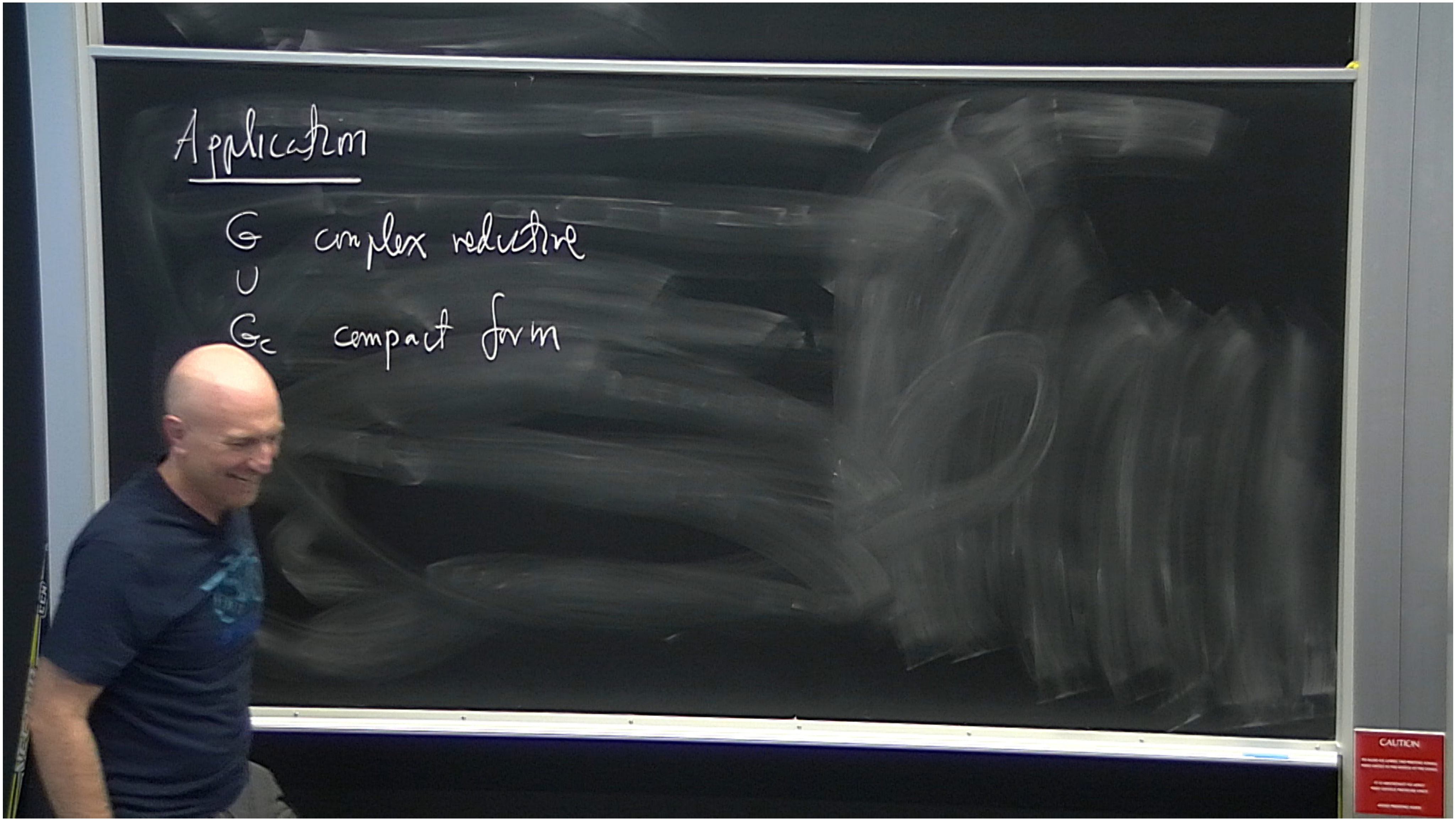
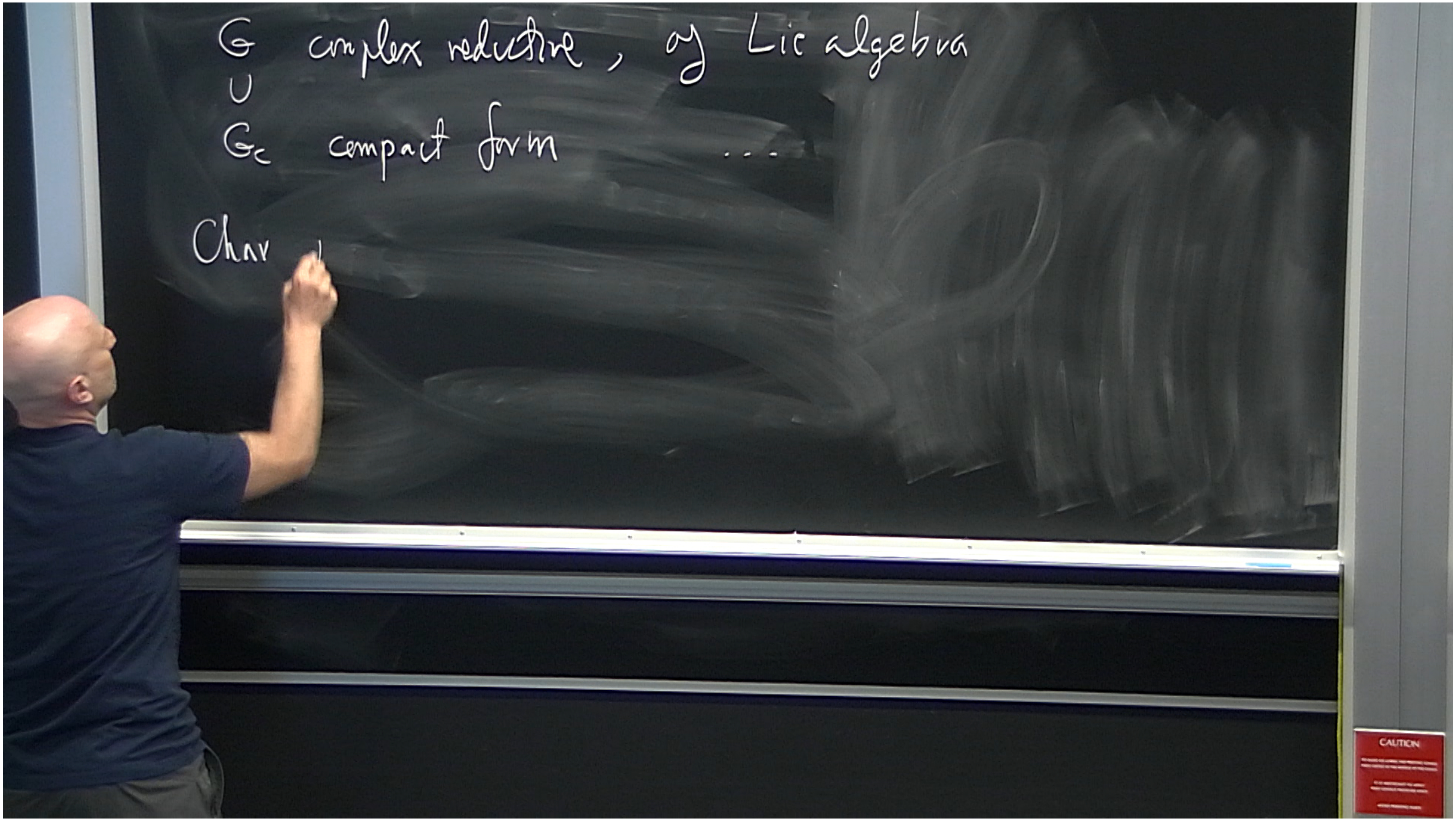
Title: Cutting and gluing branes

Date: Aug 13, 2018 12:00 PM

URL: http://pirsa.org/18080044

Abstract: Iâ€™ll discuss some results and expectations about the behavior of branes in Betti geometric Langlands under cutting and gluing Riemann surfaces.


Application
G complex reductive
∪
G_c compact form
CAUTION


G complex reductive, $\mathfrak{g}$ Lie algebra
$\cup$
G_c compact form ...
Char

G complex reductive, $\mathfrak{g}$ Lie algebra

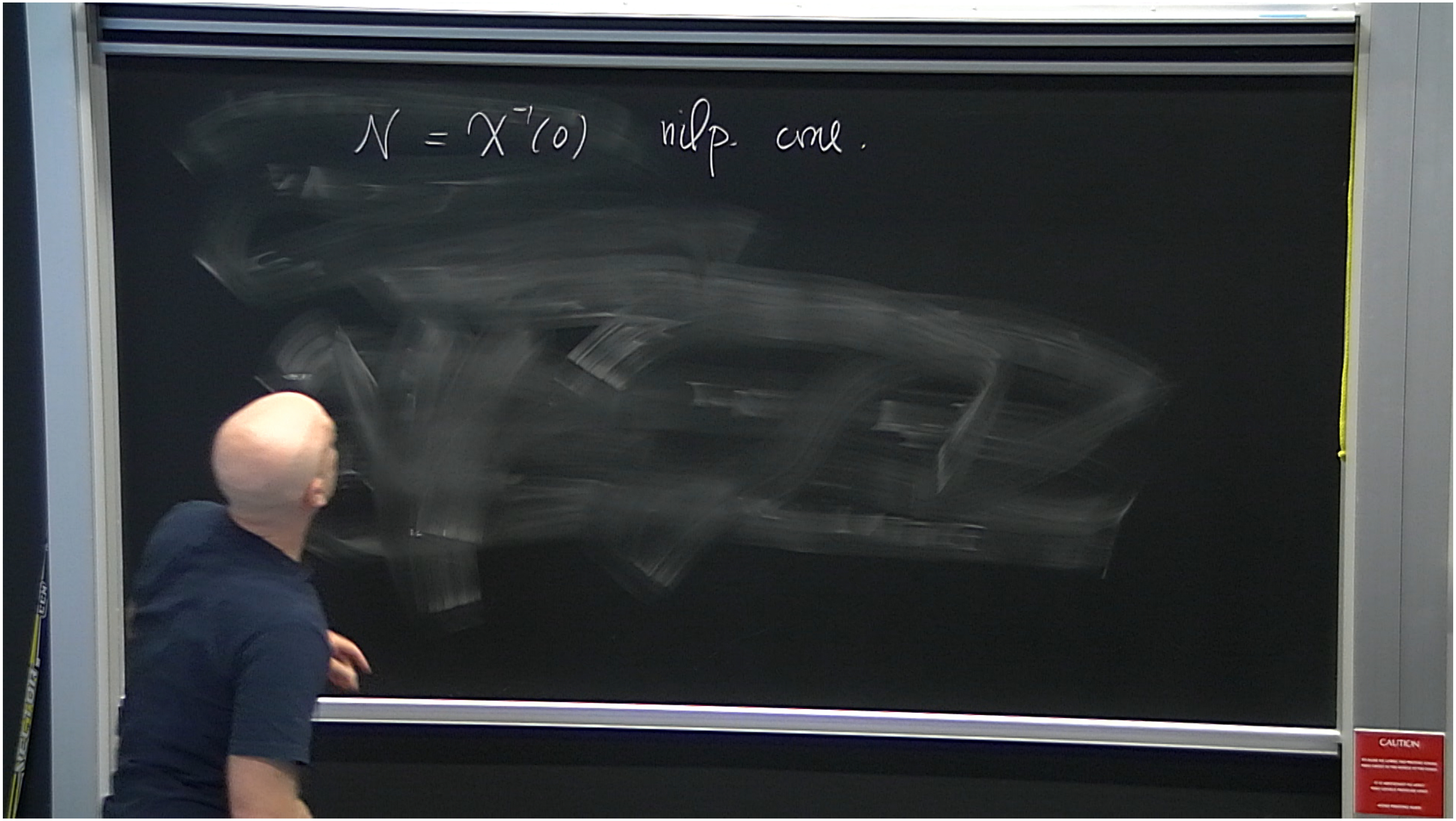
$\cup$

G_c compact form ...

Char polynomial:

$$\chi : \mathfrak{g} \longrightarrow \mathfrak{c} = \mathfrak{h} /\!/ W$$

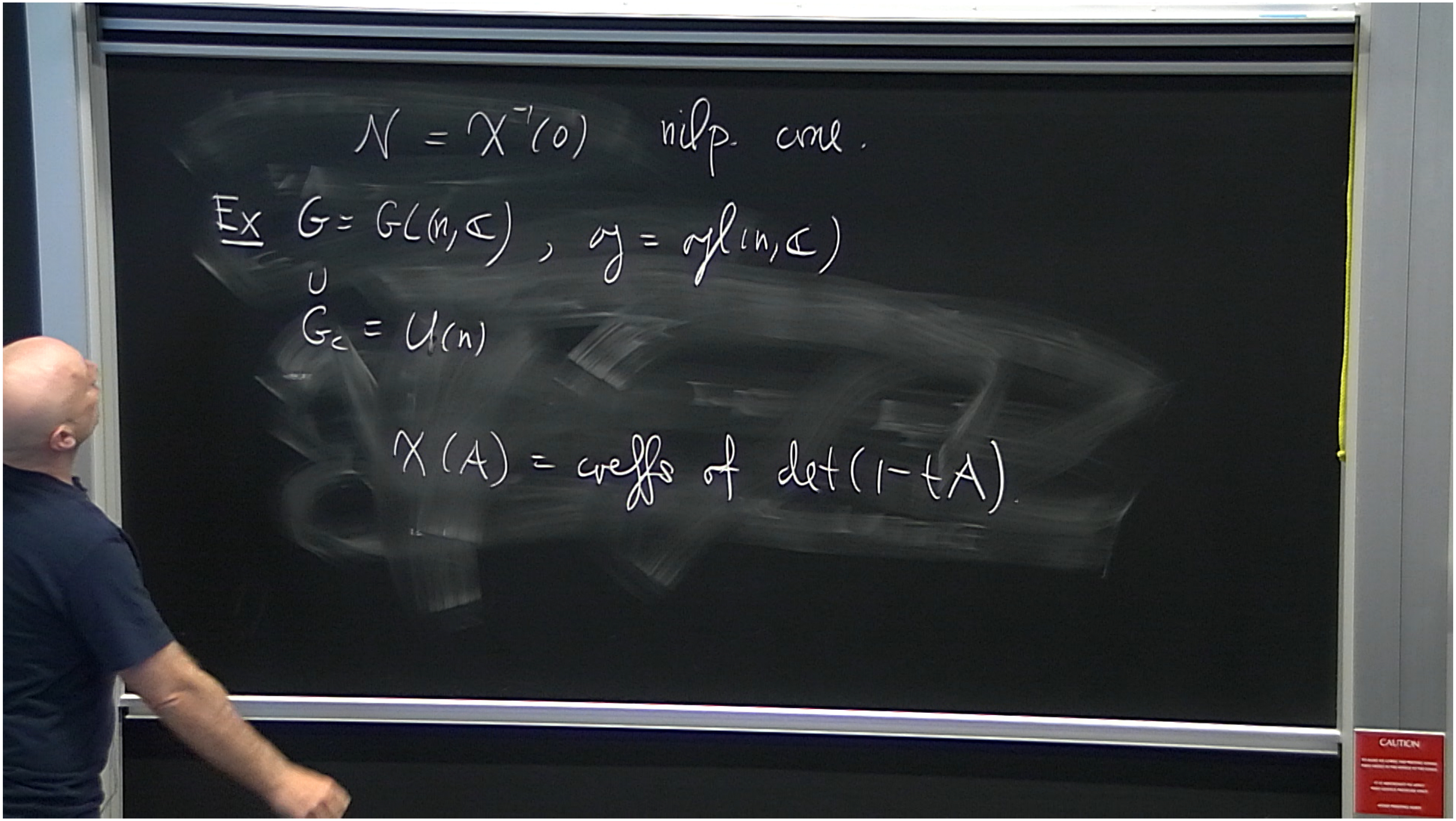
"multisets of e-values"


$\mathcal{N} = \chi^{-1}(0)$ nilp. cone.

$$N = \chi^{-1}(0) \quad \text{nilp. cone.}$$

E.g. $G = GL(n,\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$

$$\begin{matrix} G \\ \cup \\ G_c = U(n) \end{matrix}$$


$N = \chi^{-1}(0)$ nilp. cone.
Ex $G = GL(n,\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$
$\cup$
$G_c = U(n)$
$\chi(A) =$ coeffs of $\det(1 - tA)$.

$\mathcal{N} = \chi^{-1}(0)$ nilp. cone.

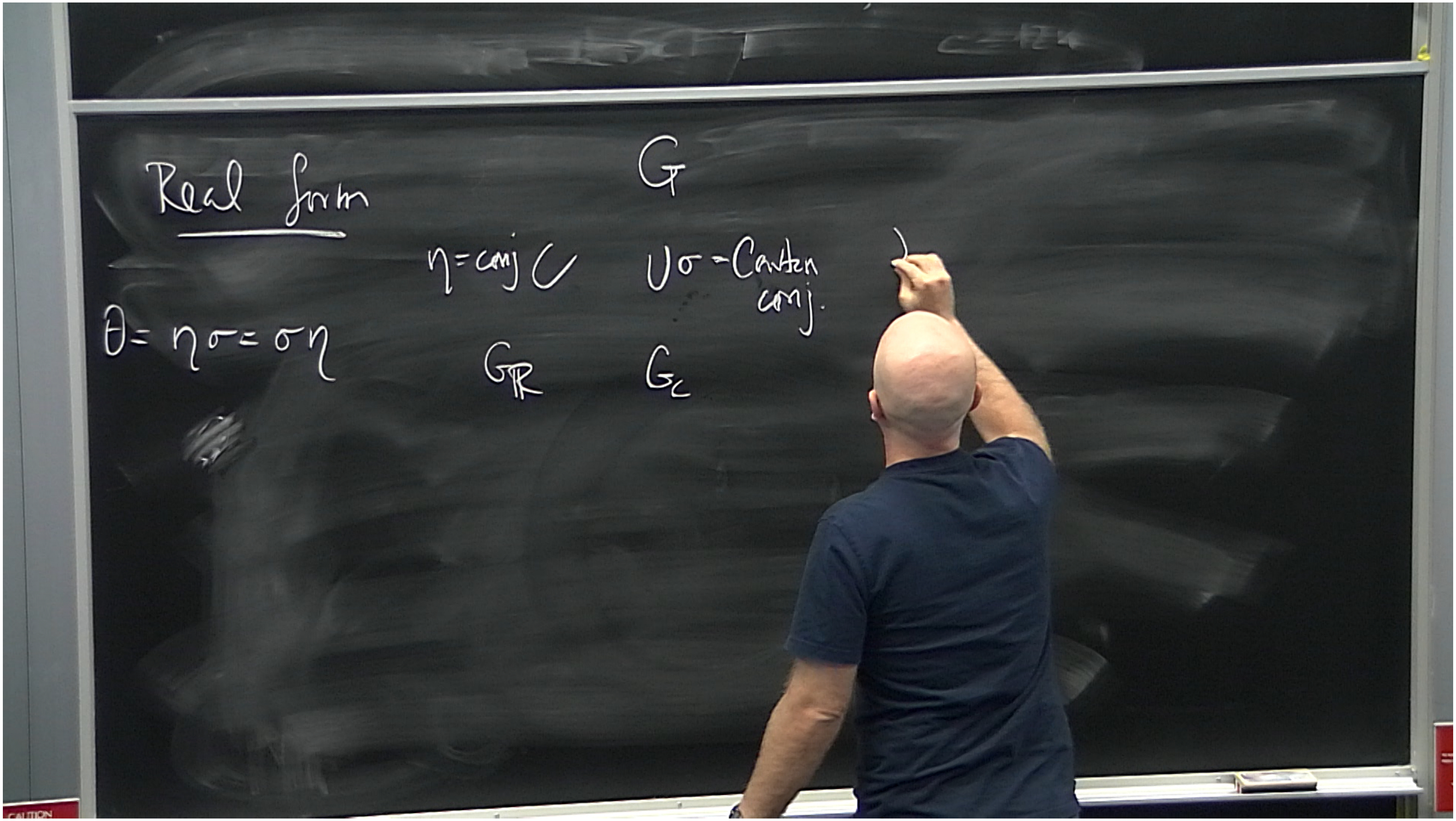
<u>Ex</u> $G = GL(n, \mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$

$\cup$

$G_c = U(n)$

$\chi(A) =$ coeffs of $\det(1 - tA)$

$\mathcal{N} =$ matrices with e-values $= 0$.

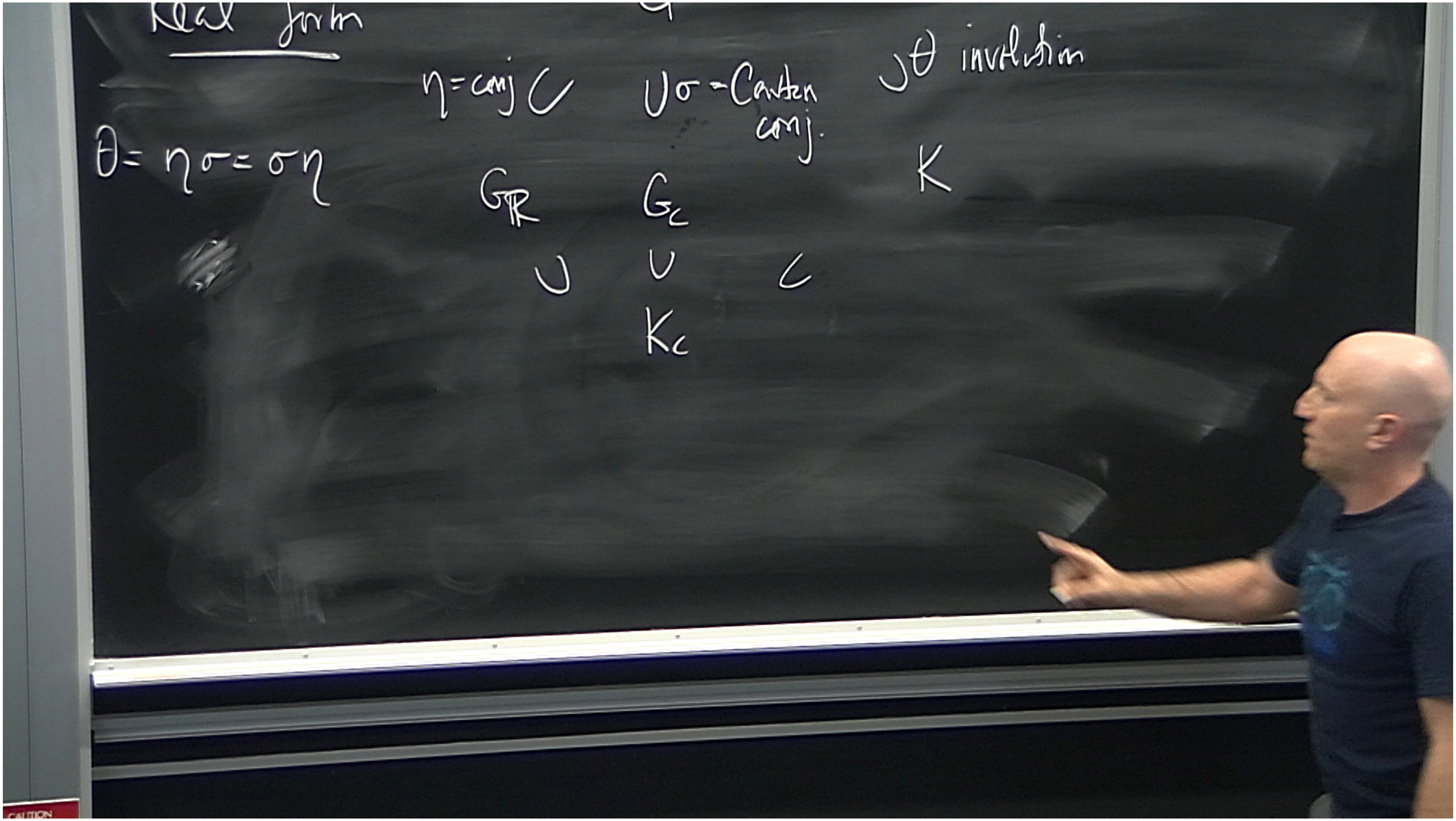

Real form
G
η = conj $\mathbb{C}$
U σ = Cartan conj.
$\theta = \eta\sigma = \sigma\eta$
$G_{\mathbb{R}}$
G_c

