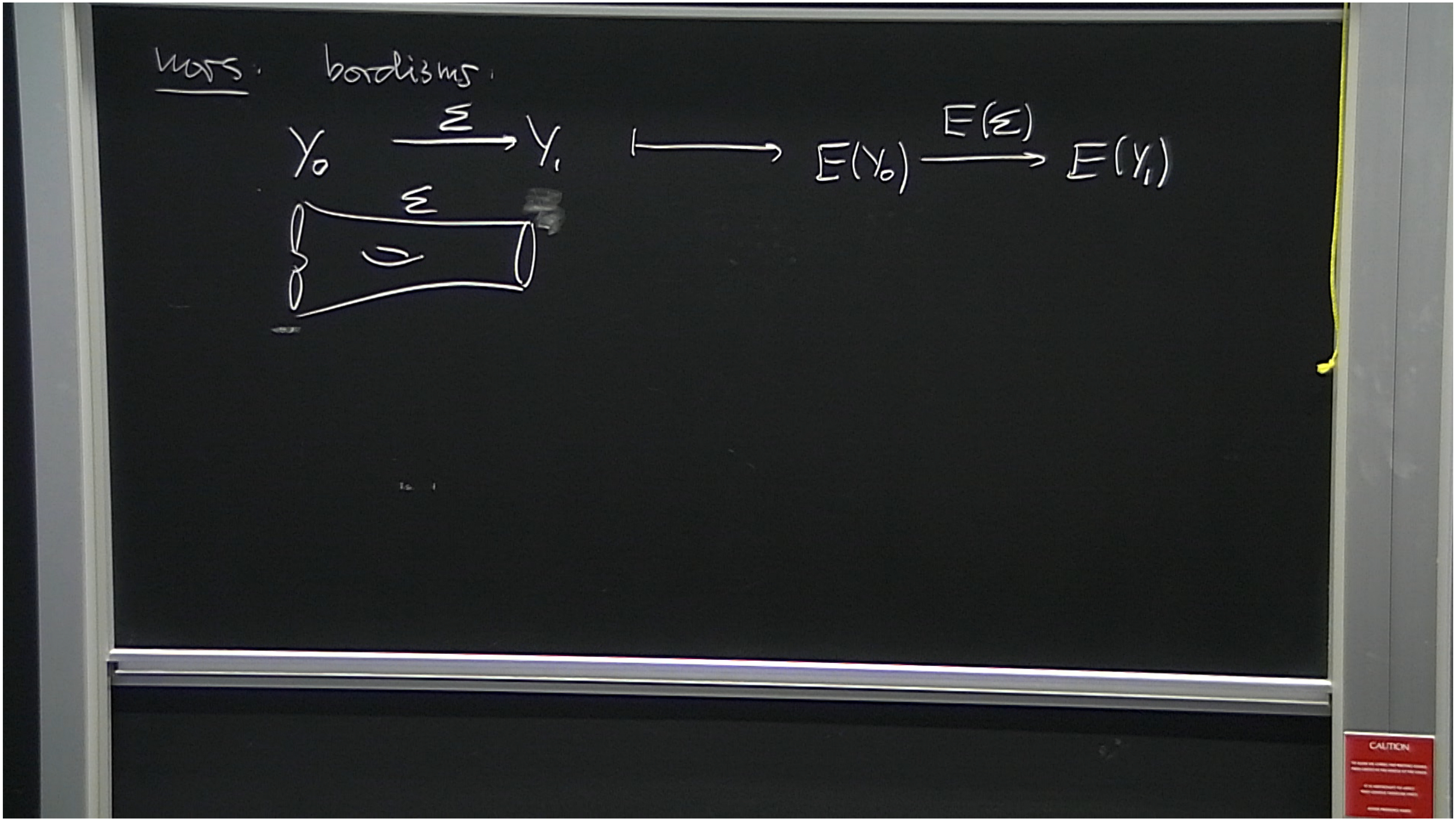
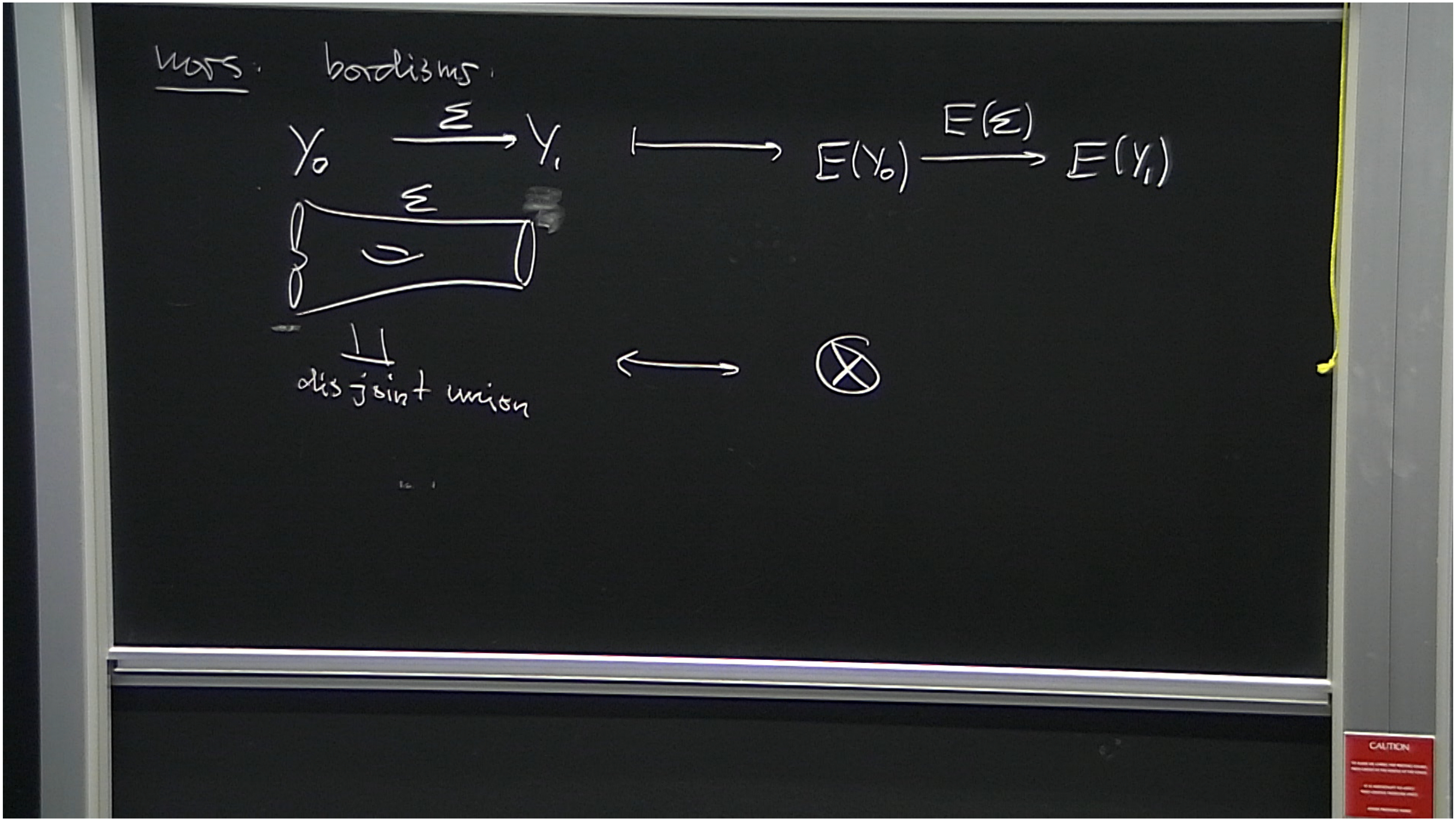

joint w/ Matthias Kreck & Peter Teichner
d-dim top field theory is a symmetric
monoidal functor


joint w/ Matthias Kreck & Peter Teichner
d-dim top field theory is a symmetric monoidal functor
$\mathrm{Bord}_d \xrightarrow{F} \mathrm{sVect}$
CAUTION

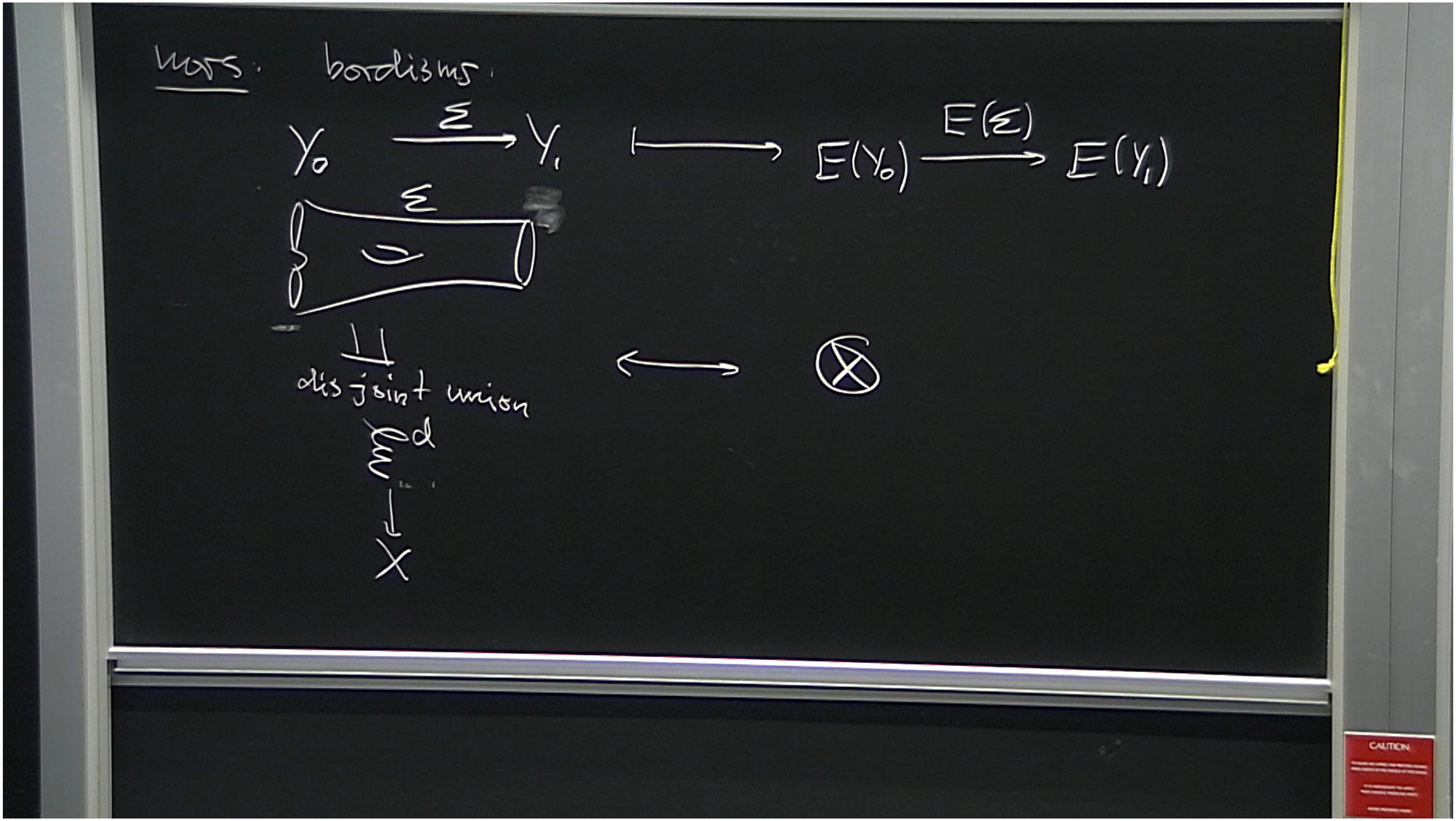

monoidal functor
Bord d
F
sVect
objects:
closed (d-1)-mfd Y
F(Y) super vector space
CAUTION

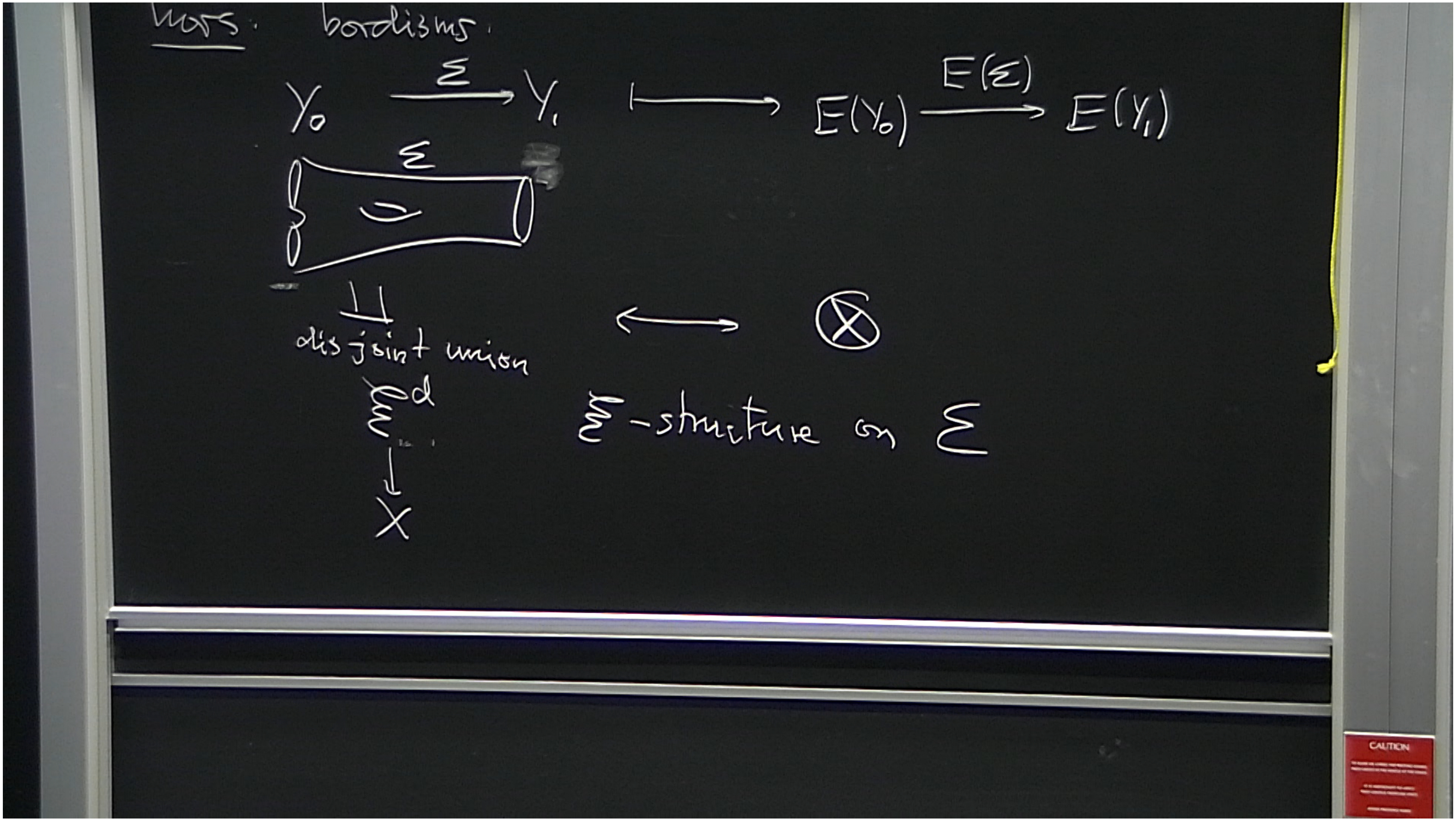

wors: bordisms:
$Y_0 \xrightarrow{\Sigma} Y_1$
Σ
CAUTION

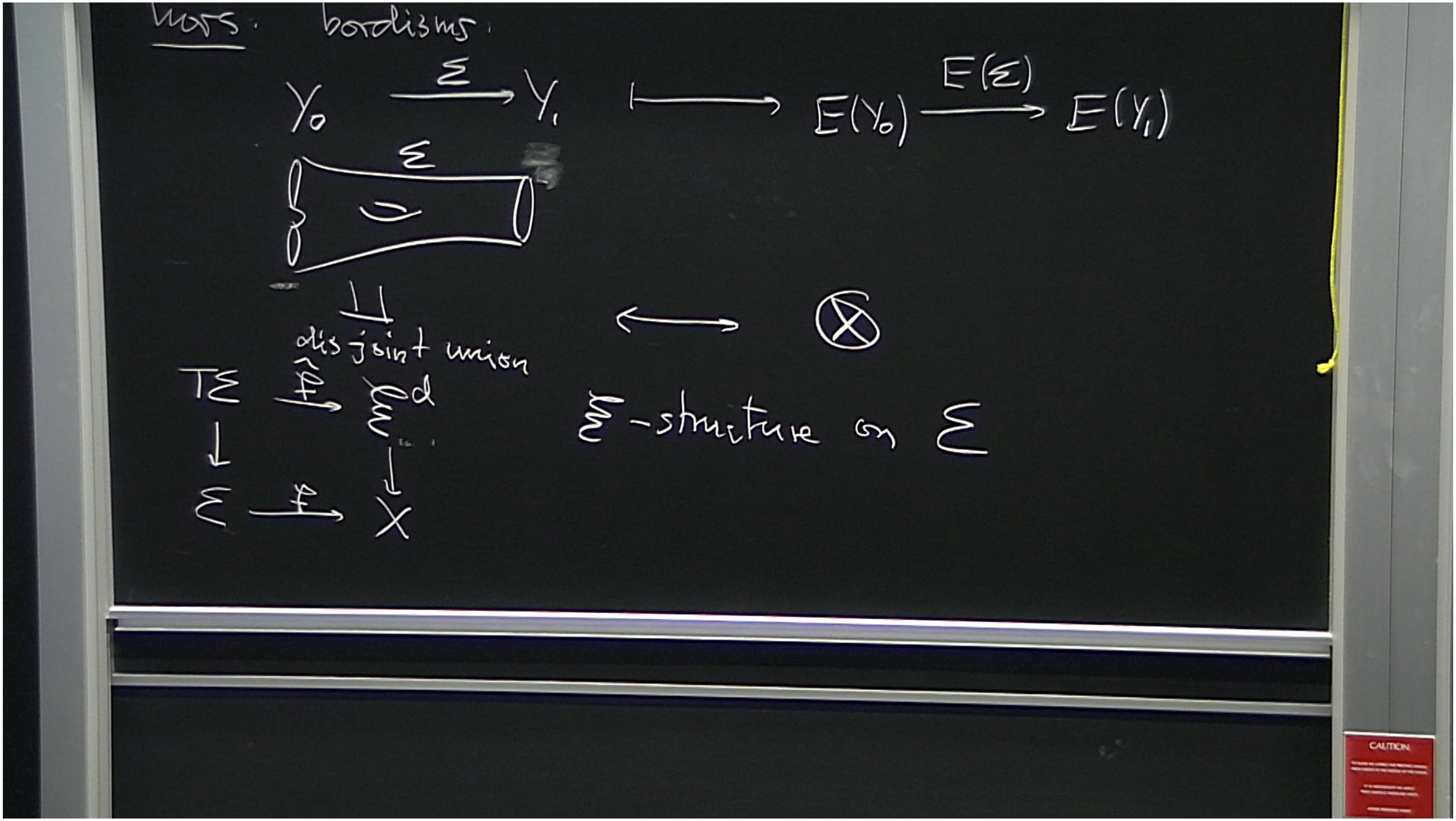

mors: bordisms:
$Y_0 \xrightarrow{\Sigma} Y_1 \longmapsto E(Y_0) \xrightarrow{E(\Sigma)} E(Y_1)$
Σ
CAUTION

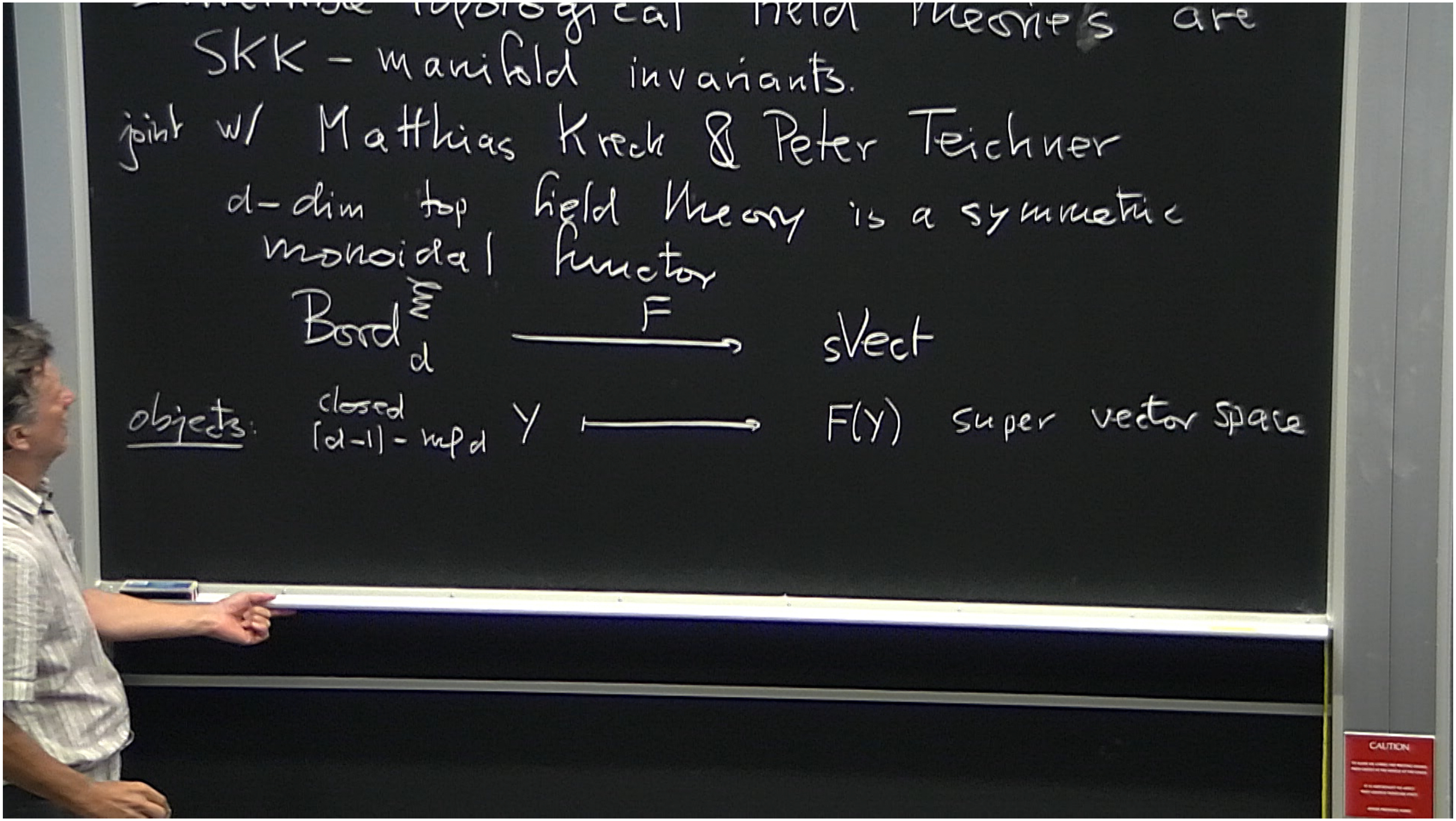


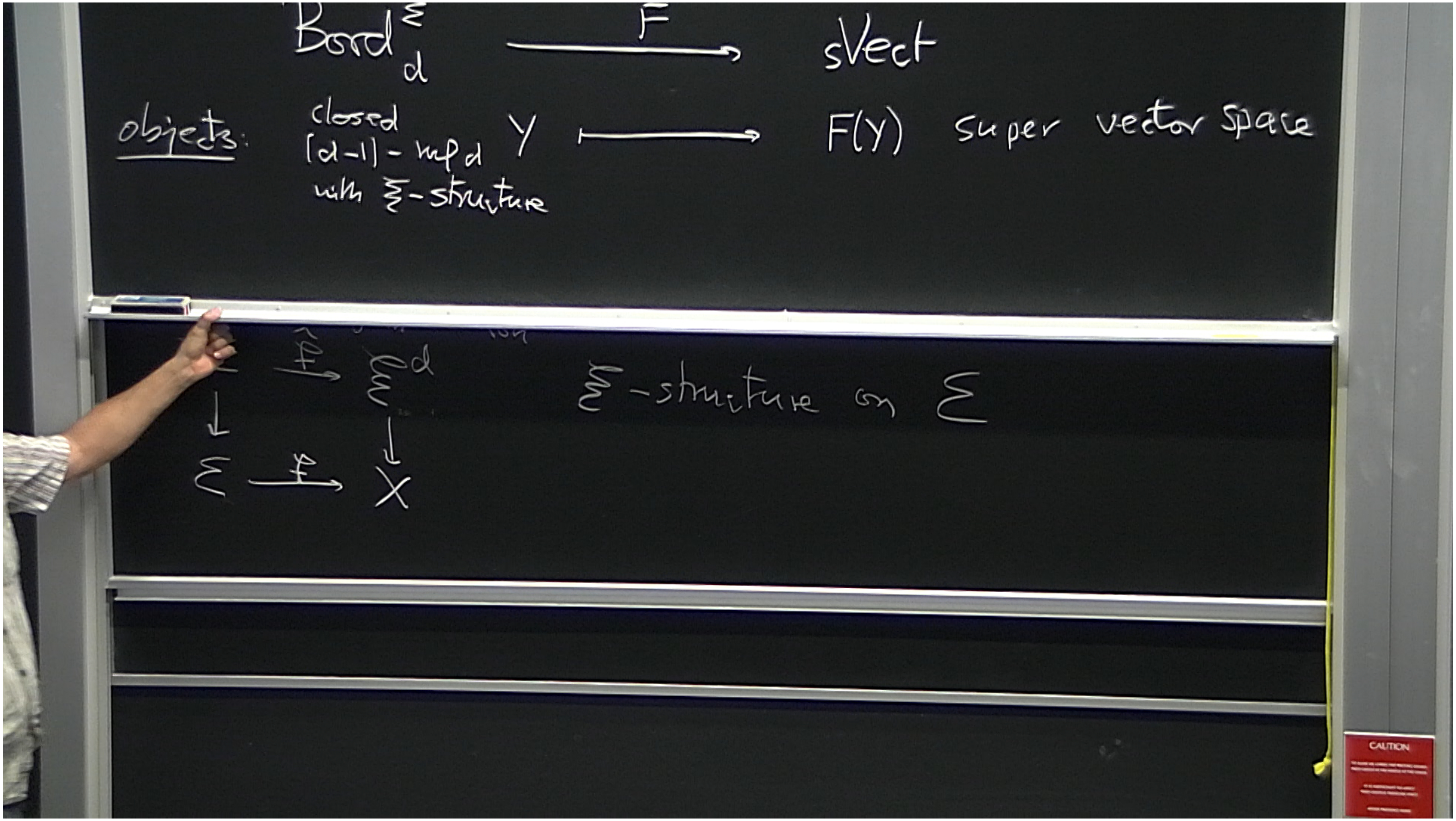
mors: bordisms:
Σ
Y₀ → Y₁ ⟼ E(Y₀) → E(Y₁)
E(Σ)
Σ
⊔
disjoint union
⊗
CAUTION


mors: bordisms:
$Y_0 \xrightarrow{\Sigma} Y_1 \longmapsto E(Y_0) \xrightarrow{E(\Sigma)} E(Y_1)$
Σ
$\sqcup$
disjoint union
$\longleftrightarrow \otimes$
X

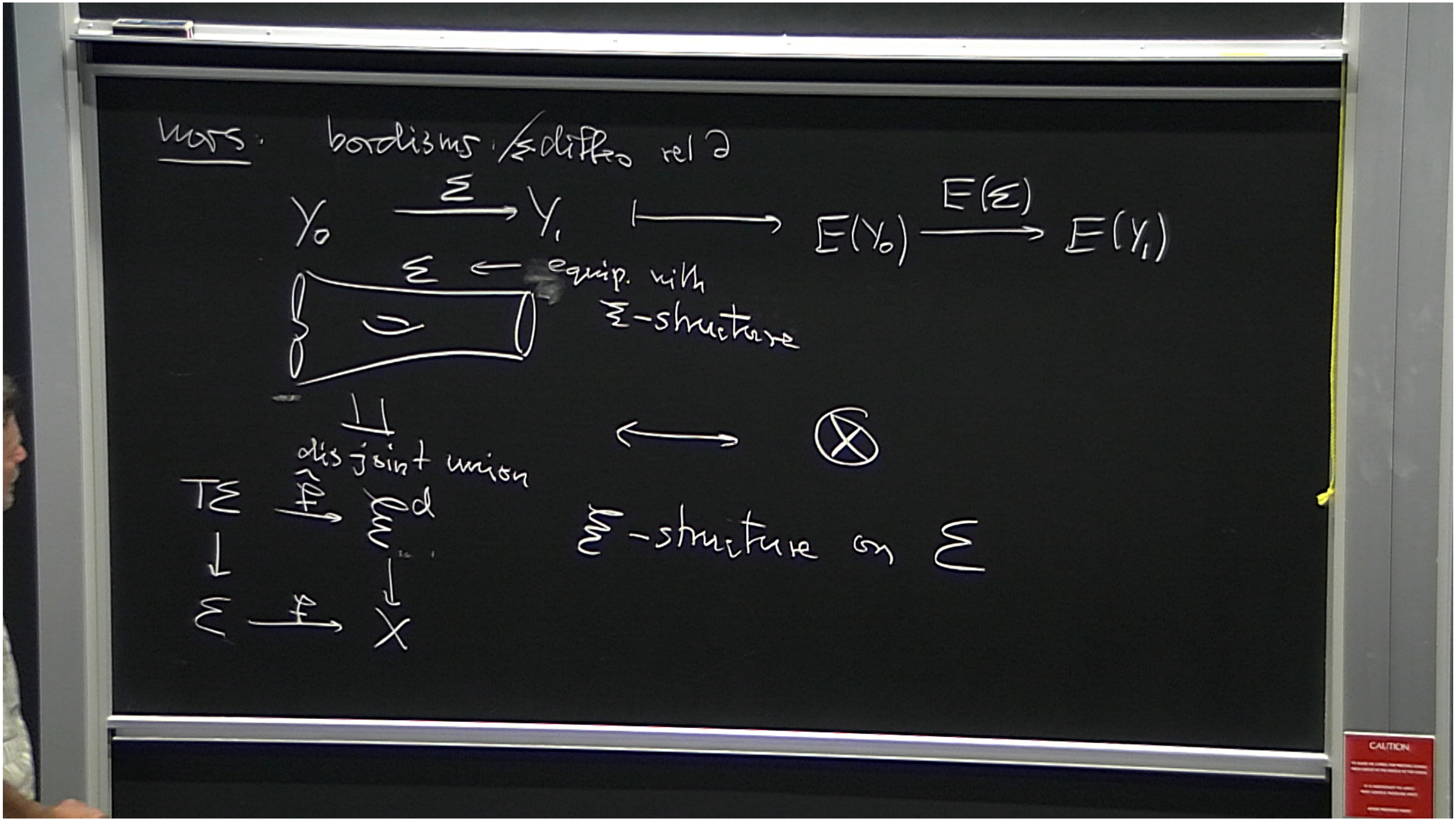

mors: bordisms:
Σ
Y0 → Y1 ⟼ E(Y0) → E(Y1)
E(Σ)
Σ
⊔
disjoint union
⟷
⊗
Σ-structure on Σ
X

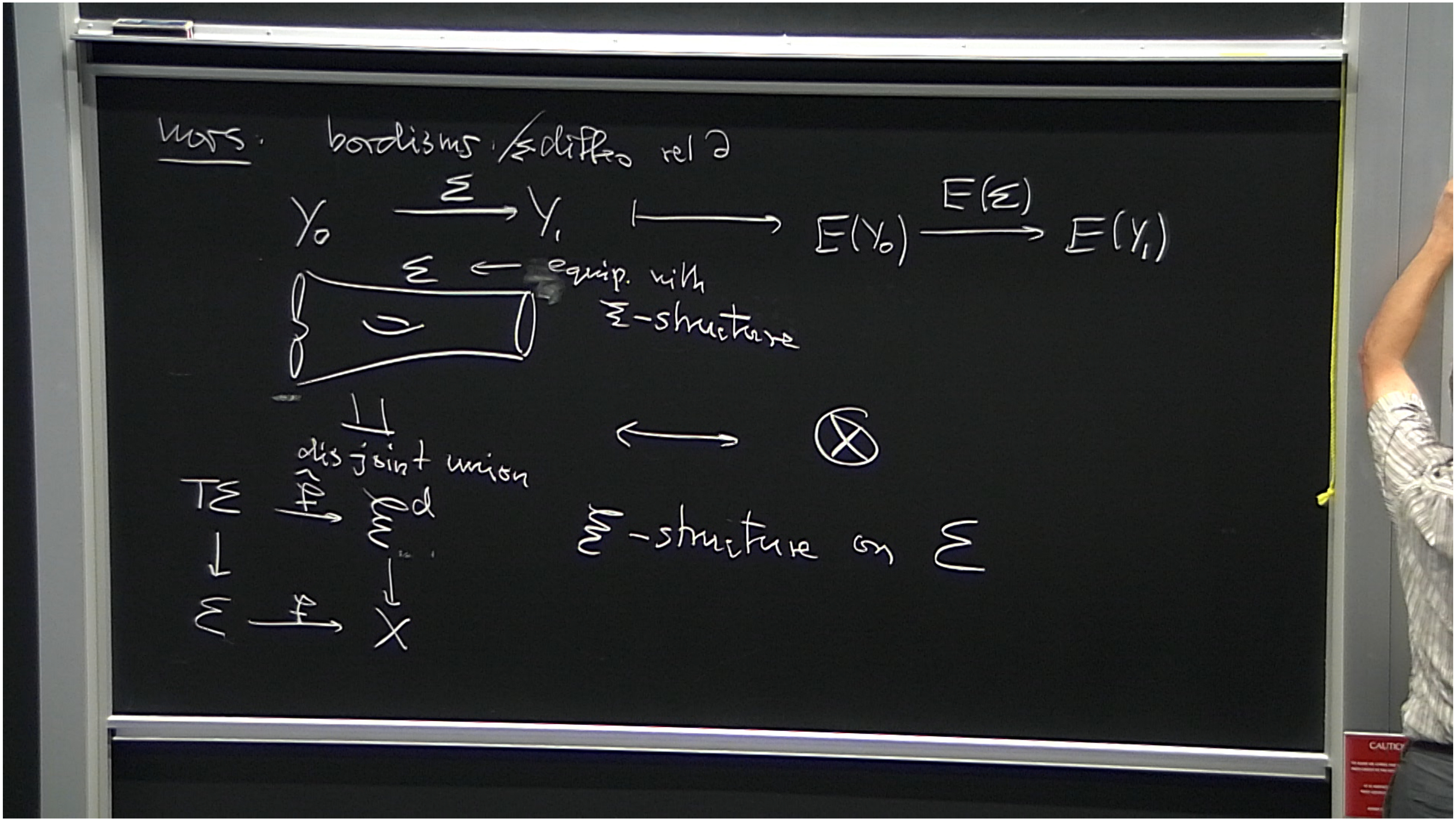

Mors: bordisms:
Y_0 —Σ→ Y_1 ⟼ E(Y_0) —E(Σ)→ E(Y_1)
Σ
⊔
disjoint union
⟷ ⊗
TΣ —f̃→ Ẽ ⊂ d
Σ —f→ X
Ẽ-structure on Σ

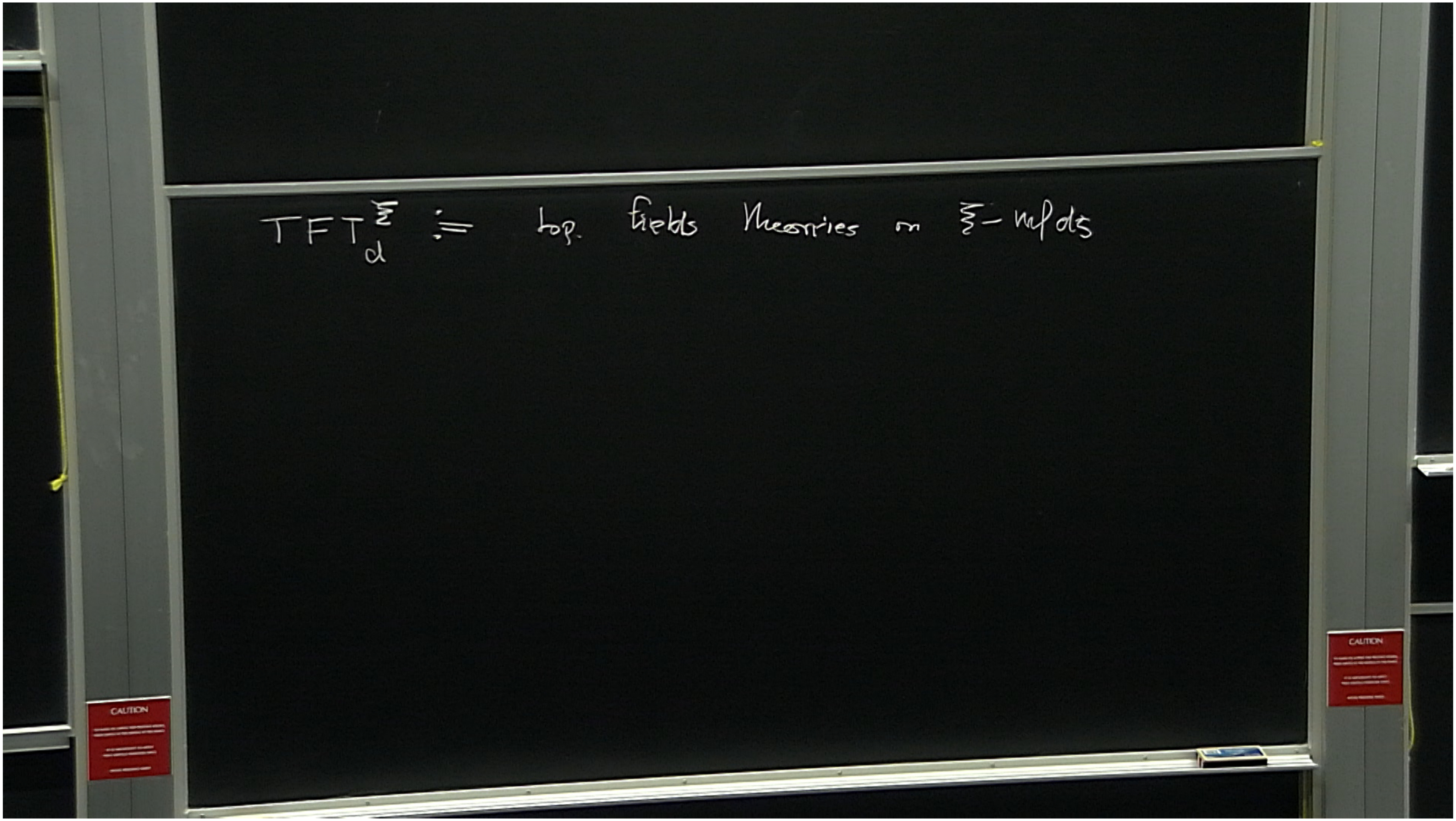

SKK - manifold invariants.
joint w/ Matthias Kreck & Peter Teichner
d-dim top field theory is a symmetric monoidal functor
Bord$_d^\Xi$ $\xrightarrow{F}$ sVect
objects: closed (d-1)-mfd Y ⟼ F(Y) super vector space



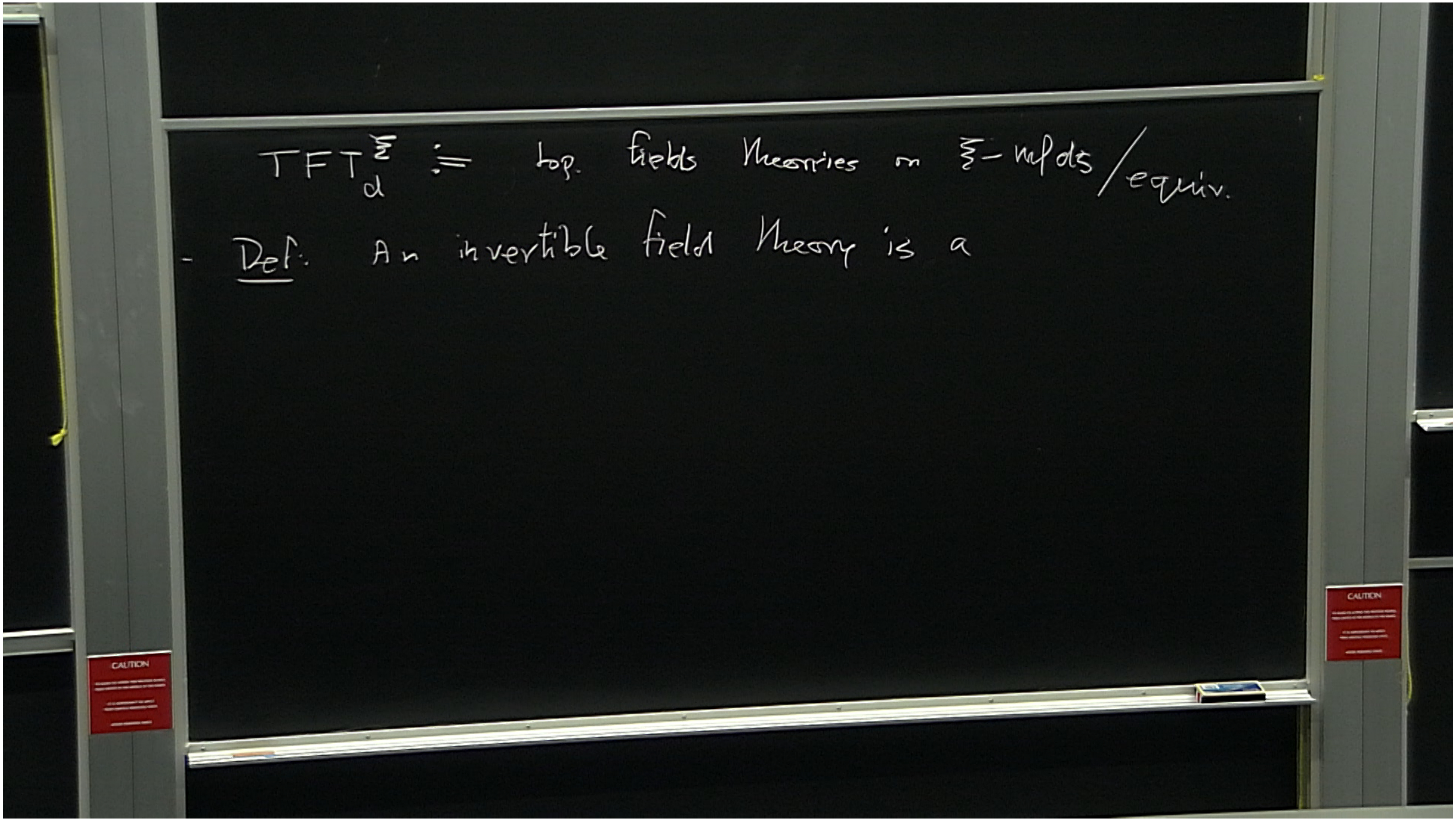
Bord$_d^\xi$ $\xrightarrow{F}$ sVect
objects: closed (d-1)-mfd with ξ-structure Y ⟼ F(Y) super vector space
ξ-structure on Σ
Σ $\xrightarrow{f}$ X

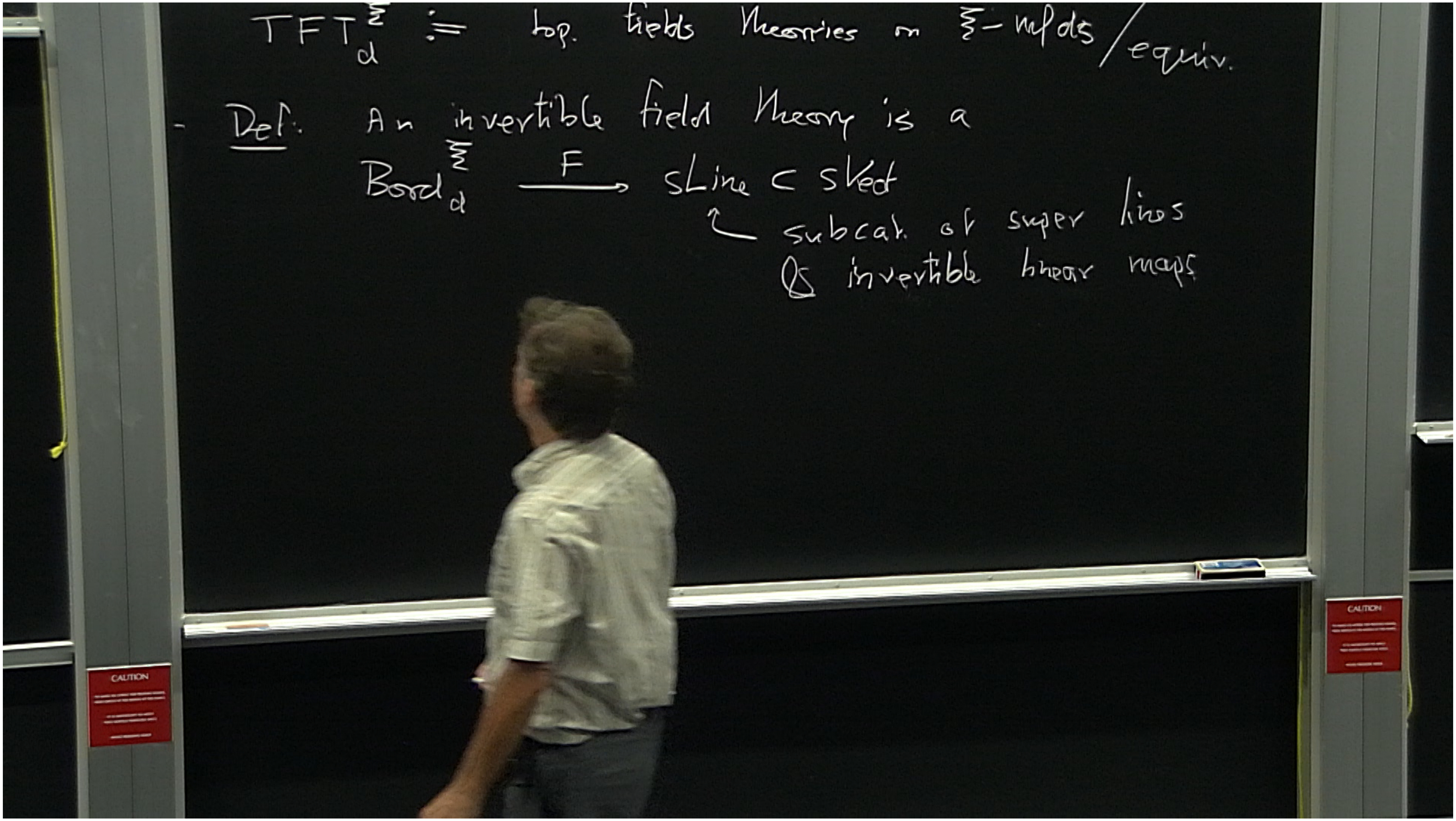

mors: bordisms / diffeo rel ∂
E(Σ)
disjoint union
-structure on Σ

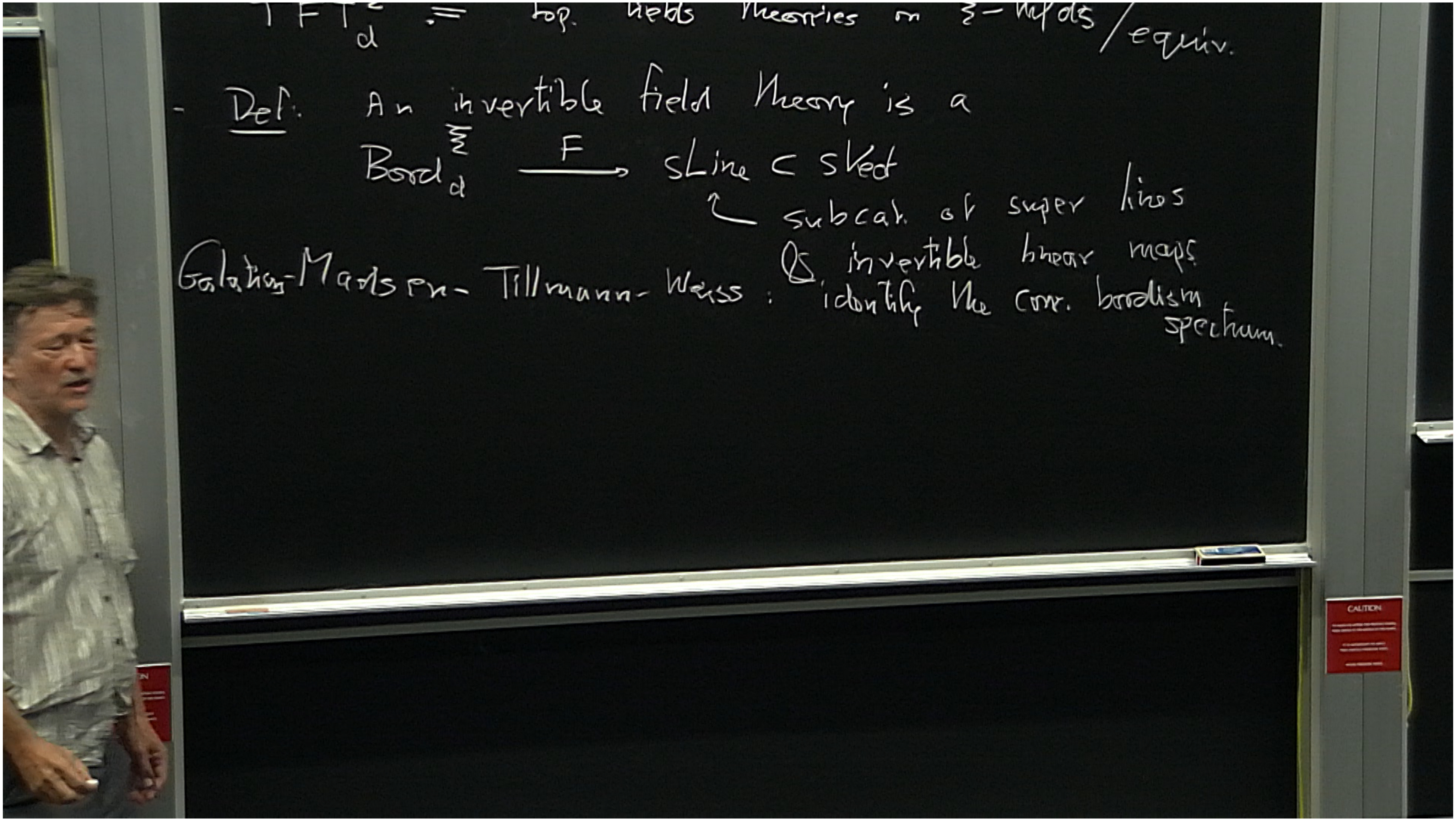

mors: bordisms / ≅ diffeo rel ∂
Σ
Y0 → Y1 ⟼ E(Y0) → E(Y1)
E(Σ)
Σ ← equip. with
Ξ-structure
⊔
disjoint union
⊗
TΣ
f̂
Ξd
f
X
Ξ-structure on Σ

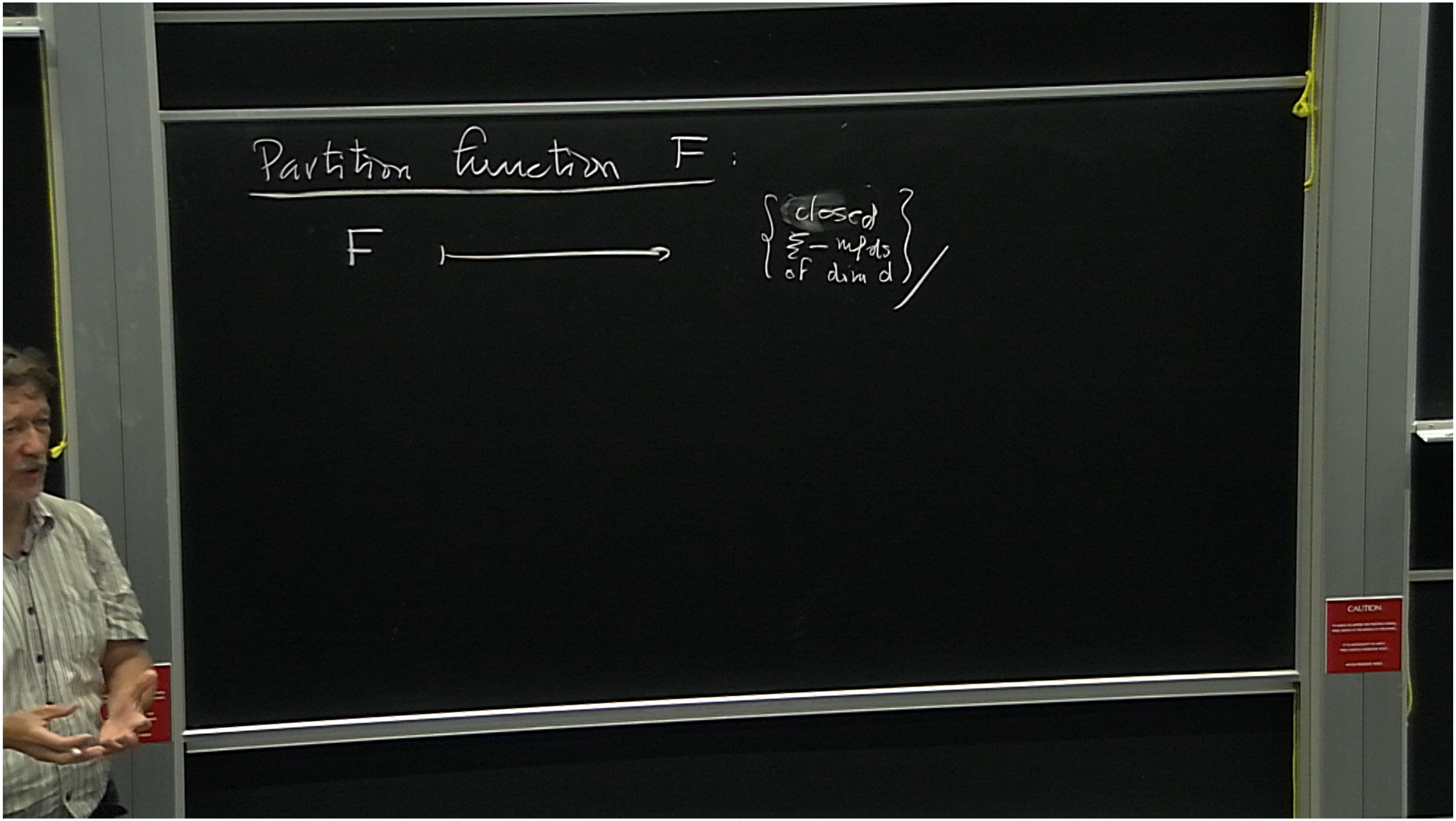


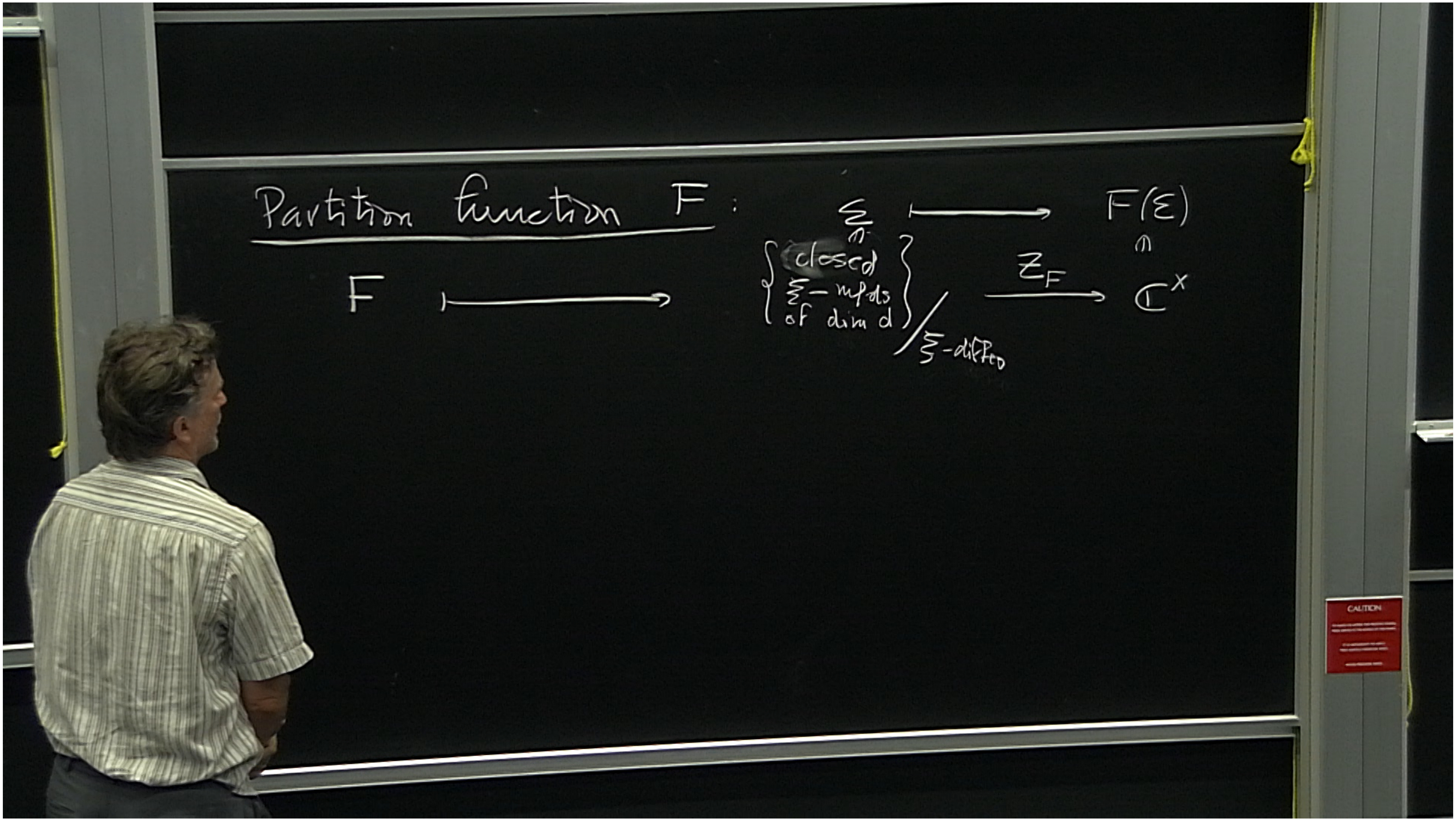
TFT_d^{Ξ} := top. fields theories on Ξ-mfds