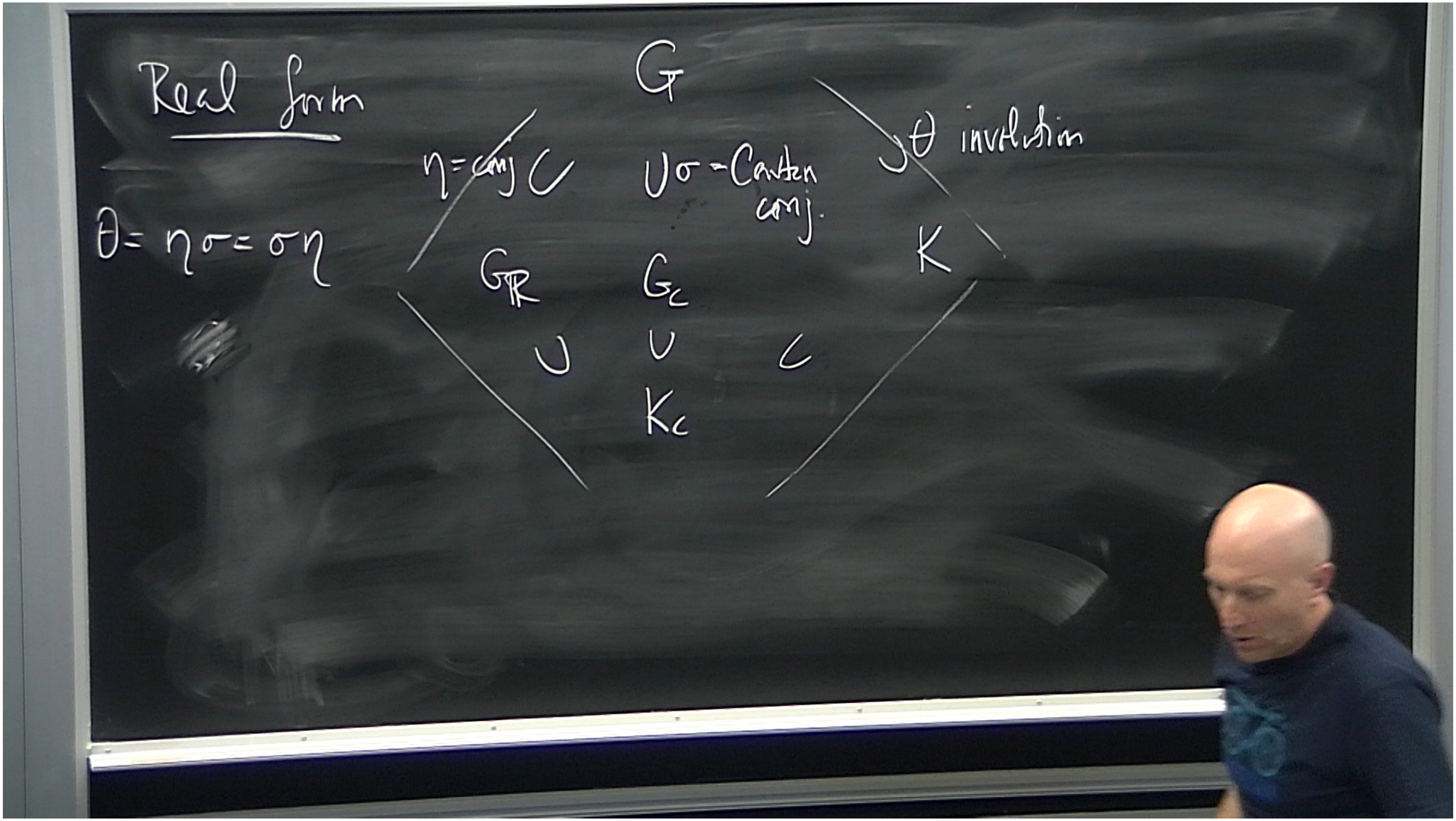
Real form

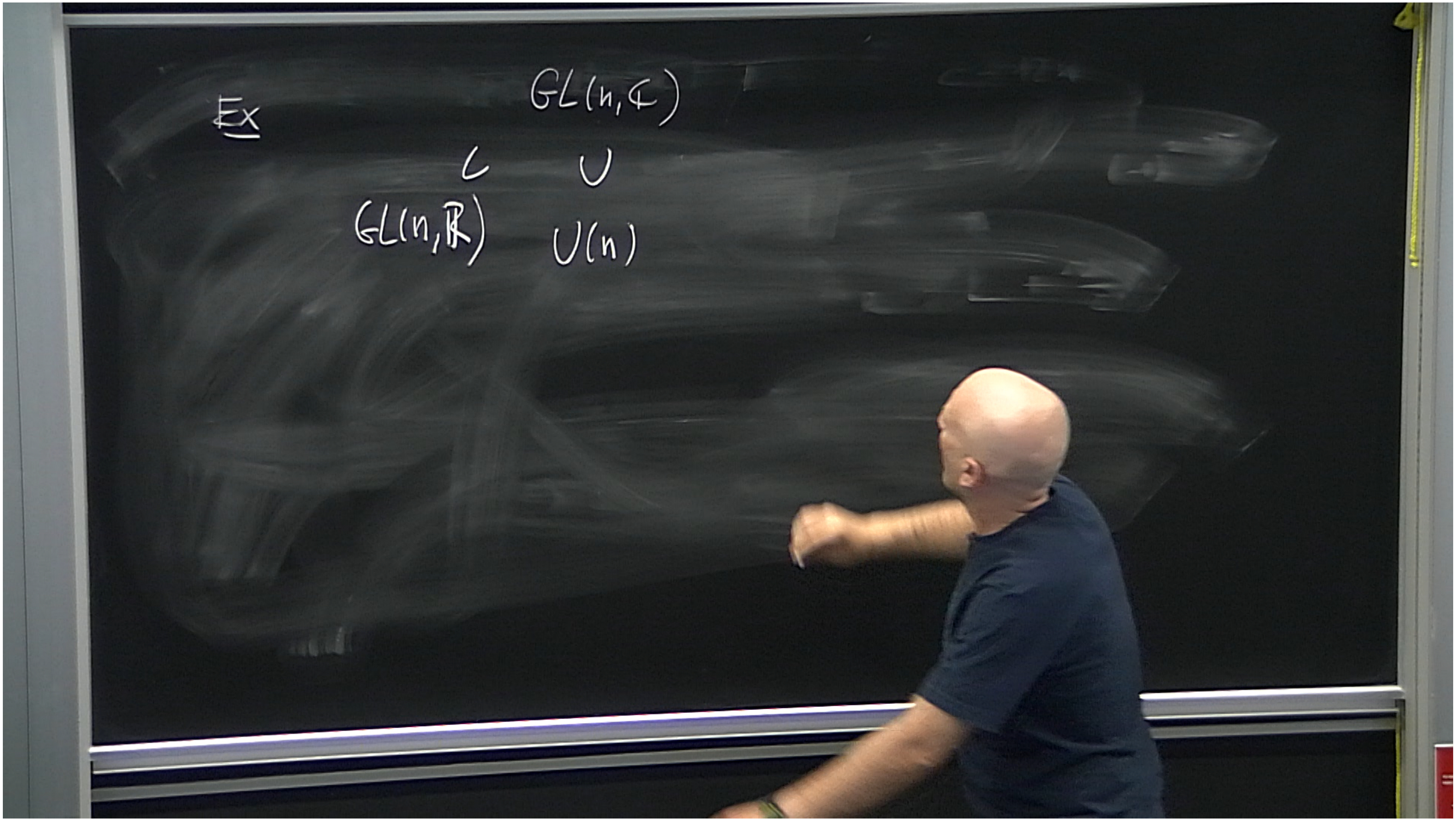
$$G$$

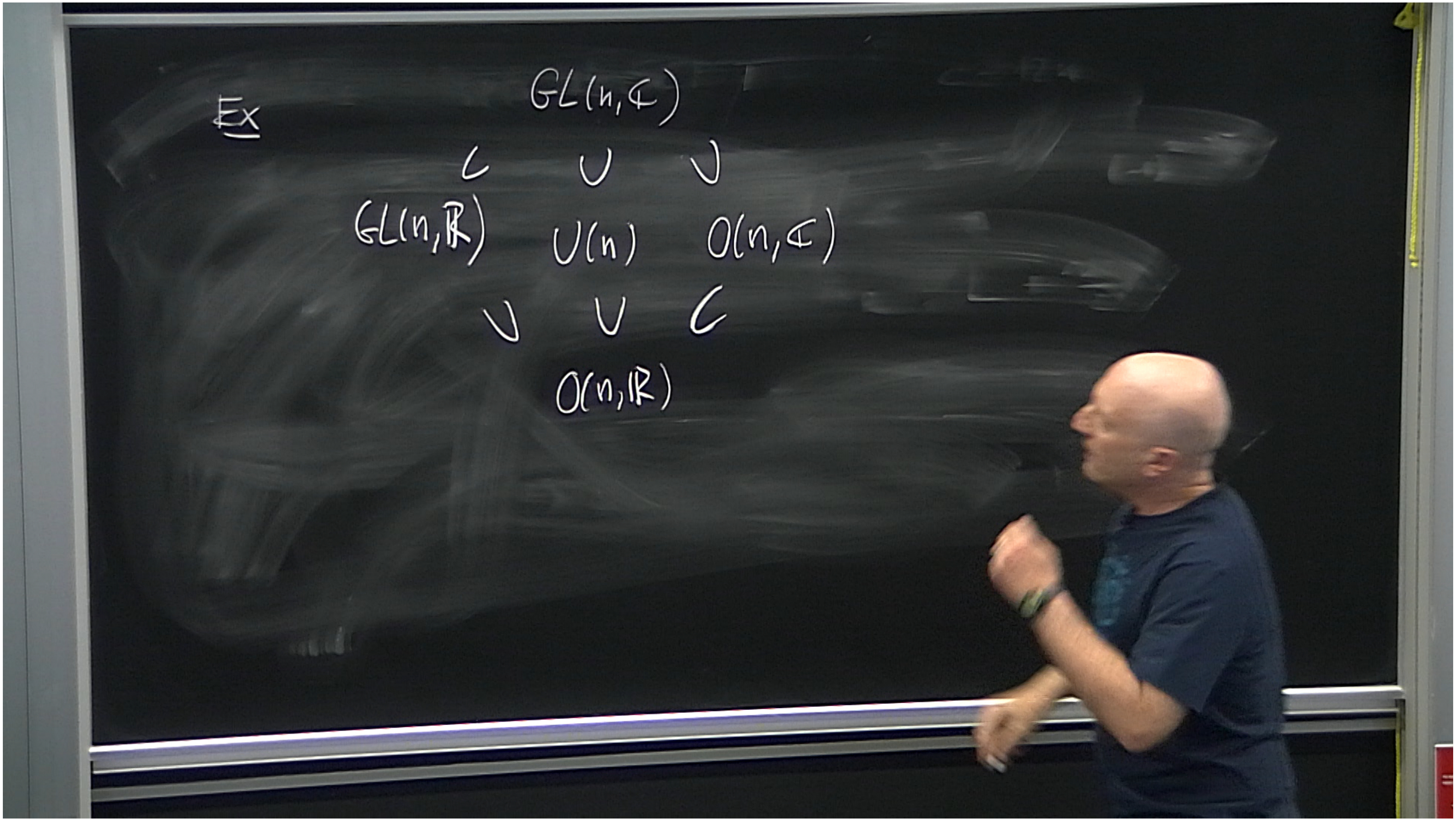
η = conj $\mathbb{C}$ $\quad$ $U \; \sigma$ = Cartan conj. $\quad$ θ involution

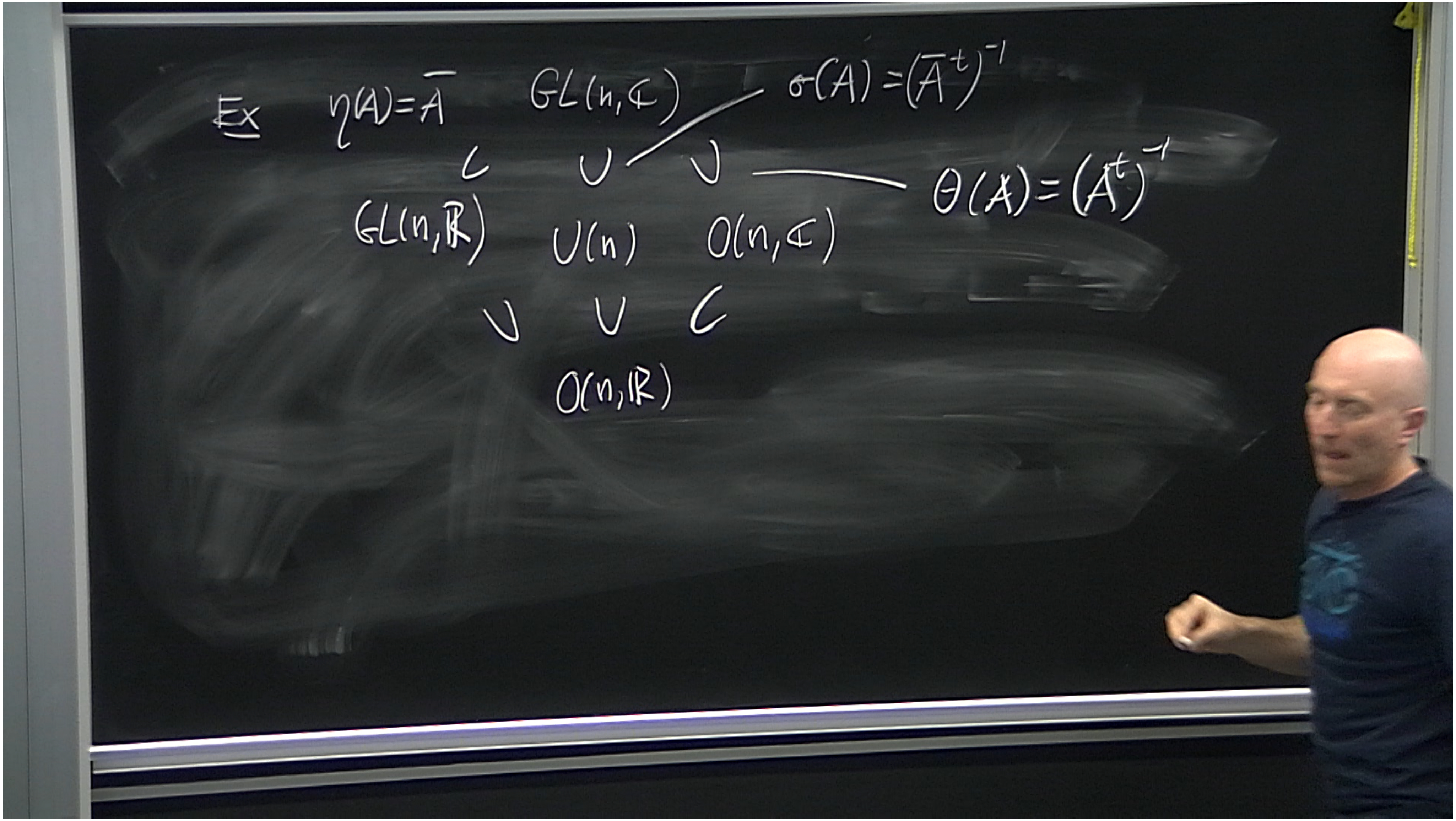
$\theta = \eta\sigma = \sigma\eta$ $\quad$ $G_{\mathbb{R}}$ $\quad$ G_c $\quad$ K


Real form
η = conj $\mathbb{C}$
$\cup\ \sigma$ = Cartan conj.
$\cup\ \theta$ involution
$\theta = \eta\sigma = \sigma\eta$
$G_{\mathbb{R}}$ $G_{\mathbb{C}}$ K
$\cup$ $\cup$ $\subset$
$K_{\mathbb{C}}$


Real form
G
θ involution
η = conj
U σ - Cartan conj.
θ = ησ = ση
G_R
G_C
K
U
K_C


Ex
$GL(n,\mathbb{C})$
$\cup \quad \cup$
$GL(n,\mathbb{R}) \quad U(n)$


Ex
$GL(n,\mathbb{C})$
$GL(n,\mathbb{R})$ $U(n)$ $O(n,\mathbb{C})$
$O(n,\mathbb{R})$


Ex
$\eta(A) = \bar{A}$
$GL(n,\mathbb{C})$
$\sigma(A) = (\bar{A}^t)^{-1}$
$\theta(A) = (A^t)^{-1}$
$GL(n,\mathbb{R})$
$U(n)$
$O(n,\mathbb{C})$
$O(n,\mathbb{R})$

Ex $\eta(A) = \bar{A}$ $\qquad$ $\sigma(A) = (\bar{A}^t)^{-1}$ $\qquad$ $\theta(A) = (A^t)^{-1}$

$$
\begin{array}{ccccc}
 & & GL(n,\mathbb{C}) & & \\
 & \subset & \cup & \supset & \\
GL(n,\mathbb{R}) & & U(n) & & O(n,\mathbb{C}) \\
 & \supset & \cup & \subset & \\
 & & O(n,\mathbb{R}) & &
\end{array}
$$

<u>Focus</u>: η,
conjugation

Ex $\quad \eta(A) = \bar{A} \qquad \sigma(A) = (\bar{A}^t)^{-1} \qquad \theta(A) = (A^t)^{-1}$

$$\begin{array}{ccccc} & & GL(n,\mathbb{C}) & & \\ & \subset & \cup & \cup & \\ GL(n,\mathbb{R}) & & U(n) & & O(n,\mathbb{C}) \\ & \cup & \cup & \subset & \\ & & O(n,\mathbb{R}) & & \end{array}$$

Focus: η, $-\theta$ on $\mathfrak{g}$.

conjugation

Ex $\quad \eta(A) = \bar{A} \qquad \sigma(A) = (\bar{A}^{t})^{-1} \qquad \theta(A) = (A^{t})^{-1}$

$$
\begin{array}{ccccc}
 & & GL(n,\mathbb{C}) & & \\
 & \subset & \cup & \cup & \\
GL(n,\mathbb{R}) & & U(n) & & O(n,\mathbb{C}) \\
 & \cup & \cup & \subset & \\
 & & O(n,\mathbb{R}) & &
\end{array}
$$

Focus: η, $-\theta$ on $\mathfrak{g}$.

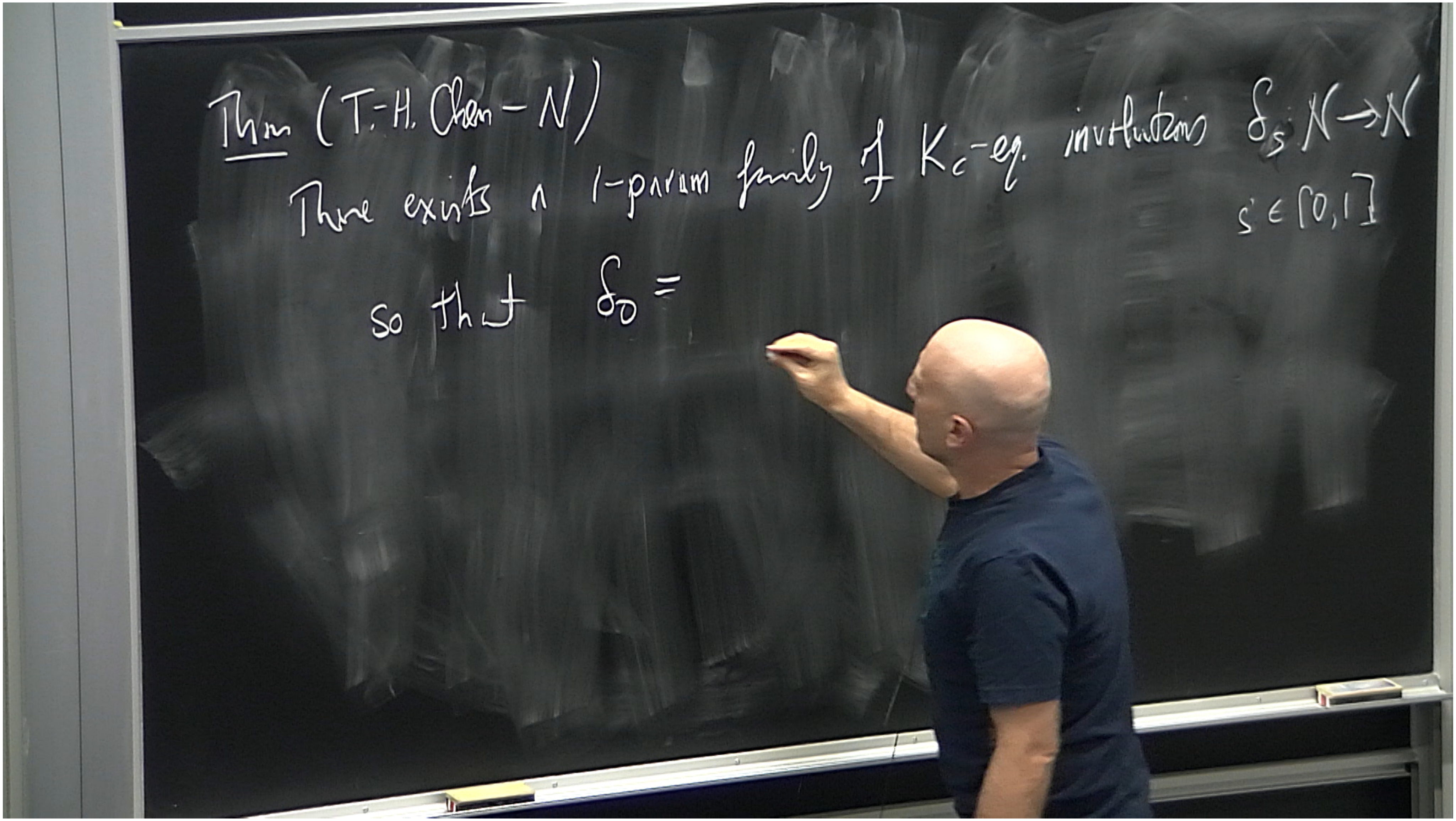
conjugation

involution

in part on N

Thm (T.-H. Chen – N)

There exists a 1-param family of K_c-eq. involutions $\delta_\sigma : \mathcal{N} \to \mathcal{N}$


Thm (T.-H. Chen – N)
There exists a 1-param family of K_c-eq. involutions $\delta_s : N \to N$
$s \in [0,1]$
so that $\delta_0 =$

Thm (T.-H. Chen – N)

There exists a 1-param family of K_c-eq. involutions $\delta_s : N \to N$, $s \in [0,1]$

so that $\delta_0 = \eta$

$\delta_1 = -\theta$.

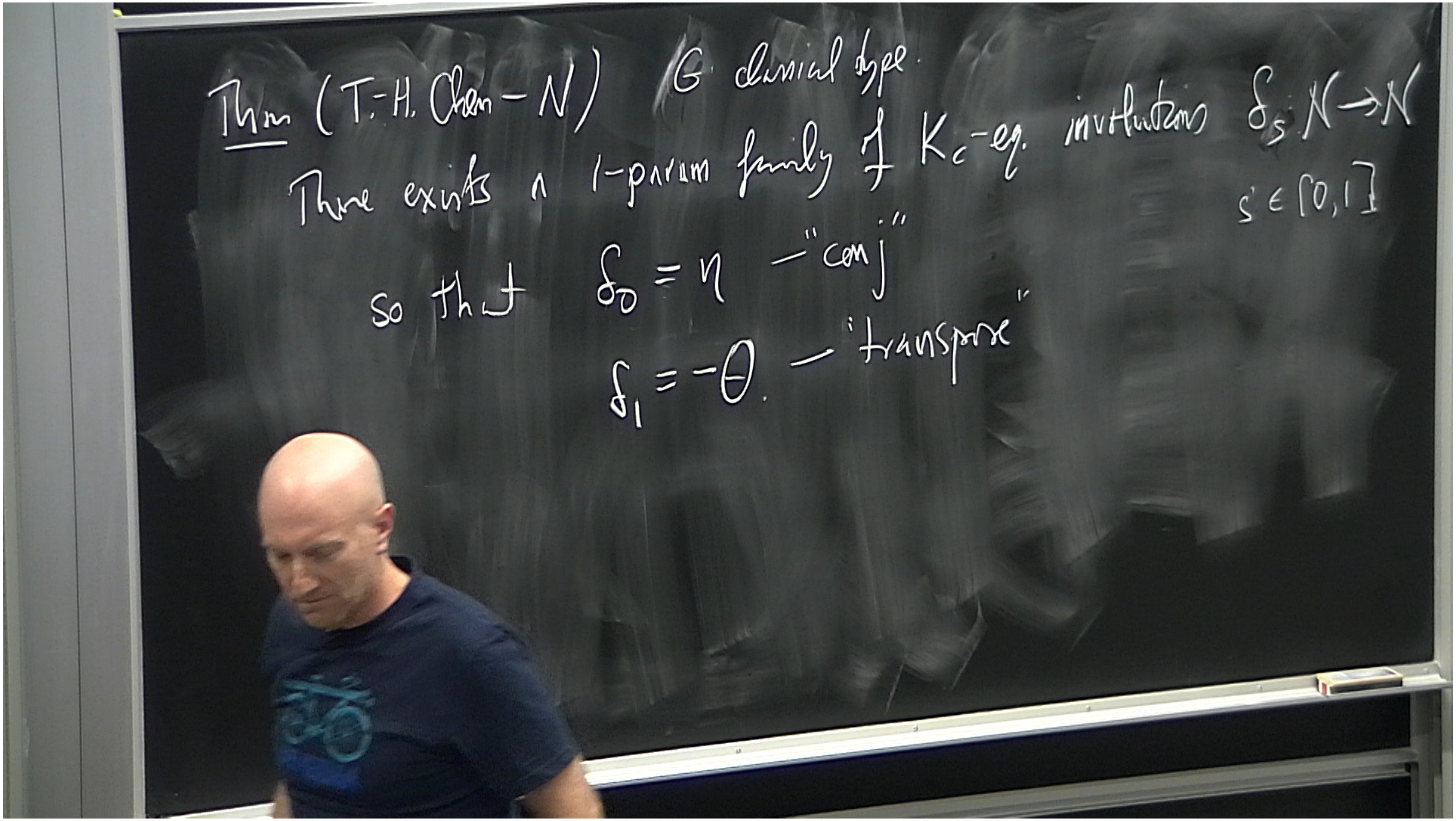
<u>Thm</u> (T.-H. Chen – N)

There exists a 1-param family of K_c-eq. involutions $\delta_s : \mathcal{N} \to \mathcal{N}$

$s \in [0,1]$

so that $\delta_0 = \eta$ — "conj"

$\delta_1 = -\theta$. — "transpose"

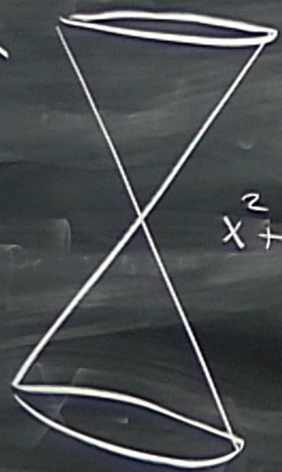

Thm (T.-H. Chen – N)   G classical type.
There exists a 1-param family of K_c-eq. involutions $\delta_s : N \to N$
$s \in [0,1]$
so that $\delta_0 = \eta$ – "conj"
$\delta_1 = -\theta$ – "transpose"