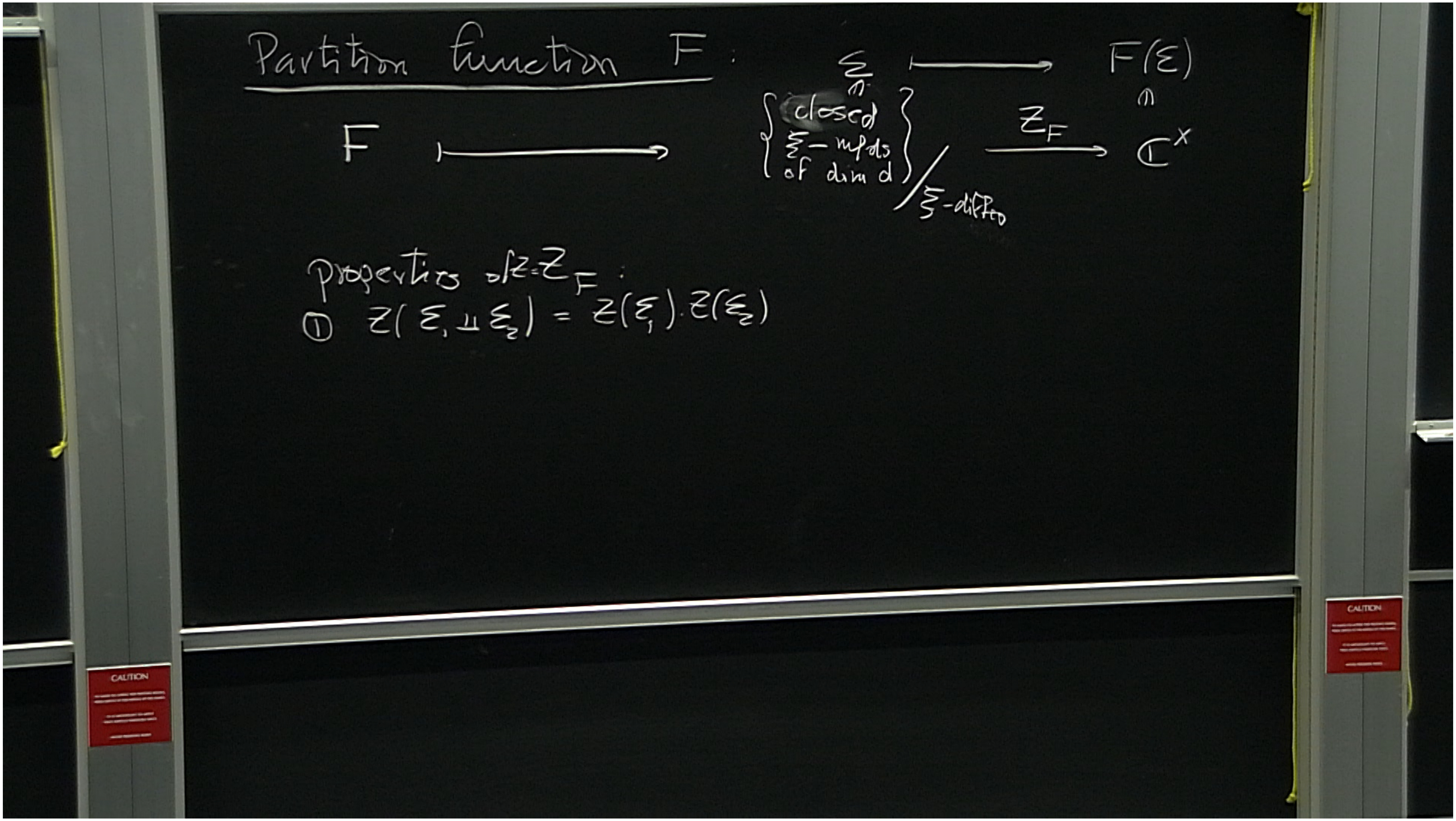

TFT_d^ξ :≅ top. fields theories on ξ-mflds / equiv.
- Def: An invertible field theory is a


TFT_d^Ξ := top. fields theories on Ξ-mfds / equiv.
Def: An invertible field theory is a
Bord_d^Ξ —F→ sLine ⊂ sVect
subcat. of super lines
& invertible linear maps


TFT d ξ = top. fields theories on ξ-mfds / equiv.
Def: An invertible field theory is a
Bord d ξ F sLine ⊂ sVect
subcat. of super lines & invertible linear maps
Galatius-Madsen-Tillmann-Weiss: identify the corr. bordism spectrum.


Partition function F :
F ⟼
closed
Σ – mfds
of dim d


Partition function F :
Σ ⟼ F(Σ)
F ⟼
{closed Σ-mfds of dim d} / Σ-diffeo
Z_F
ℂ^×
CAUTION

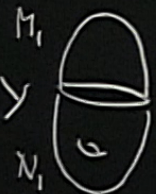

Partition function F :
Σ ⟼ F(Σ)
F ⟼ {closed Σ-mfds of dim d}/Σ-diffeo
Z_F → ℂ^×
properties of Z=Z_F :
① Z(Σ₁ ⊔ Σ₂) = Z(Σ₁)·Z(Σ₂)

(of dim d) / Σ-diffeo

properties of $Z = Z_F$:

① $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \cdot Z(\Sigma_2)$

②



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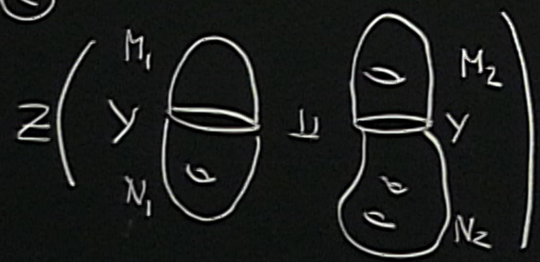

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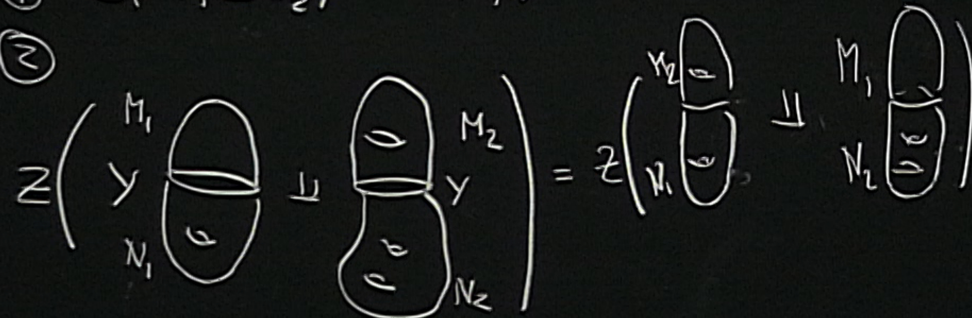

Partition function F : $\Sigma \longmapsto F(\Sigma)$

$$F \longmapsto \left\{ \left\{ \begin{matrix} \text{closed} \\ \Sigma\text{-mfds} \\ \text{of dim } d \end{matrix} \right\} \Big/ \Sigma\text{-diffeo} \xrightarrow{\ Z_F\ } \mathbb{C}^{\times} \Big/ \begin{matrix} ① \\ + \\ ② \end{matrix} \right\}$$

properties of $Z = Z_F$:

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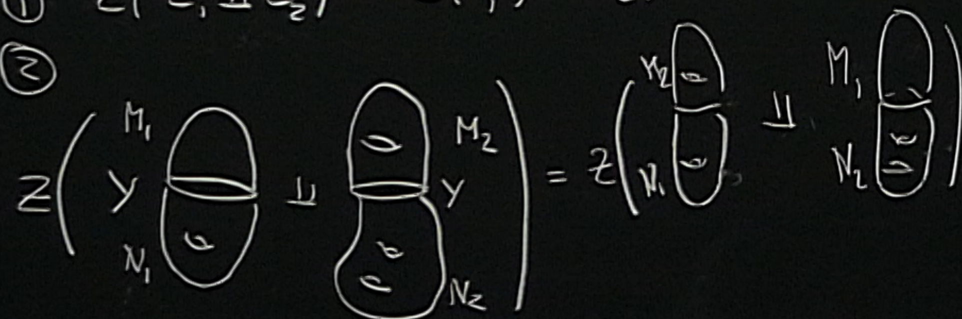
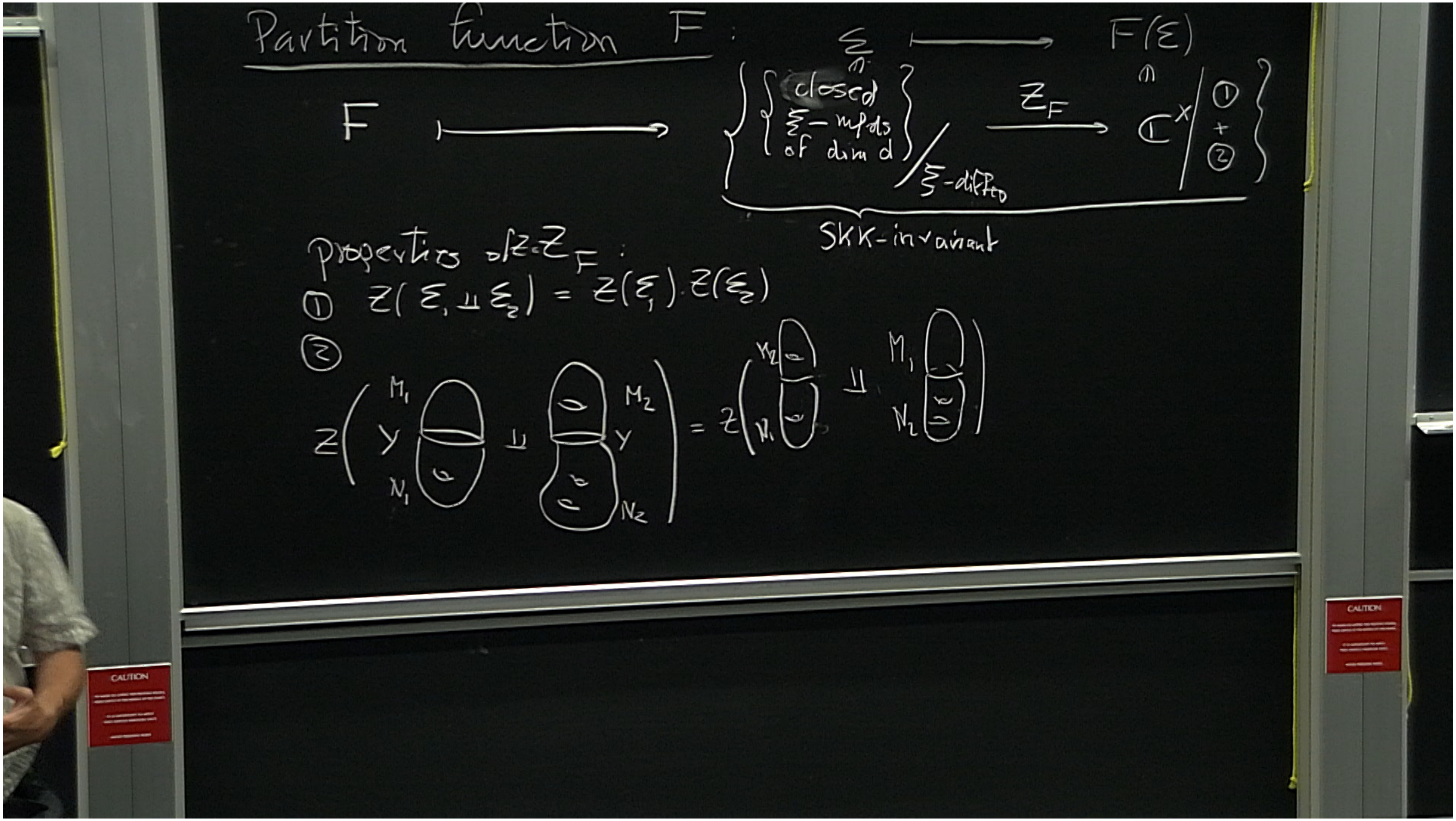
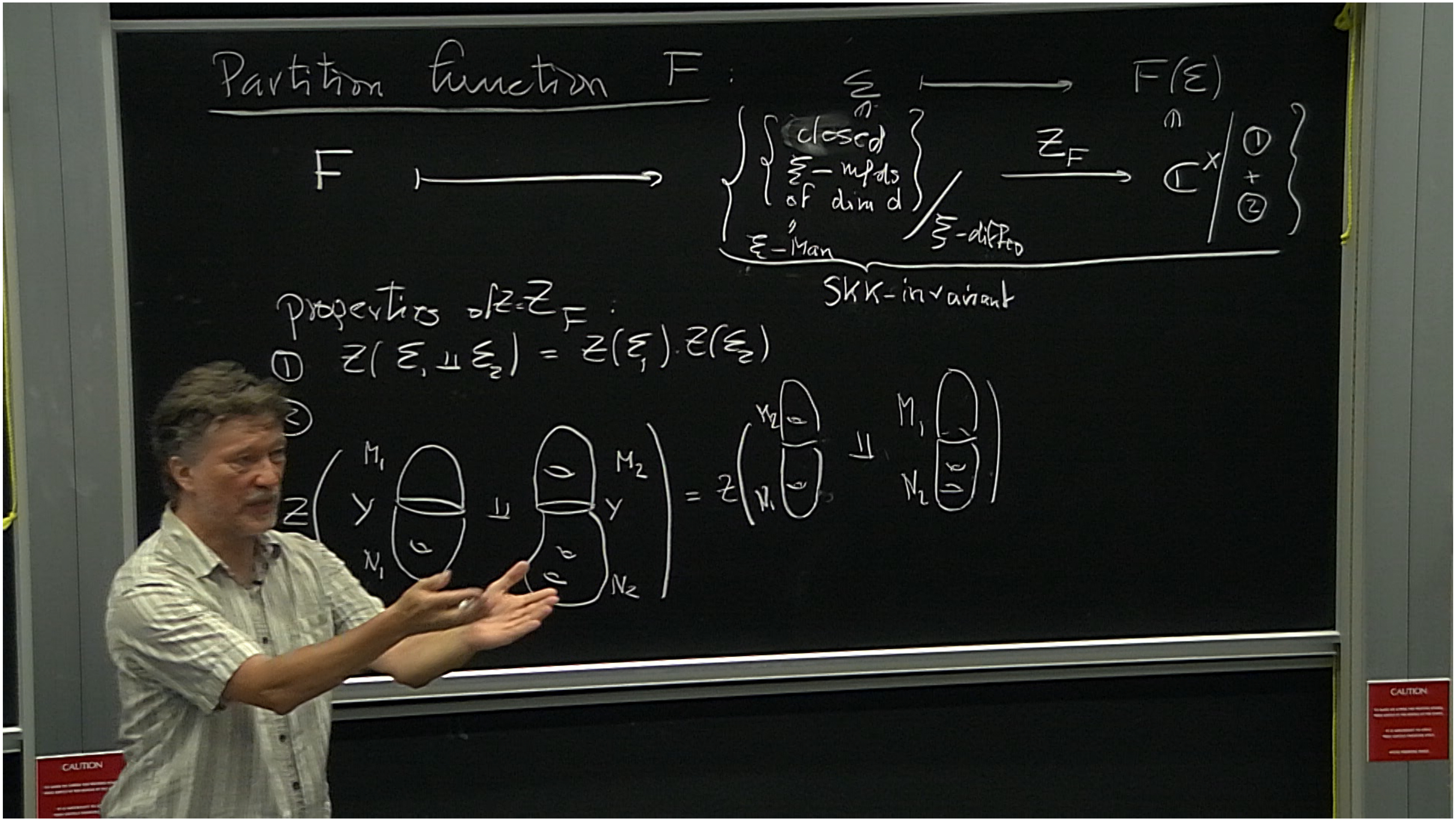
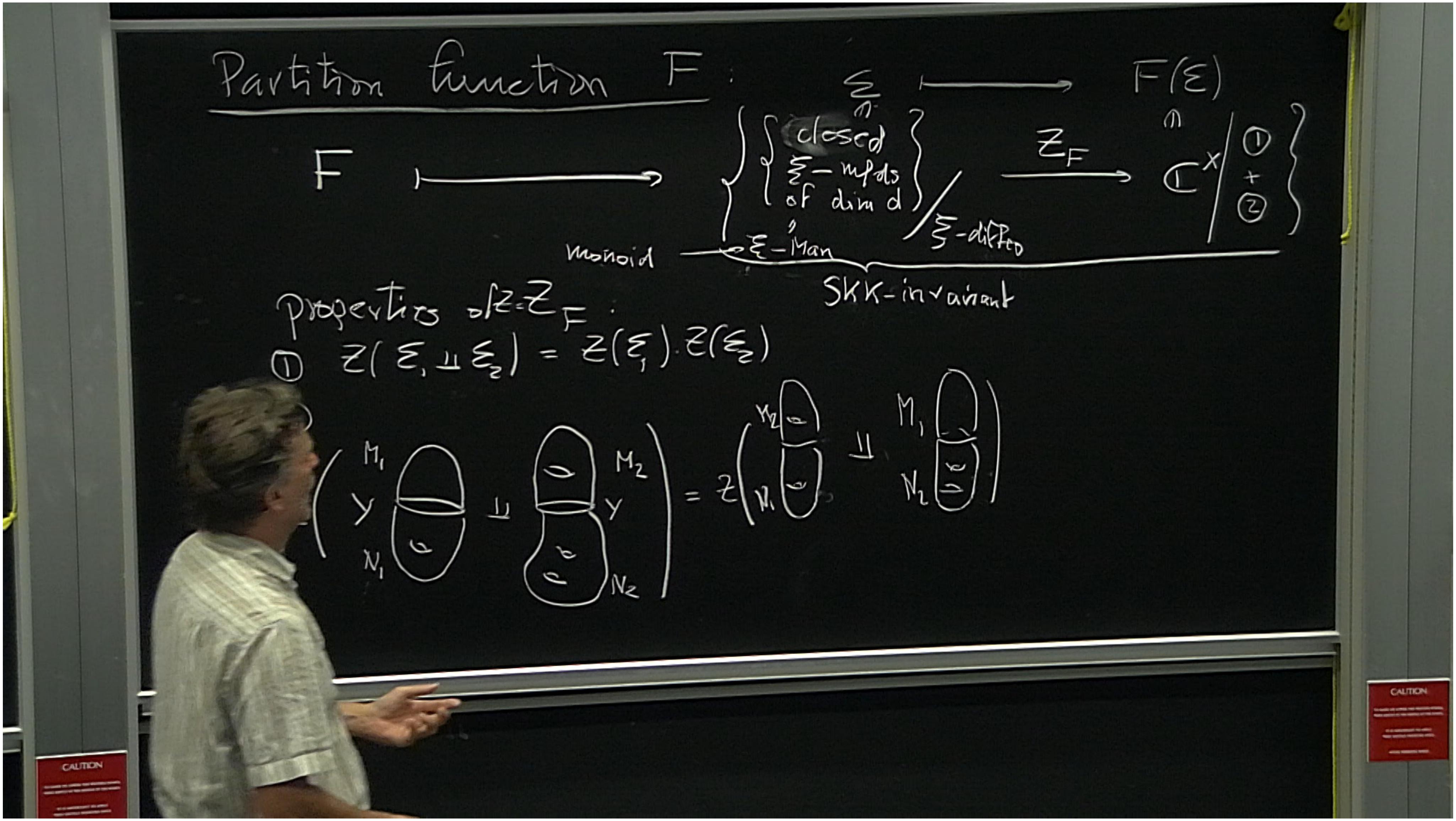
②

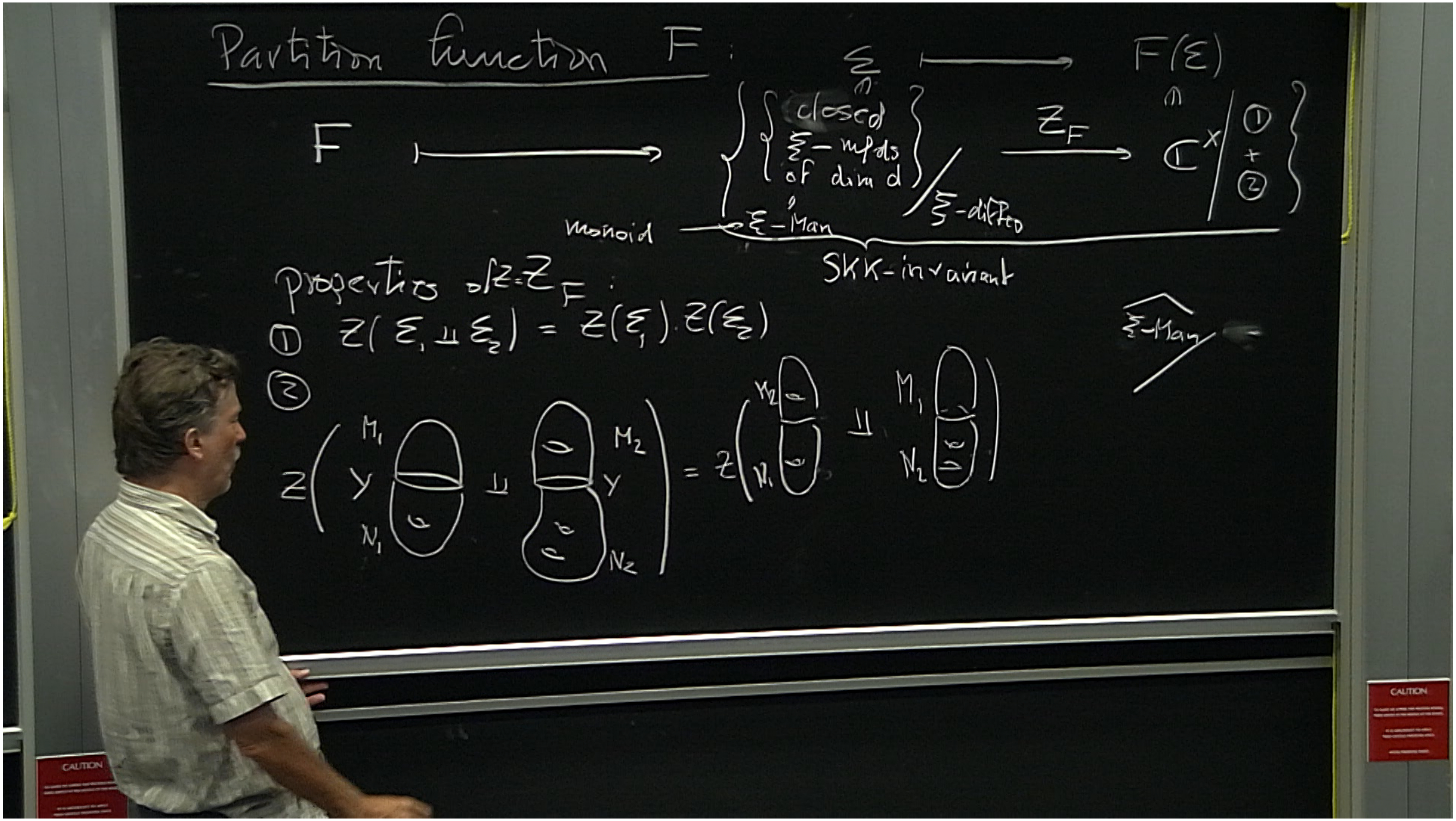


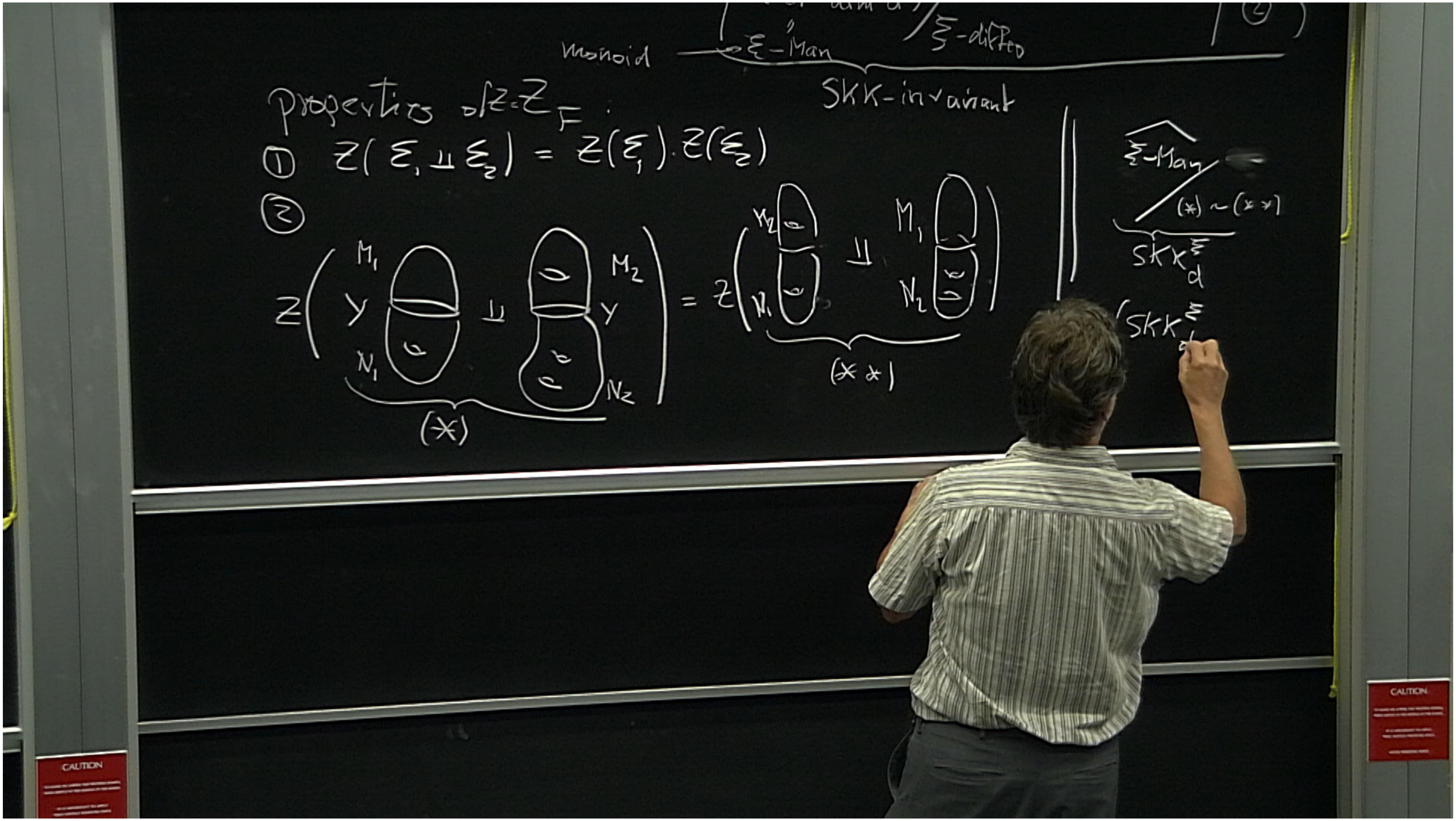

Partition function F : Σ ⟼ F(Σ)
F ⟼ { {closed Σ-mfds of dim d} / Σ-diffeo → Z_F → ℂ^× / ① + ② }
SKK-invariant
properties of Z=Z_F :
① Z(Σ₁ ⊔ Σ₂) = Z(Σ₁)·Z(Σ₂)
②
M₁ Y N₁ ⊔ M₂ Y N₂
M₂ N₁ ⊔ M₁ N₂



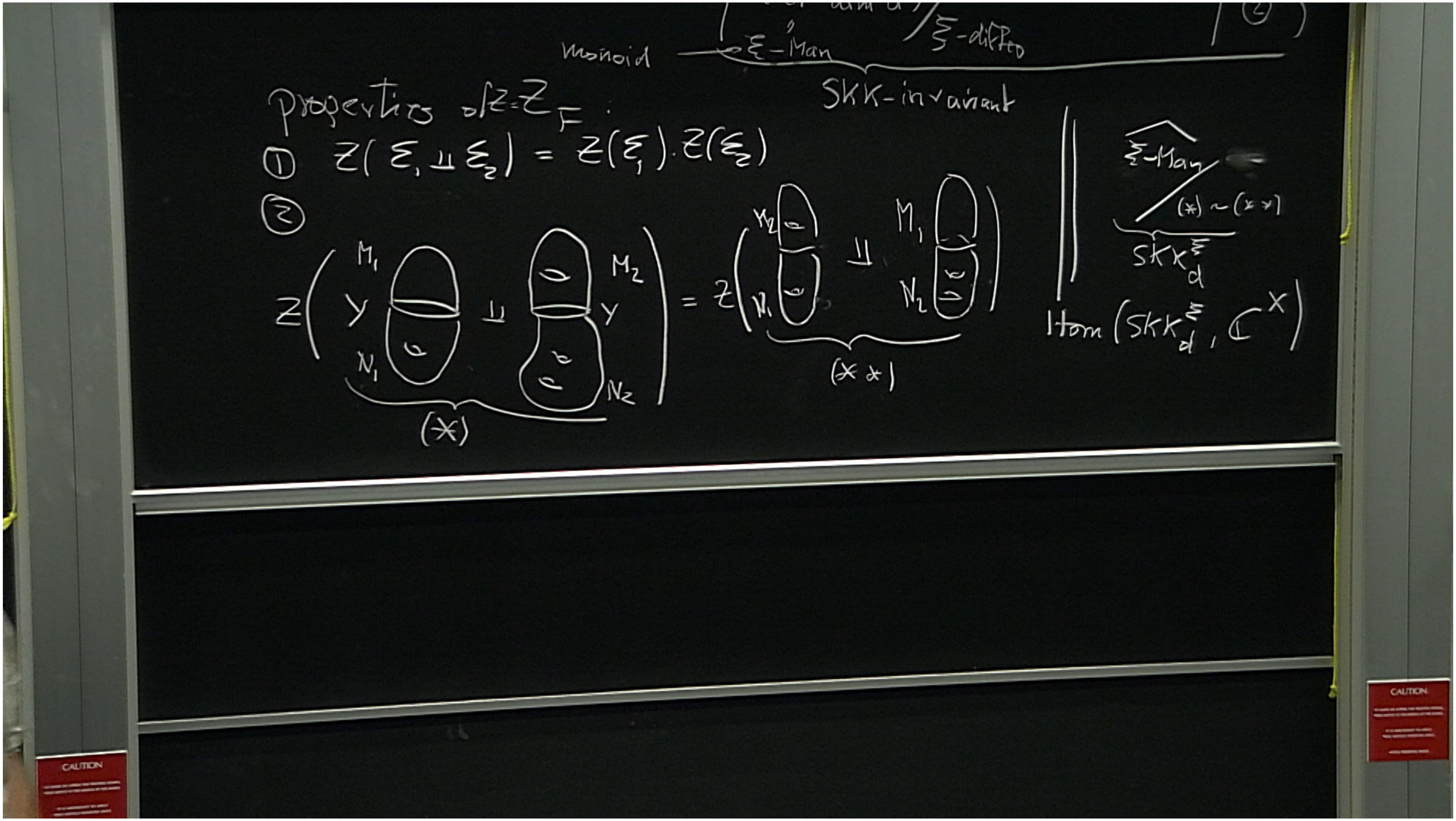
Partition function F :
Σ ⟼ F(Σ)
F ⟼ { {closed Σ-mfds of dim d} / Σ-diffeo → C^× / ① + ② }
Z_F
Σ-Man
SKK-invariant
properties of Z = Z_F :
① Z(Σ₁ ⊔ Σ₂) = Z(Σ₁) · Z(Σ₂)
②
Z(M₁ Y N₁ ⊔ M₂ Y N₂) = Z(M₂ N₁ ⊔ M₁ N₂)
CAUTION

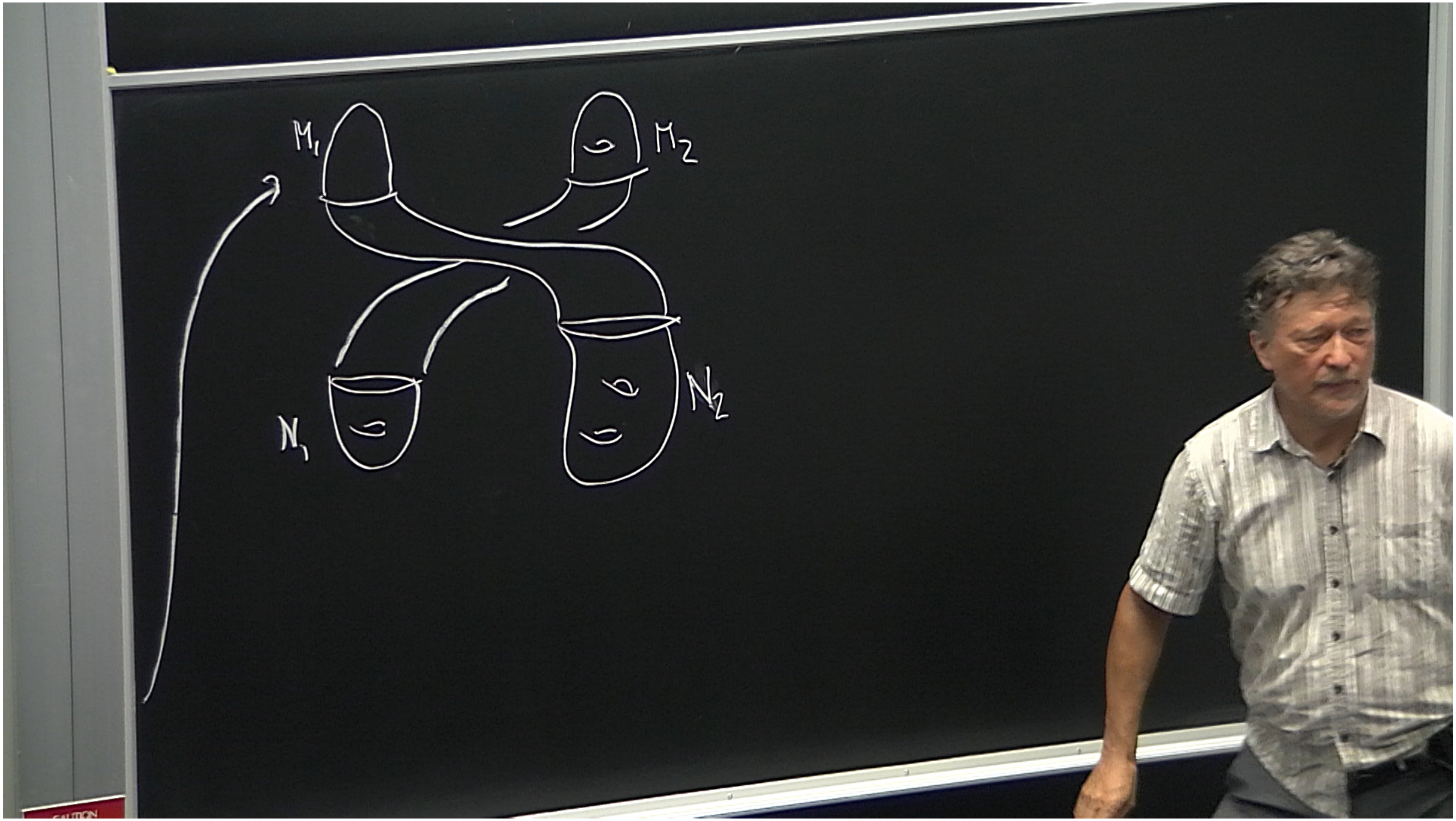

Partition function F :
F
closed
Σ-mfds
of dim d
Σ-diffeo
Z_F
monoid
Σ-Man
SKK-invariant
properties of Z=Z_F :
① Z(Σ₁ ⊔ Σ₂) = Z(Σ₁)·Z(Σ₂)
M₁
Y
N₁
M₂
N₂
= Z


Partition function F
F ⟼ Z_F
closed Σ-mfds of dim d / Σ-diffeo
monoid Σ-Man
SKK-invariant
properties of Z = Z_F :
① Z(Σ₁ ⊔ Σ₂) = Z(Σ₁) · Z(Σ₂)
②
Σ-Man

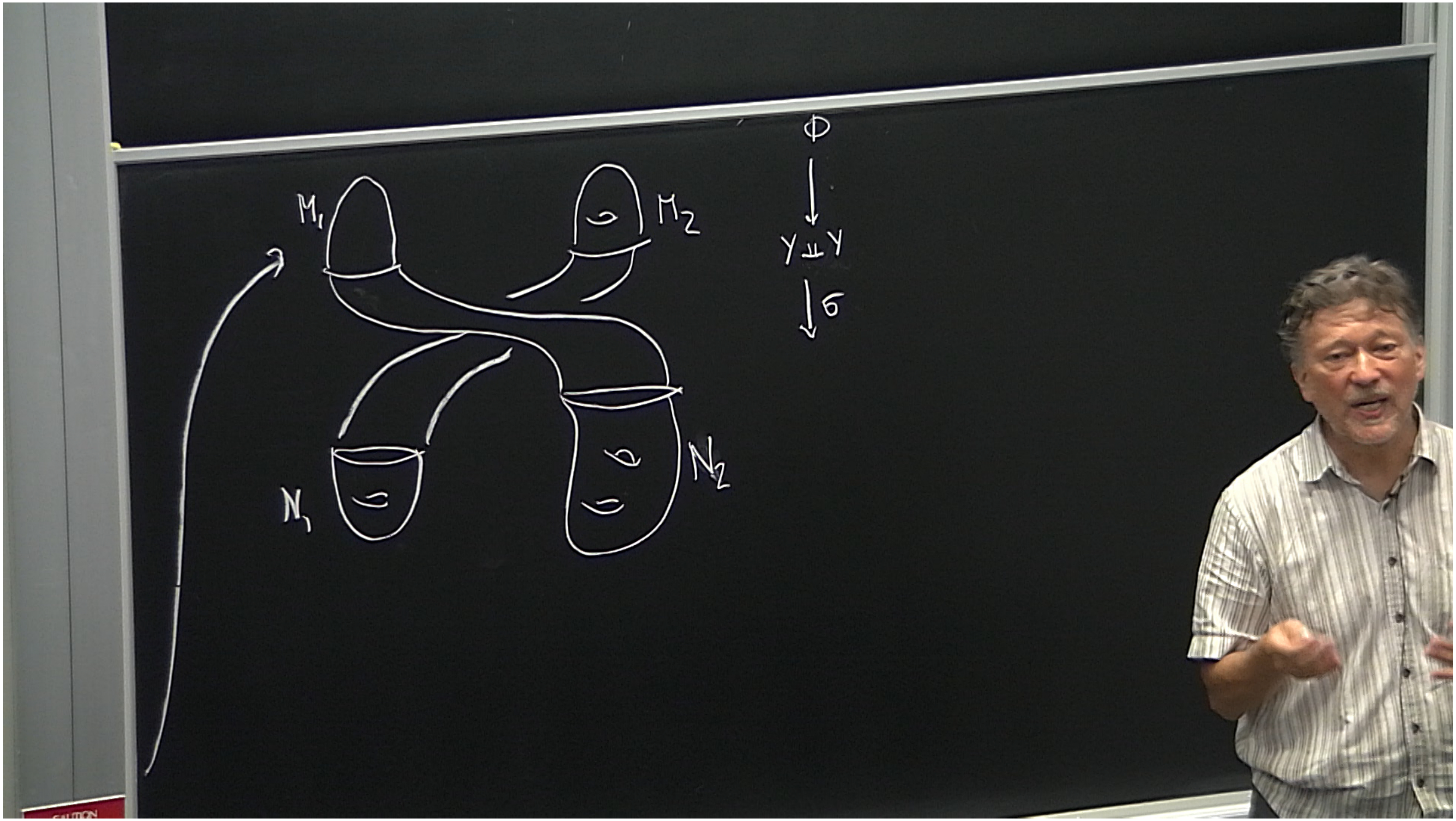


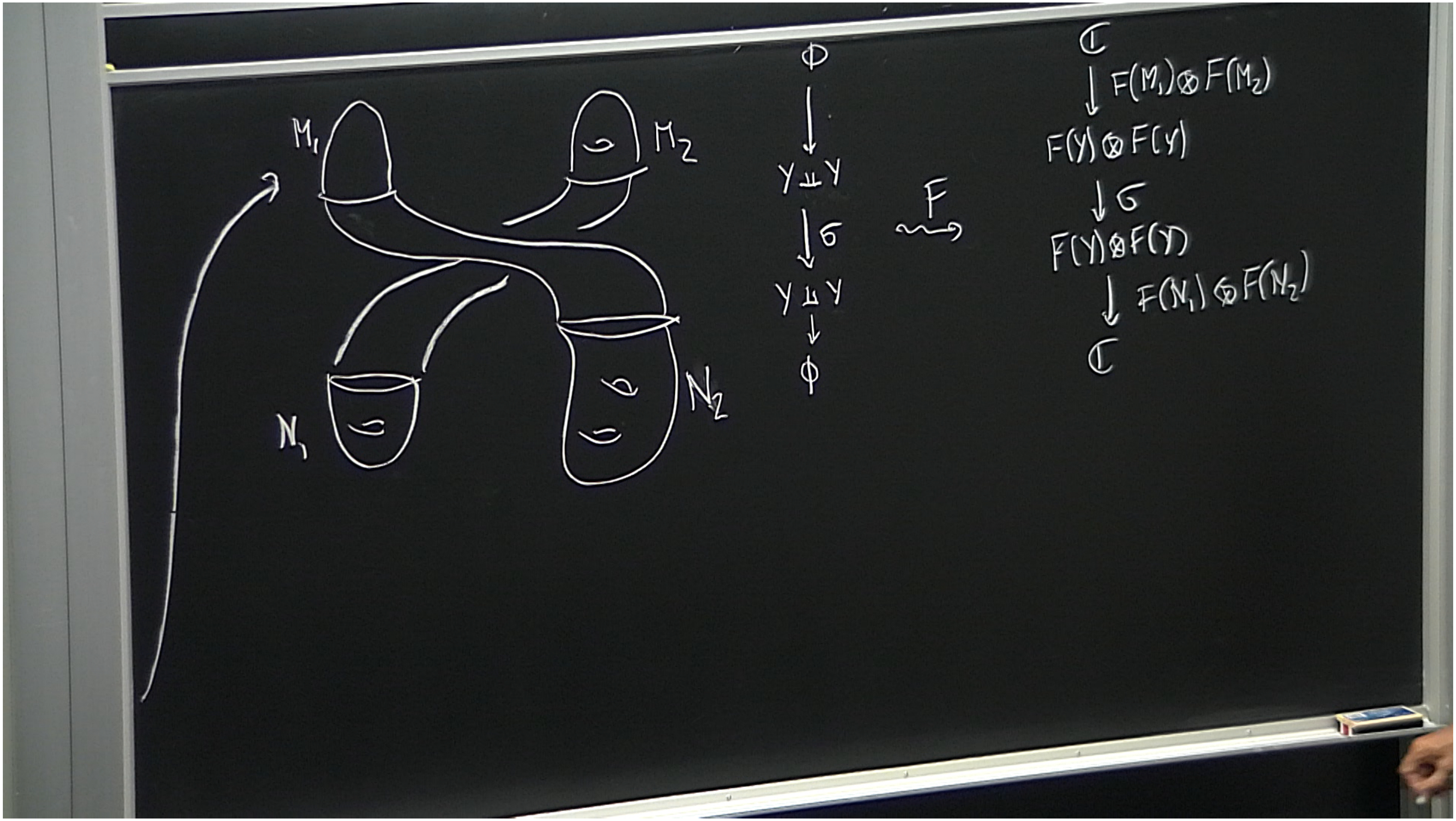
monoid
Σ-Man
Σ-diffeo
SKK-invariant
properties of Z = Z_F :
① Z(Σ₁ ⊔ Σ₂) = Z(Σ₁) · Z(Σ₂)
②
M₁ Y N₁ M₂ Y N₂
(*)
(**)
Σ-Man / (*) ~ (**)
SKK_d^Σ
CAUTION

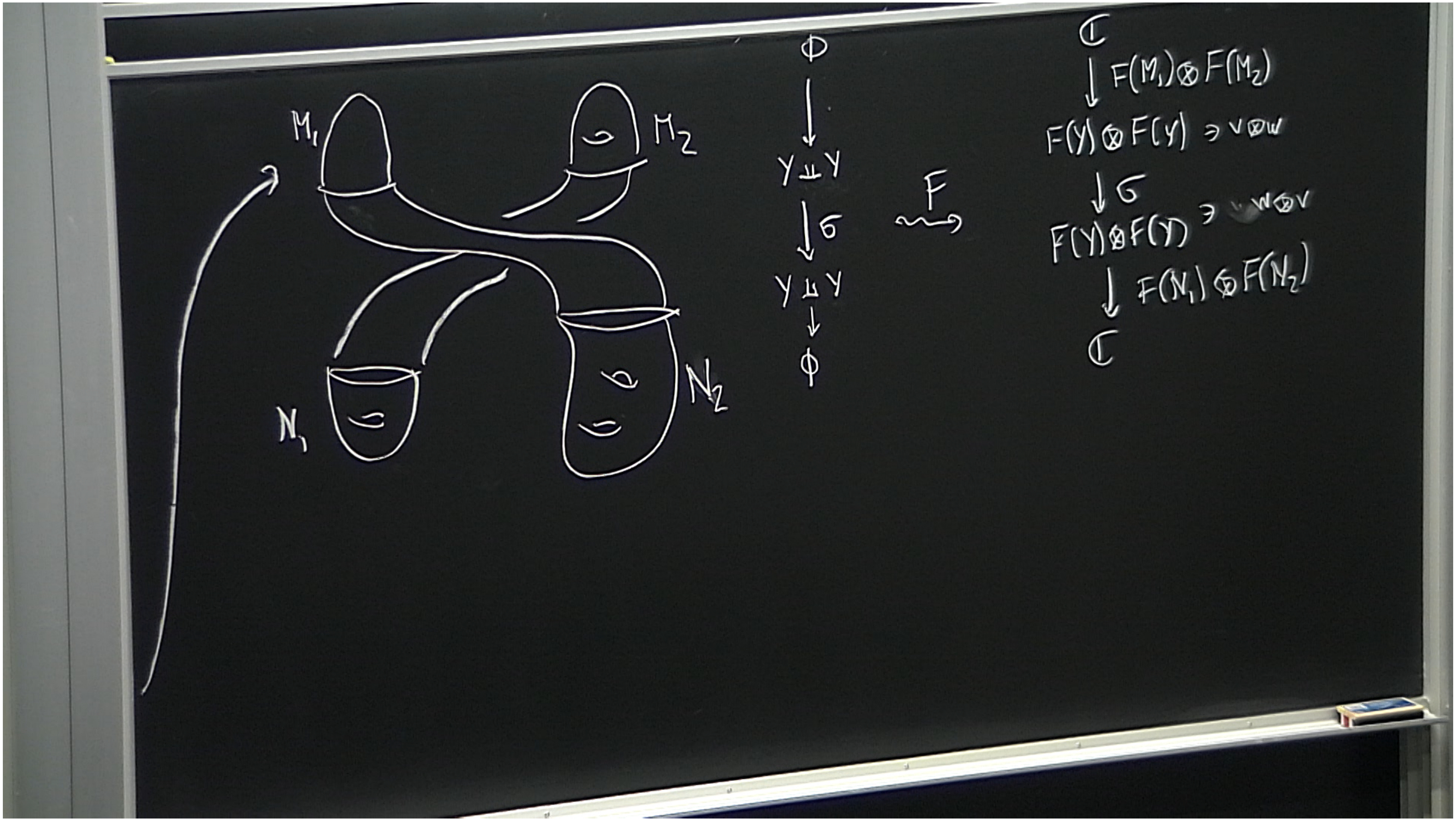

monoid ⟶ Σ-Man / Σ-diffeo
SKK-invariant
properties of Z = Z_F :
① Z(Σ₁ ⊔ Σ₂) = Z(Σ₁)·Z(Σ₂)
②
M₁ Y N₁ ⊔ M₂ Y N₂
(∗)
M₂ N₁ ⊔ M₁ N₂
(∗∗)
Σ-Man / (∗) ~ (∗∗)
SKK_d^Σ
Hom(SKK_d^Σ, ℂ^×)
CAUTION