Cor There exists K_c-eq homeomorphism

$$\begin{array}{ccc} \mathcal{N}_{\mathbb{R}} & \simeq & \mathcal{N}_{\mathfrak{p}} \\ \| & & \| \\ \mathcal{N} \cap \mathfrak{g}_{\mathbb{R}} & & \mathcal{N} \cap \mathfrak{p} \\ \uparrow & & \uparrow \\ \eta\text{-fixed pts} & & -\theta\text{-fixed pts} \end{array}$$

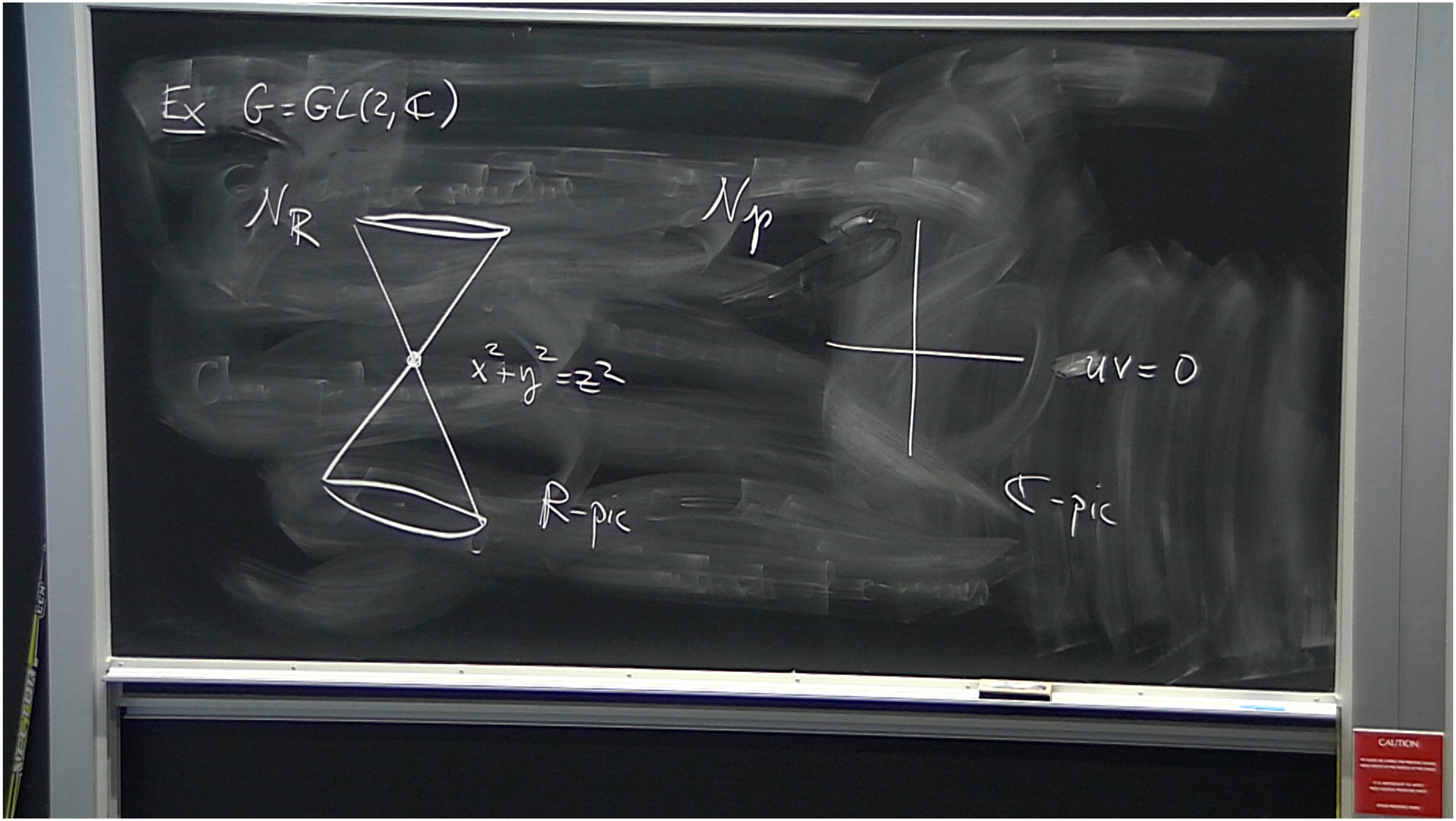
Ex $G = GL(2,\mathbb{C})$

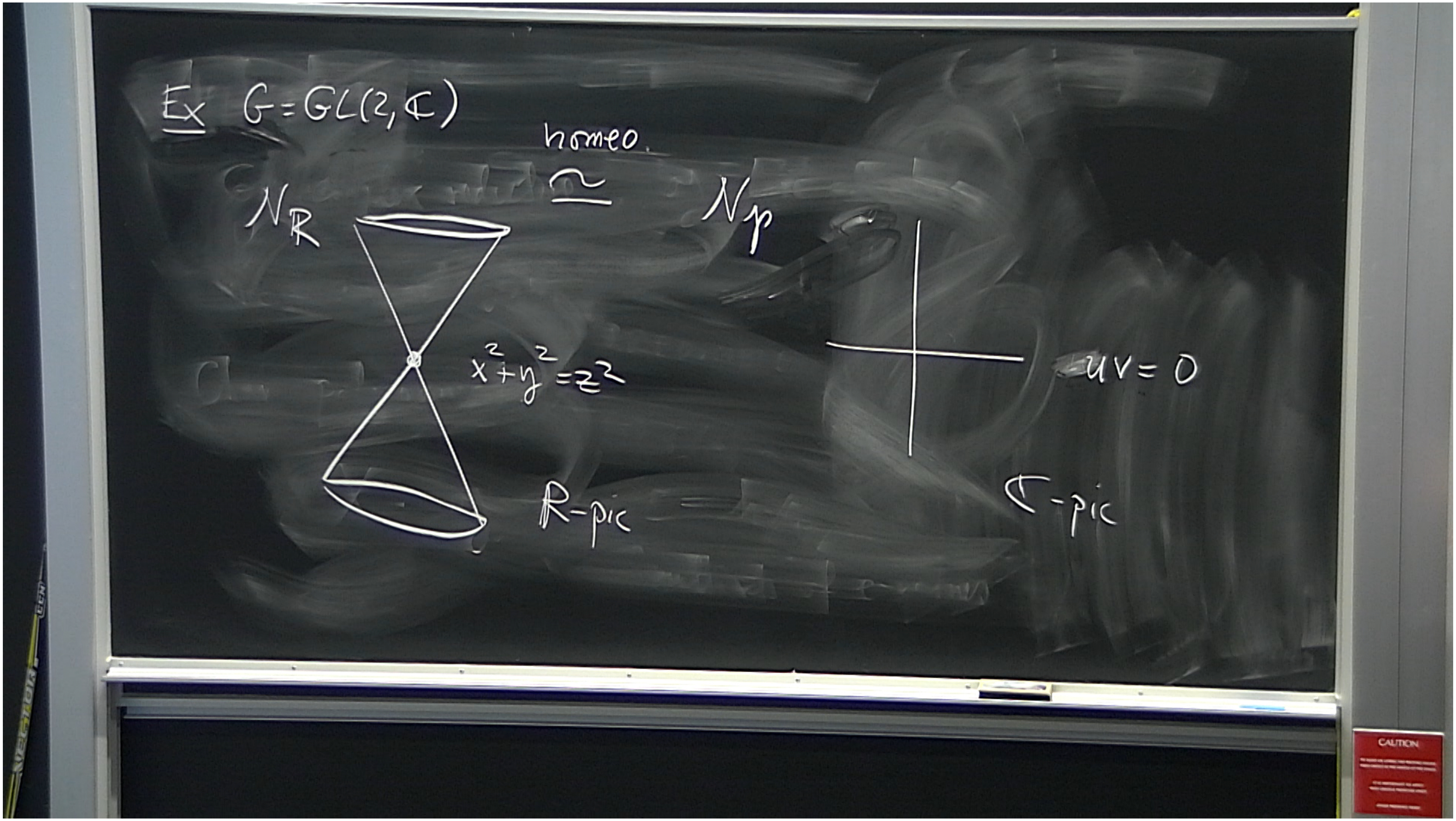
$\mathcal{N}_{\mathbb{R}}$



$x^2+y^2=z^2$

$\dots$


Ex G = GL(2, ℂ)
N_R
x² + y² = z²
R-pic
N_p
uv = 0
ℂ-pic


Ex G = GL(2, ℂ)
homeo.
≃
N_R
N_p
x²+y²=z²
uv = 0
ℝ-pic
ℂ-pic

Cor There exists $K_{\mathbb{C}}$-eq homeomorphism

$$\begin{array}{ccc} \mathcal{N}_{\mathbb{R}} & \simeq & \mathcal{N}_{\mathfrak{p}} \\ \| & & \| \\ \mathcal{N} \cap \mathfrak{g}_{\mathbb{R}} & & \mathcal{N} \cap \mathfrak{p} \end{array}$$

$\mathfrak{g}_{\mathbb{R}}$: η-fixed pts

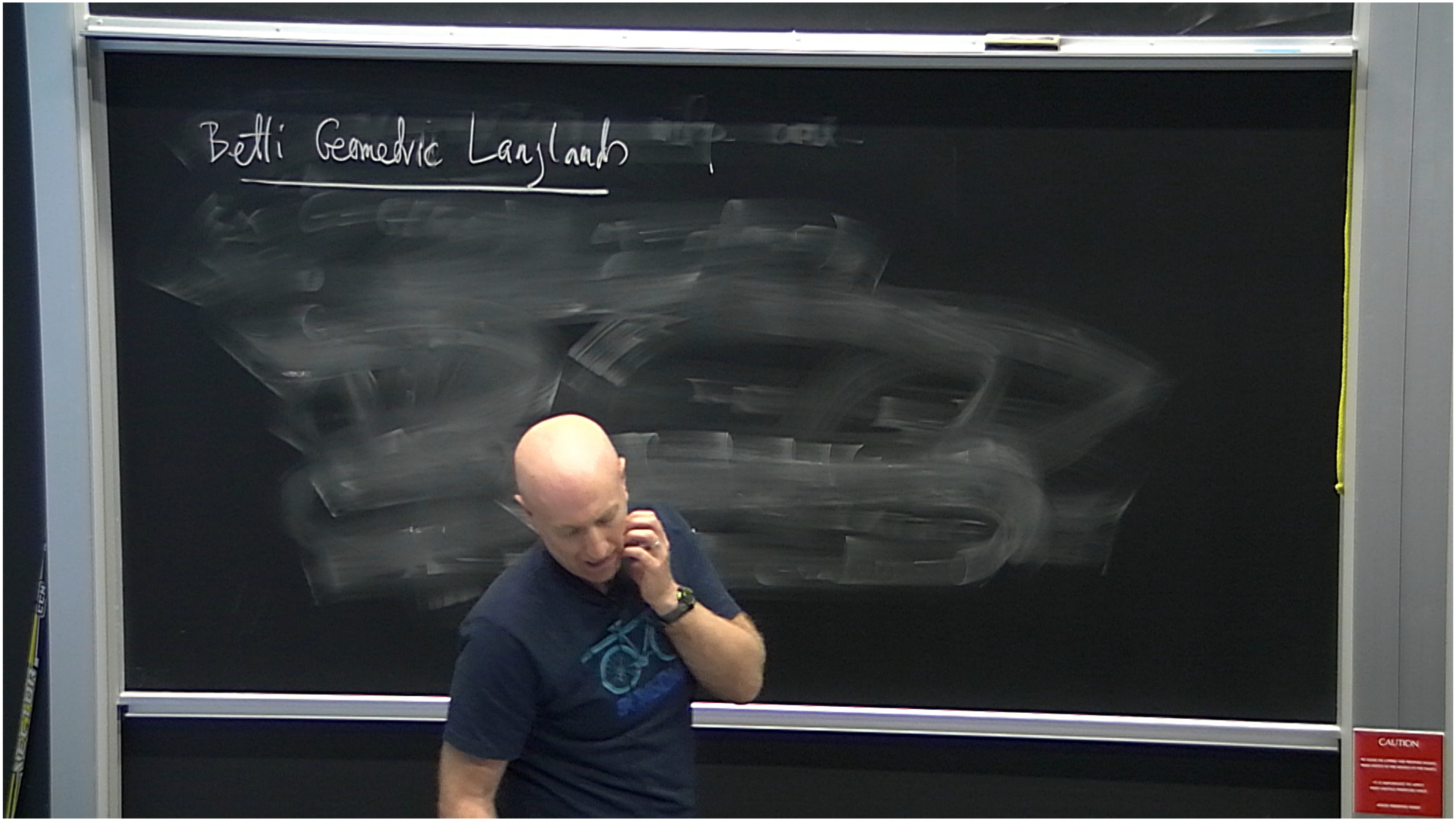
$\mathfrak{p}$: $-\theta$-fixed pts

Precursor

Kostant-Sekiguchi corr:

between

$$|\mathcal{N}_{\mathbb{R}}/G_{\mathbb{R}}| \simeq |\mathcal{N}_{\mathfrak{p}}/K|$$


Betti Geometric Langlands

Betti Geometric Langlands

Analogy: de Rham Geometric Langlands (CFT)

Betti Geometric Langlands (TFT?)

Spectral side: alg's match connections:

de Rham: flat conn

Betti: rep of π_1

Rough Conj (2d MS) X sm proj curve / $\mathbb{C}$

A-side

B-side

$\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(X))$

Rough Conj (2d MS) X sm proj curve $/\mathbb{C}$

A-side

B-side

$$\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G(X)) \simeq \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(X))$$

$\mathcal{N}$ ← nilp sing supp.

Rough Conj (2d MS) X sm proj curve $/\mathbb{C}$

A-side B-side

$$\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G(X)) \simeq \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(X))$$

$\mathcal{N}$ ← nilp sing supp.

$$\text{D-mod}^{\mathrm{reg}}(\mathrm{Bun}_G(X))$$

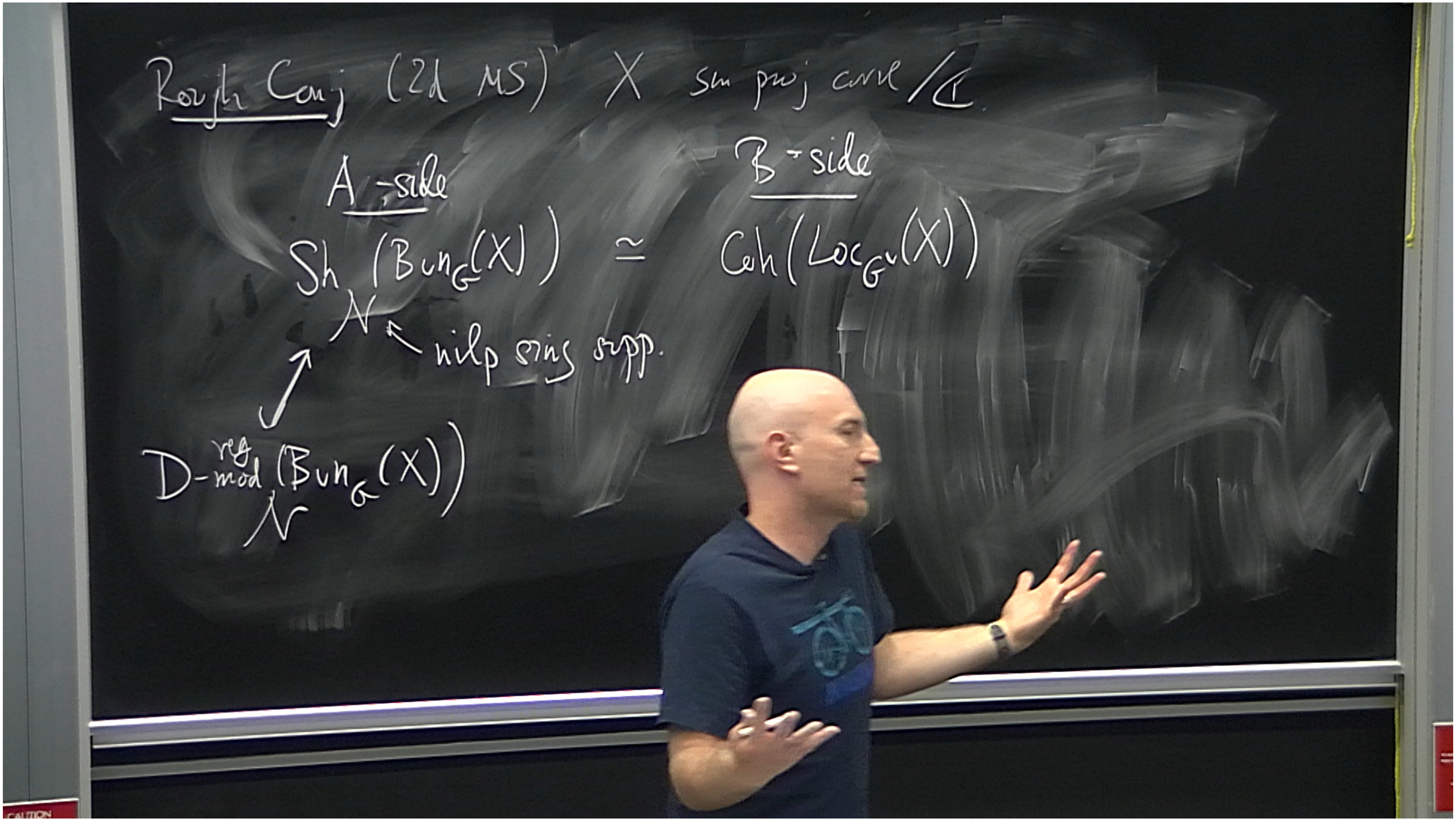
Rough Conj (2d MS) X sm proj curve $/\mathbb{C}$

A-side | B-side

$$Sh_{\mathcal{N}}(Bun_G(X)) \simeq Coh(Loc_{G^\vee}(X))$$

nilp sing supp.

$$D\text{-mod}^{reg}_{\mathcal{N}}(Bun_G(X))$$


Rough Conj (2d MS) X sm proj curve /ℂ
A-side
B-side
Sh_N (Bun_G(X)) ≃ Coh(Loc_G∨(X))
nilp sing supp.
D-mod^reg_N (Bun_G(X))

Rough Conj (2d MS) $\quad X$ sm proj curve $/\mathbb{C}$

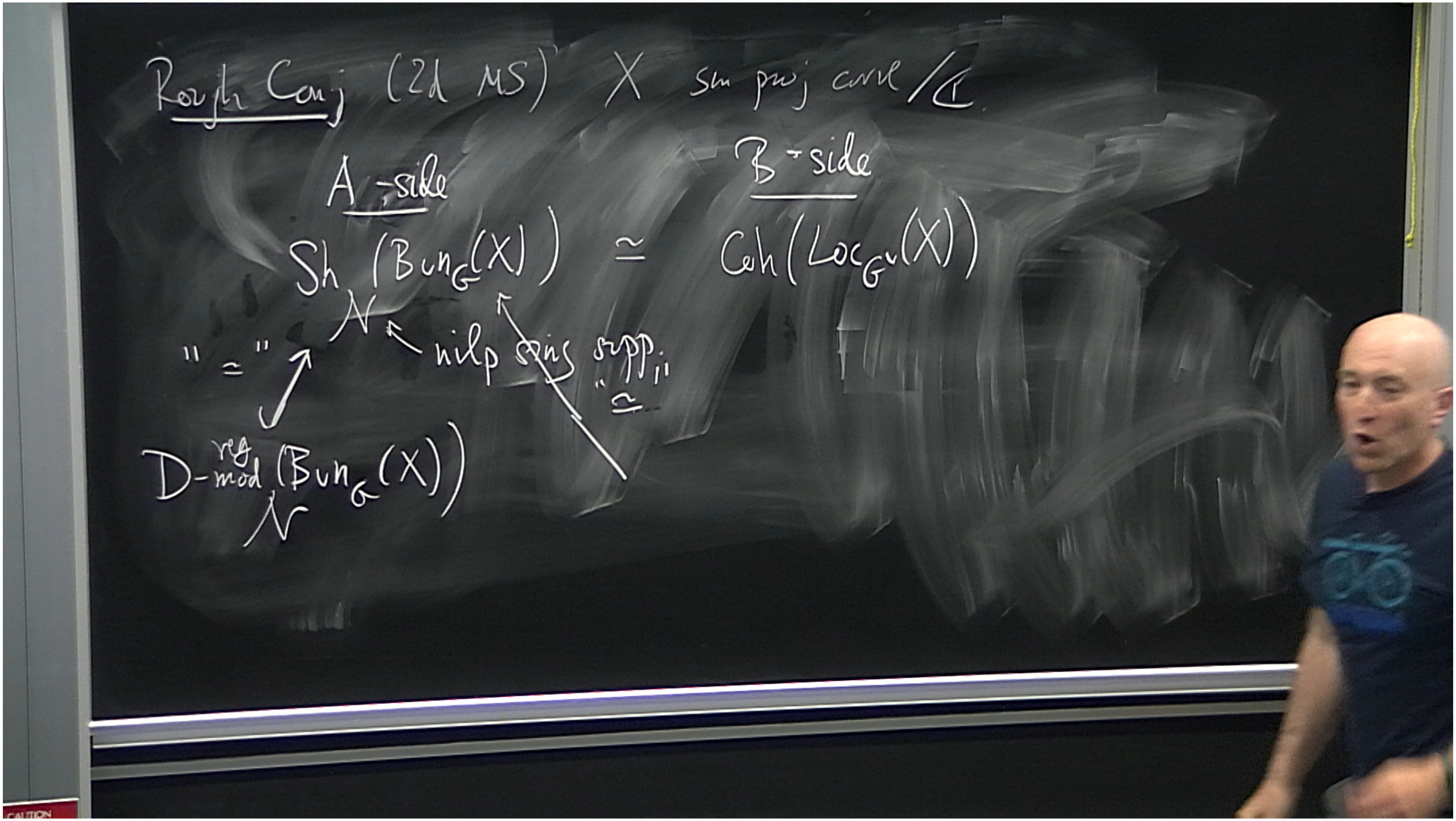
A-side $\qquad\qquad$ B-side

$$\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G(X)) \simeq \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(X))$$

$\mathcal{N}$ ← nilp sing supp.

" = "

$$\text{D-mod}^{\mathrm{reg}}_{\mathcal{N}}(\mathrm{Bun}_G(X))$$


Rough Conj (2d MS) X sm proj curve / ℂ
A-side
B-side
Sh_N(Bun_G(X)) ≃ Coh(Loc_G∨(X))
nilp sing supp
"≃"
"≃"
D-mod^reg_N(Bun_G(X))

Rough Conj (2d MS) $\quad X$ sm proj curve $/\mathbb{C}$

A-side $\qquad\qquad\qquad$ B-side

$$Sh_{\mathcal{N}}(Bun_G(X)) \simeq Coh(Loc_{G^\vee}(X))$$

nilp sing supp

"$\simeq$" $\qquad$ "$\simeq$"

$$D\text{-}mod^{reg}_{\mathcal{N}}(Bun_G(X)) \qquad WFuk(\mathcal{M}_G(X))$$

Rough Conj (2d MS) X sm proj curve $/\mathbb{C}$

A-side | **B-side**

$$Sh_{\mathcal{N}}(Bun_G(X)) \simeq Coh(Loc_{G^\vee}(X))$$

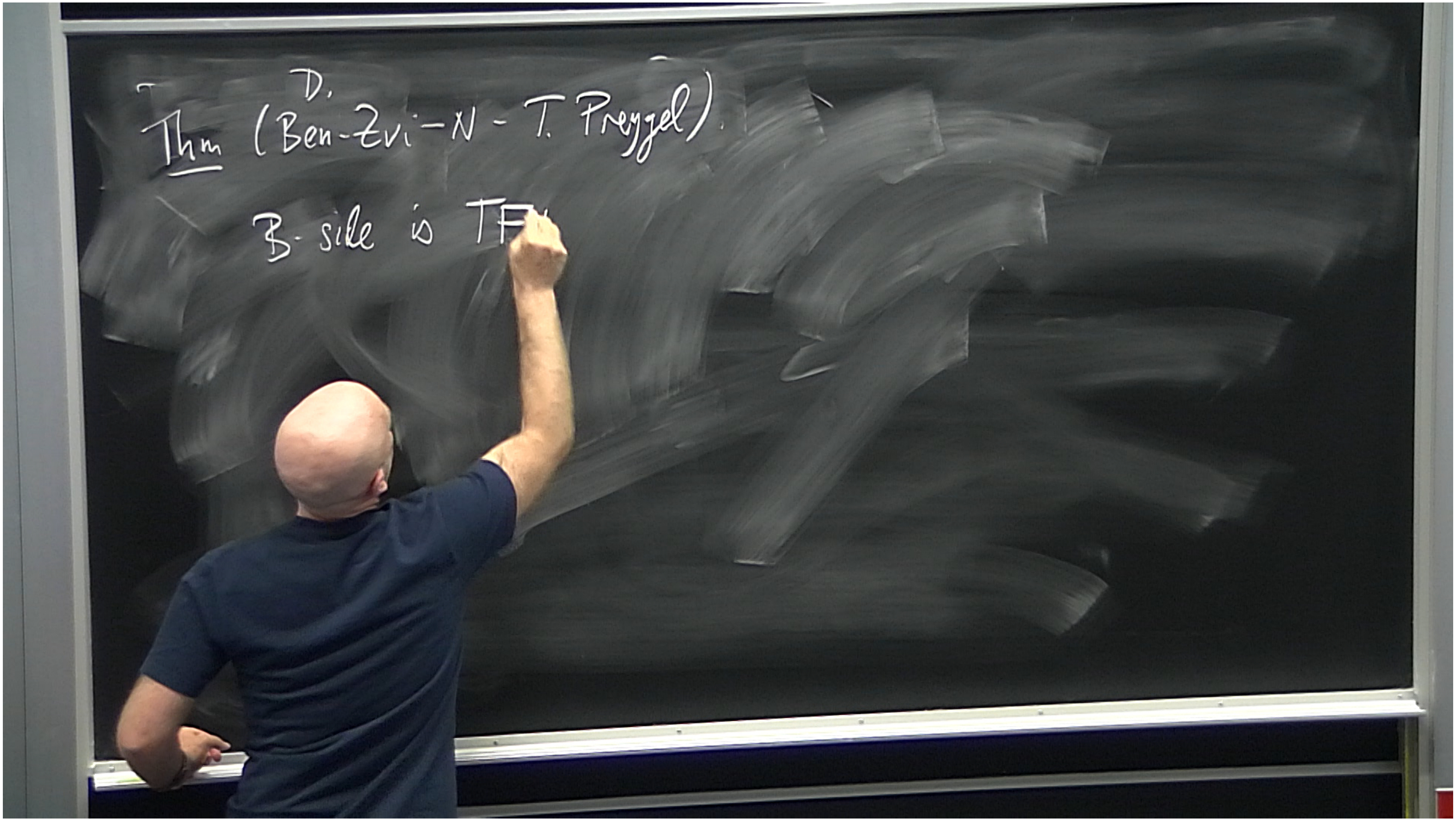
$\mathcal{N}$ ← nilp sing supp.

"$\simeq$"

"$\simeq$" community in progress...

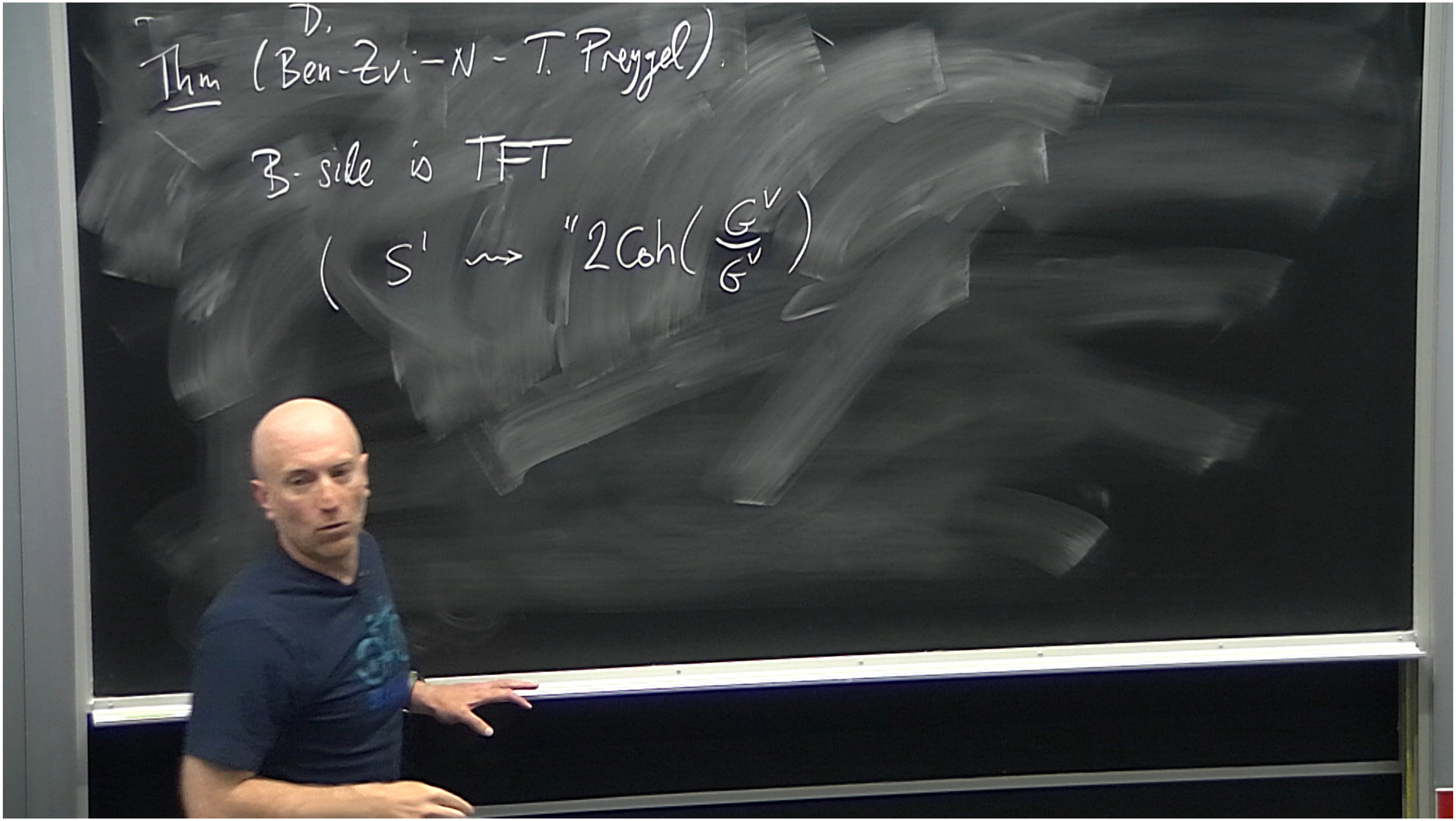
$$D\text{-}mod^{reg}_{\mathcal{N}}(Bun_G(X))$$

$$WFuk(\mathcal{M}_G(X))$$


D.
Thm (Ben-Zvi - N - T. Preygel).
B- side is TF

Thm (D. Ben-Zvi – N – T. Preygel).

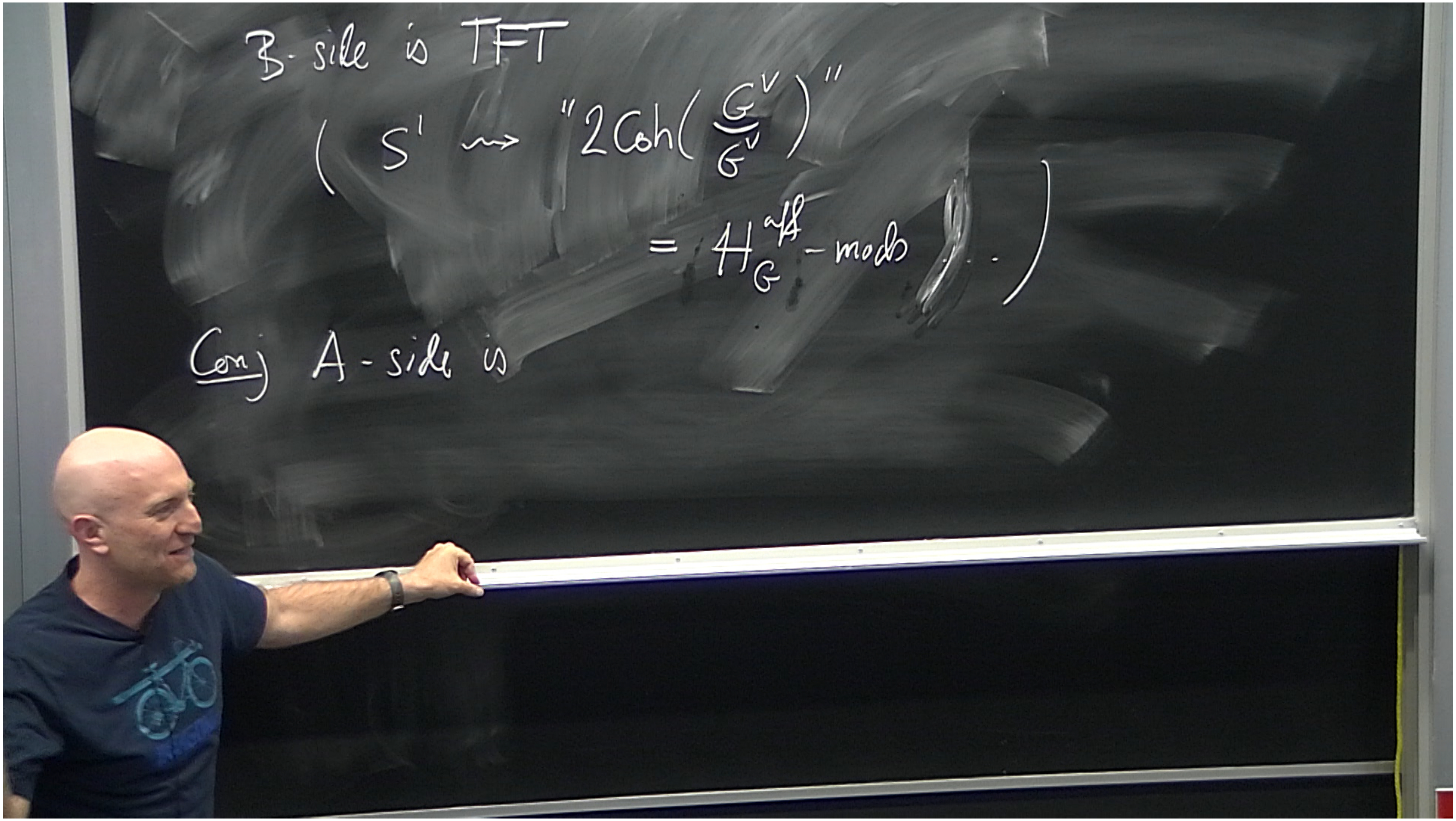
B-side is TFT


D.
Thm (Ben-Zvi – N – T. Preygel).
B-side is TFT
(S¹ ⇝ "2Coh(G^∨ / G^∨)

Thm (Ben-Zvi – N – T. Preygel).

B-side is TFT

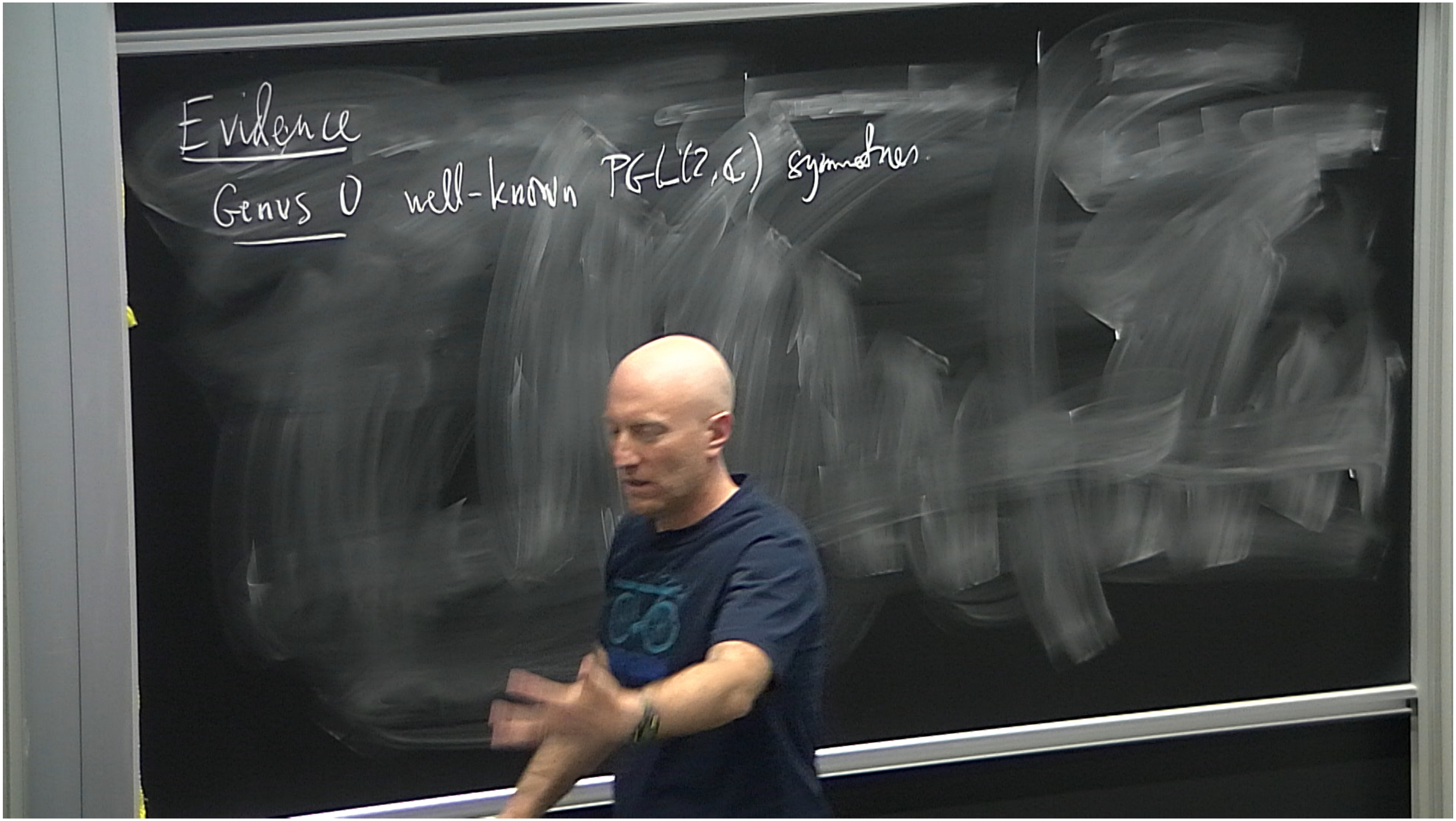
$$\left(S^1 \rightsquigarrow \text{“}2\mathrm{Coh}\left(\frac{G^\vee}{G^\vee}\right)\text{”} = \mathcal{H}_G^{\mathrm{aff}}\text{-mods} \ \ldots \right)$$


B-side is TFT
($S^1 \rightsquigarrow$ "2Coh($\frac{G^\vee}{G^\vee}$)"
$= \mathcal{H}_G^{aff}$-mods ...)
Conj A-side is

B-side is TFT

$$\left(S^1 \rightsquigarrow \text{“}2\mathrm{Coh}\left(\frac{G^\vee}{G^\vee}\right)\text{”} = \mathcal{H}_G^{\mathrm{aff}}\text{-mods} \ \ldots \right)$$

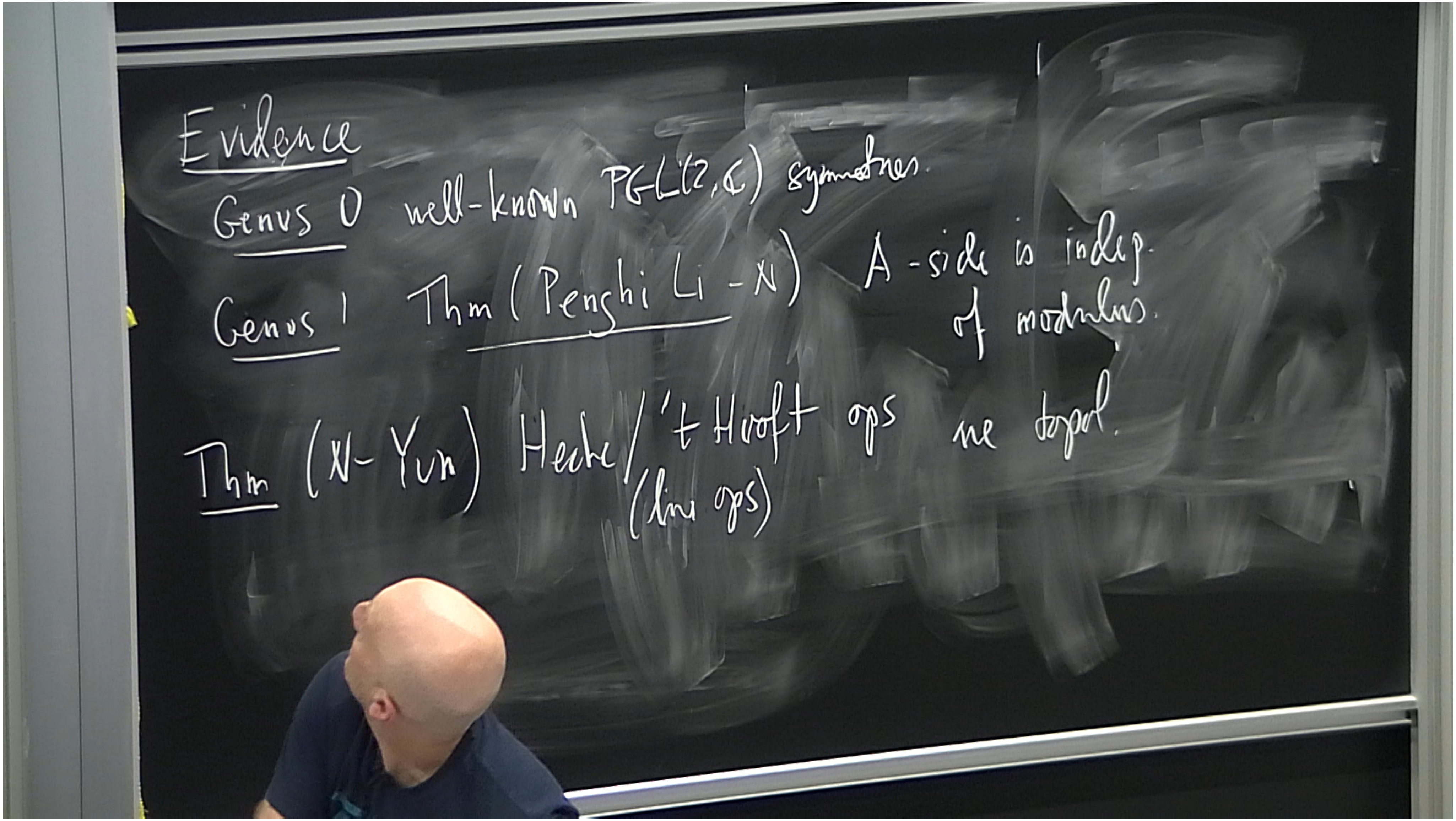
Conj A-side is TFT.

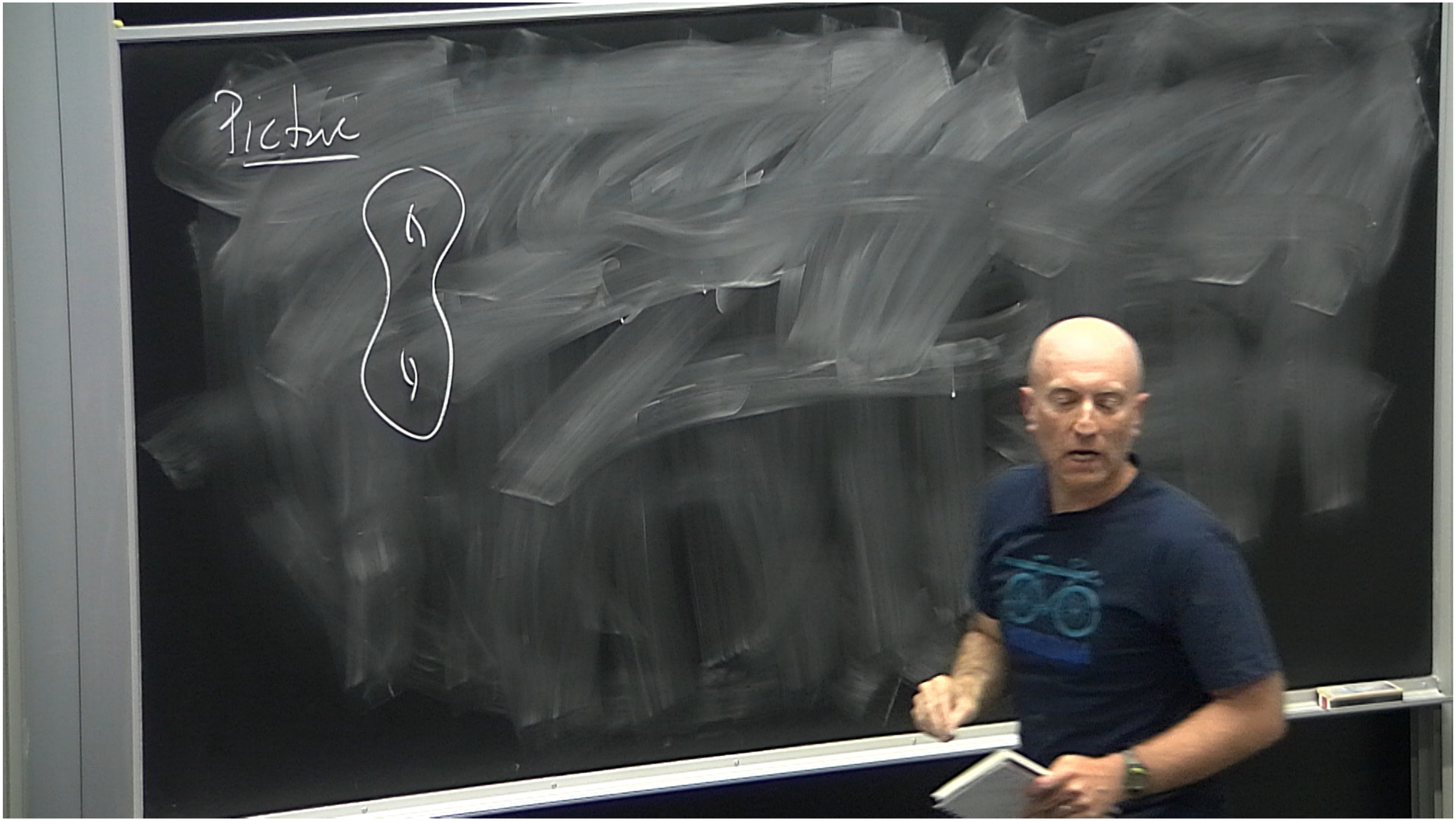

Evidence
Genus 0 well-known PGL(2,ℂ) symmetries.