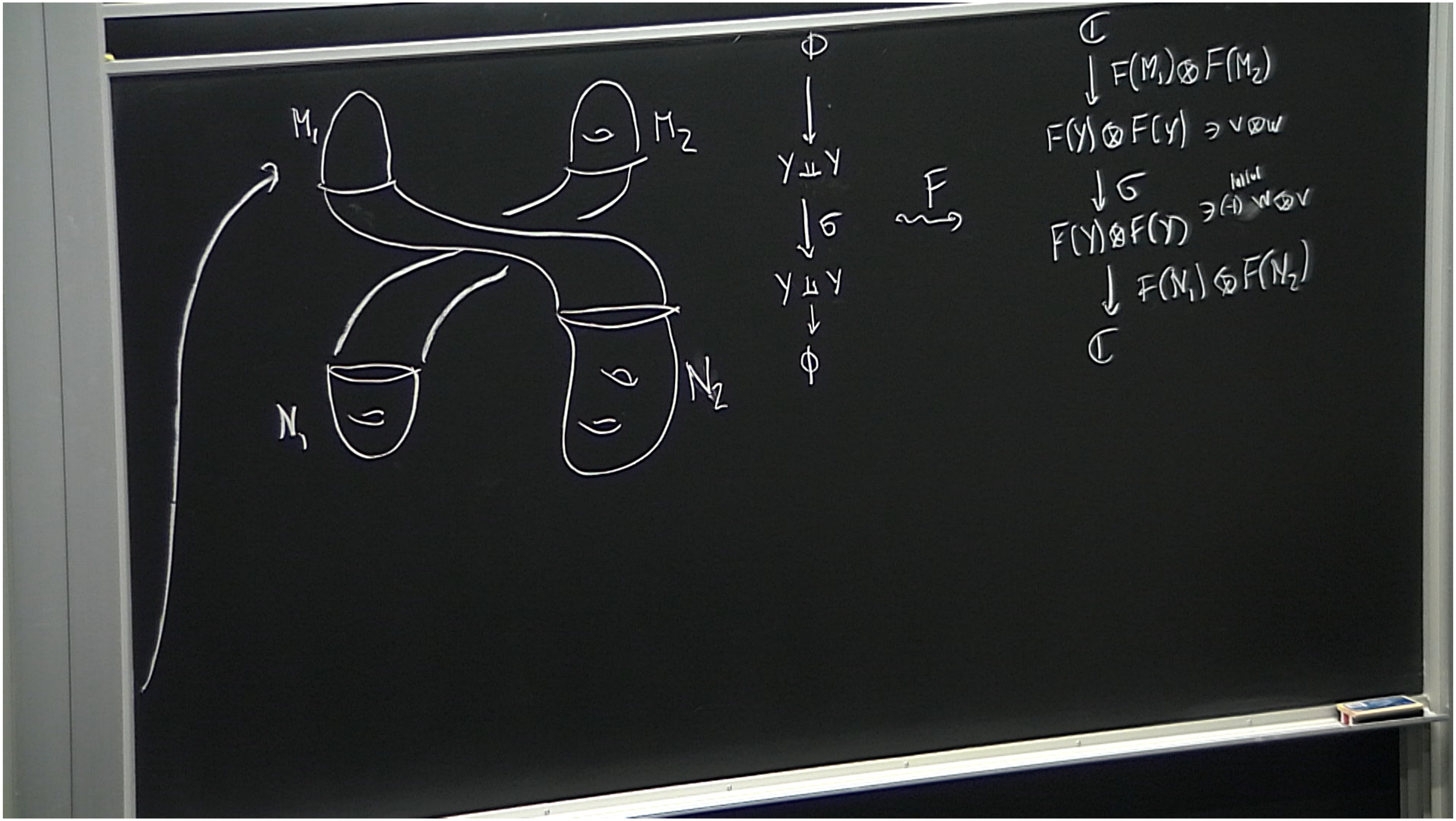


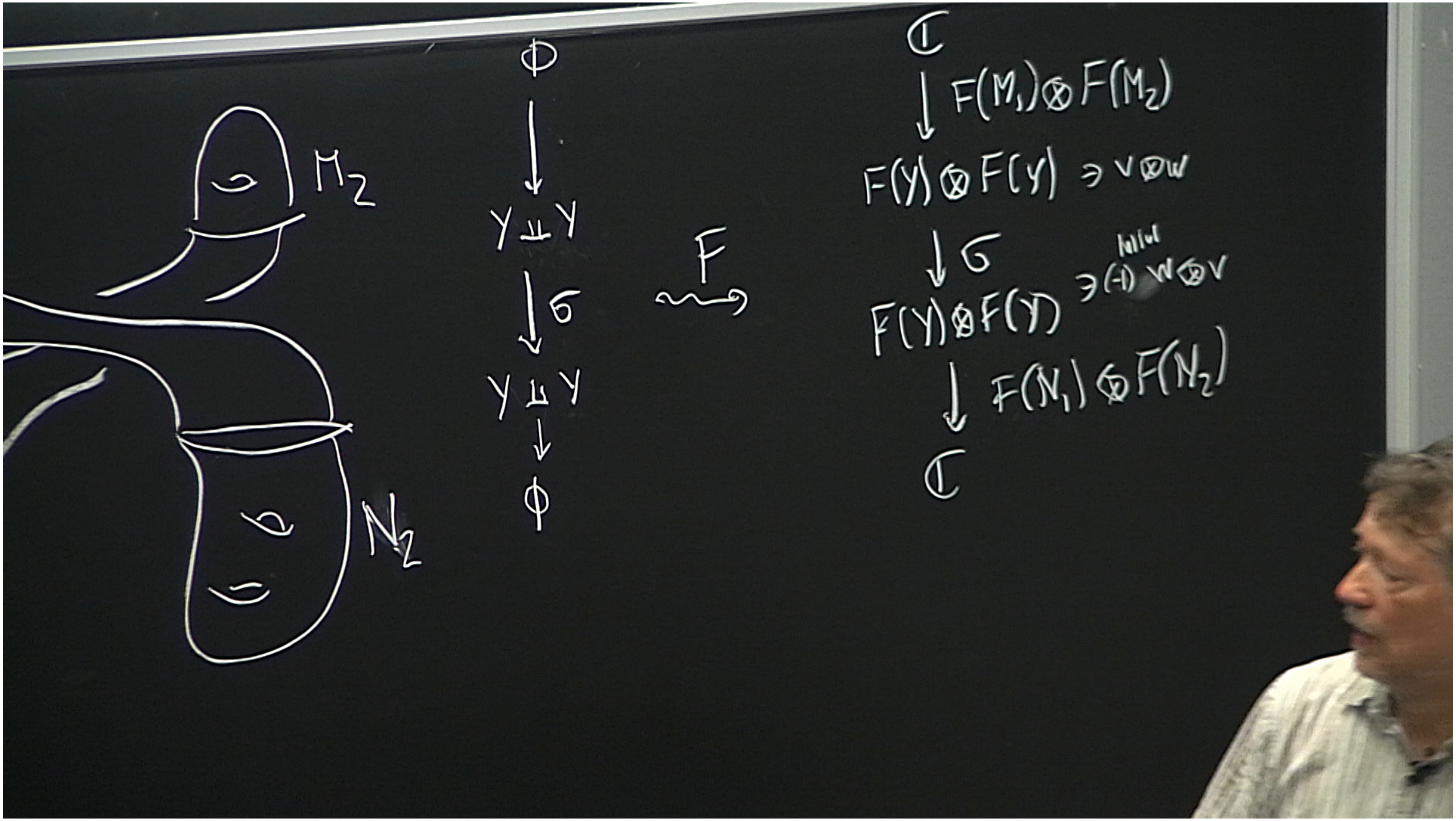
M1
M2
N1
N2

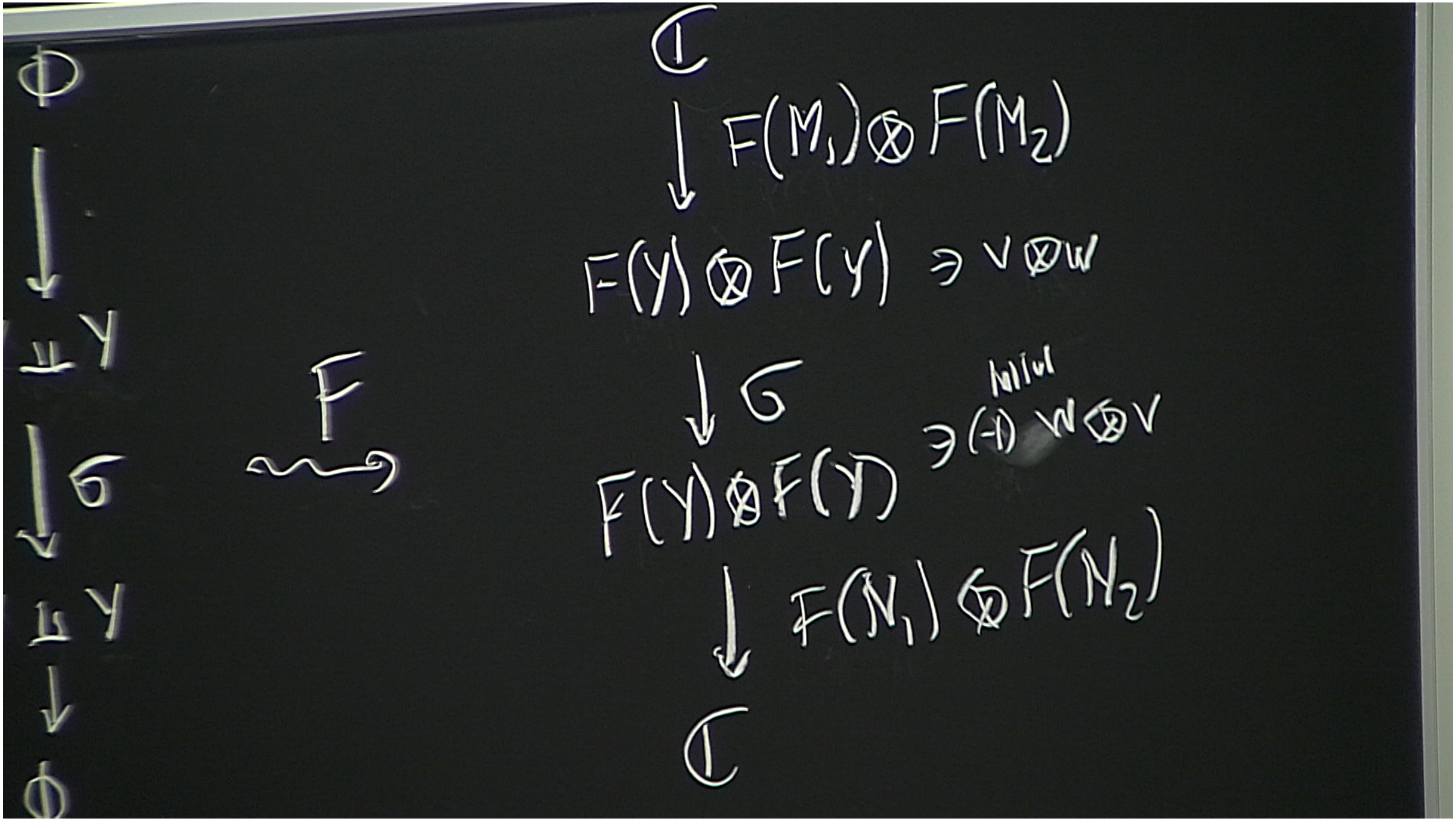

M1
M2
N1
N2
Φ
Y ⊔ Y
σ


M1
M2
N1
N2
Φ
Y ⊔ Y
σ
Y ⊔ Y
Φ
F
ℂ
F(M1) ⊗ F(M2)
F(Y) ⊗ F(Y)
σ
F(Y) ⊗ F(Y)
F(N1) ⊗ F(N2)
ℂ


M1
M2
N1
N2
Φ
Y ⊔ Y
σ
Y ⊔ Y
Φ
F
ℂ
F(M1) ⊗ F(M2)
F(Y) ⊗ F(Y) ∋ v⊗w
σ
F(Y) ⊗ F(Y) ∋ w⊗v
F(N1) ⊗ F(N2)
ℂ


M1
M2
N1
N2
∅
Y ⊔ Y
σ
Y ⊔ Y
∅
F
ℂ
F(M1) ⊗ F(M2)
F(Y) ⊗ F(Y) ∋ v ⊗ w
σ
F(Y) ⊗ F(Y) ∋ (-1)^|v||w| w ⊗ v
F(N1) ⊗ F(N2)
ℂ


M2
N2
ϕ
Y ⊔ Y
σ
Y ⊔ Y
ϕ
F
ℂ
F(M1) ⊗ F(M2)
F(Y) ⊗ F(Y) ∋ v ⊗ w
σ
F(Y) ⊗ F(Y) ∋ (-1) w ⊗ v
F(N1) ⊗ F(N2)
ℂ


$\mathbb{C}$
$F(M_1) \otimes F(M_2)$
$F(Y) \otimes F(Y) \ni v \otimes w$
σ
$\ni (-1)^{|v||w|}\, w \otimes v$
$F(Y) \otimes F(Y)$
$F(N_1) \otimes F(N_2)$
$\mathbb{C}$
F
$\emptyset$
$\sqcup Y$
σ
$\sqcup Y$
$\emptyset$

Thm.:

($\times$ ← invertible)

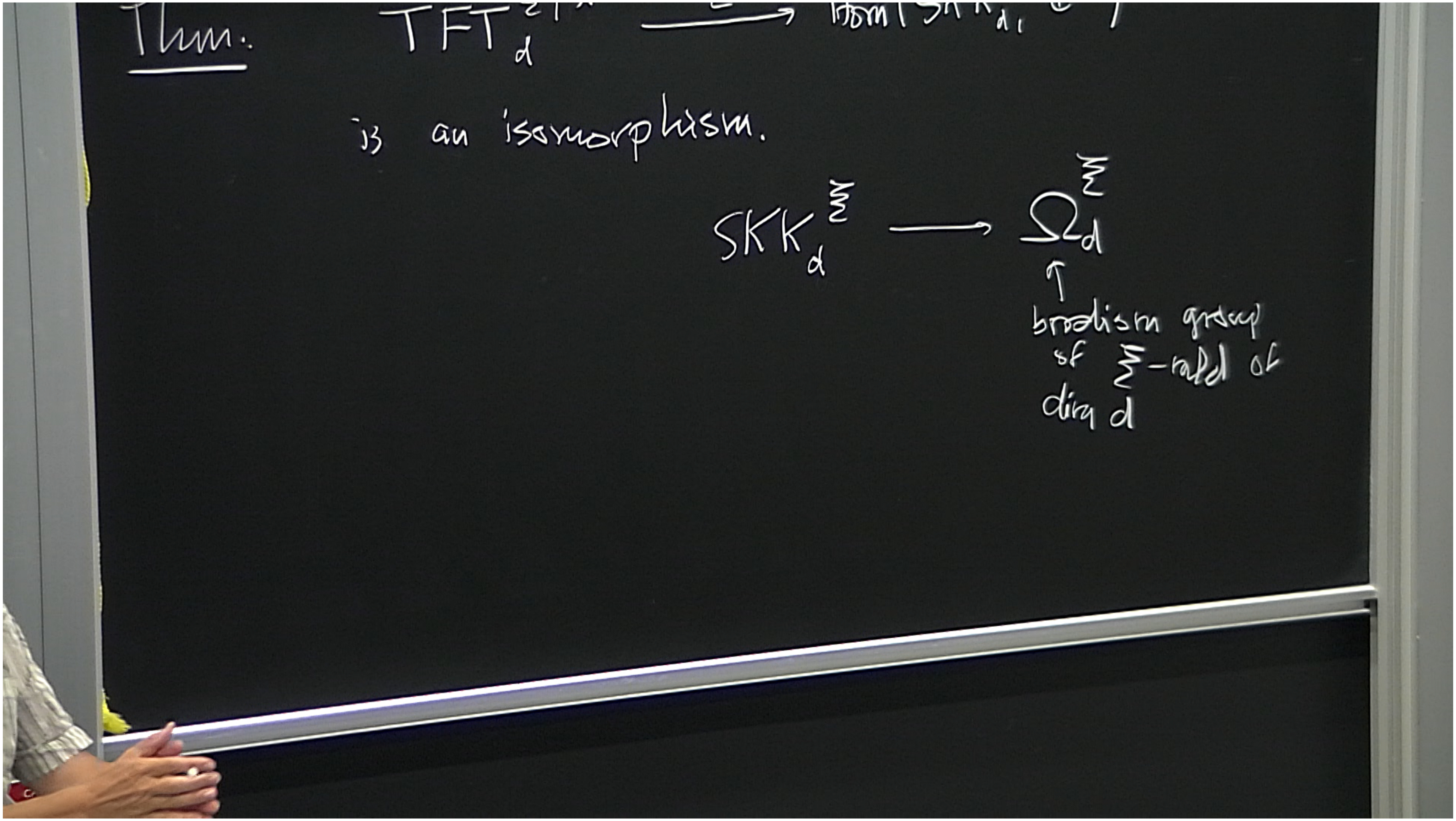
$$\mathrm{TFT}_d^{\xi, \times} \xrightarrow{Z} \mathrm{Hom}(\mathrm{SKK}_{d}^{\xi}, \mathbb{C}^\times)$$

Thm.

(× = invertible)

$$\mathrm{TFT}_d^{\Xi,\,\times} \xrightarrow{\;Z\;} \mathrm{Hom}(\mathrm{SKK}_{d}^{\Xi},\, \mathbb{C}^\times)$$

is an isomorphism.


Thm.:
is an isomorphism.
SKK → Ω
bordism group
dim d

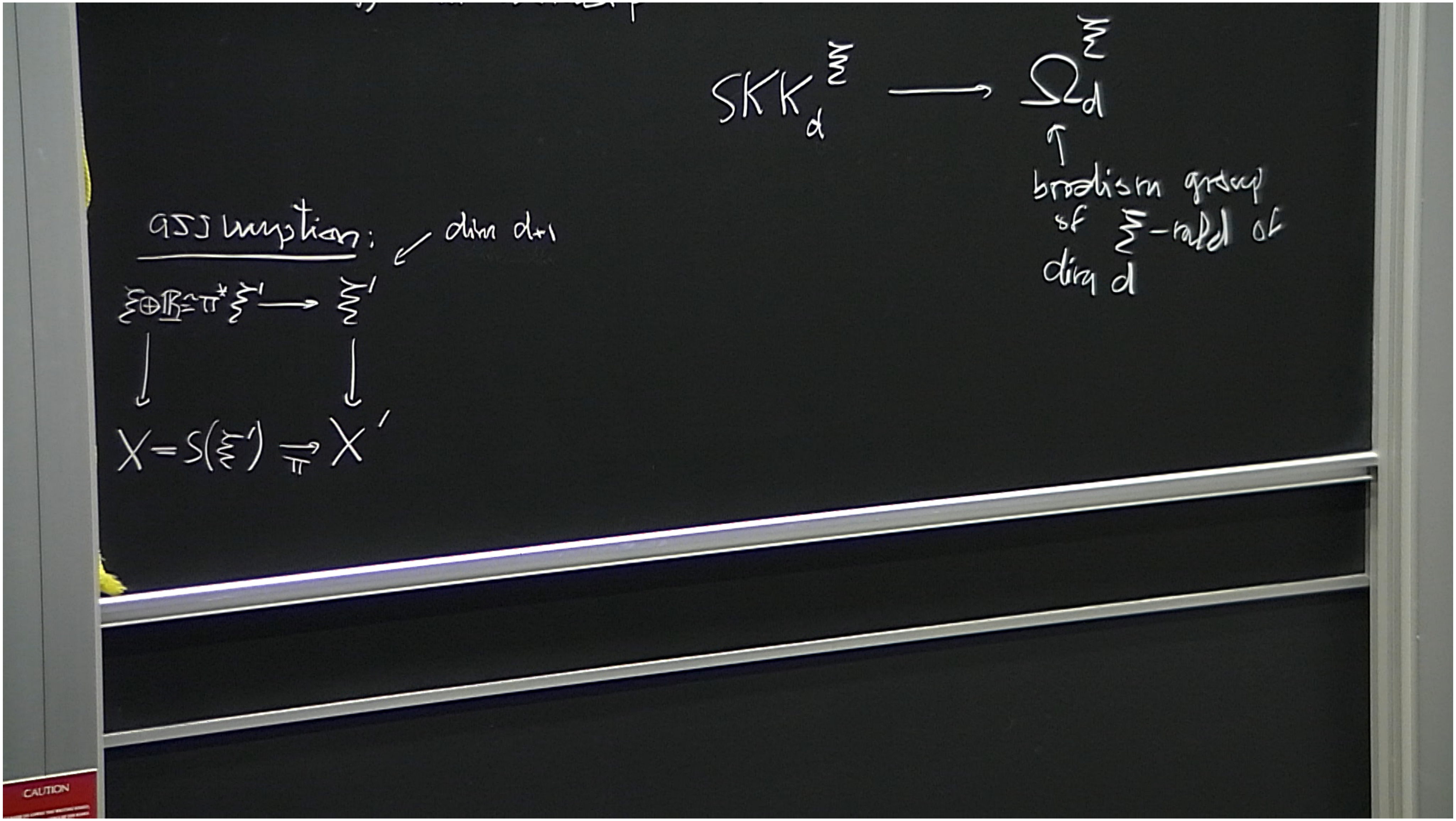
$$SKK_d^{\xi} \longrightarrow \Omega_d^{\xi}$$

↑ bordism group of ξ-mfld of dim d

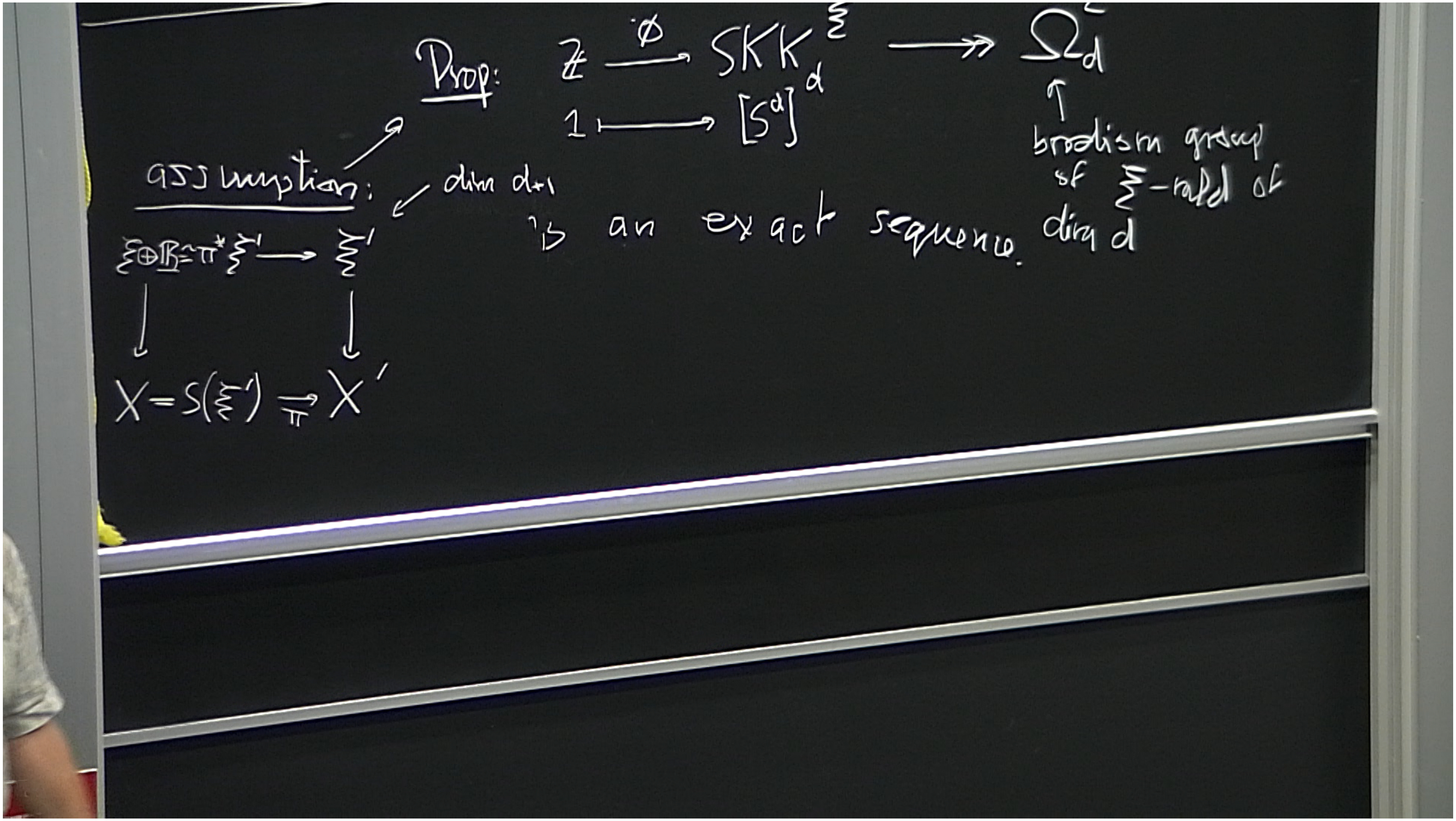
assumption: ← dim $d+1$

$$\xi' \downarrow$$

$$X = S(\xi') \to X'$$



$SKK_d^{\xi} \longrightarrow \Omega_d^{\xi}$
bordism group of ξ-mfld of dim d
assumption:
dim d+1
$\xi \oplus \underline{\mathbb{R}} = \pi^* \xi' \longrightarrow \xi'$
$X = S(\xi') \xrightarrow{\pi} X'$
CAUTION



Prop: $\mathbb{Z} \xrightarrow{\phi} SKK_d^{\xi} \twoheadrightarrow \Omega_d^{\xi}$
$1 \longmapsto [S^d]$
is an exact sequence.
bordism group of ξ-mfld of dim d
assumption:
dim d+1
$\xi \oplus \underline{\mathbb{R}} \cong \pi^* \xi' \longrightarrow \xi'$
$X = S(\xi') \underset{\pi}{\rightarrow} X'$

is an isomorphism.

Prop: $\mathbb{Z} \xrightarrow{\phi} SKK_d^{\xi} \twoheadrightarrow \Omega_d^{\xi'}$

$1 \longmapsto [S^d]$

↑ bordism group of ξ'-mfld of dim d

is an exact sequence, and

if d even: ϕ is injective

if d odd: $2[S^d] = 0$

<u>assumption:</u> dim d+1

$$\begin{array}{ccc} \xi \oplus \underline{\mathbb{R}} = \pi^* \xi' & \longrightarrow & \xi' \\ \downarrow & & \downarrow \\ X = S(\xi') & \xrightarrow{\pi} & X' \end{array}$$

is an isomorphism.

Prop: $\mathbb{Z} \xrightarrow{\phi} SKK_d^{\xi} \twoheadrightarrow \Omega_d^{\xi'}$

$1 \longmapsto [S^d]$

is an exact sequence, and

if d even: ϕ is injective

if d odd: $2[S^d] = 0$

$[S^d] \neq 0 \iff \exists$ closed ξ'-mfd W with $\chi(W)$ odd.

($\Omega_d^{\xi'}$: bordism group of ξ-mfd of dim d)

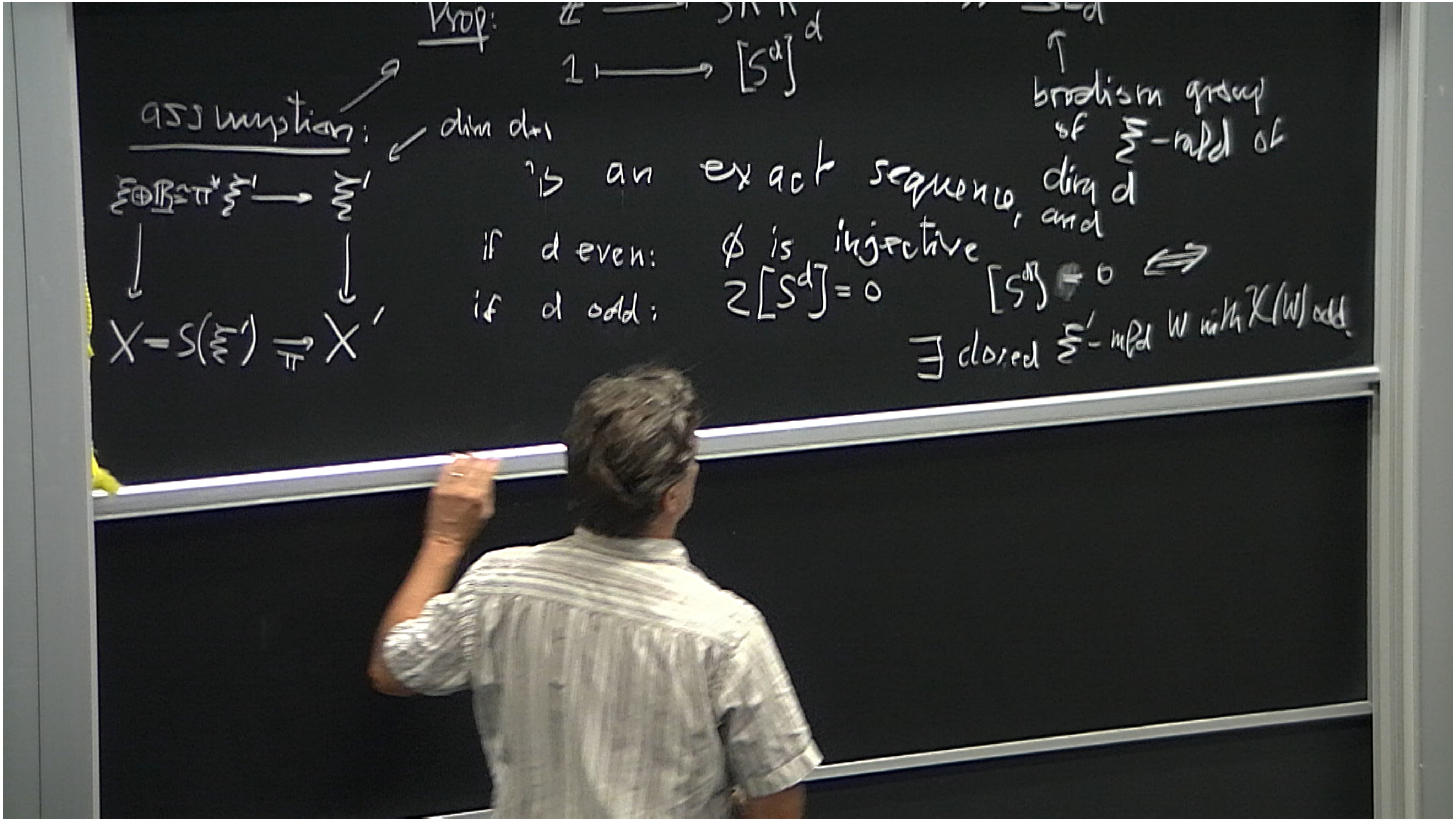
assumption: (dim $d+1$)

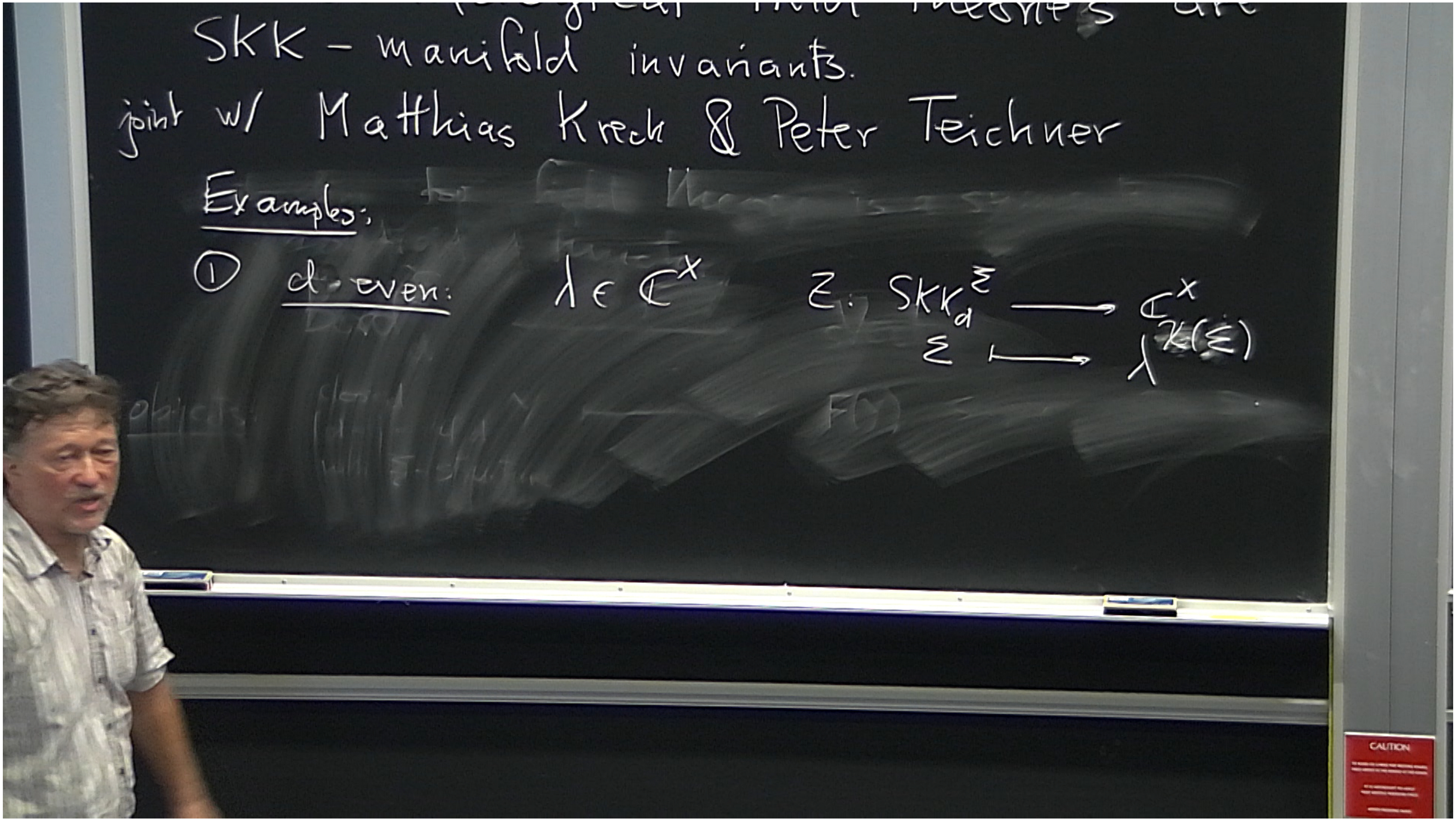
$$\begin{array}{ccc} \xi \oplus \underline{\mathbb{R}} \cong \pi^*\xi' & \longrightarrow & \xi' \\ \downarrow & & \downarrow \\ X = S(\xi') & \xrightarrow{\pi} & X' \end{array}$$