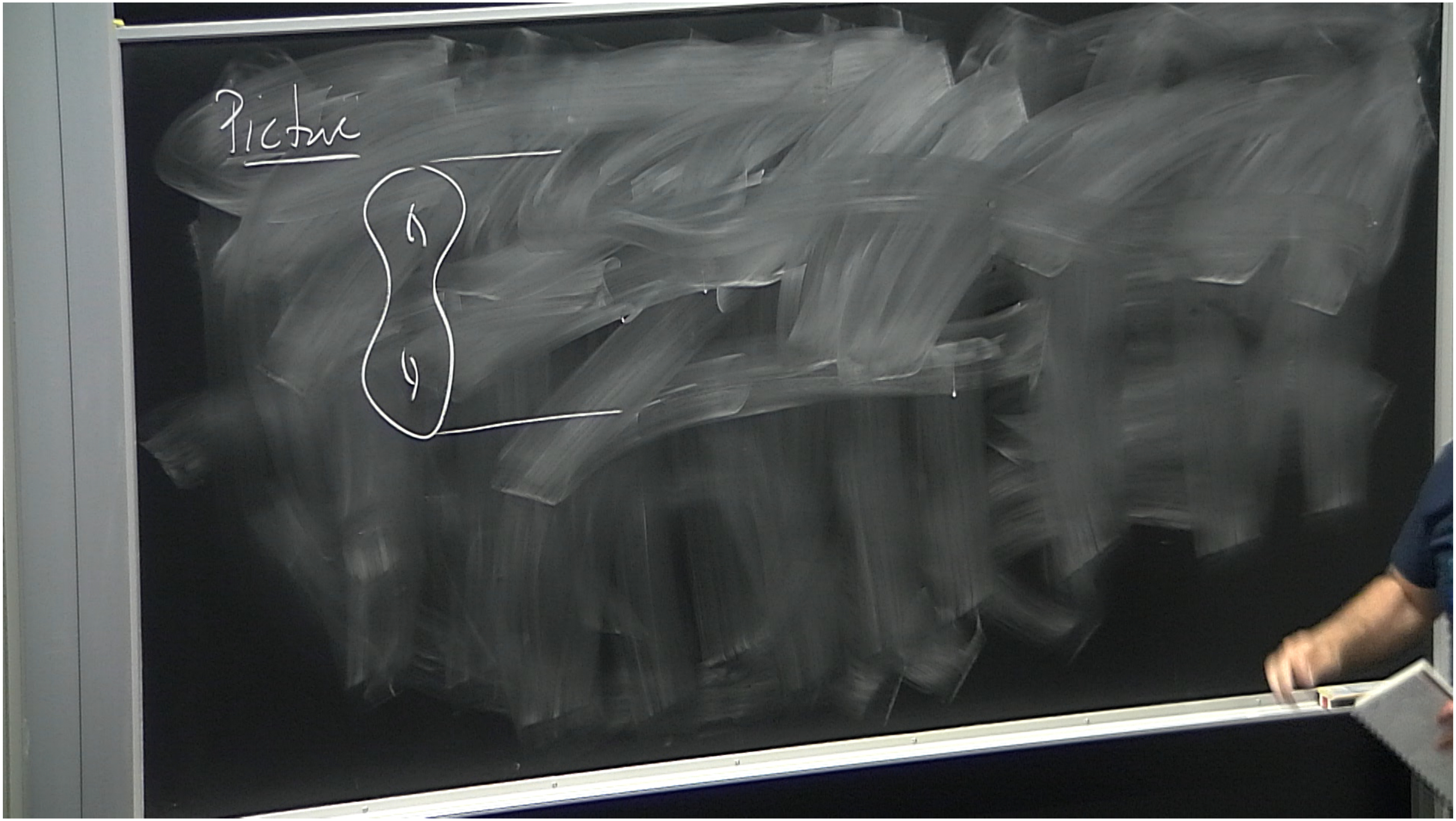
Evidence

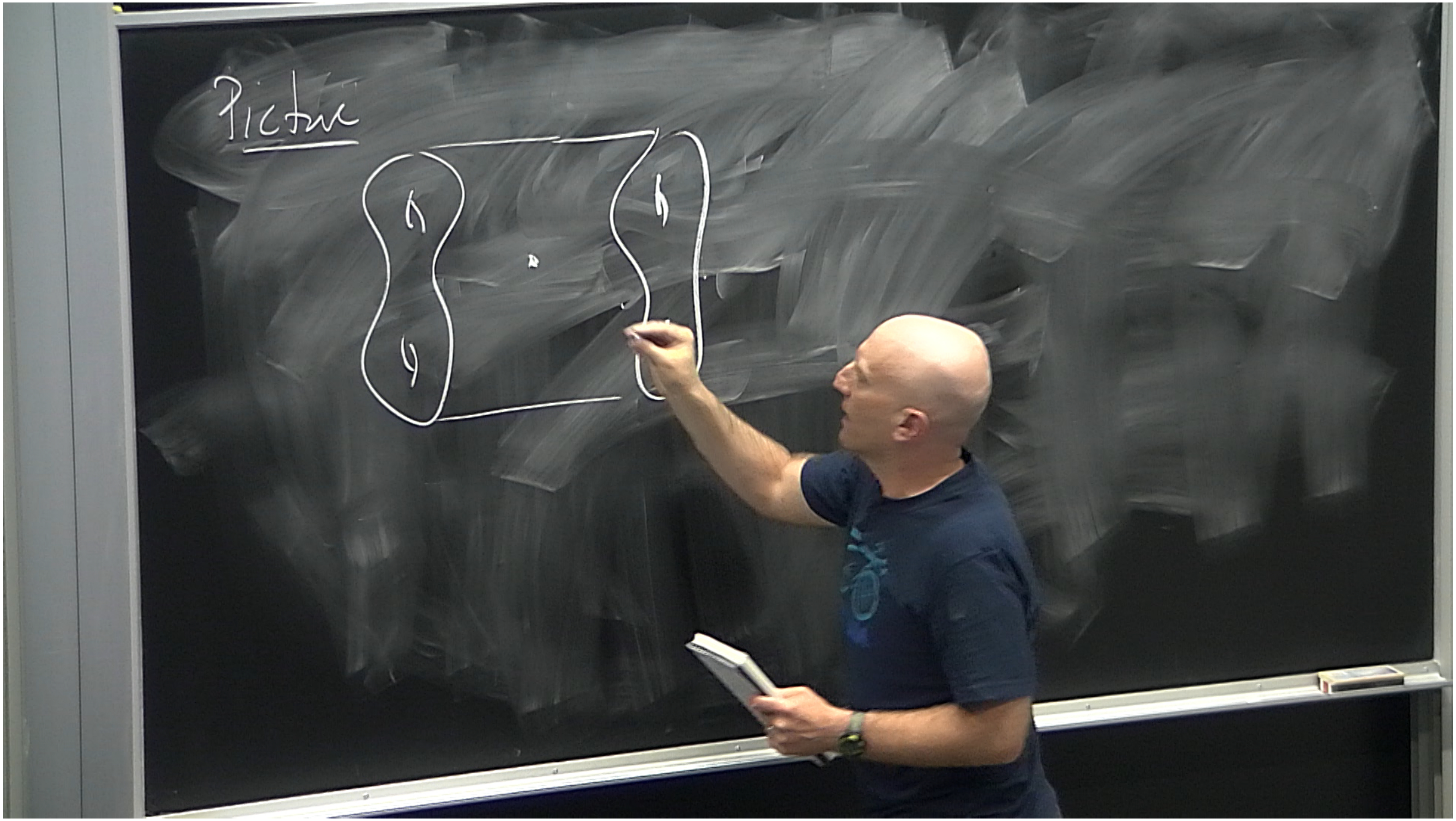
Genus 0 well-known $PGL(2,\mathbb{C})$ symmetries.

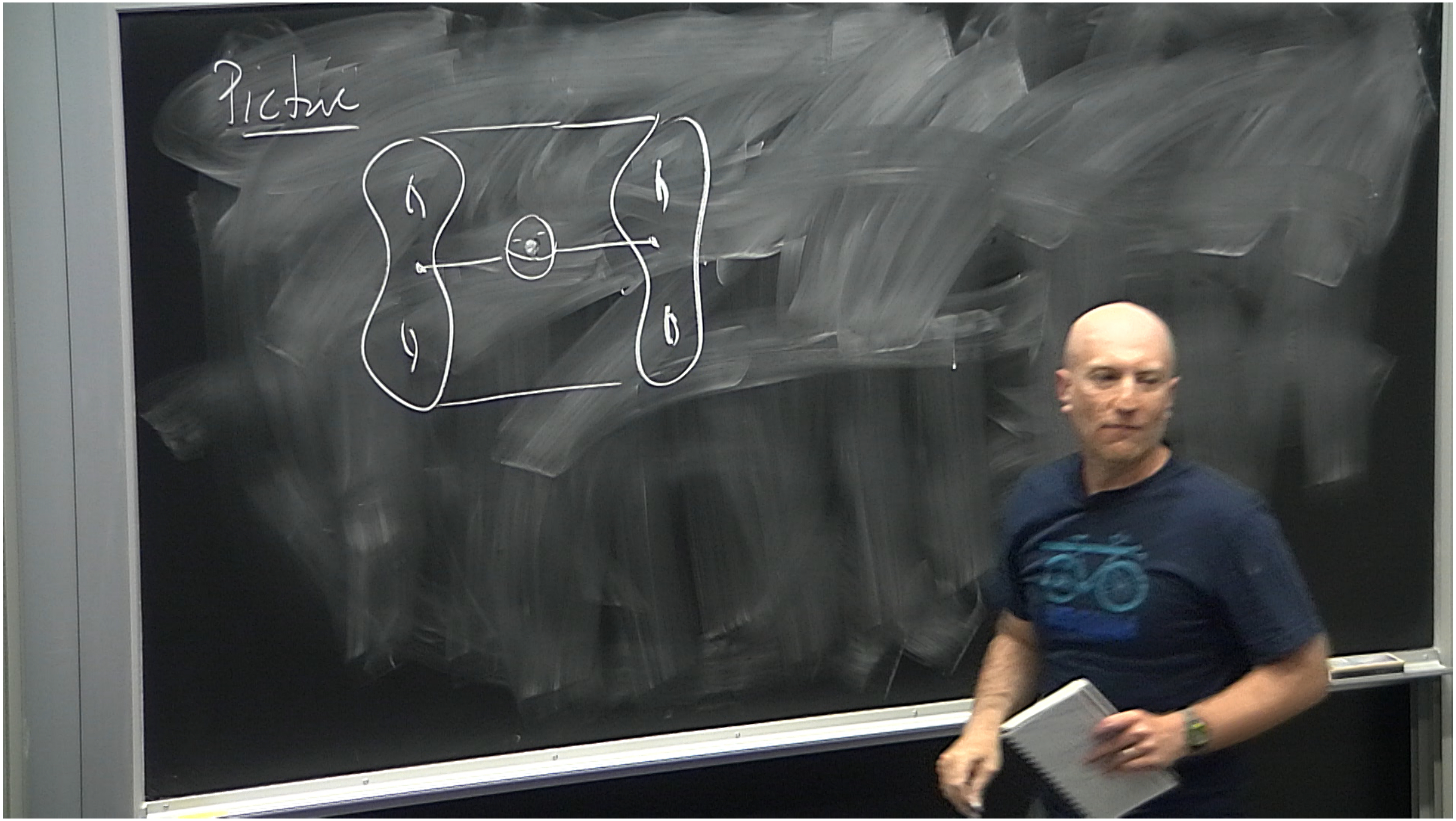
Genus 1 Thm (Penghi Li - X) A-side is indep. of modulus.

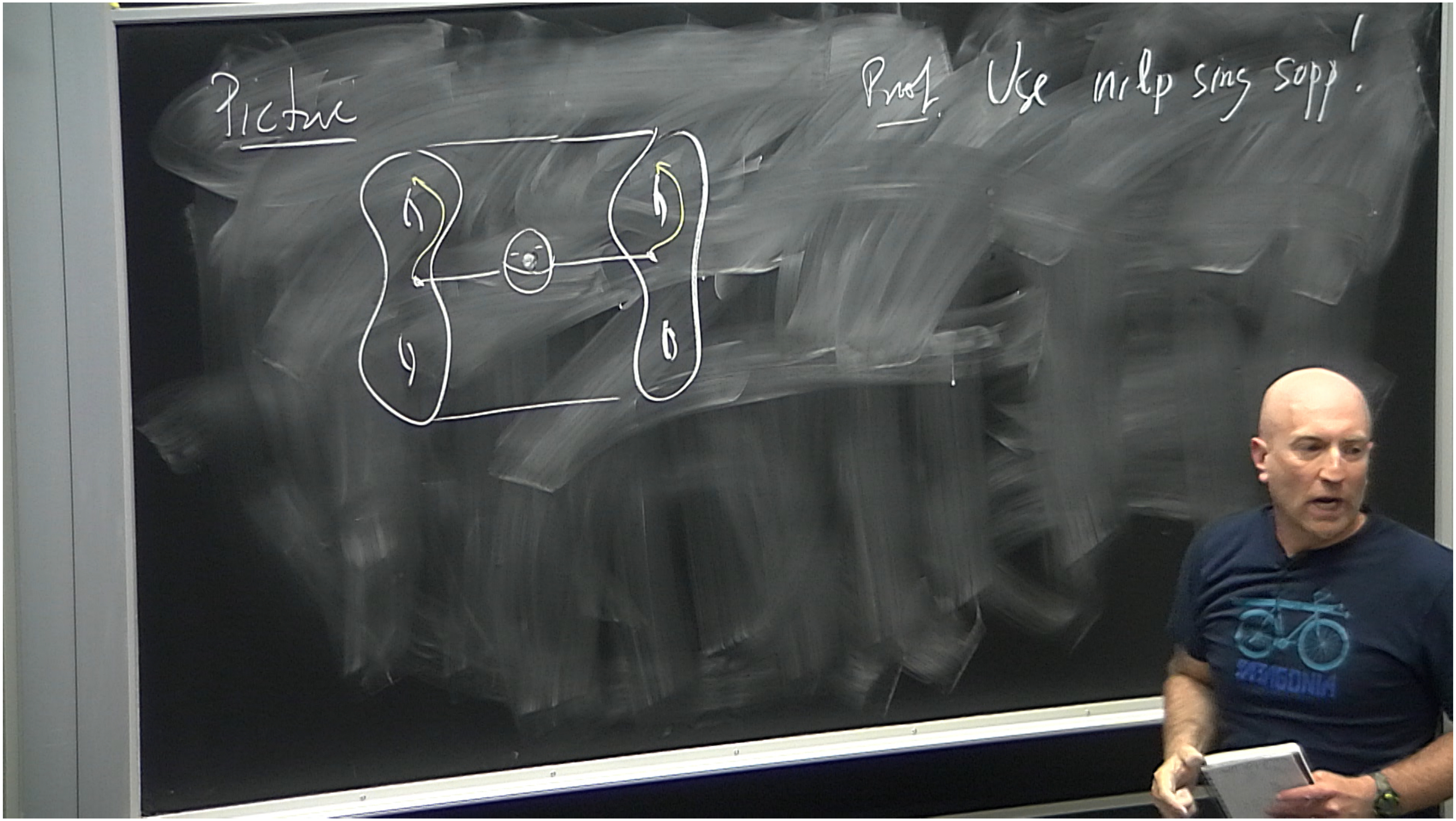

Evidence
Genus 0 well-known PGL(2,C) symmetries.
Genus 1 Thm (Penghi Li - X) A-side is indep. of modulus.
Thm (X-Yun) Hecke/'t Hooft ops are topol.
(line ops)

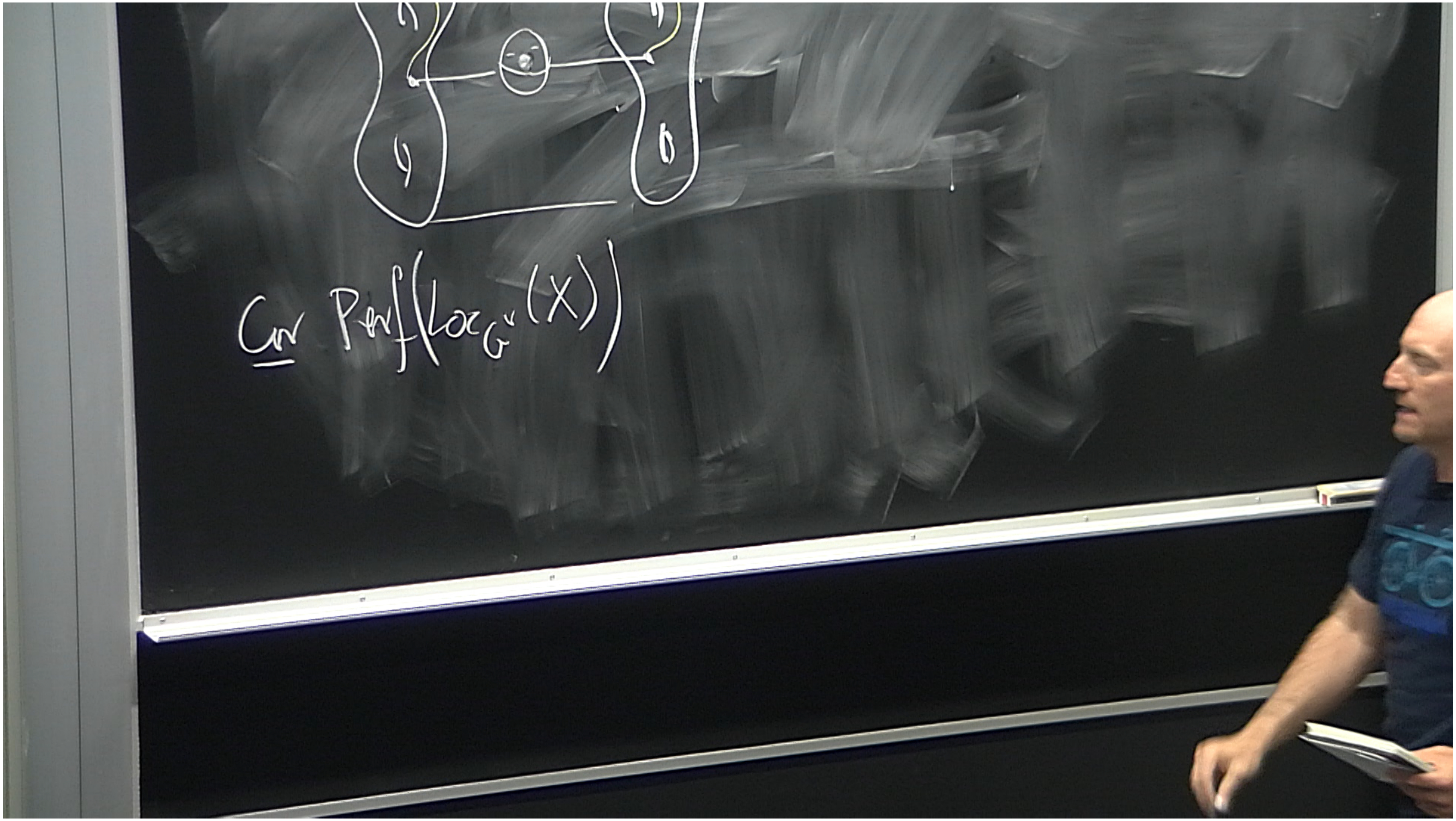

Picture

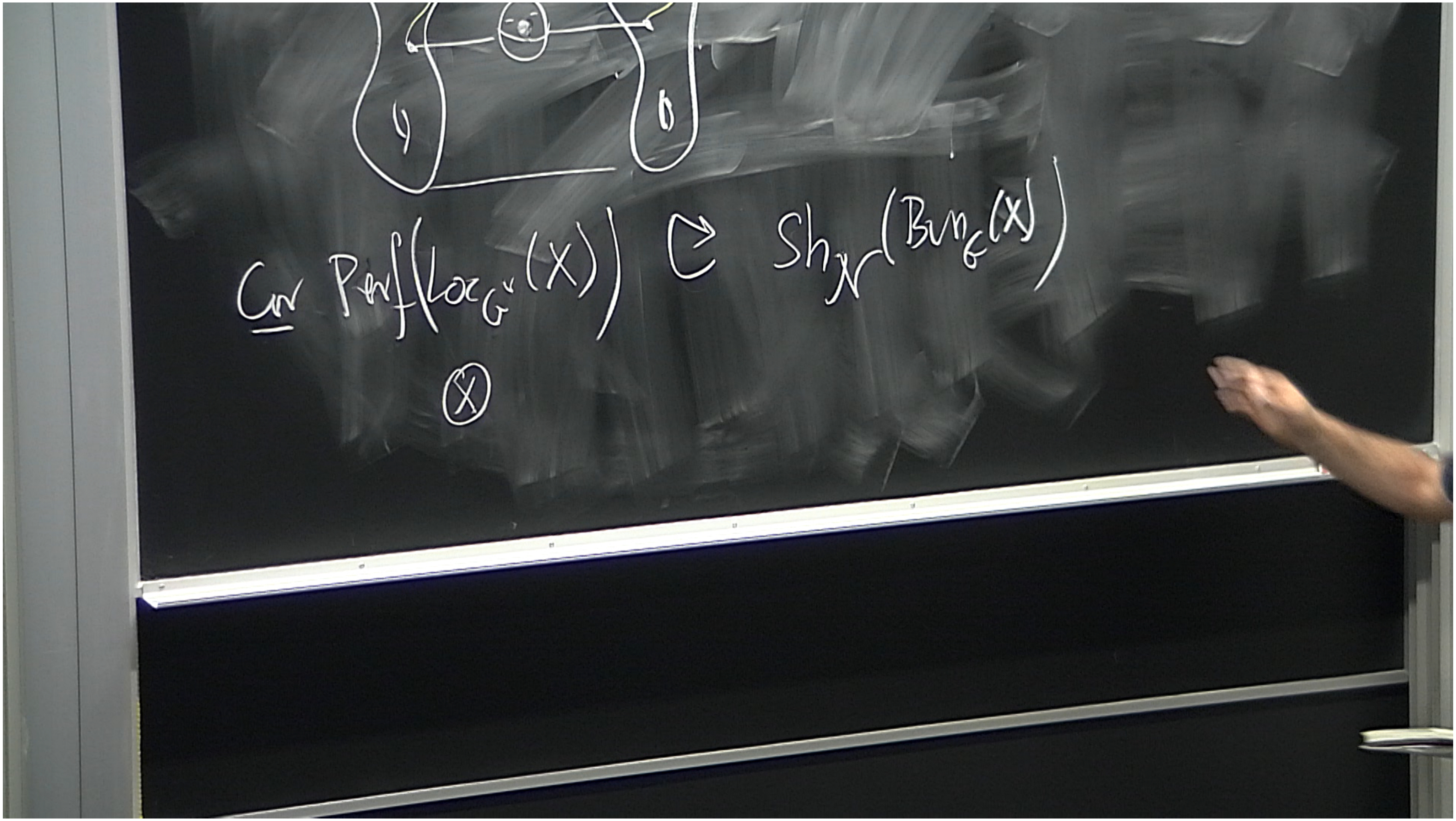

Picture


Picture


Picture

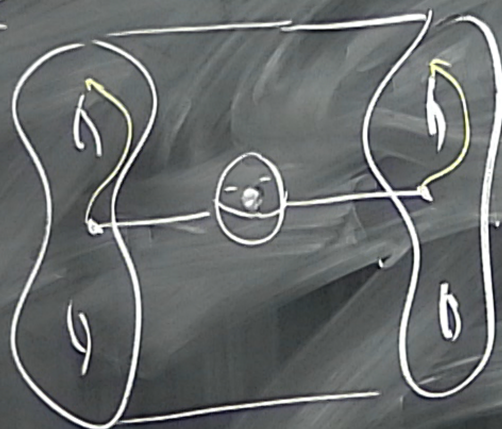

Picture
Proof. Use nilp sing supp!


$\underline{Coh}$ Perf$(Loc_{G^\vee}(X))$


Coh Perf(Loc_{G^∨}(X)) ⟳ Sh_N(Bun_G(X))
⊗

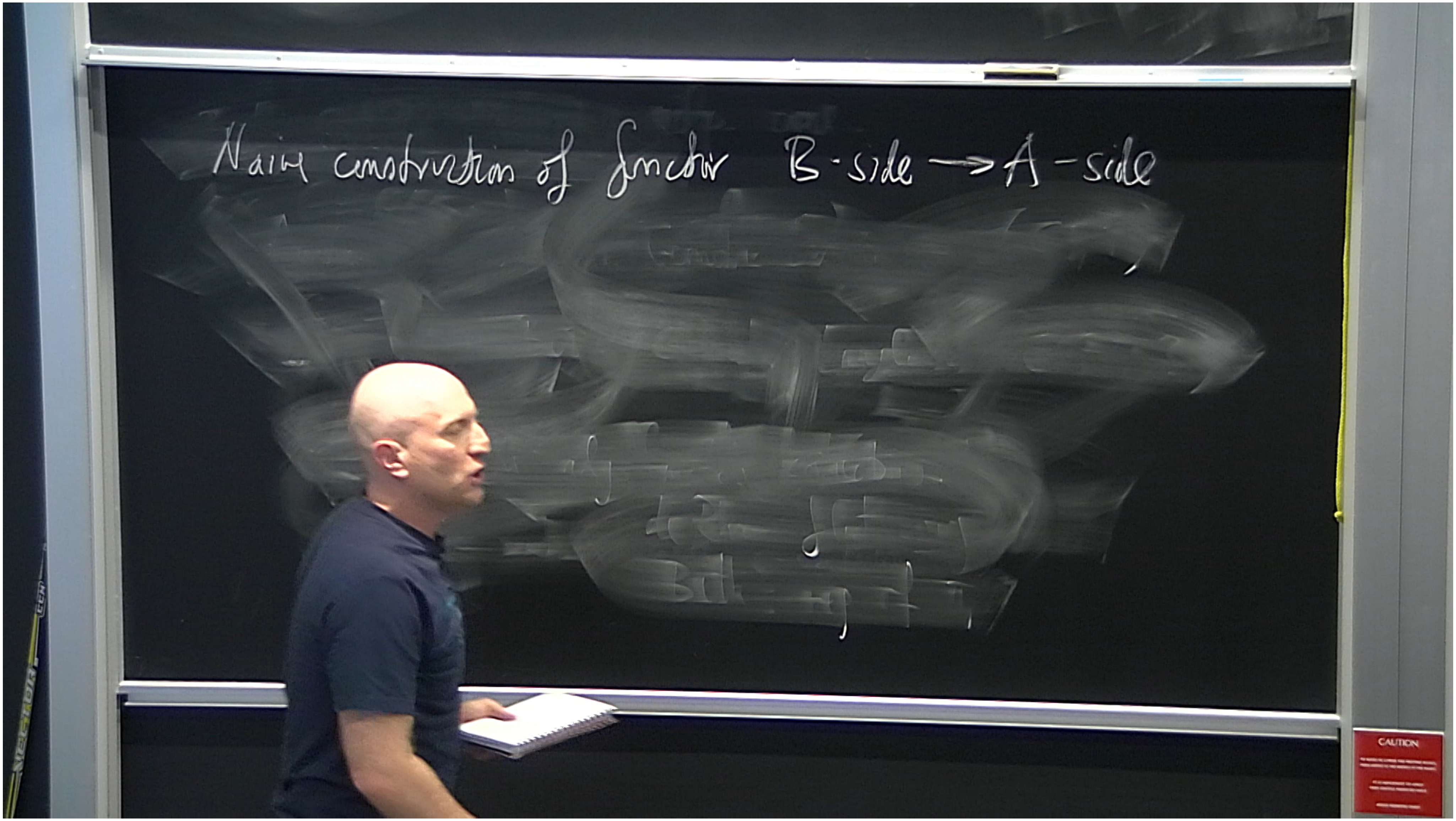
Picture

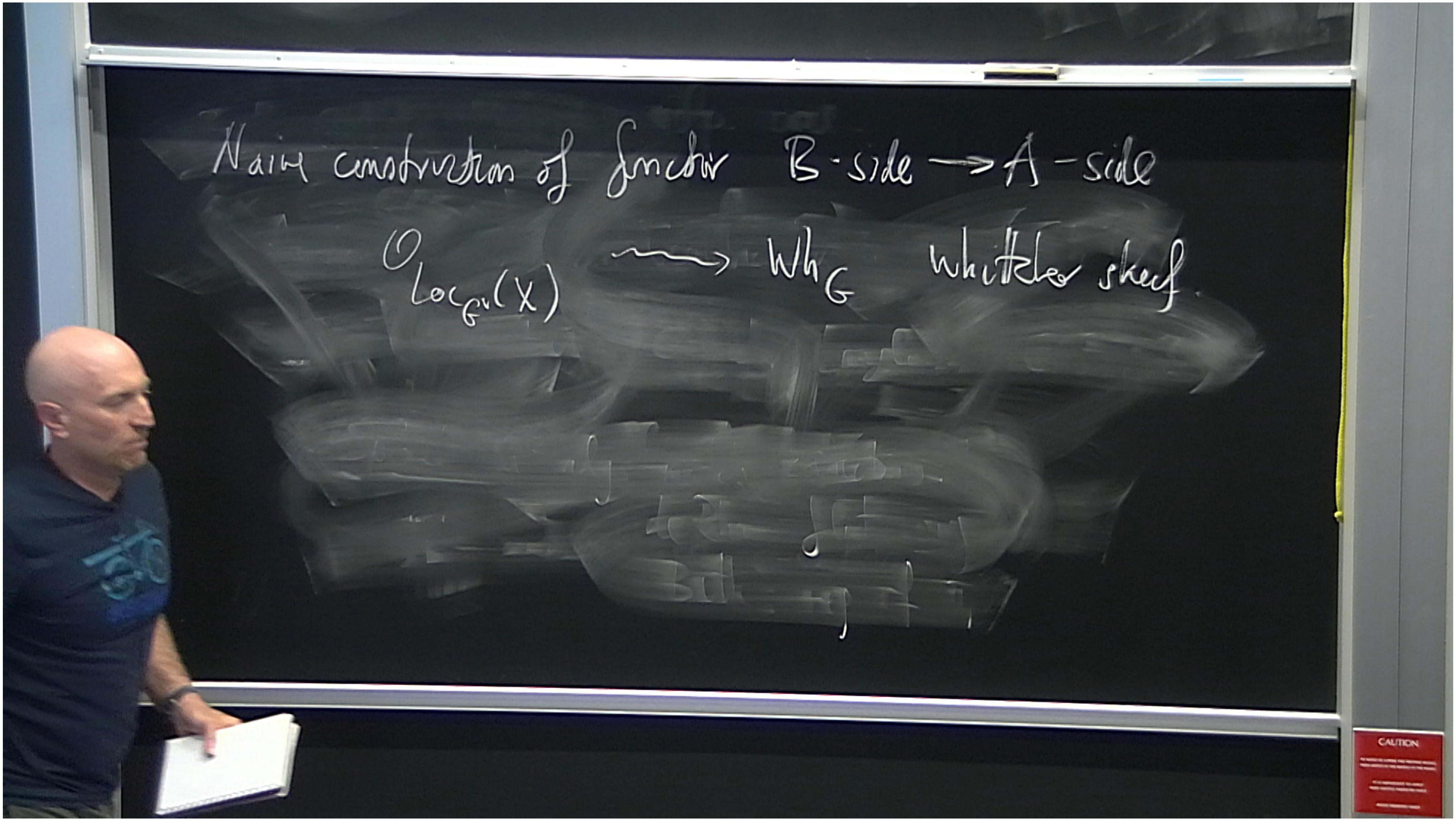
Prop. Use nilp sing supp!