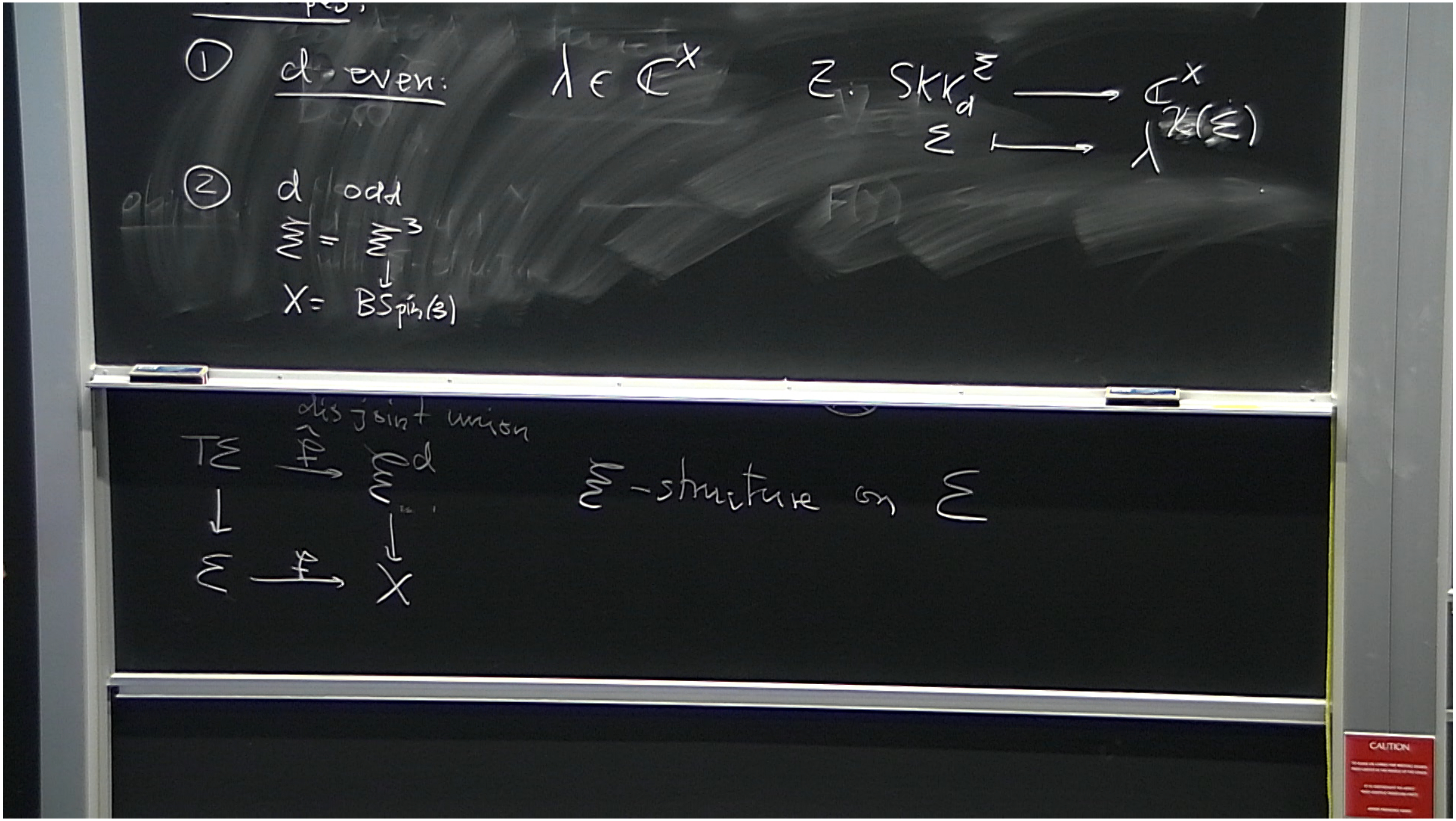
Φ

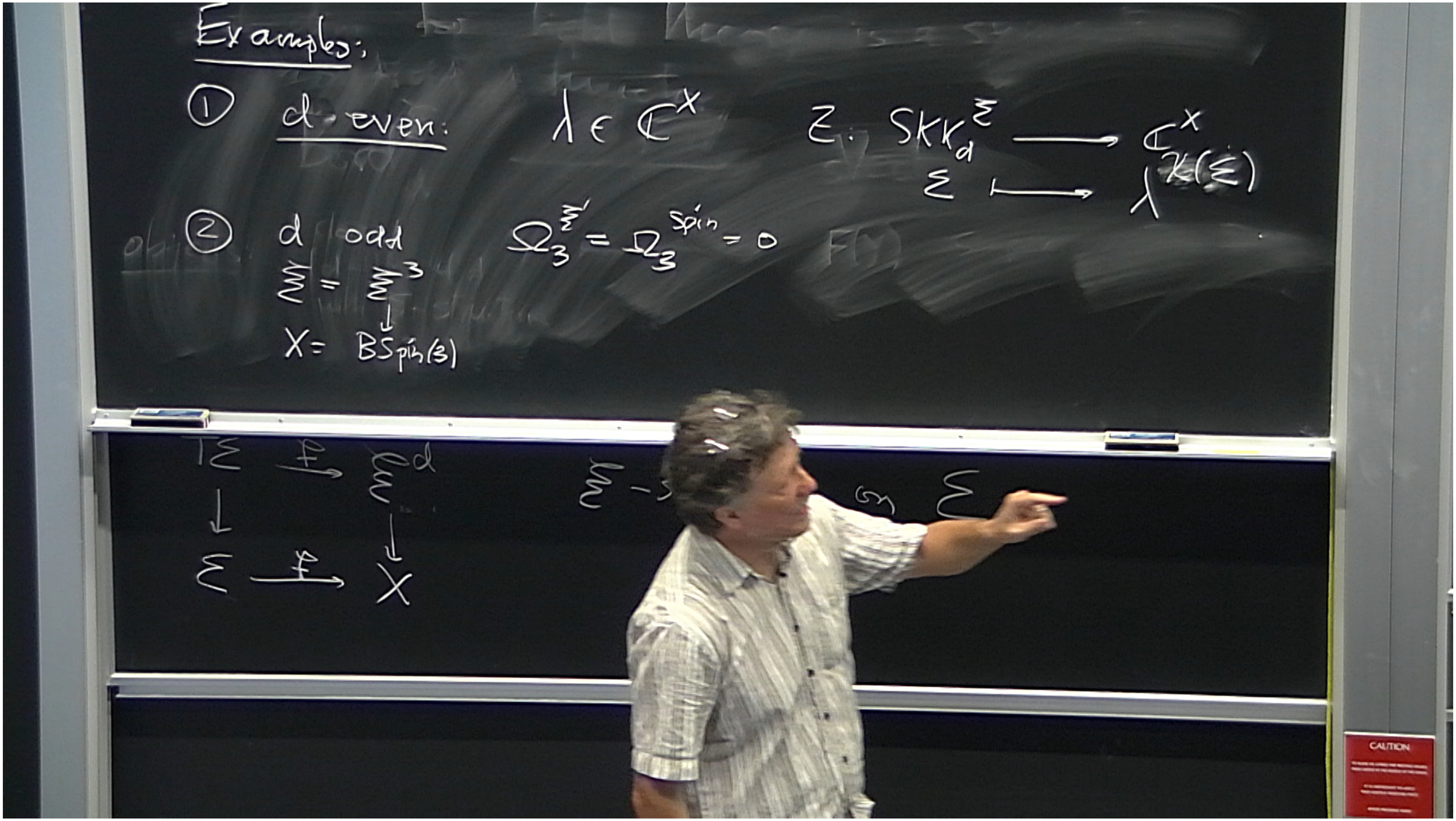
$\mathbb{C}$

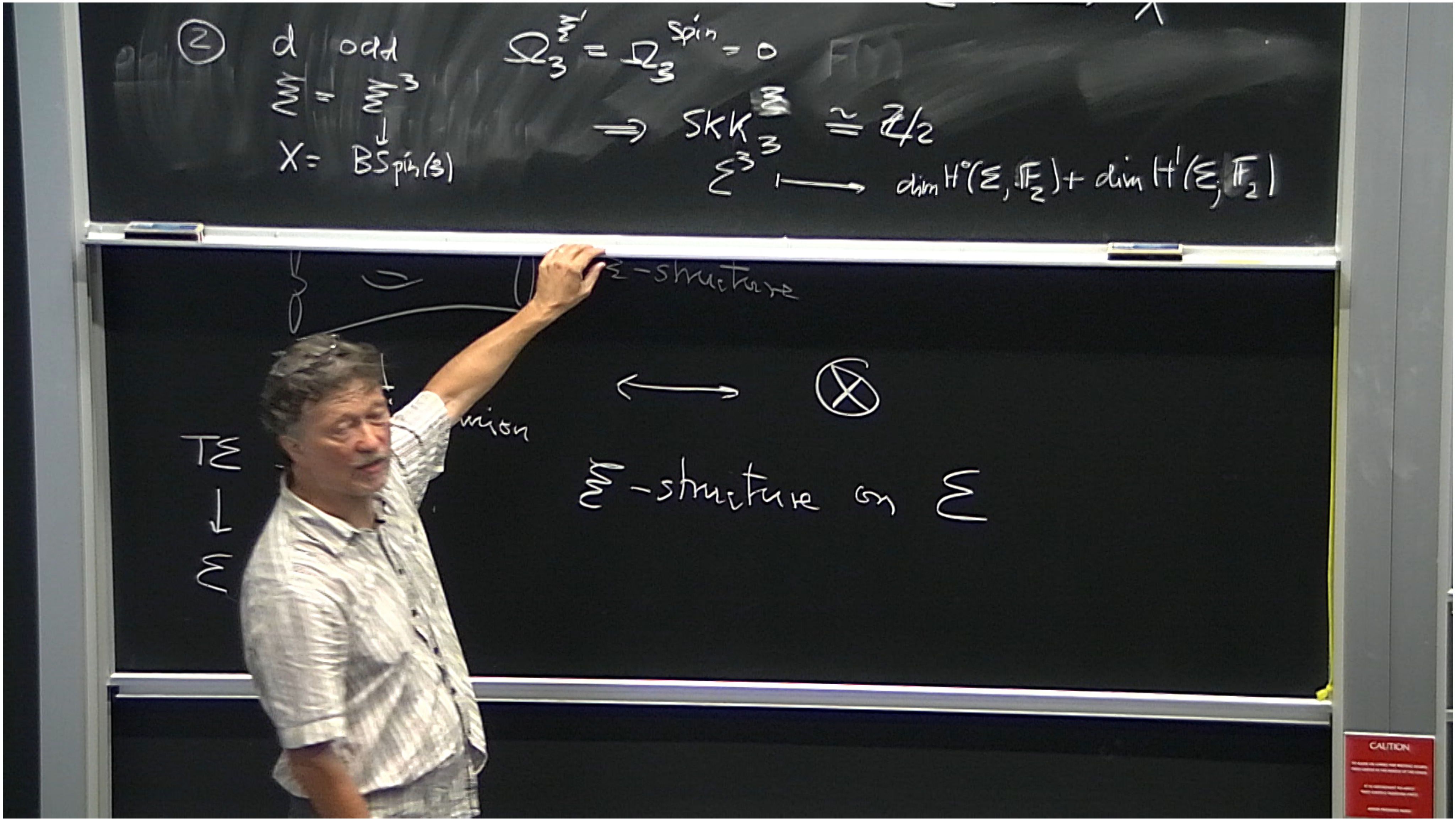
$F(M_1) \otimes F(M_2)$

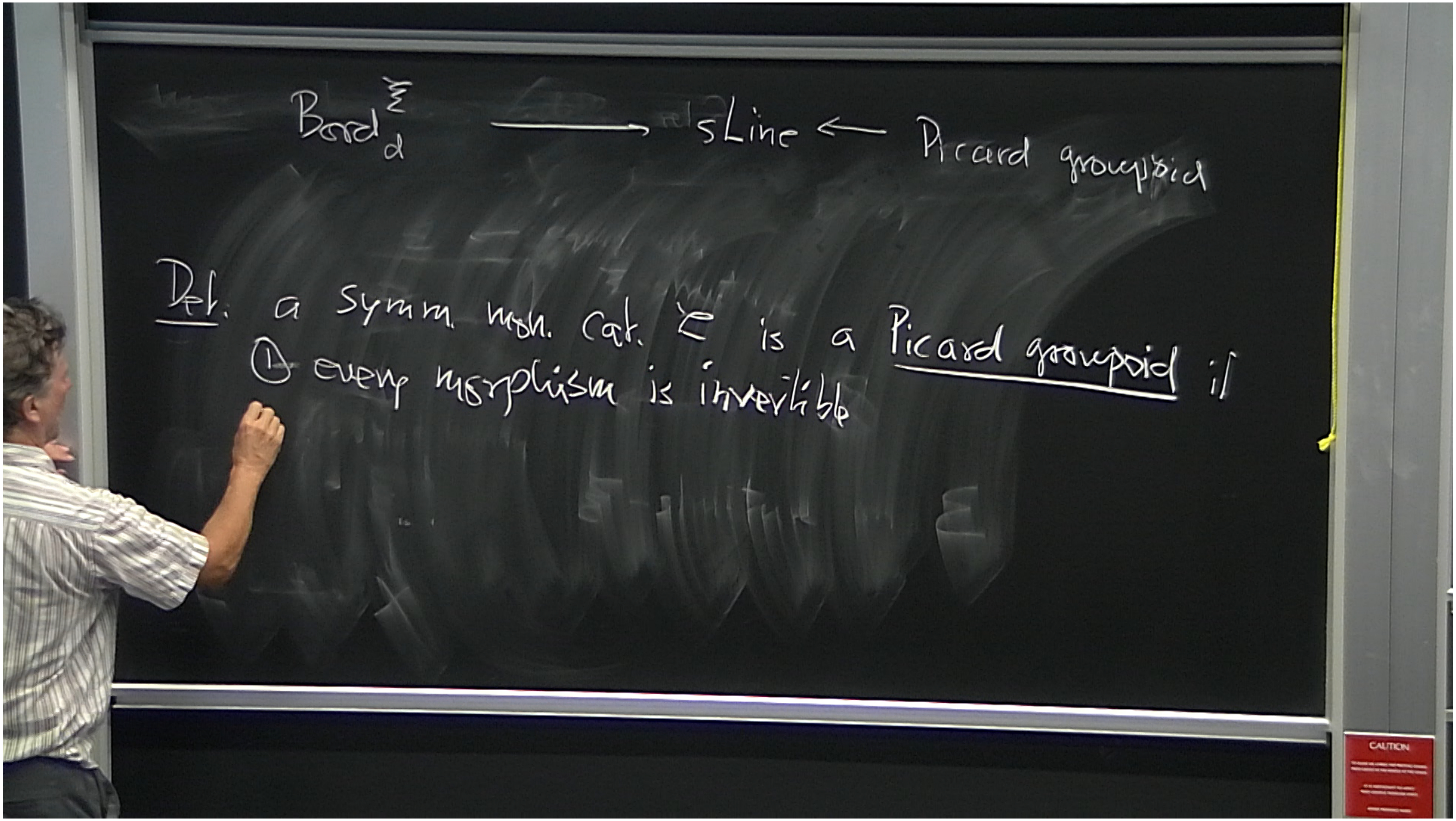

Prop:
1 ⟼ [S^d]
bordism group of ξ-mfd of dim d
assumption:
dim d+1
ξ⊕R = π*ξ' ⟶ ξ'
X = S(ξ') ⟶π X'
is an exact sequence, and
if d even: φ is injective
if d odd: 2[S^d] = 0
[S^d] ≠ 0 ⟺
∃ closed ξ'-mfd W with χ(W) odd


SKK - manifold invariants.
joint w/ Matthias Kreck & Peter Teichner
Examples:
① d even: λ ∈ ℂ^×
Z: SKK_d^Σ ⟶ ℂ^×
Σ ⟼ λ^χ(Σ)
CAUTION


① d even: λ ∈ ℂ^×
Z: Skk_d^Σ ⟶ ℂ^×
Σ ⟼ λ^χ(Σ)
② d odd
Ξ = Ξ^3
X = BSpin(3)
disjoint union
TΣ —f̂→ Ξ^d
Σ —f→ X
Ξ-structure on Σ


Examples:
① d even: λ ∈ ℂ^×
Z: SKK_d^Σ ⟶ ℂ^×
Σ ⟼ λ^χ(Σ)
② d odd Ω_3^Ξ = Ω_3^spin = 0
Ξ = Ξ^3
X = BSpin(3)
TΣ → Ξ^d
Σ → X


② d odd
Ω₃^ξ' = Ω₃^spin = 0
ξ = ξ³
X = BSpin(3)
⇒ SKK₃^ξ ≅ Z/2
Σ³ ↦ dim H⁰(Σ, F₂) + dim H¹(Σ, F₂)
ξ-structure
TΣ
Σ
ξ-structure on Σ

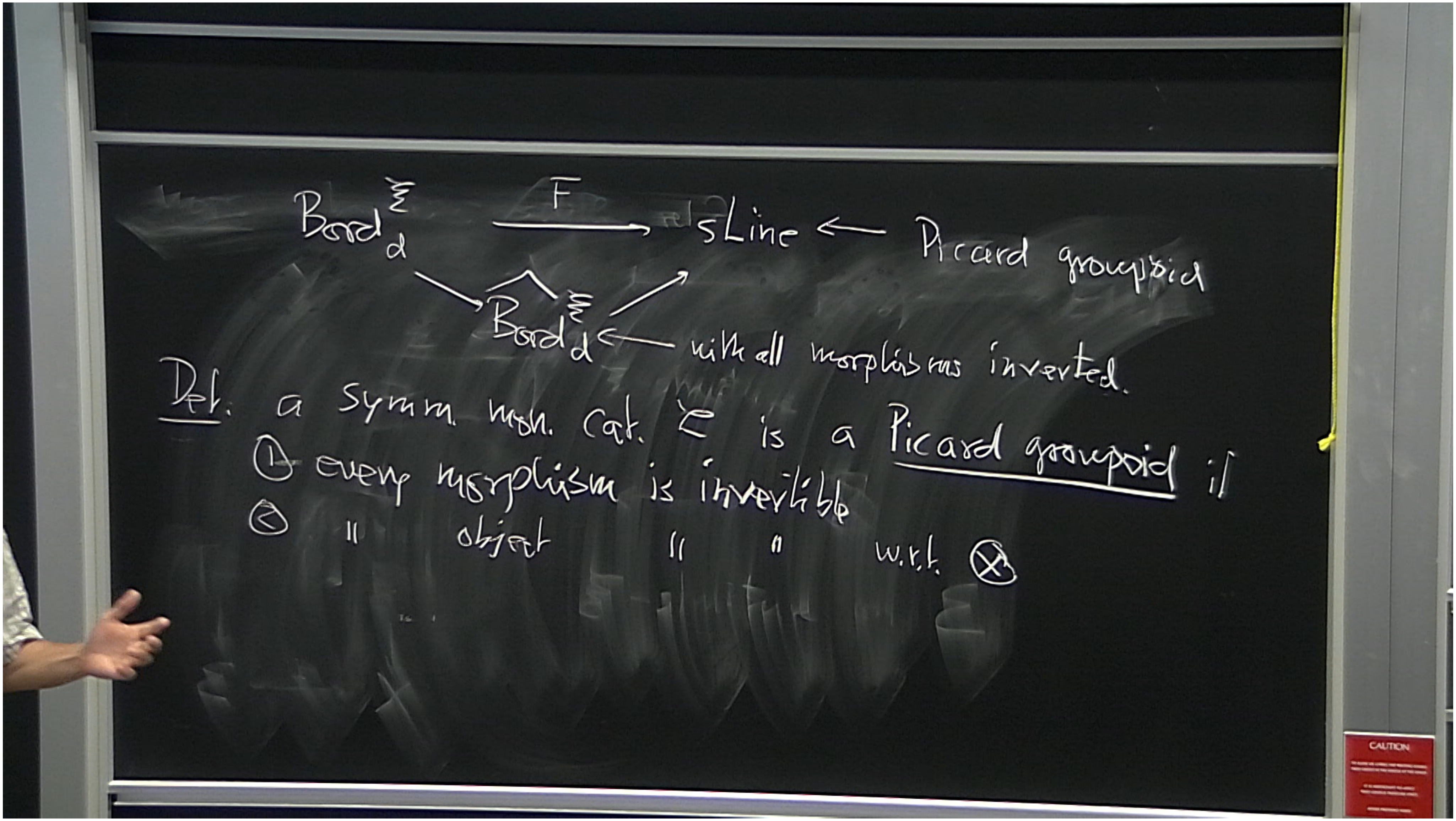

Bord_d^Σ ⟶ sLine ⟵ Picard groupoid
Def: a symm. mon. cat. ℰ is a Picard groupoid if
① every morphism is invertible
CAUTION

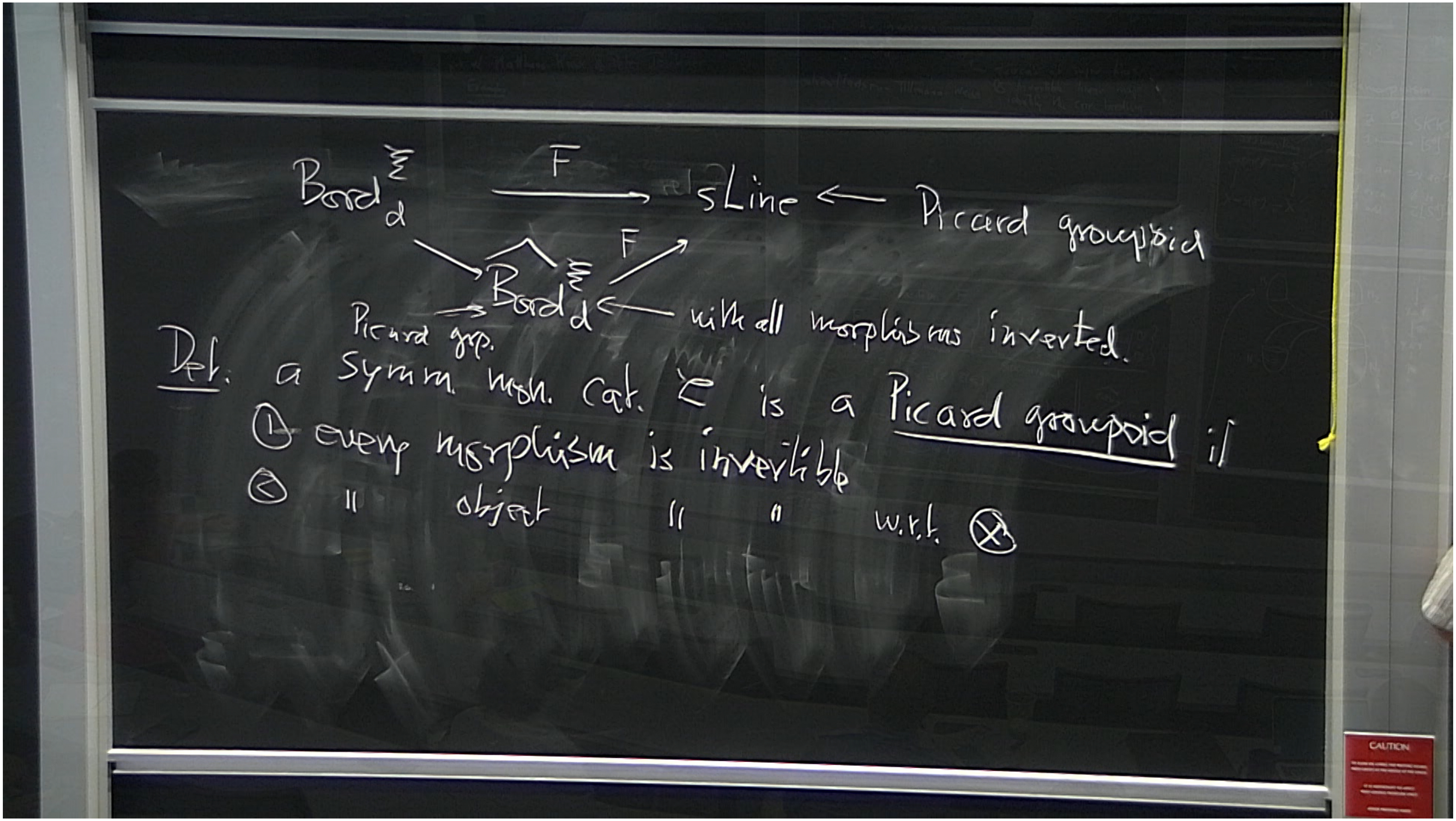
$$\mathrm{Bord}_d^{\xi} \longrightarrow \mathrm{sLine} \longleftarrow \text{Picard groupoid}$$

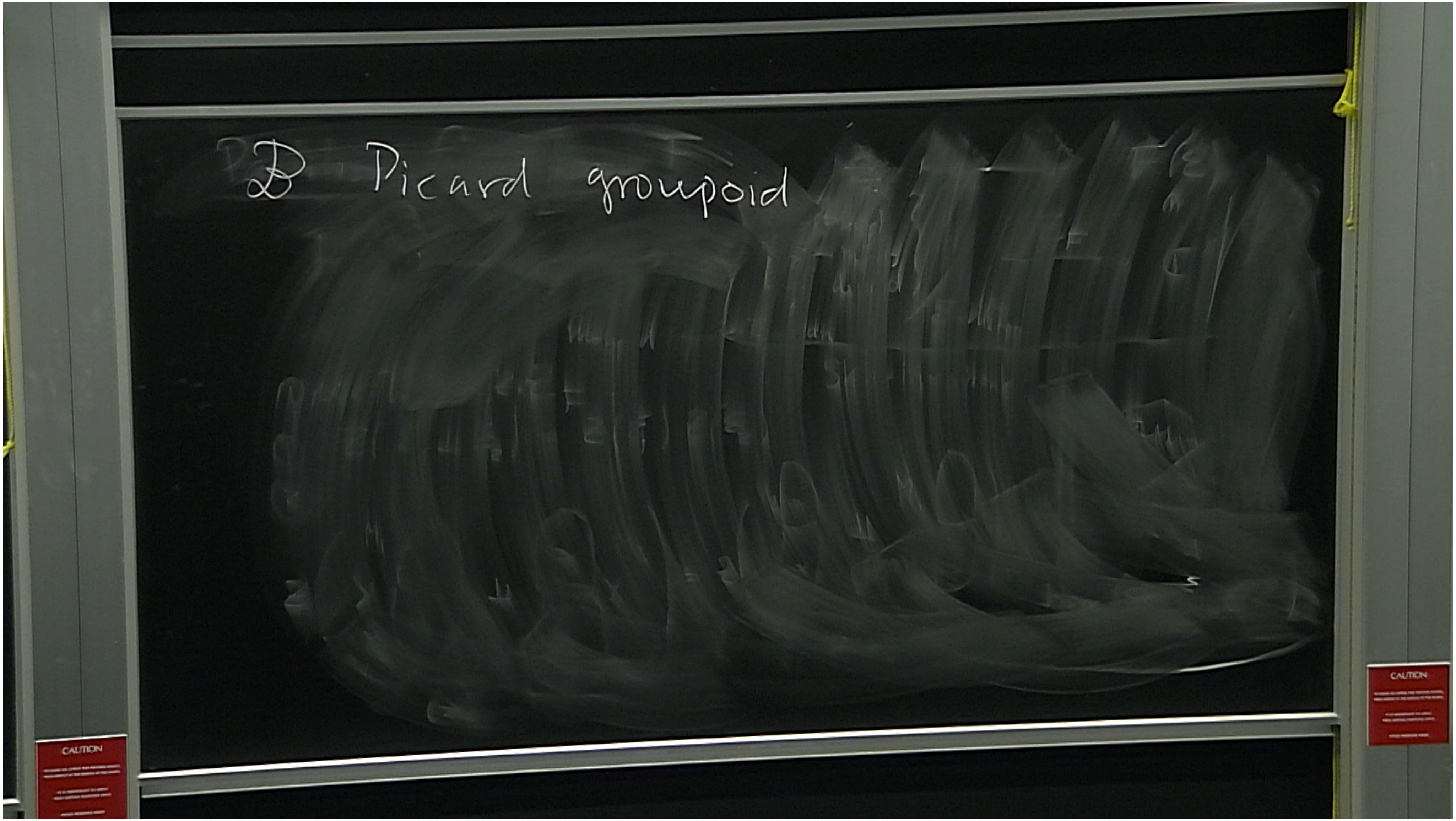
<u>Def</u>: a symm. mon. cat. $\mathcal{C}$ is a <u>Picard groupoid</u> if

① every morphism is invertible

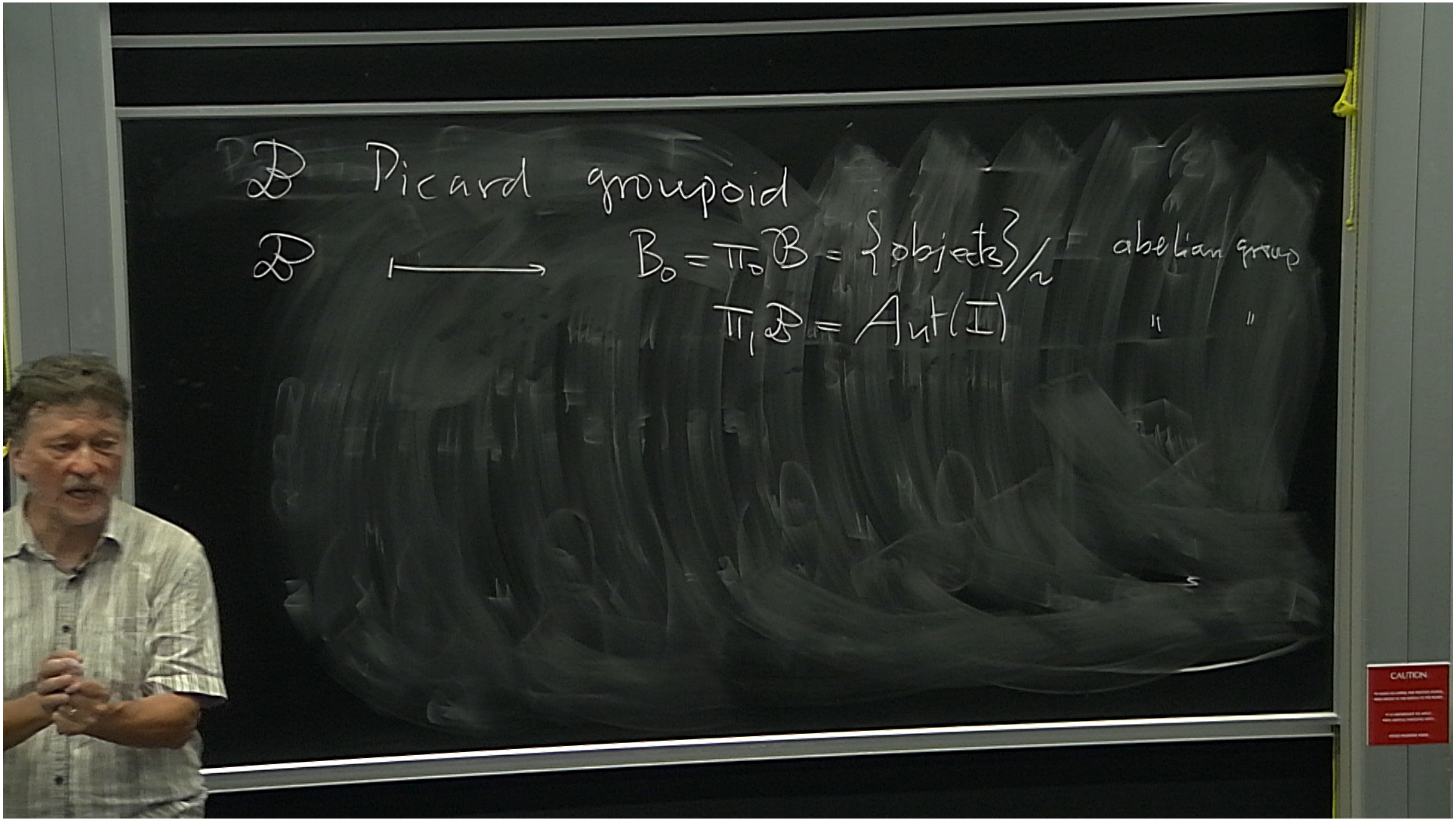
② " object " " w.r.t. $\otimes$

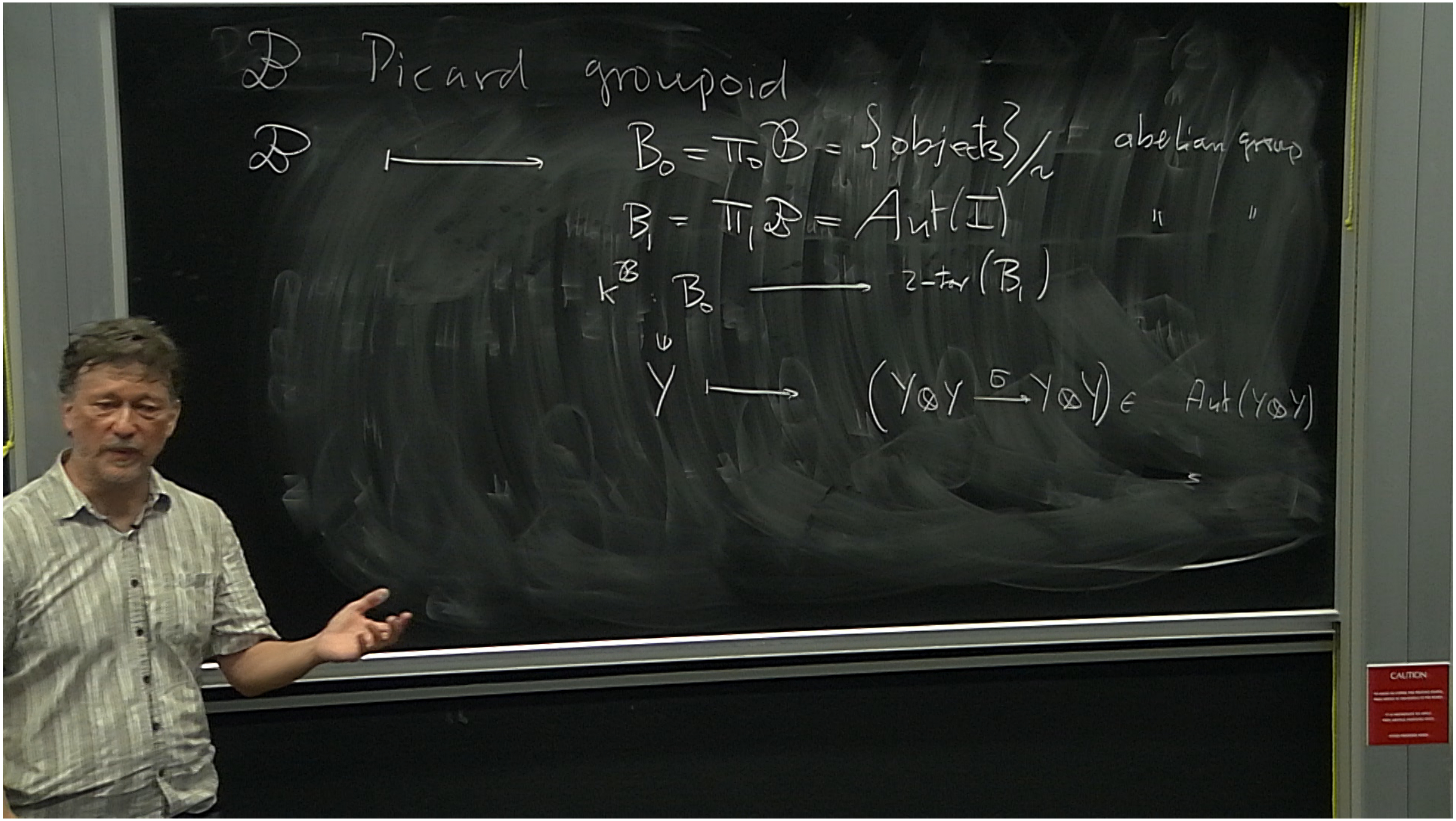

Bord_d^Σ —F→ sLine ← Picard groupoid
Bord_d^Σ ← with all morphisms inverted.
Def: a symm. mon. cat. C is a Picard groupoid if
1 every morphism is invertible
2 " object " " w.r.t. ⊗


Bord_d^Σ F sLine ← Picard groupoid
F
Picard grp. → Bord_d^Σ ← with all morphisms inverted.
Def: a symm. mon. cat. C is a Picard groupoid if
① every morphism is invertible
② " object " " w.r.t. ⊗

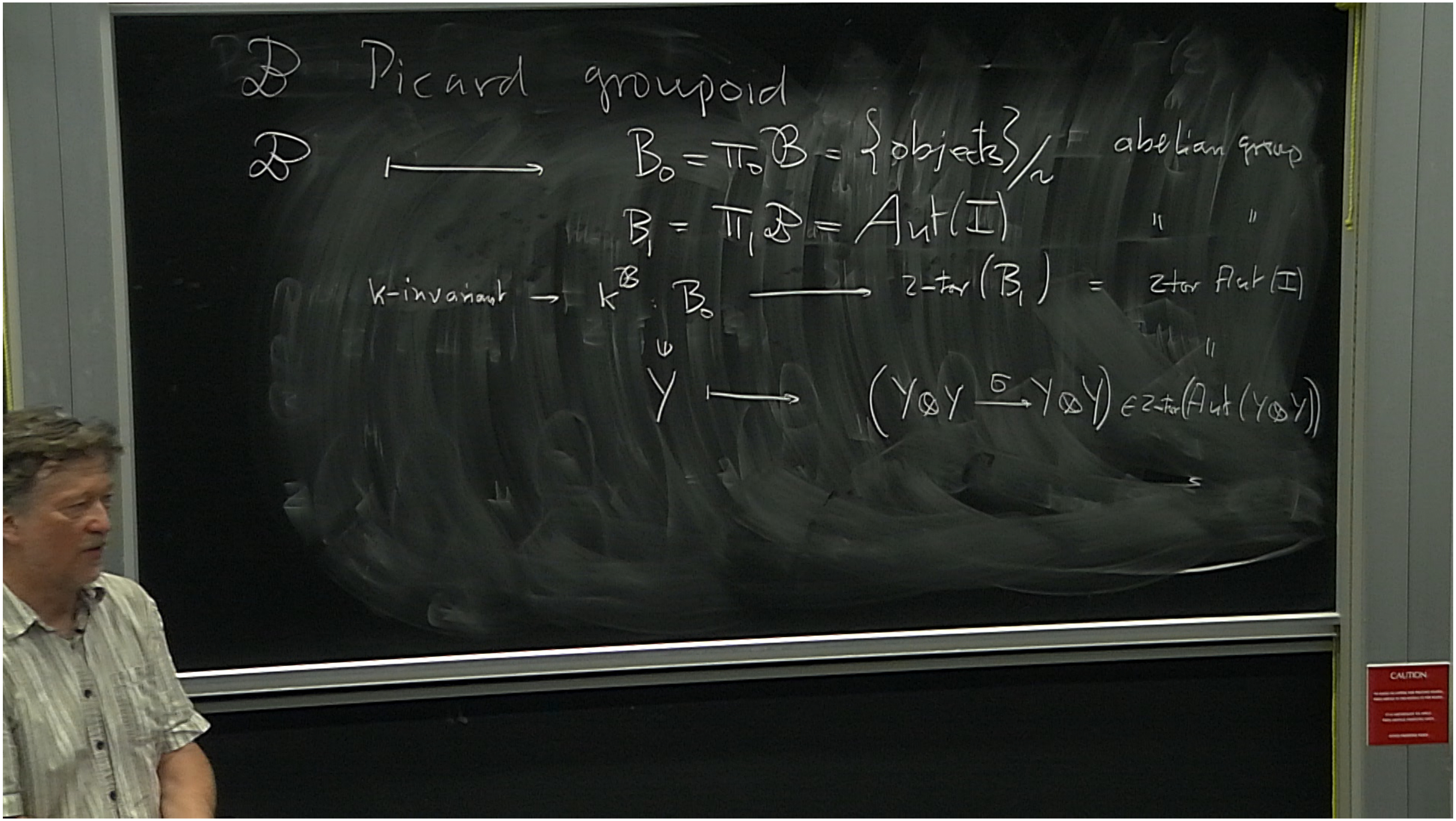


$\mathcal{B}$ Picard groupoid
CAUTION
CAUTION


𝔅 Picard groupoid
𝔅 ⟼ B₀ = π₀𝔅 = {objects}/~ abelian group
π₁𝔅 = Aut(I) " "



$\mathcal{B}$ Picard groupoid
$\mathcal{B} \longmapsto B_0 = \pi_0 \mathcal{B} = \{\text{objects}\}/\sim$ abelian group
$B_1 = \pi_1 \mathcal{B} = Aut(I)$ " "
$k^{\mathcal{B}} : B_0 \longrightarrow$ 2-tor(B_1)
$Y \longmapsto (Y \otimes Y \xrightarrow{\sigma} Y \otimes Y) \in Aut(Y \otimes Y)$


B Picard groupoid
B ⟼ B0 = π0 B = {objects}/~ abelian group
B1 = π1 B = Aut(I)
k-invariant → k^B : B0 ⟶ 2-tor(B1) = 2tor Aut(I)
Y ⟼ (Y⊗Y —σ→ Y⊗Y) ∈ 2-tor(Aut(Y⊗Y))

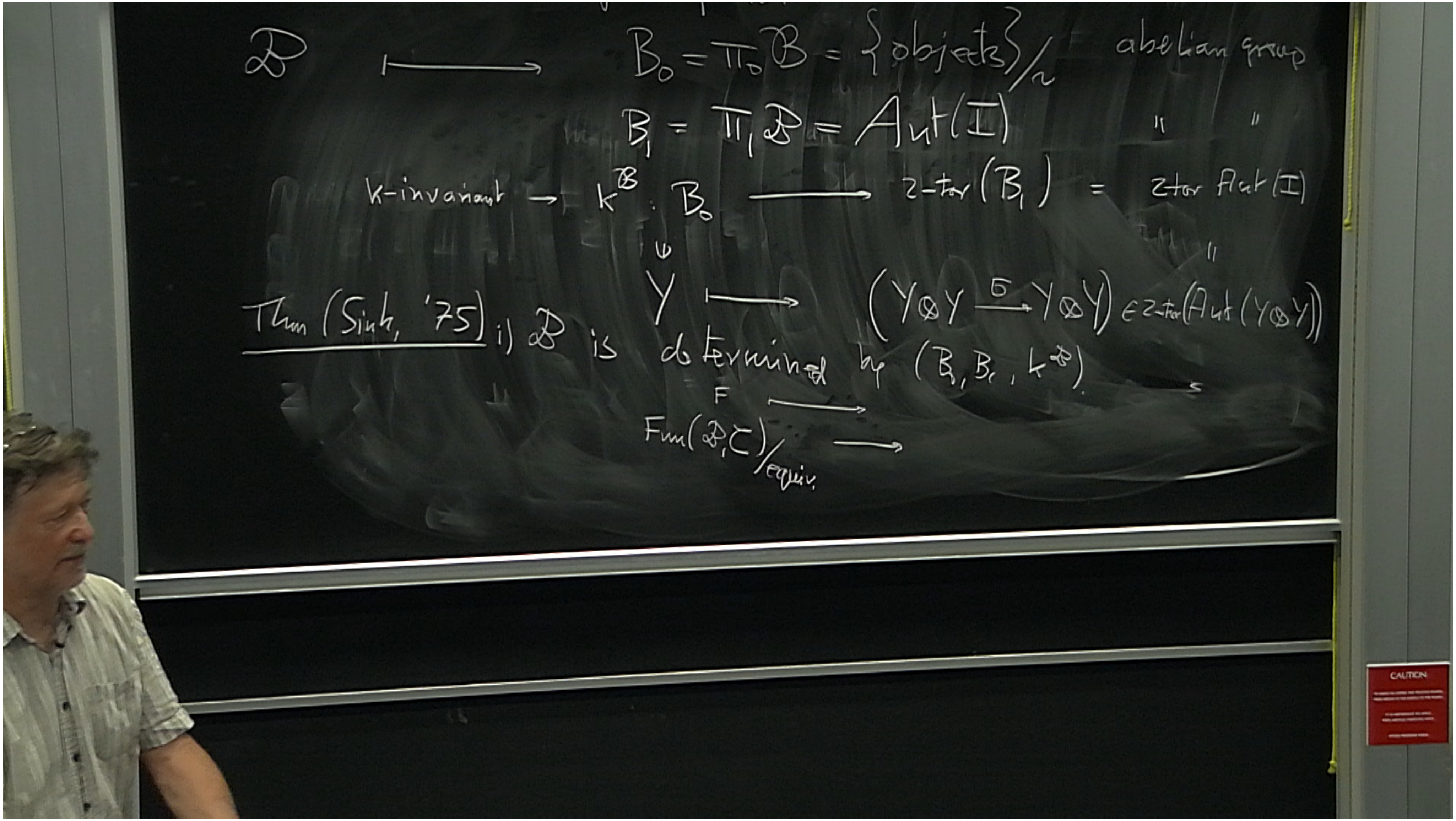
$$\mathcal{B} \longmapsto B_0 = \pi_0 \mathcal{B} = \{\text{objects}\}/_\sim \quad \text{abelian group}$$

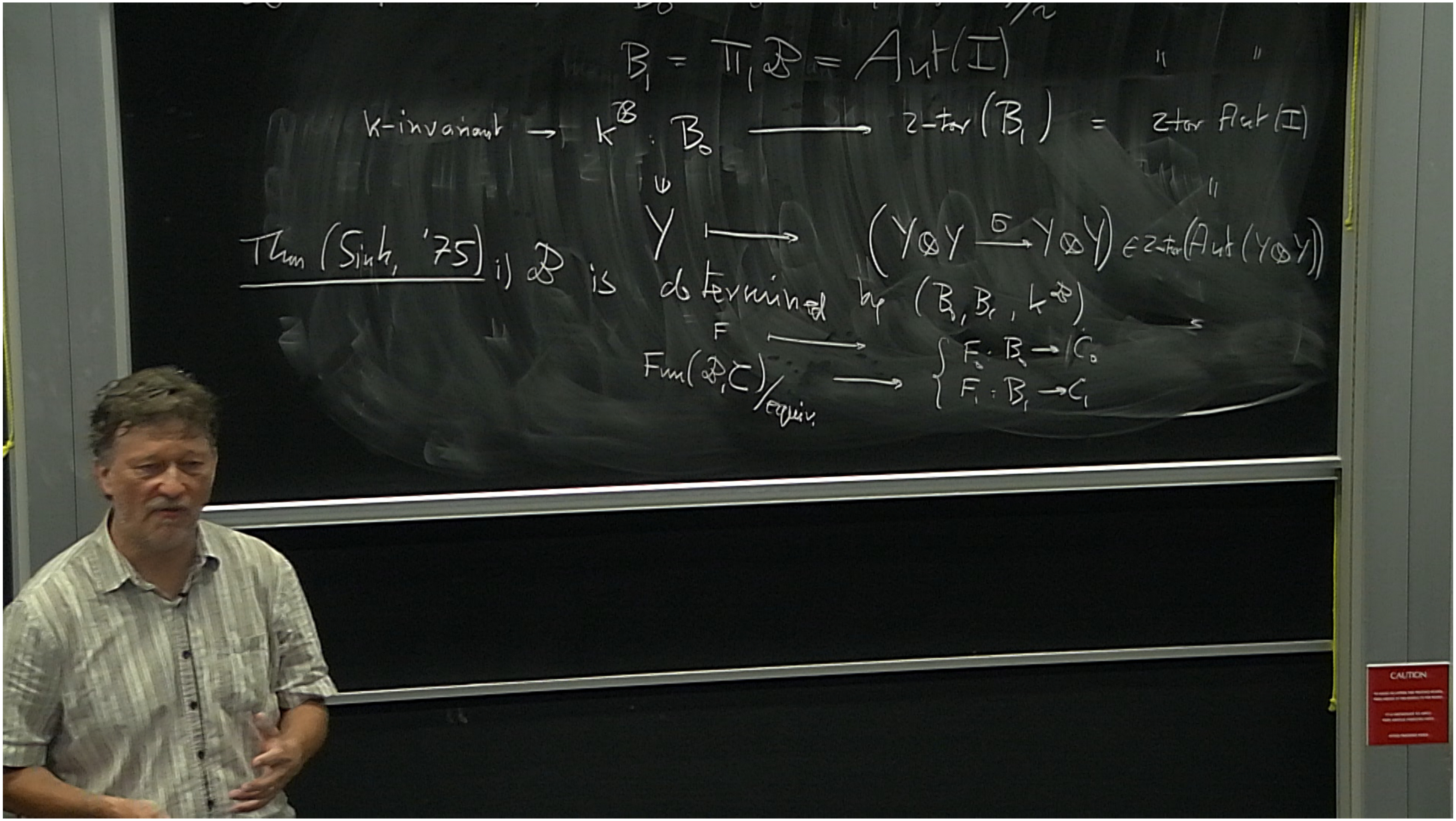
$$B_1 = \pi_1 \mathcal{B} = \mathrm{Aut}(I) \quad '' \quad ''$$

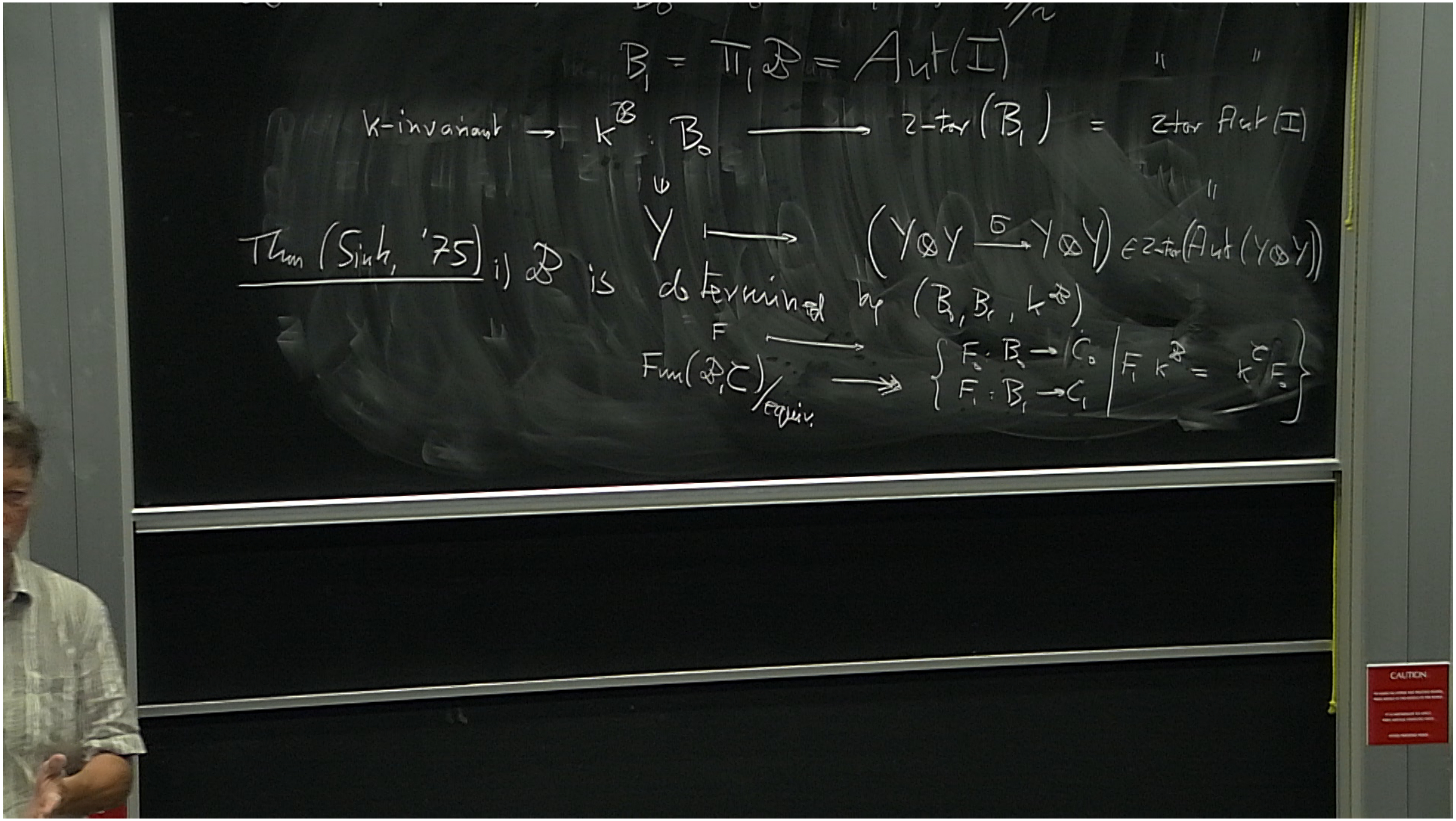
$$\text{k-invariant} \to k^{\mathcal{B}} : B_0 \longrightarrow \text{2-tor}(B_1) = \text{2-tor}\,\mathrm{Aut}(I)$$

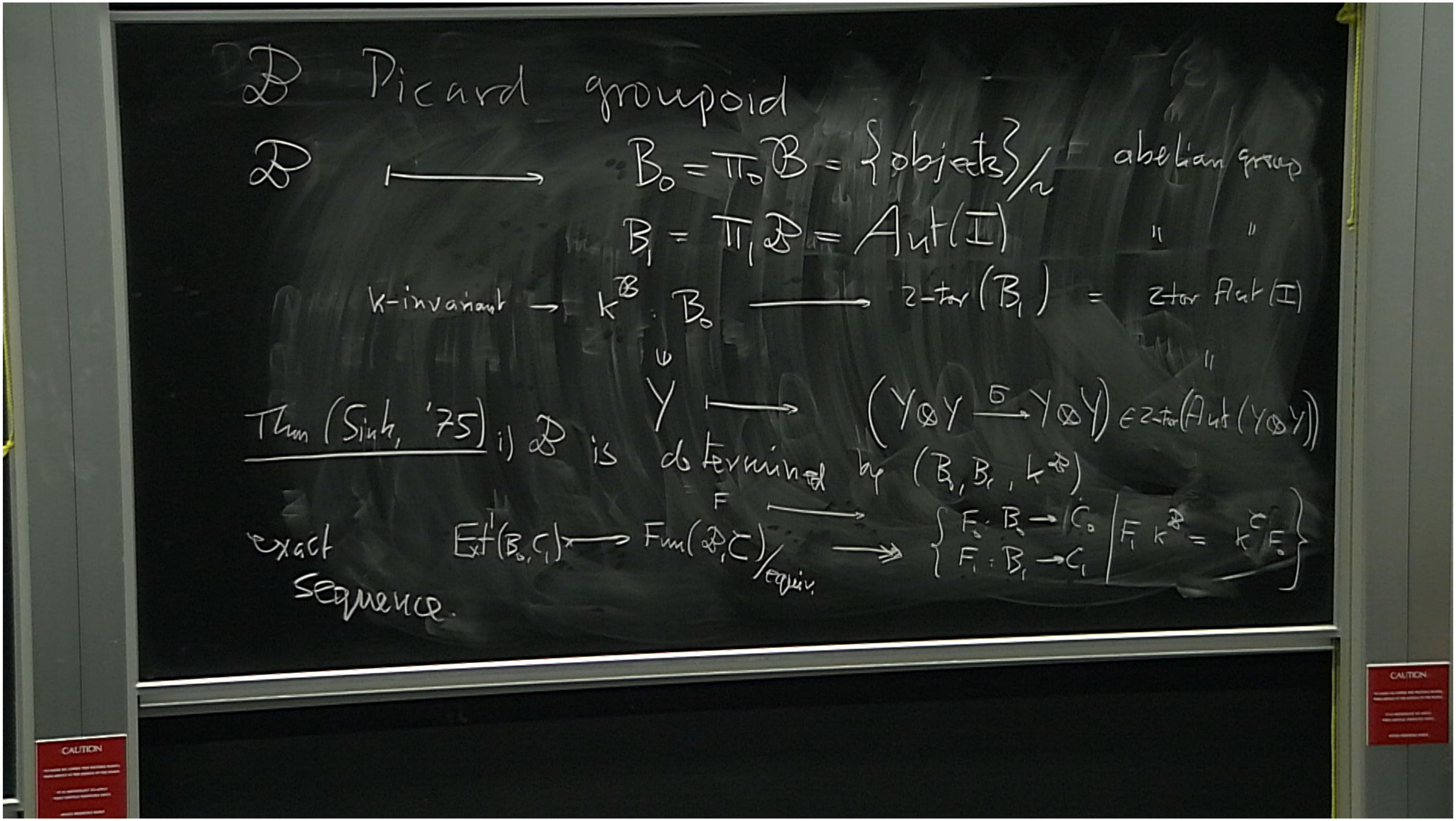
$$\begin{matrix} \psi & & & & \\ Y & \longmapsto & \left(Y \otimes Y \xrightarrow{\sigma} Y \otimes Y\right) & \in \text{2-tor}\left(\mathrm{Aut}(Y \otimes Y)\right) \end{matrix}$$