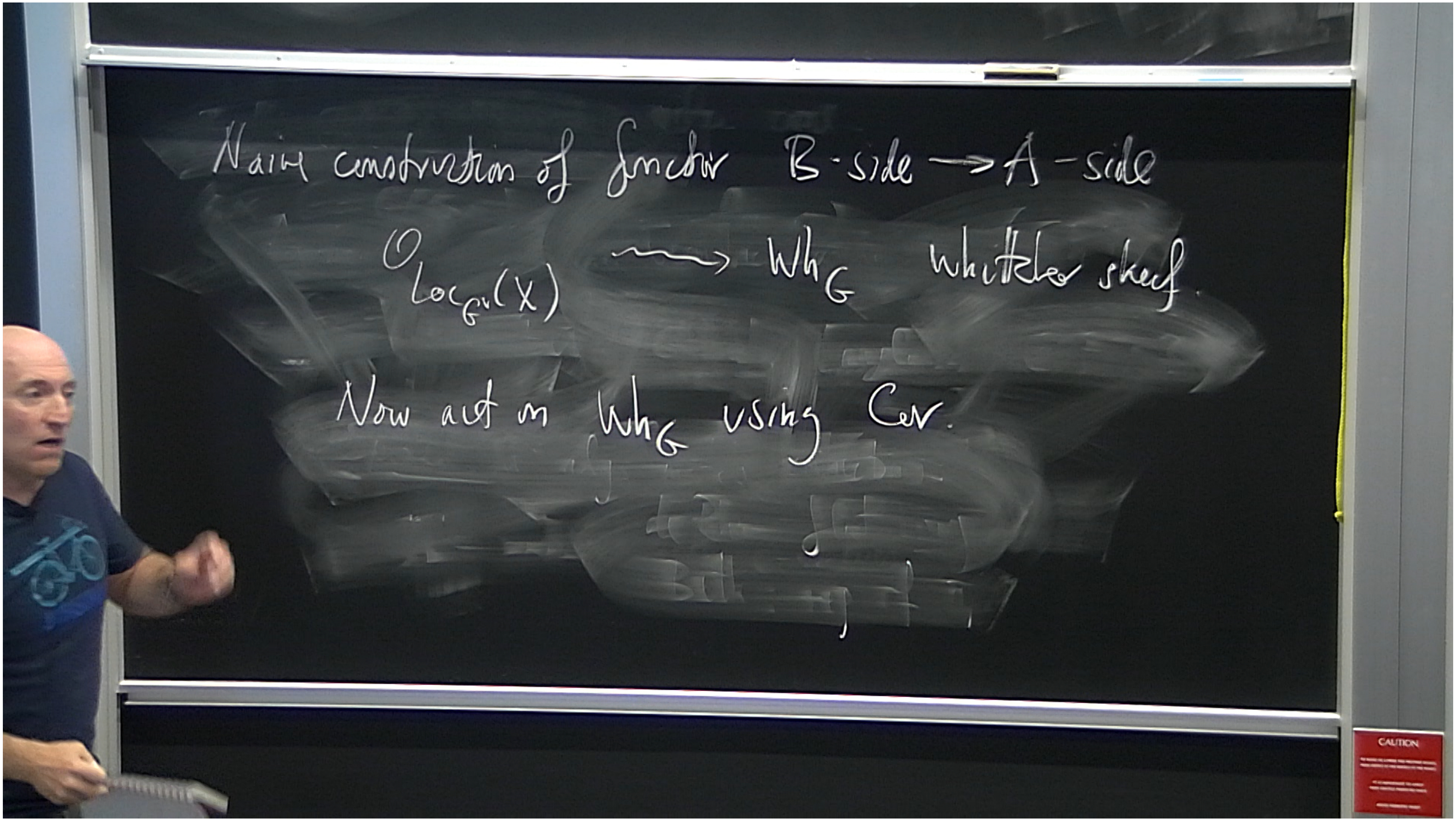


Cor $\mathrm{Perf}(\mathrm{Loc}_{G^\vee}(X)) \circlearrowright \mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G(X))$

$\otimes$


Naive construction of functor B-side → A-side


Naive construction of functor B-side → A-side
$\mathcal{O}_{Loc_{G^\vee}(X)} \rightsquigarrow Wh_G$ Whittaker sheaf.


Naive construction of functor B-side → A-side
$\mathcal{O}_{Loc_{G^\vee}(X)} \rightsquigarrow Wh_G$ whittaker sheaf.
Now act on Wh_G using Cov.

Naive construction of functor B-side $\longrightarrow$ A-side

$$\mathcal{O}_{\mathrm{Loc}_{G^\vee}(X)} \rightsquigarrow \mathrm{Wh}_G \quad \text{whittaker sheaf.}$$

Now act on Wh_G using Cov.

<u>Good news</u> $\mathcal{O} \rightsquigarrow \mathrm{Wh}$!

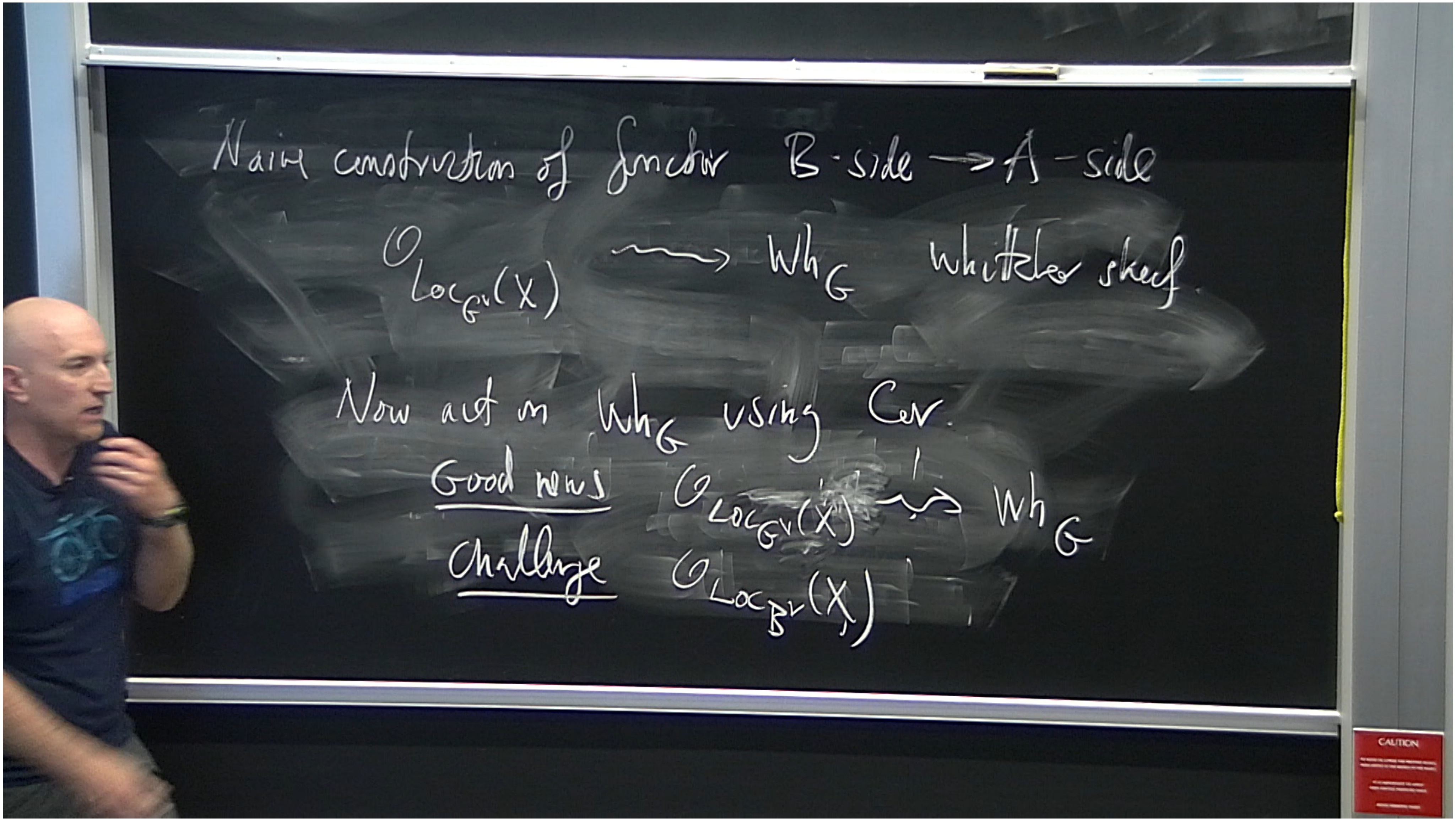
Naive construction of functor B-side $\to$ A-side

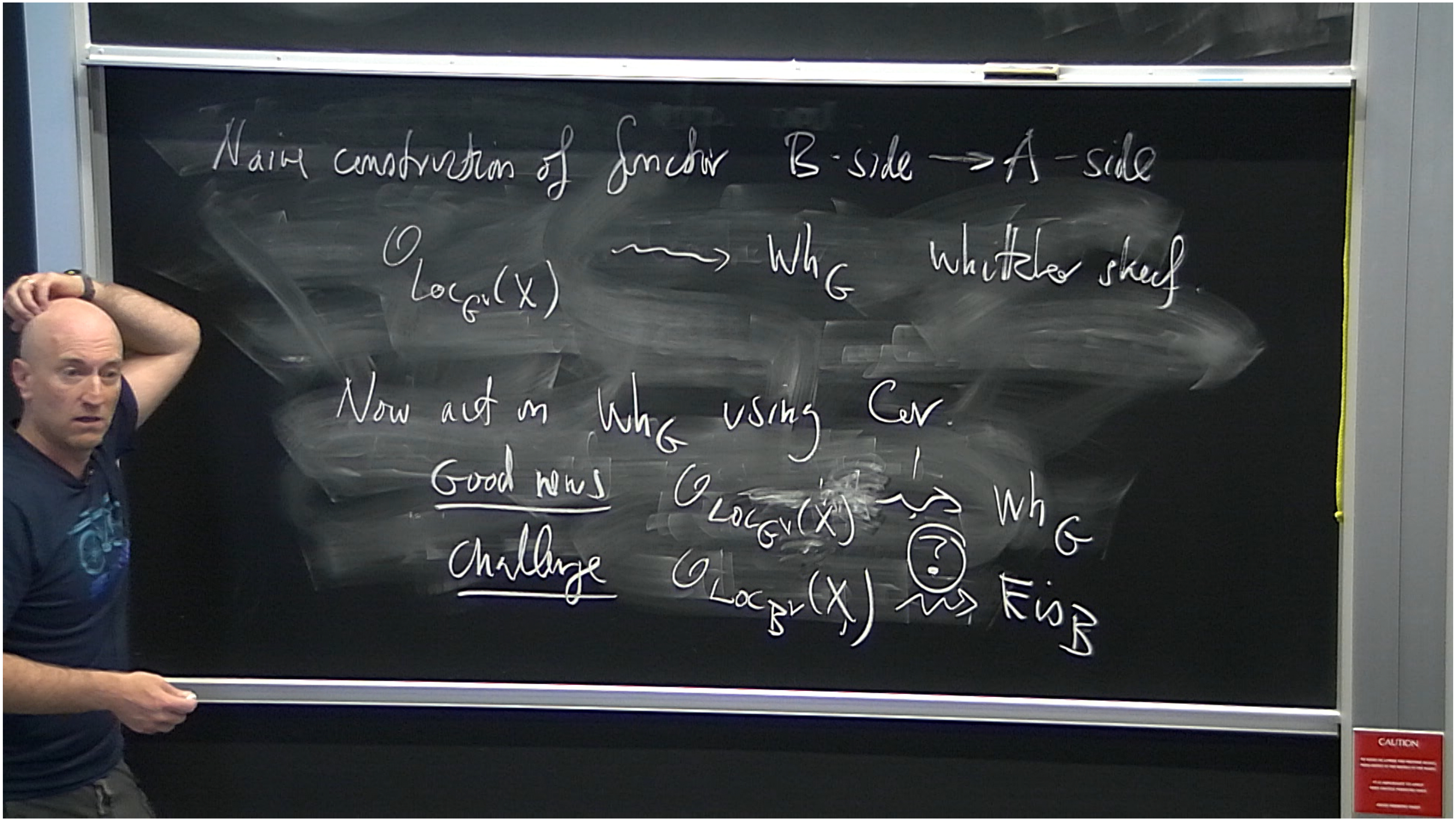
$$\mathcal{O}_{Loc_{G^\vee}(X)} \rightsquigarrow Wh_G \quad \text{Whittaker sheaf}$$

Now act on Wh_G using Cor.

Good news $\mathcal{O}_{Loc_{G^\vee}(X)} \overset{!}{\rightsquigarrow} Wh_G$

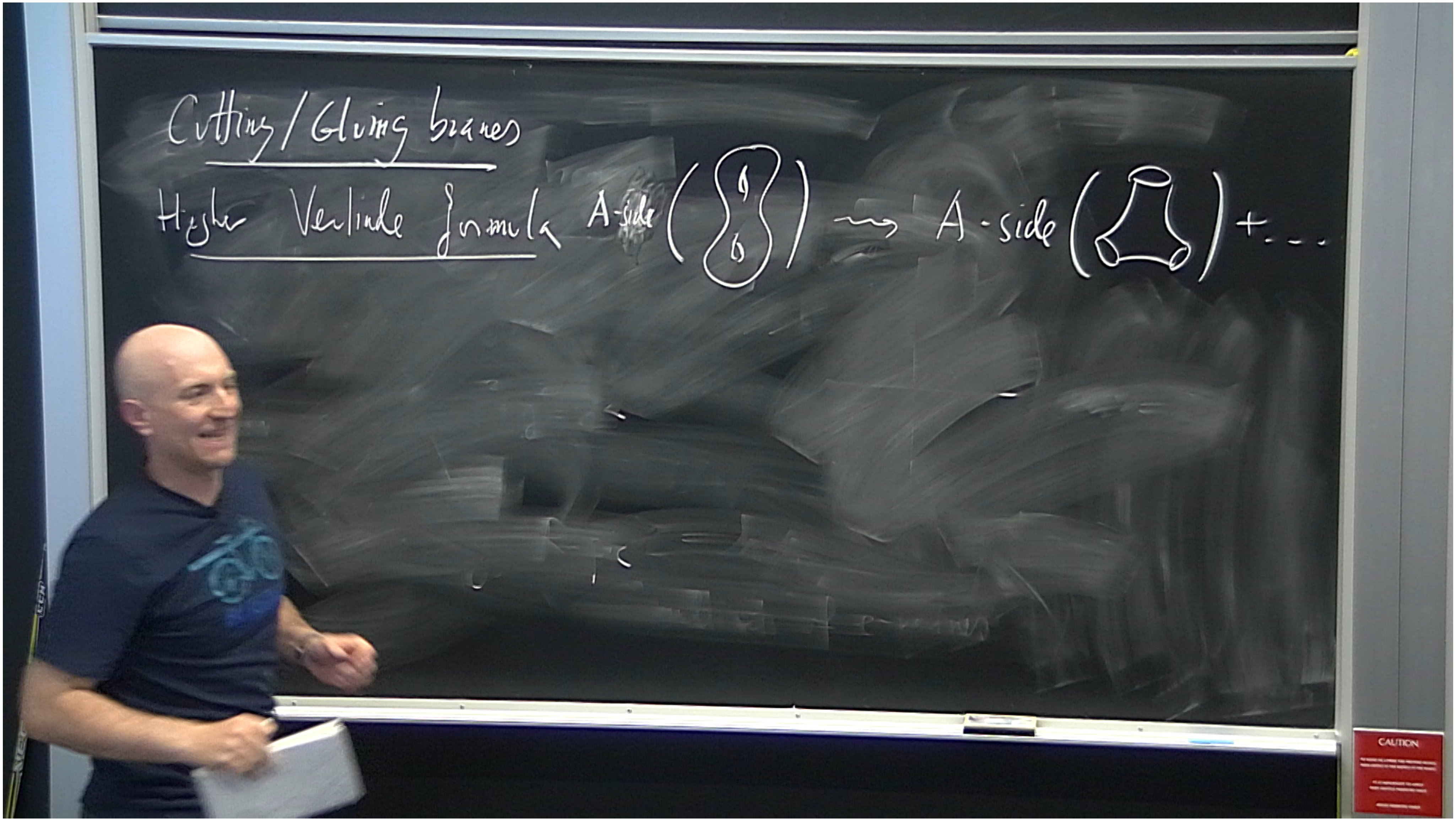
Challenge

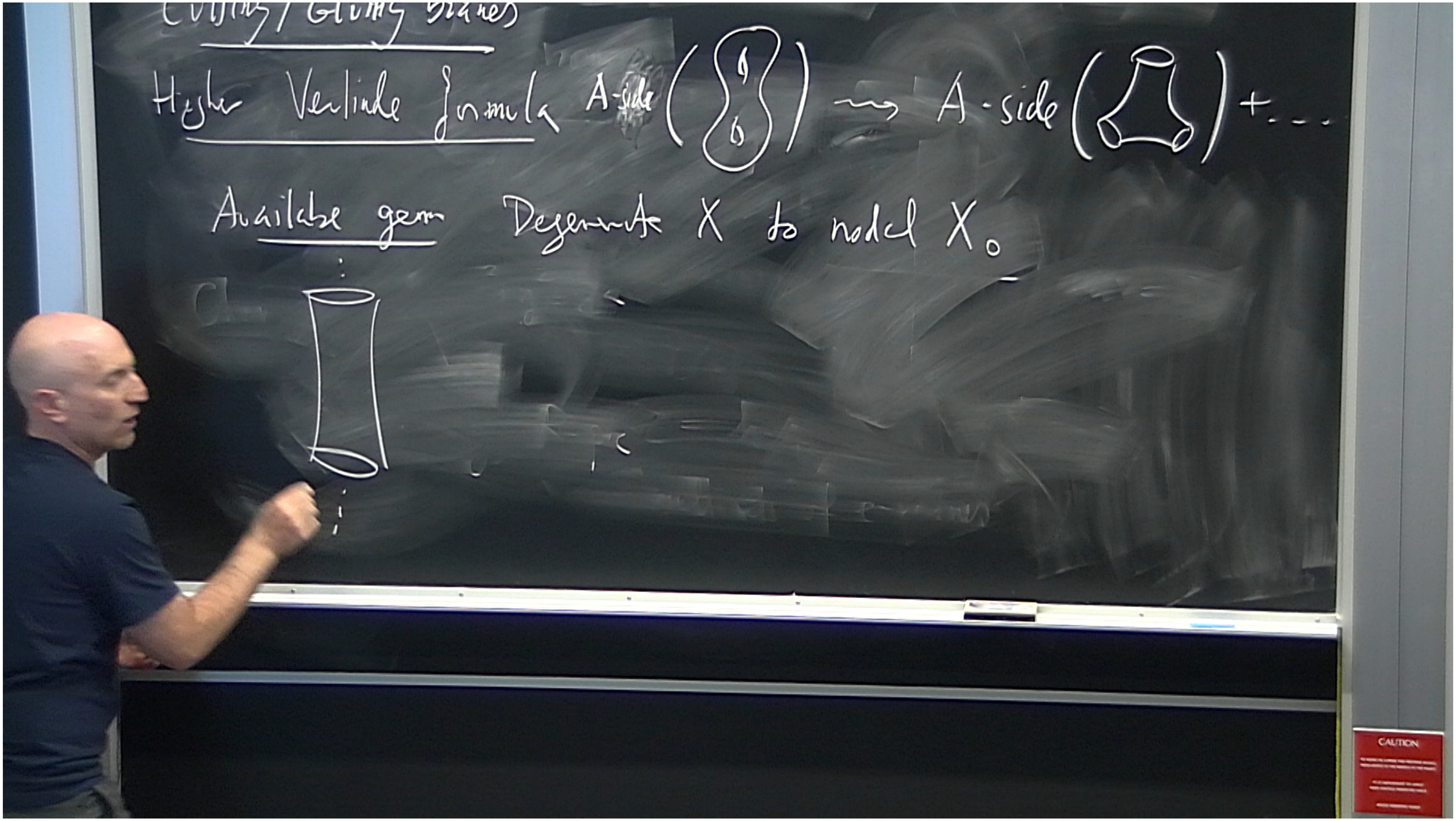

Naive construction of functor B-side → A-side
$\mathcal{O}_{Loc_{G^\vee}(X)} \rightsquigarrow Wh_G$ Whittaker sheaf
Now act on Wh_G using Ger.
Good news $\mathcal{O}_{Loc_{G^\vee}(X)} \overset{!}{\mapsto} Wh_G$
Challenge $\mathcal{O}_{Loc_{B^\vee}(X)}$

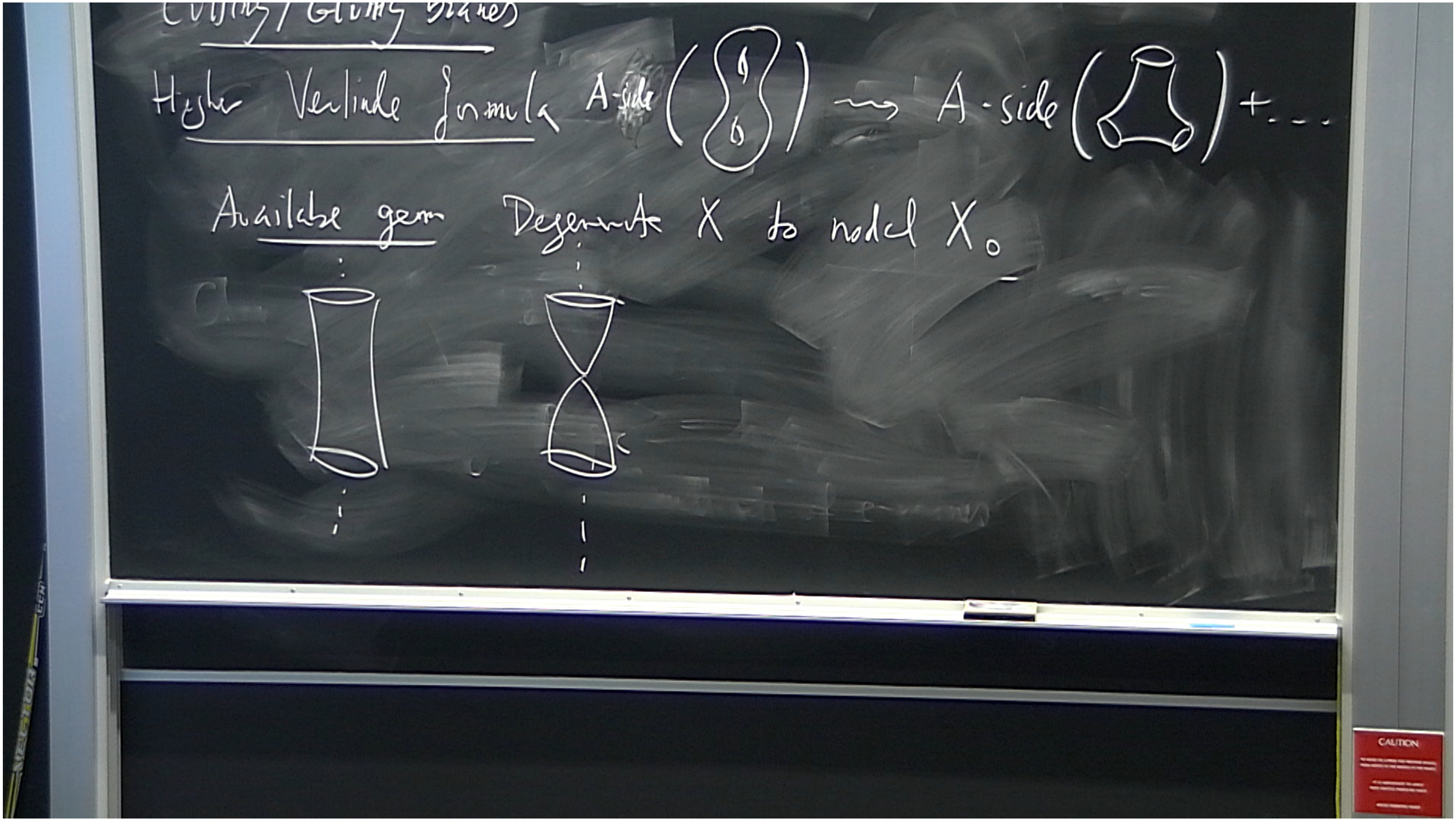

Naive construction of functor B-side → A-side
$\mathcal{O}_{Loc_{G^\vee}(X)} \rightsquigarrow Wh_G$ Whittaker sheaf.
Now act on Wh_G using Cor.
Good news $\mathcal{O}_{Loc_{G^\vee}(X)} \overset{!}{\rightsquigarrow} Wh_G$
Challenge $\mathcal{O}_{Loc_{B^\vee}(X)} \overset{?}{\rightsquigarrow} Eis_B$

Cutting / Gluing branes

Verlinde


Cutting / Gluing branes
Higher Verlinde formula
A-side
A-side
+ . . .