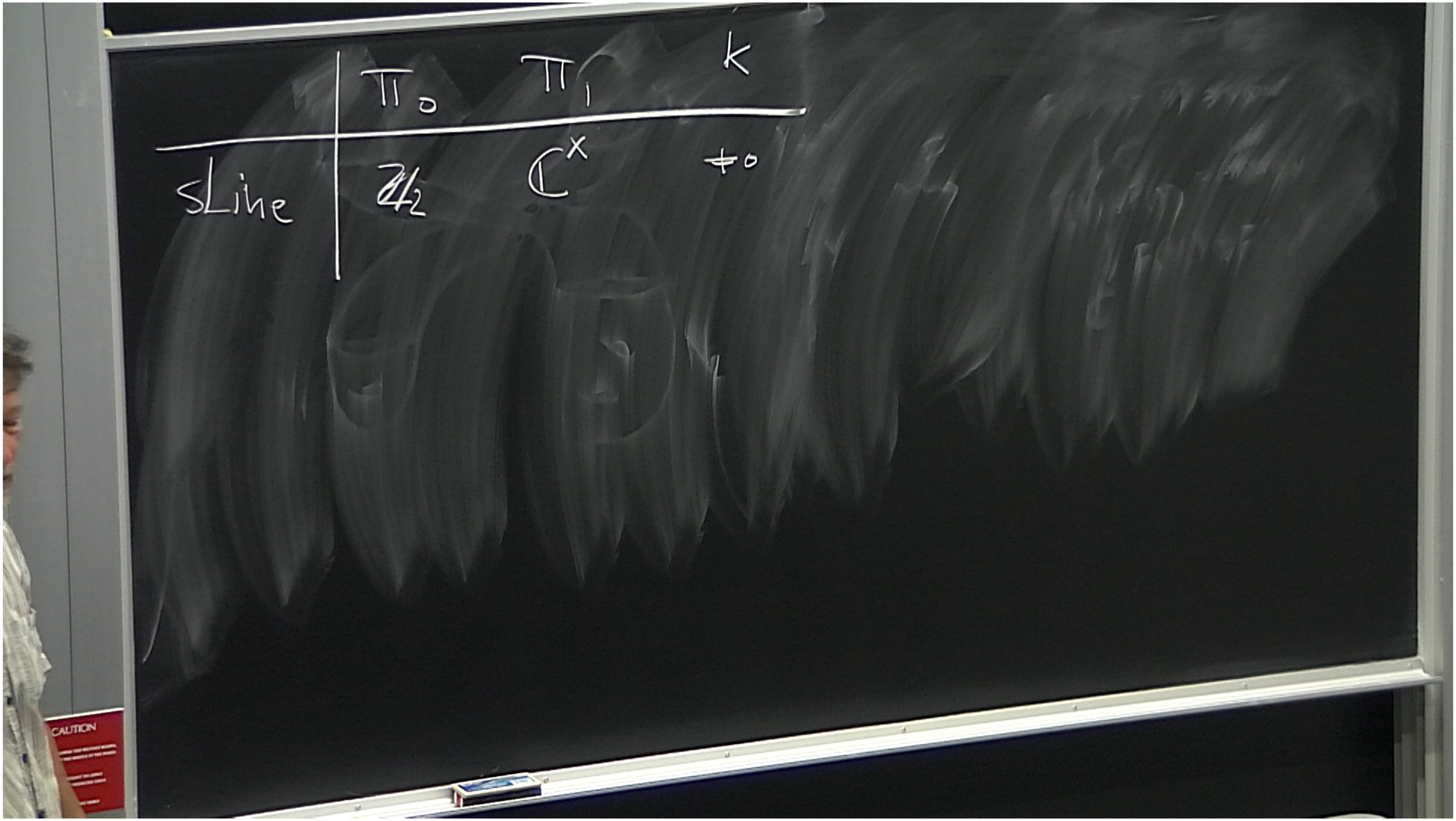
<u>Thm (Sinh, '75)</u> $\mathcal{B}$ is determined by $(B_0, B_1, k^{\mathcal{B}})$.

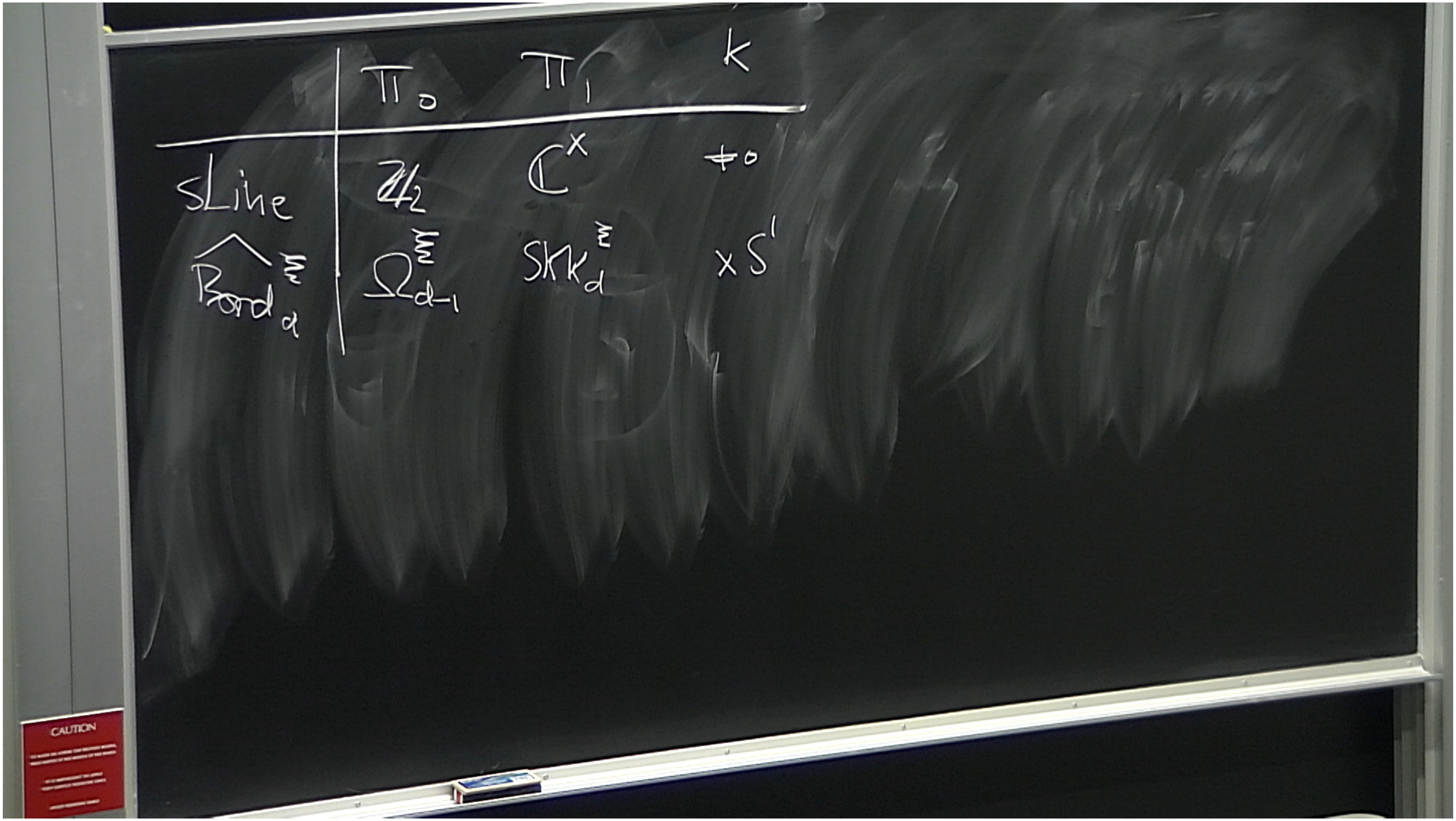

𝔅 ⟼ B₀ = π₀𝔅 = {objects}/~ abelian group
B₁ = π₁𝔅 = Aut(I)
k-invariant → k^𝔅 : B₀ ⟶ 2-tor(B₁) = 2tor Aut(I)
Y ⟼ (Y⊗Y –σ→ Y⊗Y) ∈ 2-tor(Aut(Y⊗Y))
Thm (Sinh, '75) i) 𝔅 is determined by (B₁, B₀, k^𝔅).
F
Fun(𝔅,C)/equiv.

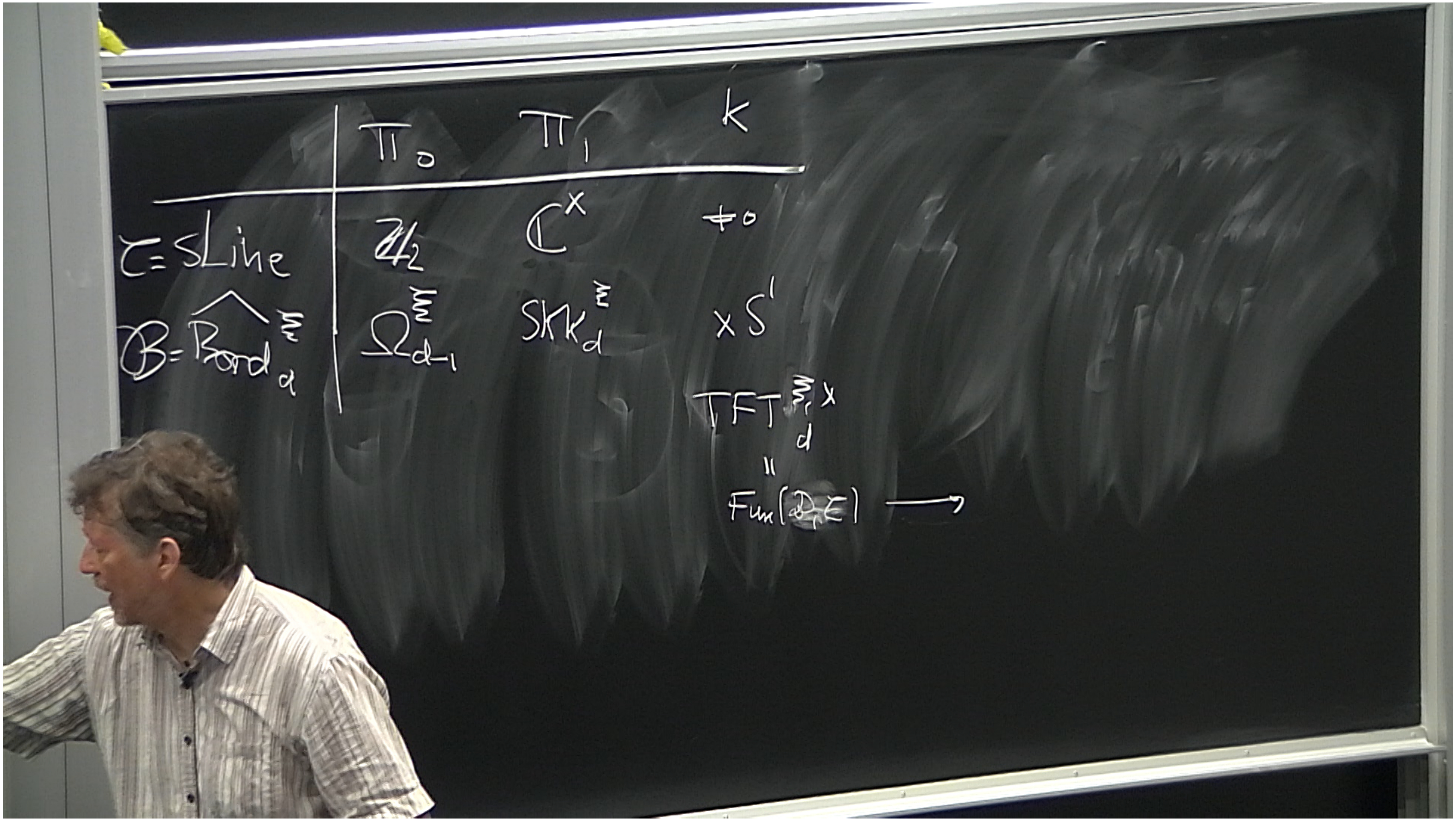

B1 = π1 𝔅 = Aut(I)
k-invariant → k𝔅 : B0 ⟶ 2-tor(B1) = 2tor Aut(I)
Y ⟼ (Y⊗Y σ⟶ Y⊗Y) ∈ 2-tor(Aut(Y⊗Y))
Thm (Sinh, '75) i) 𝔅 is determined by (B0, B1, k𝔅)
F ⟶ F0 : B0 → C0
Fun(𝔅,C)/equiv ⟶ F1 : B1 → C1

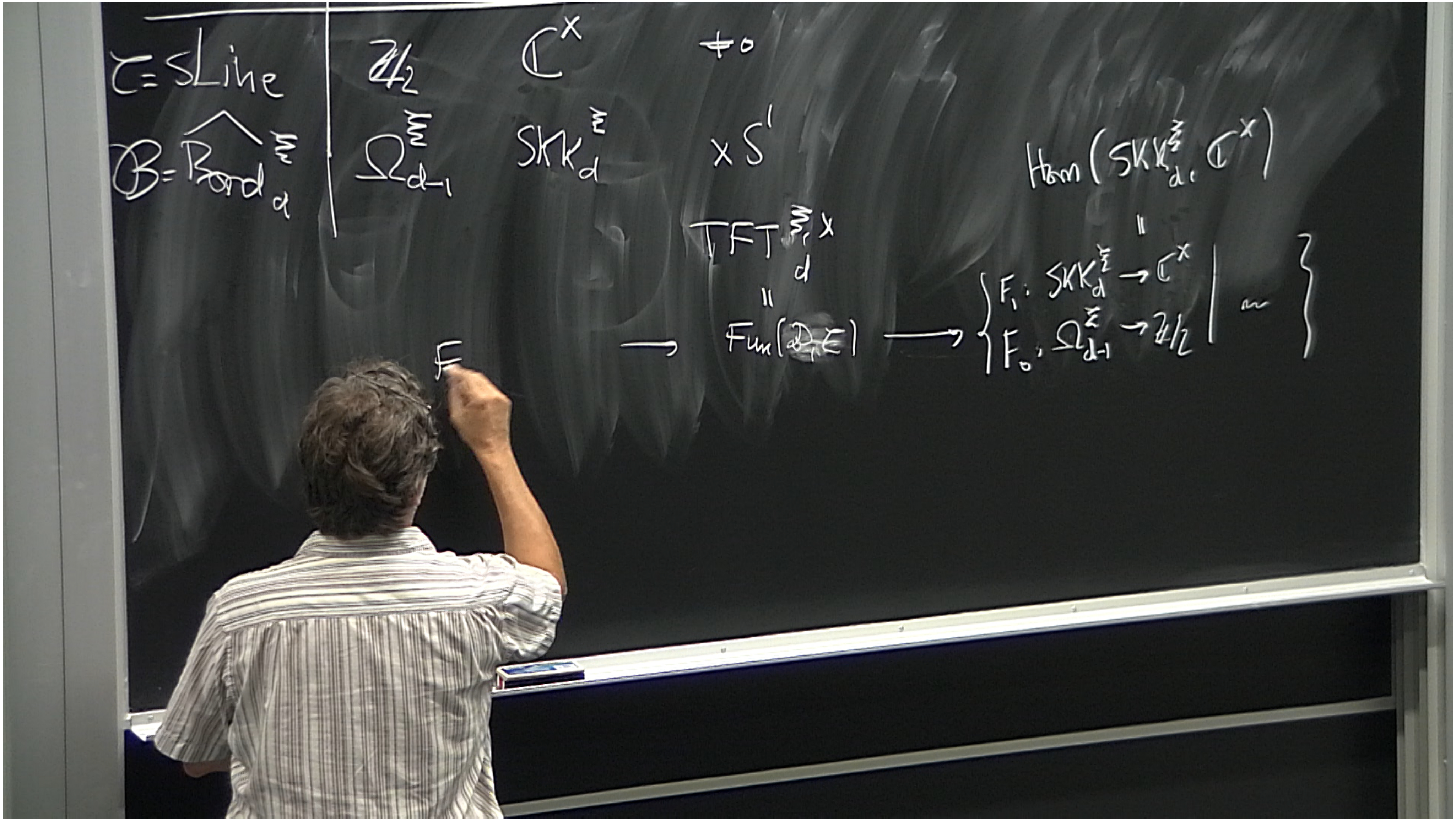

B1 = π1 𝒟 = Aut(I)
k-invariant → k^𝒟 : B0 → 2-tor(B1) = 2tor Aut(I)
Y ↦ (Y⊗Y —σ→ Y⊗Y) ∈ 2-tor(Aut(Y⊗Y))
Thm (Sinh, '75) i) 𝒟 is determined by (B0, B1, k^𝒟)
Fun(𝒟,C)/equiv. —F→ {F0 : B0 → C0, F1 : B1 → C1 | F1 k^𝒟 = k^C F0}

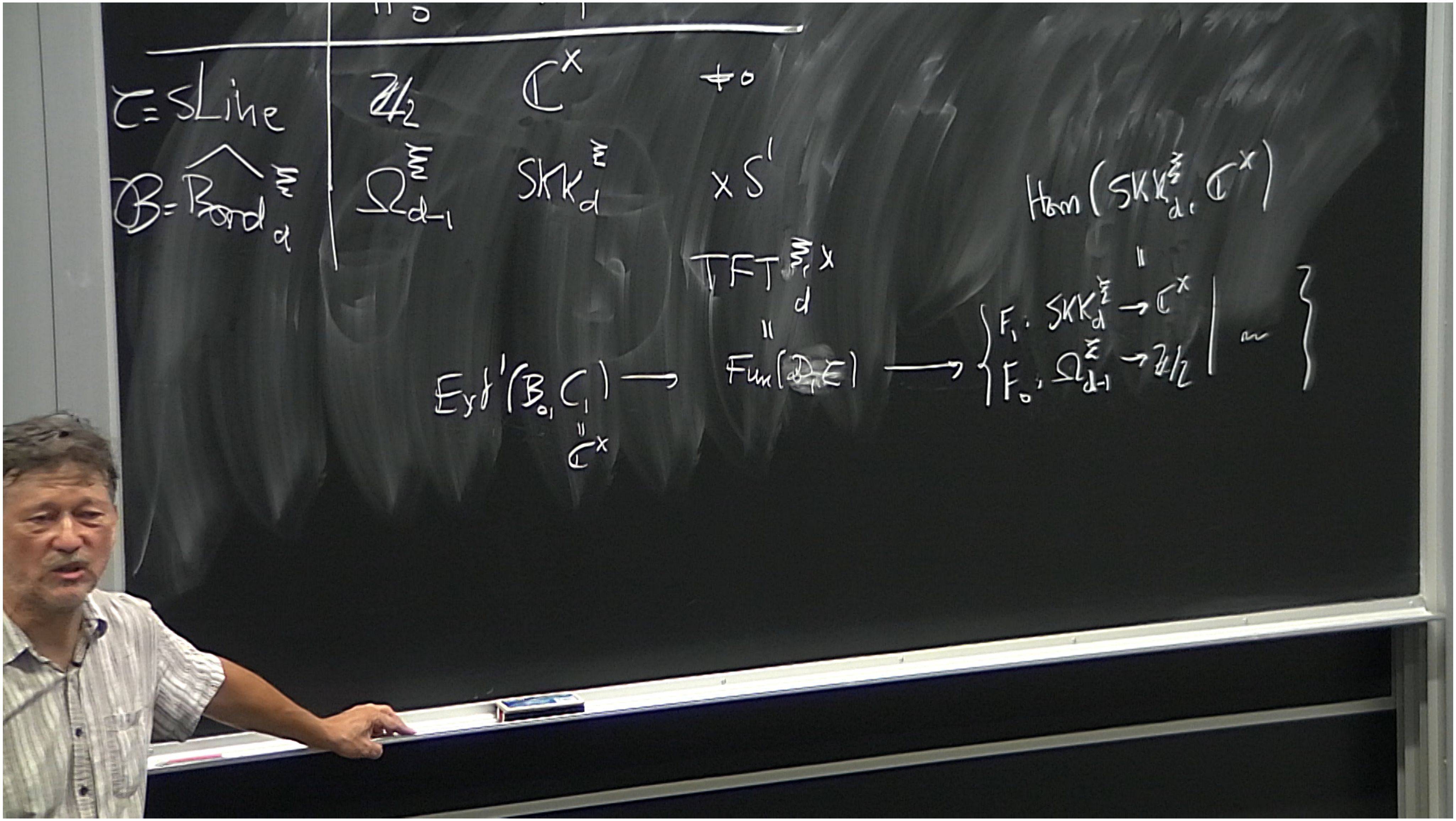

𝔅 Picard groupoid
𝔅 ⟼ B₀ = π₀𝔅 = {objects}/~ abelian group
B₁ = π₁𝔅 = Aut(I) " "
k-invariant → k^𝔅 : B₀ ⟶ 2-tor(B₁) = 2tor Aut(I)
Y ⟼ (Y⊗Y —σ→ Y⊗Y) ∈ 2-tor(Aut(Y⊗Y))
Thm (Sinh, '75) i) 𝔅 is determined by (B₀, B₁, k^𝔅)
exact sequence
Ext¹(B₀,C₁) ⟶ Fun(𝔅,C)/equiv ⟶ {F₀ : B₀ → C₀, F₁ : B₁ → C₁ | F₁ k^𝔅 = k^C F₀}

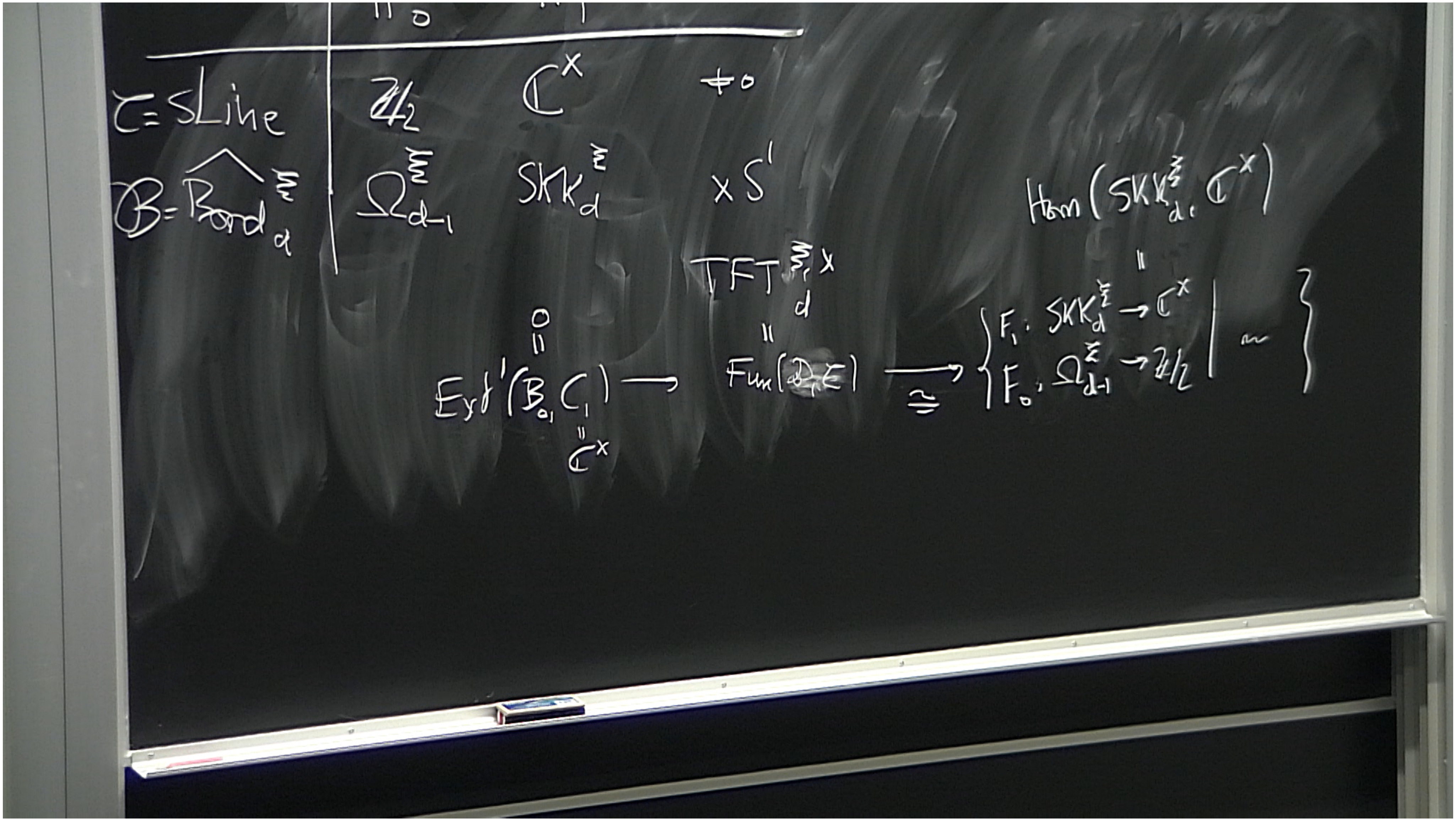

π0 π1 k
sLine ℤ/2 ℂ^× ≠0
CAUTION


	π_0	π_1	k
sLine	$\mathbb{Z}_2$	$\mathbb{C}^\times$	$\neq 0$
$\widehat{\mathrm{Bord}}_d^{fr}$	Ω_{d-1}^{fr}	Skk_d^{fr}	$\times S^1$


π0 π1 k
τ = sLine Z2 C× ≠0
B = Bord_d^ξ Ω_{d-1}^ξ SKK_d^ξ × S1
TFT_d^{ξ,×}
Fun(B, E) →


τ = sLine
B = Bord_d
Z/2
Ω_{d-1}
C^×
SKK_d
+0
× S^1
Hom(SKK_d, C^×)
TFT_d
Fun(D, E)
F_1 : SKK_d → C^×
F_0 : Ω_{d-1} → Z/2
F


τ = sLine
Z/2
C^×
B = Bord_d^ξ
Ω_{d-1}^ξ
SKK_d^ξ
× S^1
Hom(SKK_d^ξ, C^×)
TFT_d^{ξ,×}
F_1 : SKK_d^ξ → C^×
F_0 : Ω_{d-1}^ξ → Z/2
Ext^1(B_0, C_1) ⟶ Fun(B, C) ⟶
C^×


τ = sLine
B = Bord_d
Z/2
C^×
Ω_{d-1}
SKK_d
× S^1
Hom(SKK_d, C^×)
TFT_d
Ext^1(B_0, C_1) → Fun(D, E) ≅ {F_1 : SKK_d → C^× ; F_0 : Ω_{d-1} → Z/2 | ~ }