Cutting / Gluing Branes
Higher Verlinde formula A-side ( ) ⇝ A-side ( ) + ...
Available germ Degenerate X to nodal X₀


Cutting / Gluing Branes
Higher Verlinde formula A-side ( ) ⇝ A-side ( ) + ...
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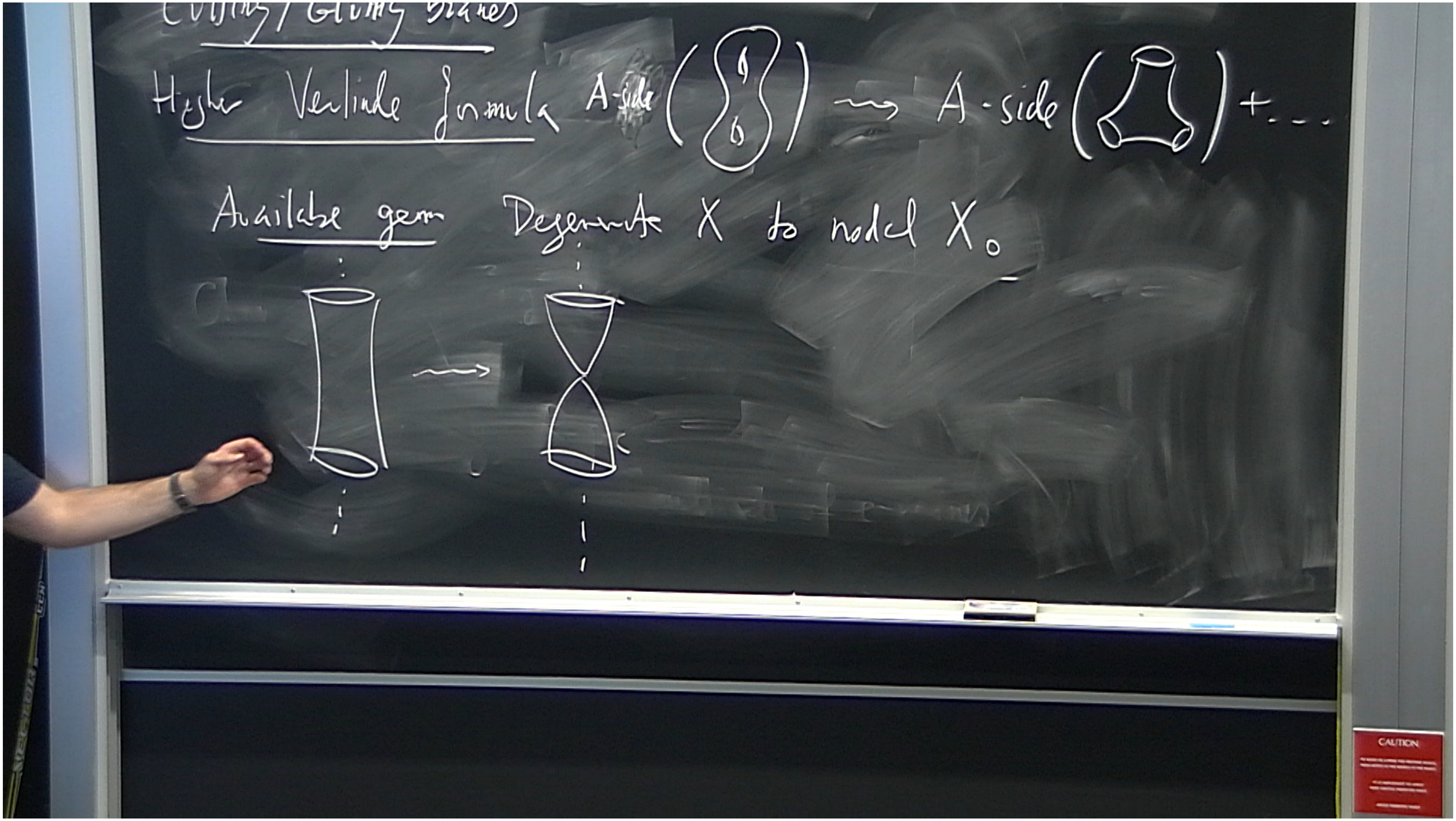
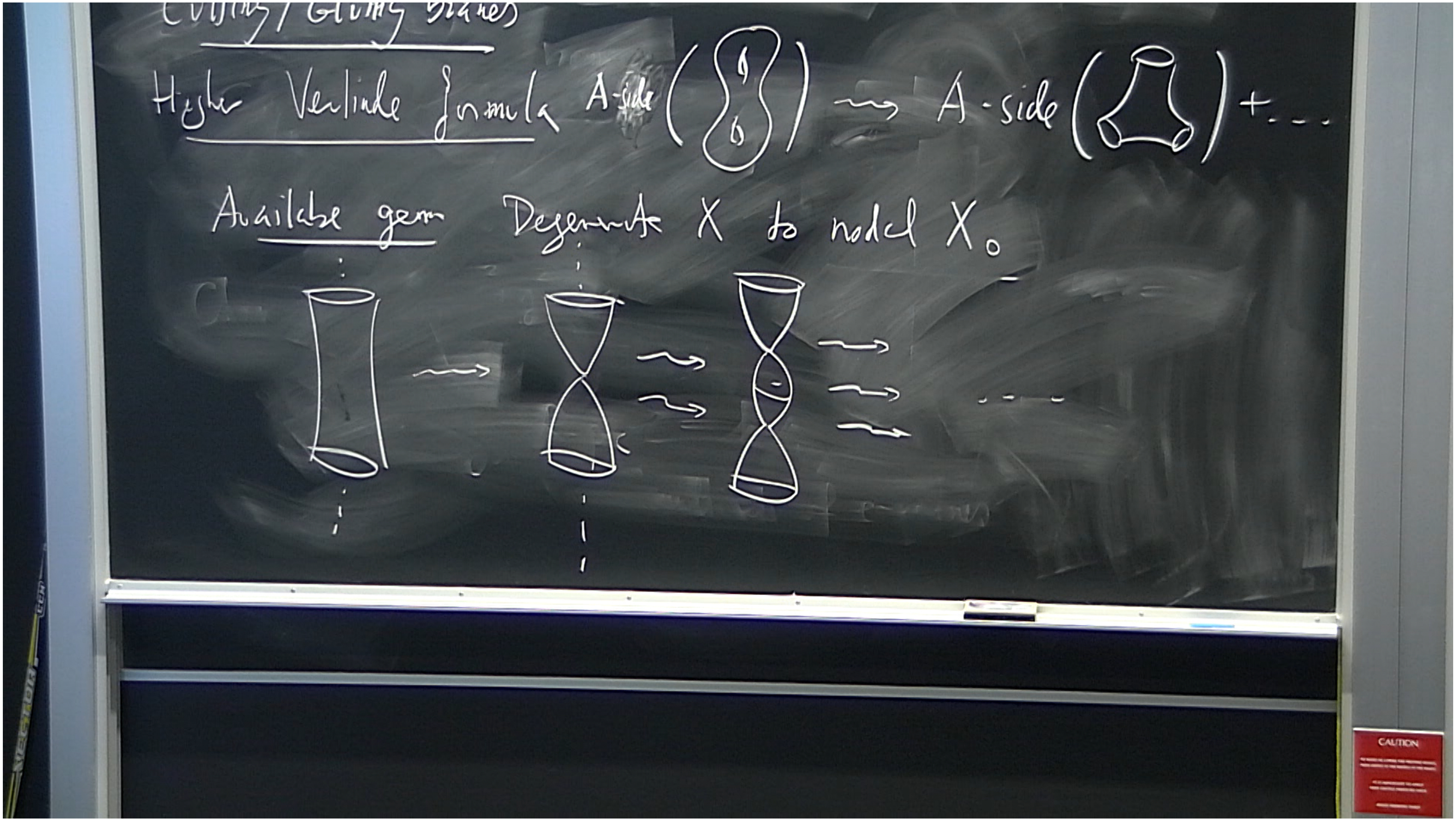
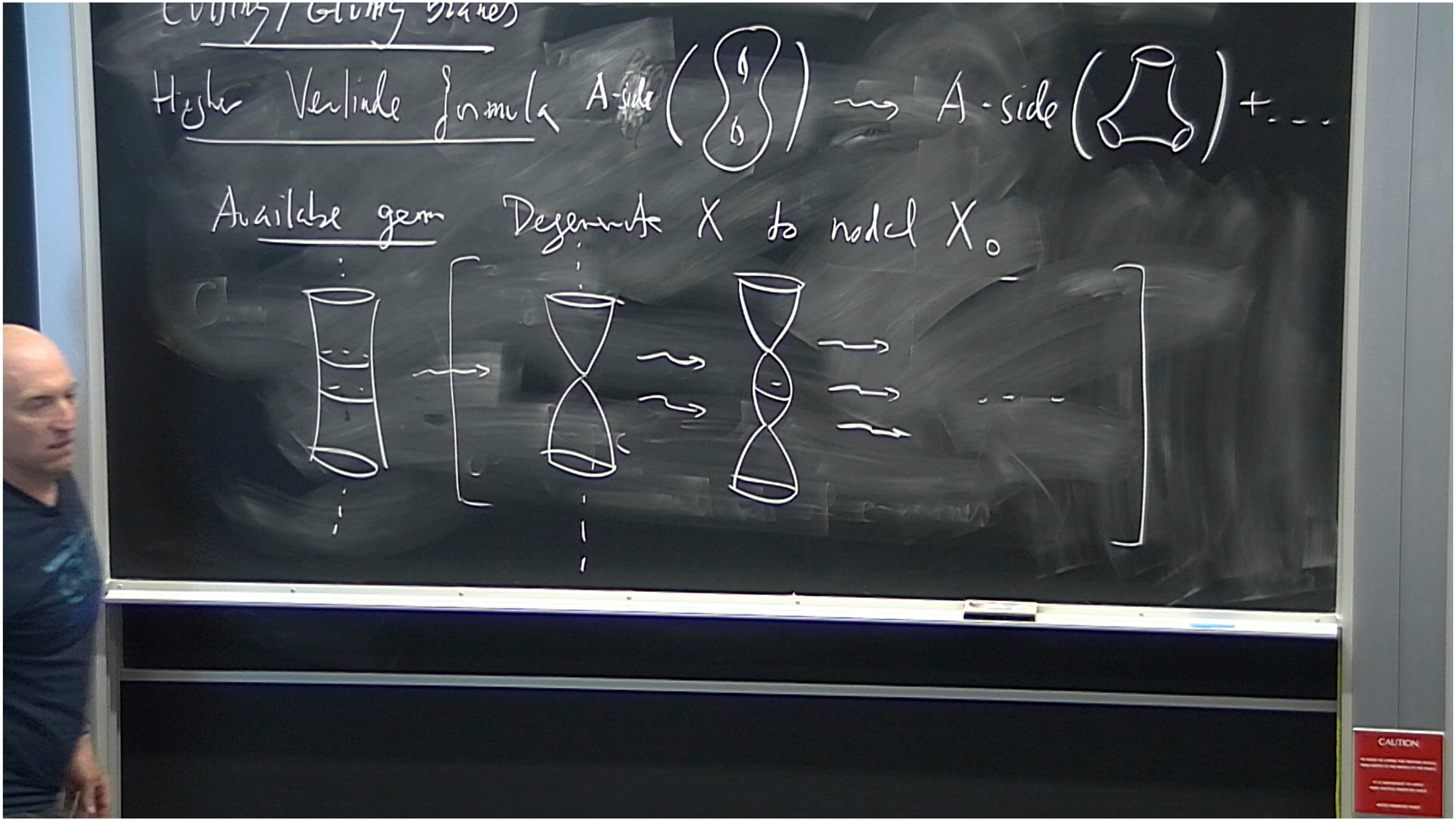
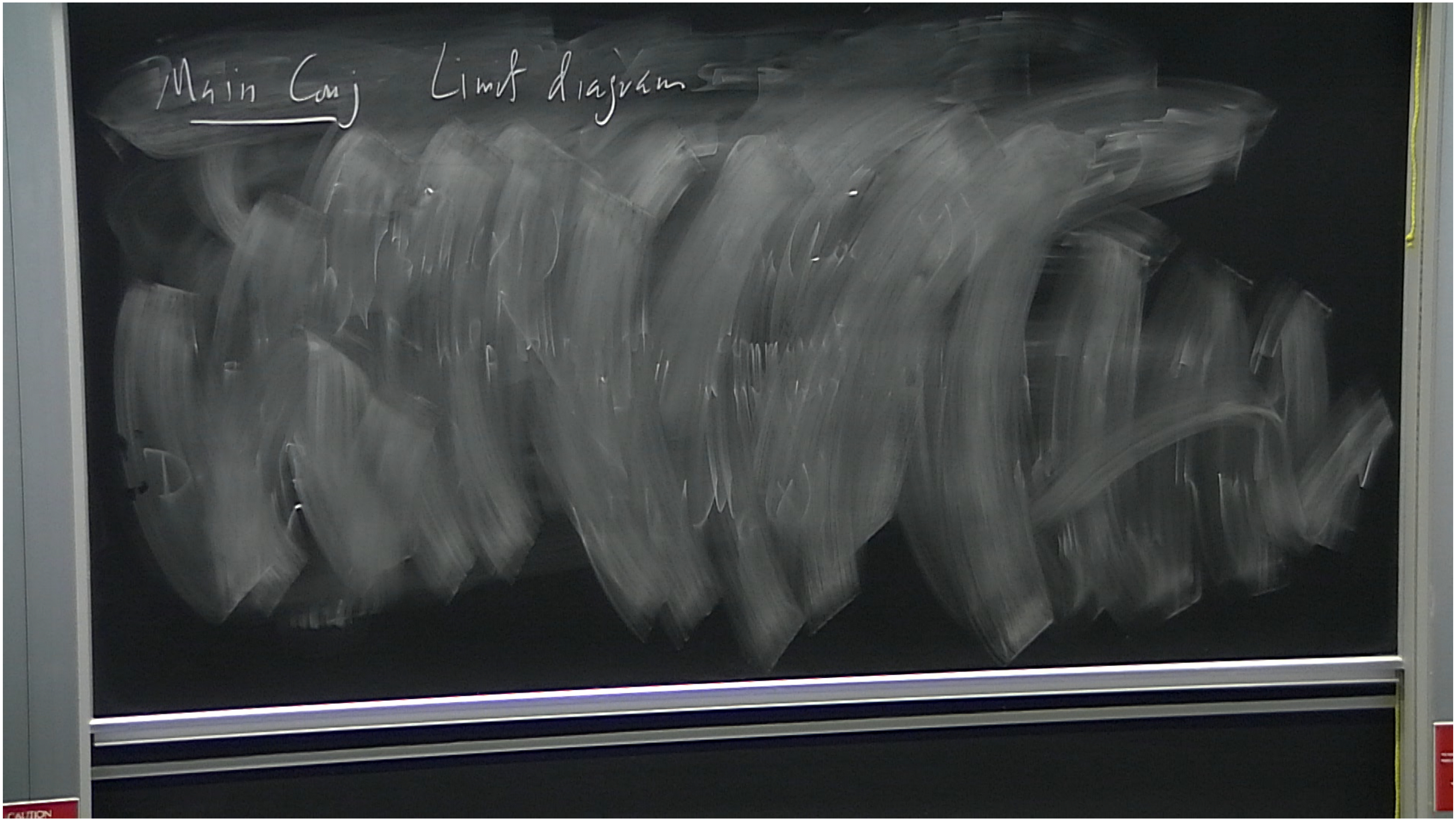
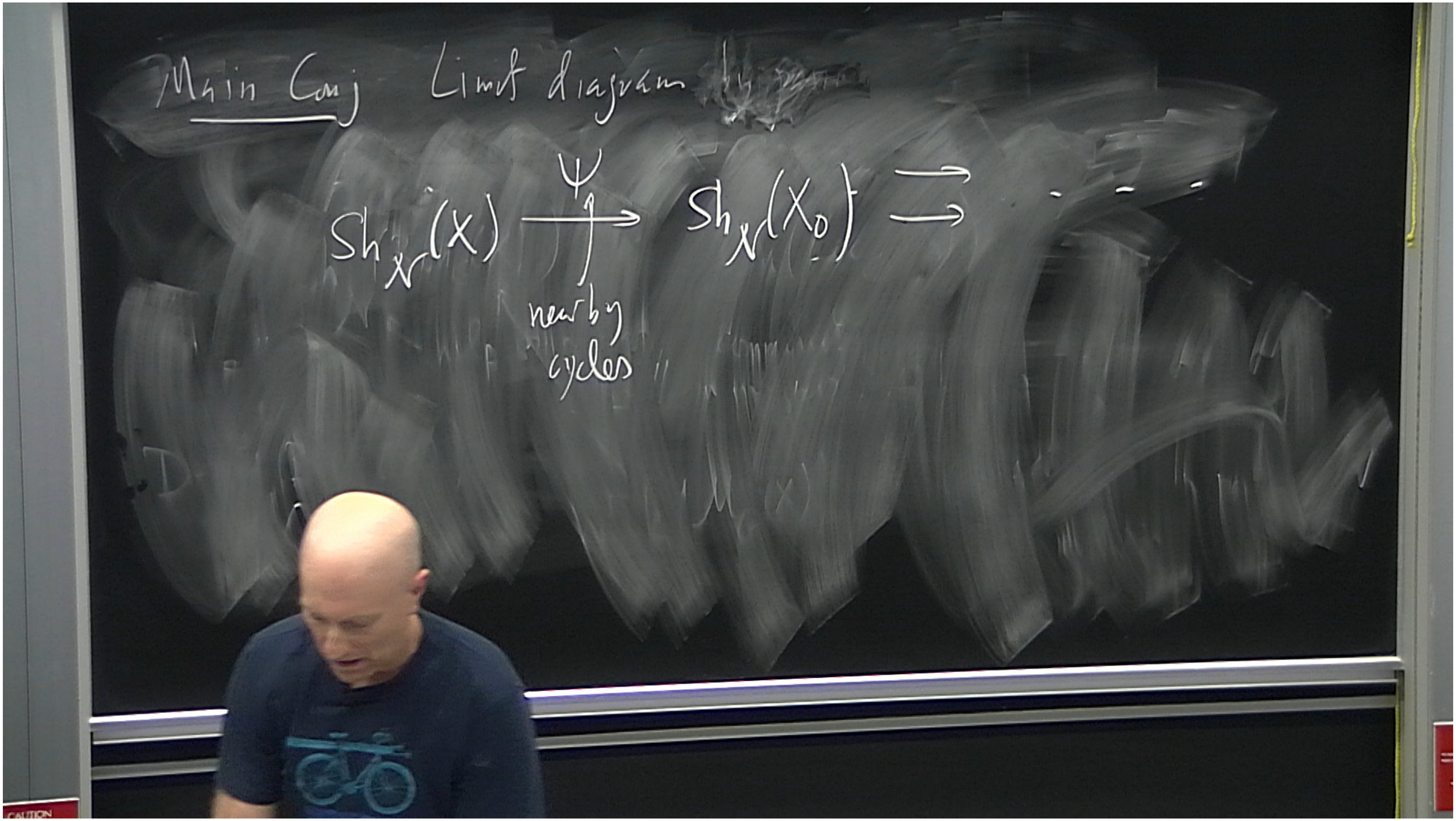
Degenerate X to nodal X₀


Higher Verlinde formula
A-side
A-side
Available germ
Degenerate X to nodal X0


Cutting / Gluing Branes
Higher Verlinde formula A-side
A-side
+ . . .
Available gen
Degenerate X to nodal X₀


Higher Verlinde formula
A-side
A-side
+ . . .
Available germ
Degenerate X to nodal X₀


Main Conj
Limit diagram


Main Conj
Limit diagram
$Sh_{\chi r}(X) \xrightarrow{\Psi} Sh_{\chi r}(X_0) \rightrightarrows \cdots$
nearby cycles

<u>Main Conj</u> Limit diagram

$$\mathrm{Sh}_{\mathcal{N}}(X) \xrightarrow[\text{nearby cycles}]{\Psi} \mathrm{Sh}_{\mathcal{N}}(X_0) \rightrightarrows \cdots$$

<u>Evidence</u>

<u>Thm</u> (Ben-Zvi - Li - N) True in genus 1.

Main Conj   Limit diagram

$$Sh_{\mathcal{N}}(X) \xrightarrow[\text{nearby cycles}]{\Psi} Sh_{\mathcal{N}}(X_0) \rightrightarrows \cdots$$

Evidence

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(Assume G adj ...?)

Main Conj Limit diagram

$$Sh_{\mathcal{N}}(X) \xrightarrow[\text{nearby cycles}]{\Psi} Sh_{\mathcal{N}}(X_0) \rightrightarrows \cdots$$

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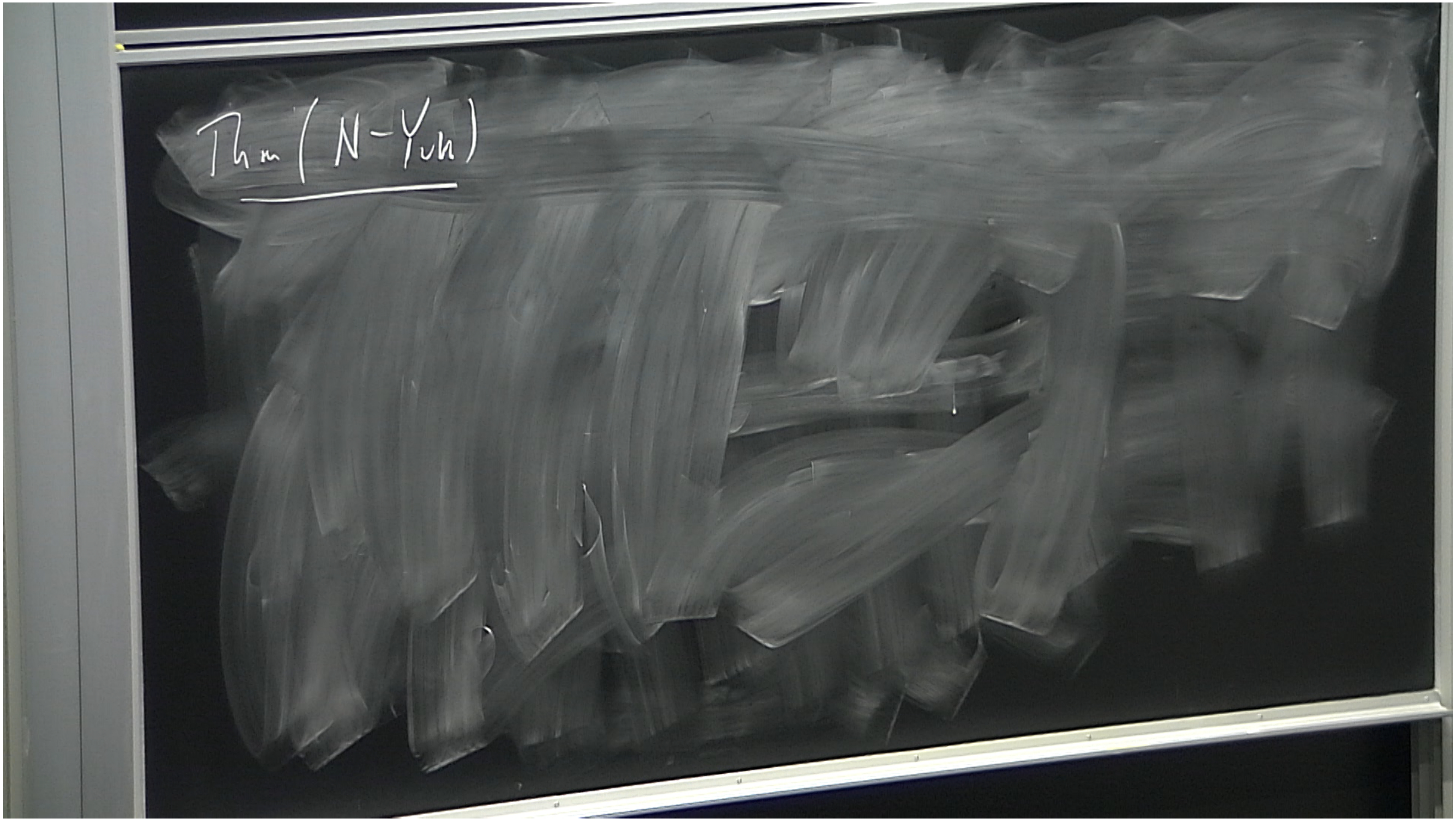
Cor Betti G.L conj in genus 1!


Thm (N-Yun)

Main Conj Limit diagram

$$\begin{tikzcd} Sh_{\mathcal{N}}(X) \arrow[r, "\Psi"] & Sh_{\mathcal{N}}(X_0) \arrow[l, bend right, "\Psi^L"'] \arrow[r, shift left] \arrow[r, shift right] & \cdots \end{tikzcd}$$

nearby cycles

Evidence

Thm (Ben-Zvi - Li - N) True in genus 1!
(Assume G adj ...?)

Cor Betti G.L conj in genus 1!

Thm (N-Yun)

$\oplus W$ ← Weyl sp.

1) $\Psi^L : Wh_G \rightsquigarrow Wh_G$

2) $\Psi^L : Eis_B \rightsquigarrow Eis_B$.

Back to application

Betti G.L makes sense for nodal [illegible] on non-N. surfaces.

Back to application

Betti G.L makes sense for real ... on non-N. surfaces.

A-side moduli: $X \circlearrowleft \alpha$ conjugation
$G \circlearrowleft \eta$ conjugation
$\rightsquigarrow \mathrm{Bun}_G(X) \circlearrowleft \alpha$ conj.

Consider real pts $\mathrm{Bun}_G(X)_{\mathbb{R}}$. topol stack.

$X/\langle\alpha\rangle$

Back to application

Betti G.L makes sense for real on non-N. surfaces.

A-side moduli: $\left.\begin{array}{l} X \circlearrowleft \alpha \text{ conjugation} \\ G \circlearrowleft \eta \text{ conjugation} \end{array}\right\} \rightsquigarrow Bun_G(X) \circlearrowleft \alpha$ conj.

Consider real pts $Bun_G(X)_{\mathbb{R}}$. topol stack.

$X/\langle\alpha\rangle$

Back to application

Betti G.L makes sense for real on non-N. surfaces.

A-side moduli: $\left.\begin{array}{l} X \circlearrowleft \alpha \text{ conjugation} \\ G \circlearrowleft \eta \text{ conjugation}\end{array}\right\} \rightsquigarrow Bun_G(X) \circlearrowleft \tilde{\alpha}$ conj.

Consider real pts $Bun_G(X)_{\mathbb{R}}$. topol stack.

Simplest case $X = \mathbb{P}^1$, $\alpha(z) = \bar{z}$ "$X/\langle\alpha\rangle = \mathbb{D}$ "

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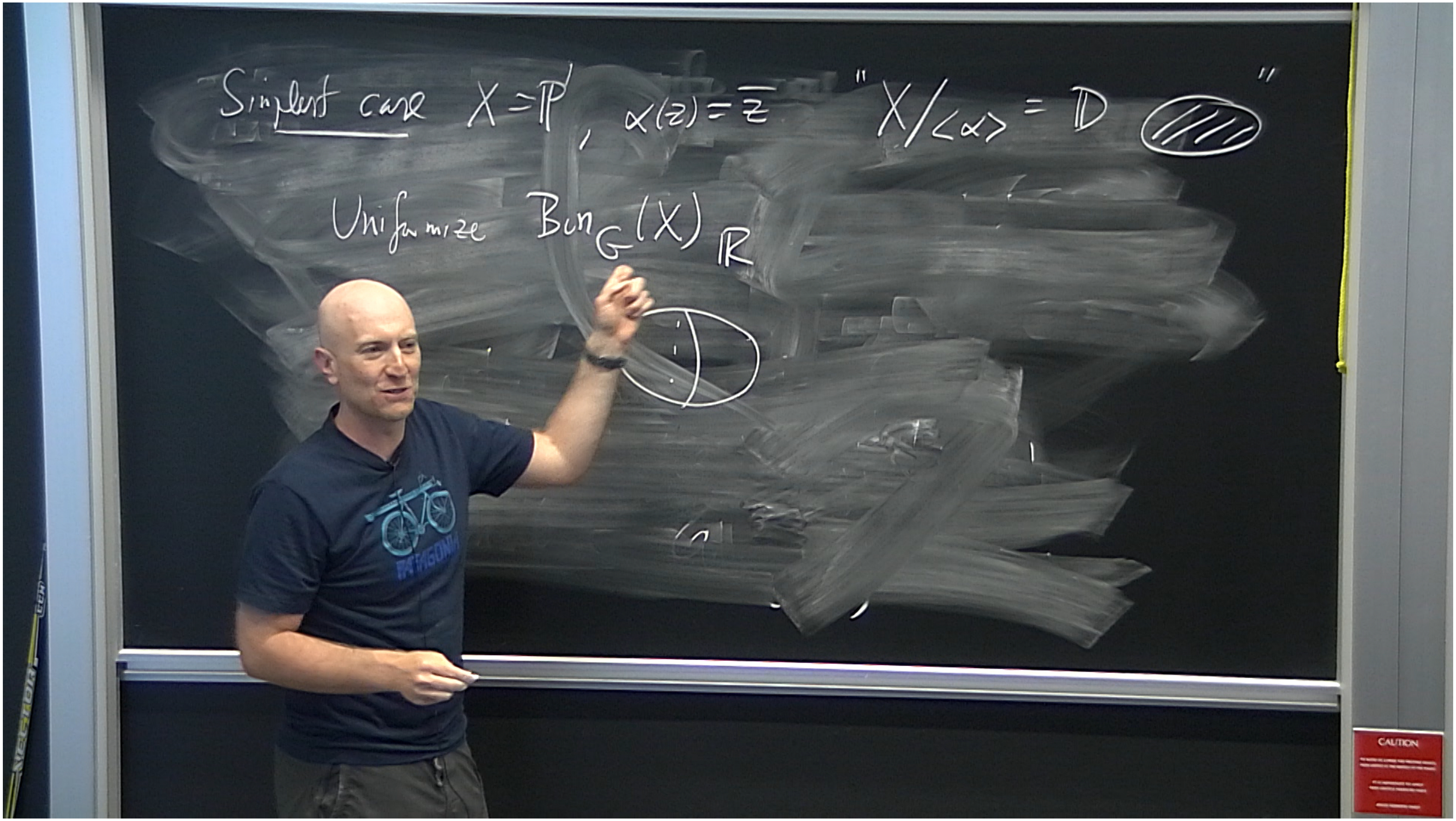
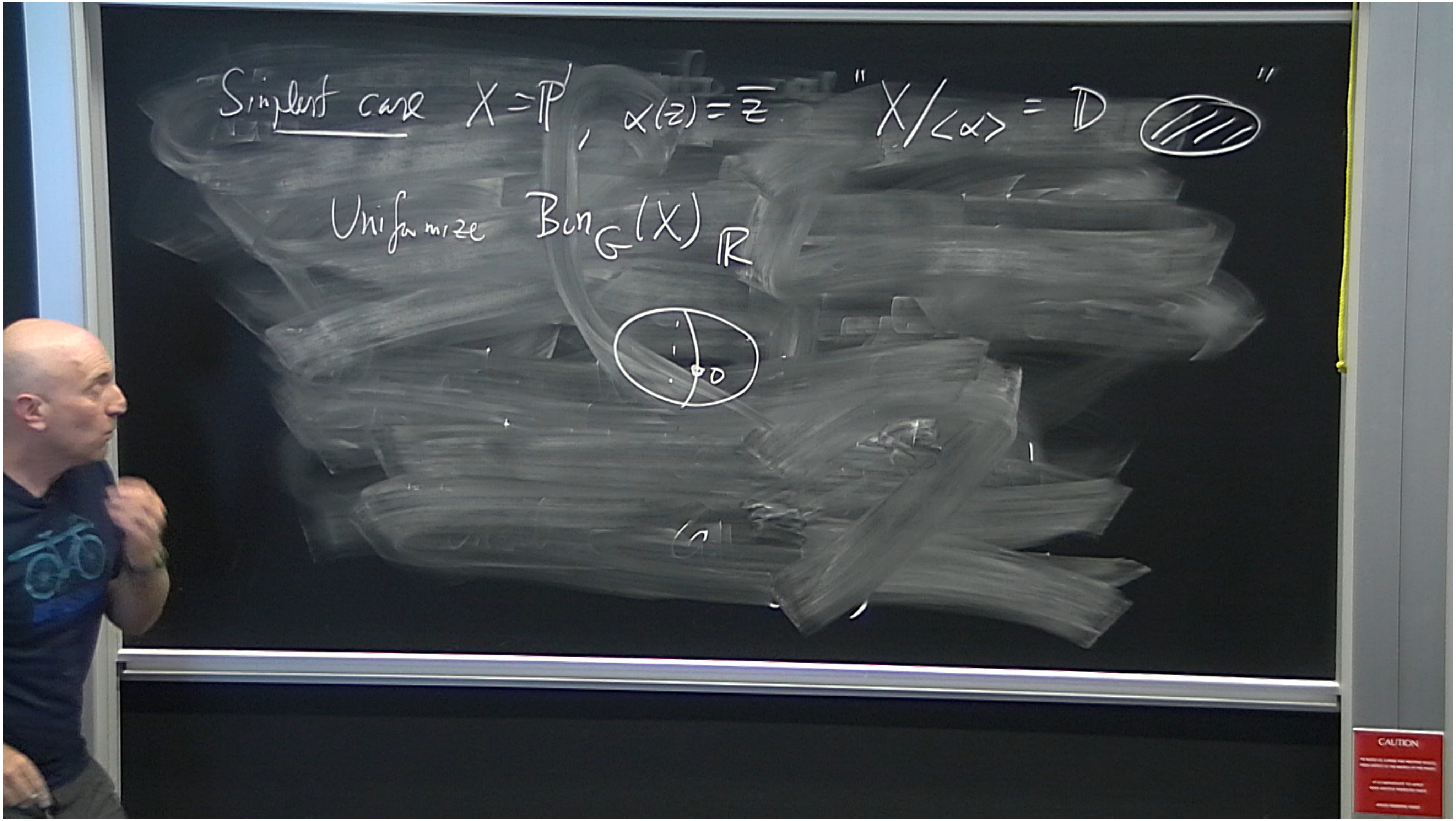
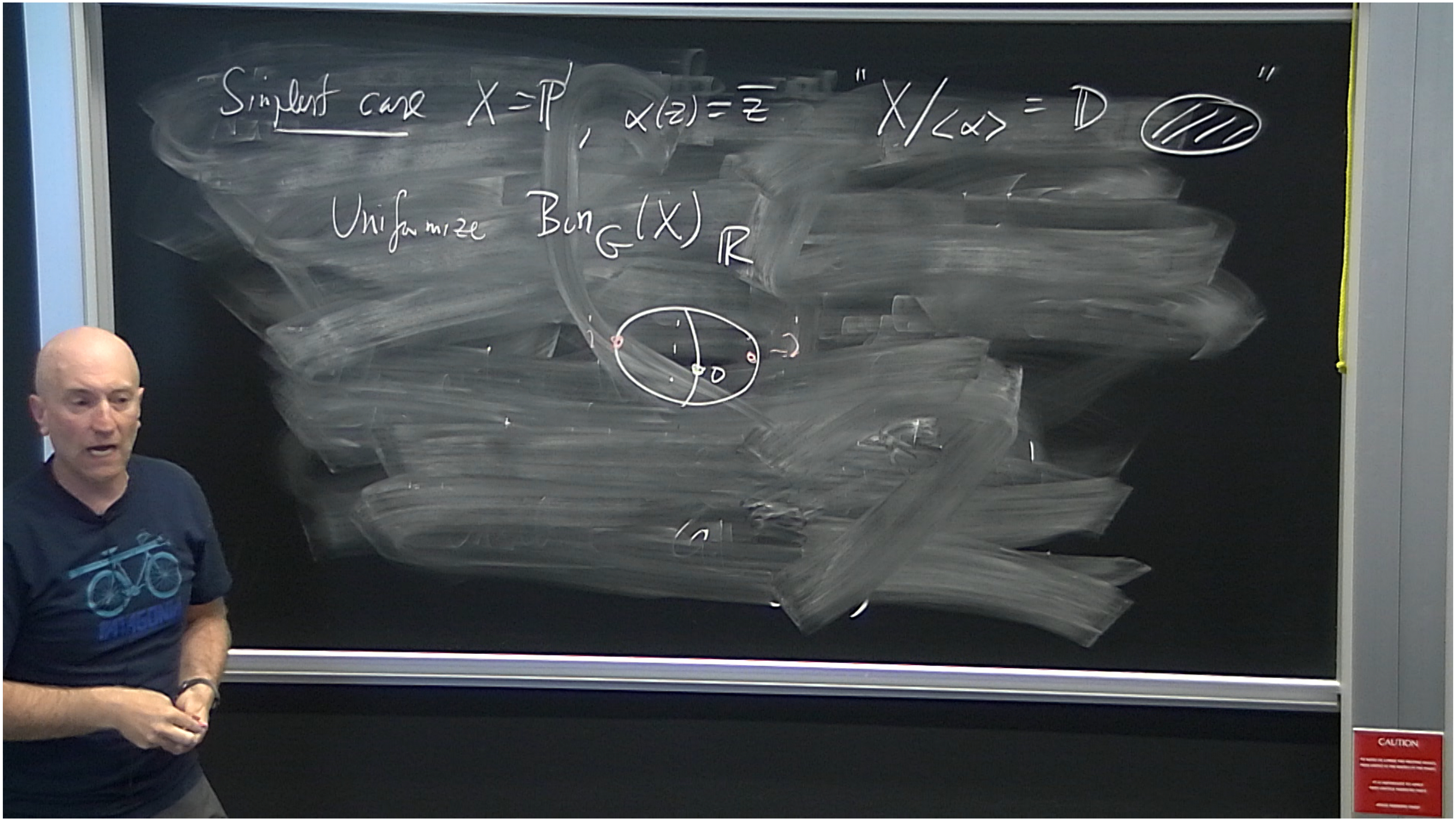
Uniformize $\mathrm{Bun}_G(X)_{\mathbb{R}}$


Simplest case X = P^1, α(z) = z̄
"X/⟨α⟩ = D"
Uniformize Bun_G(X)_R
CAUTION


Simplest case $X = \mathbb{P}^1$, $\alpha(z) = \bar{z}$
"$X/\langle \alpha \rangle = \mathbb{D}$"
Uniformize $\mathrm{Bun}_G(X)_{\mathbb{R}}$
0


Simplest case $X = \mathbb{P}^1$, $\alpha(z) = \bar{z}$
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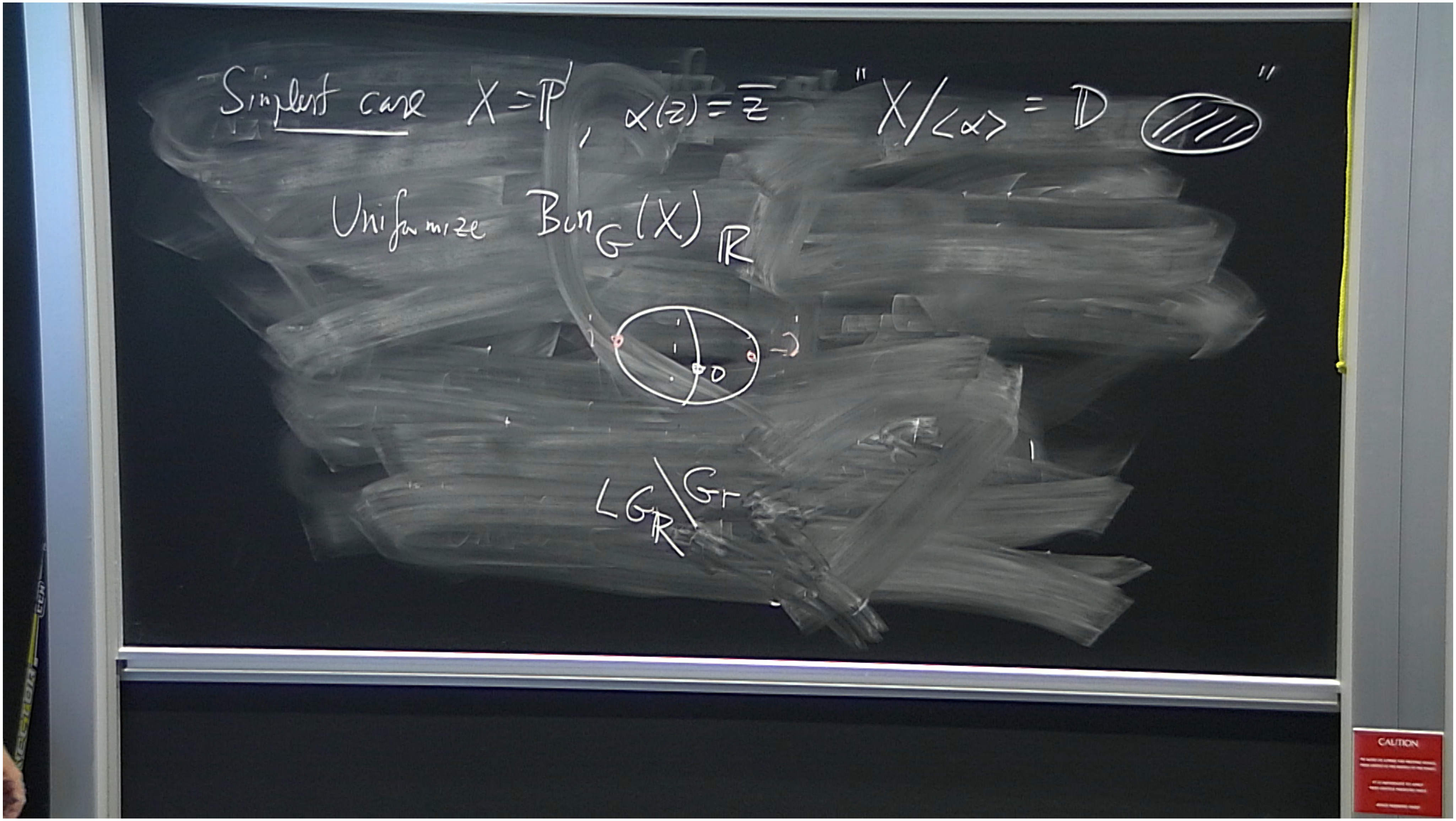
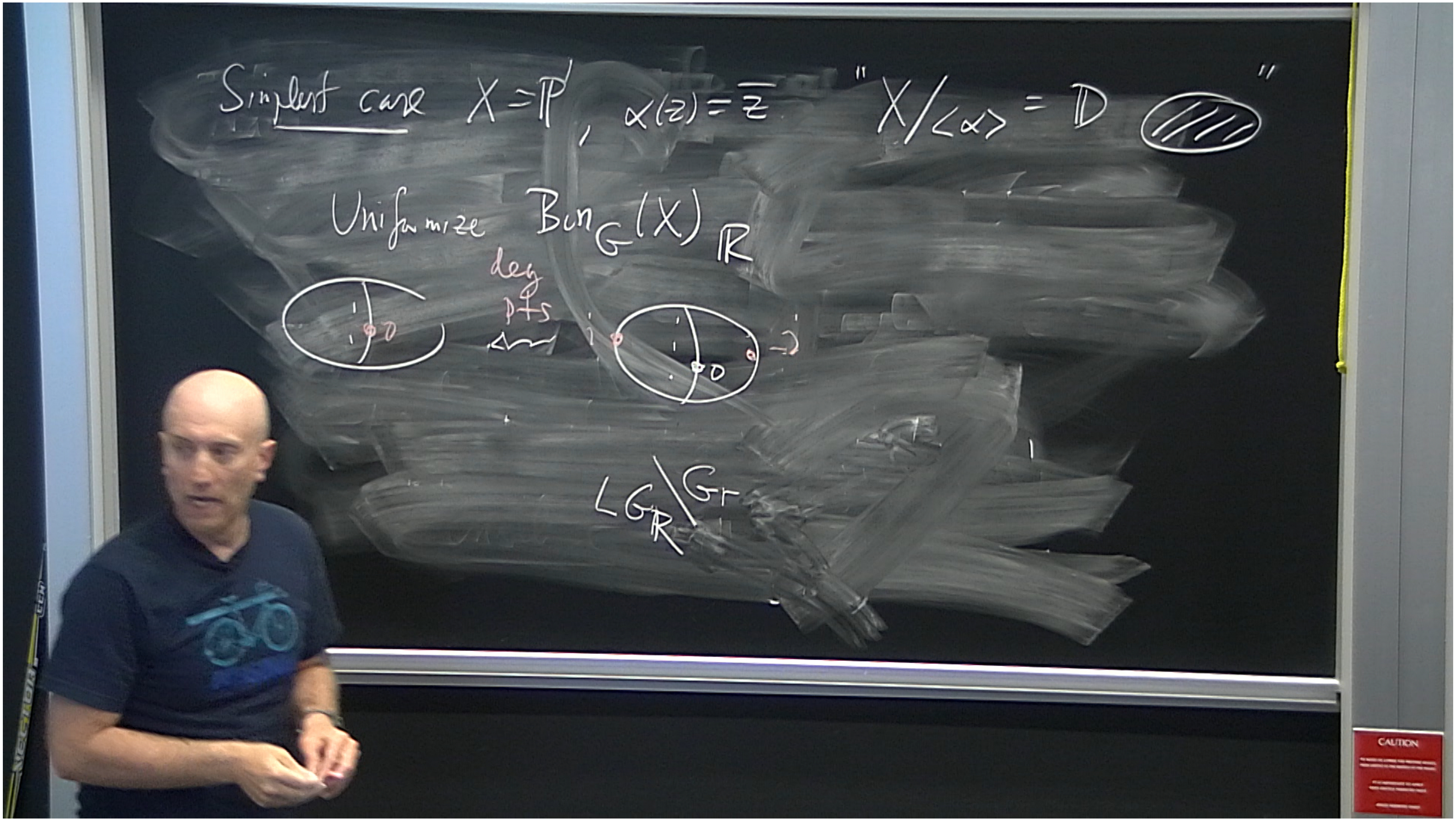
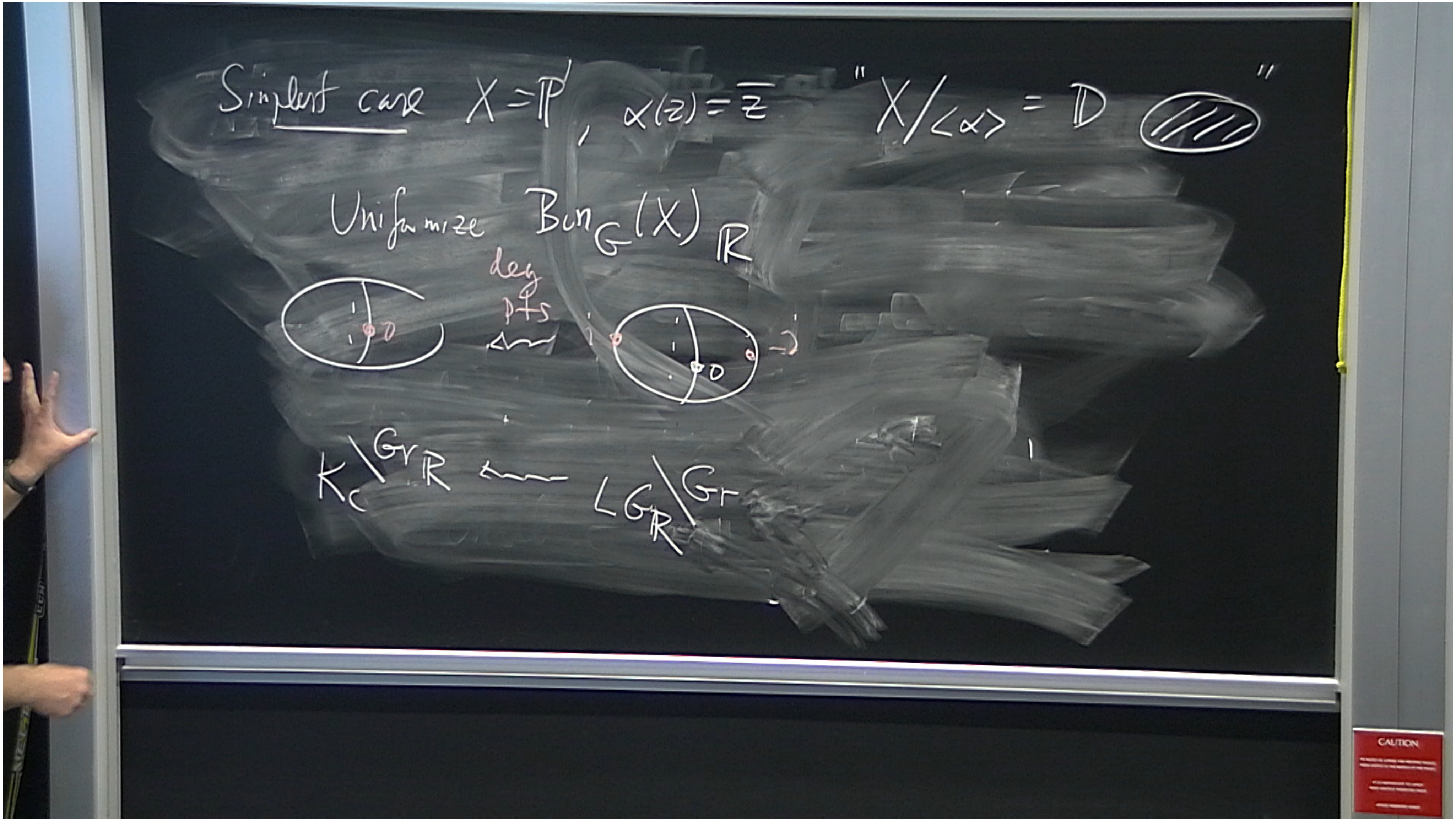
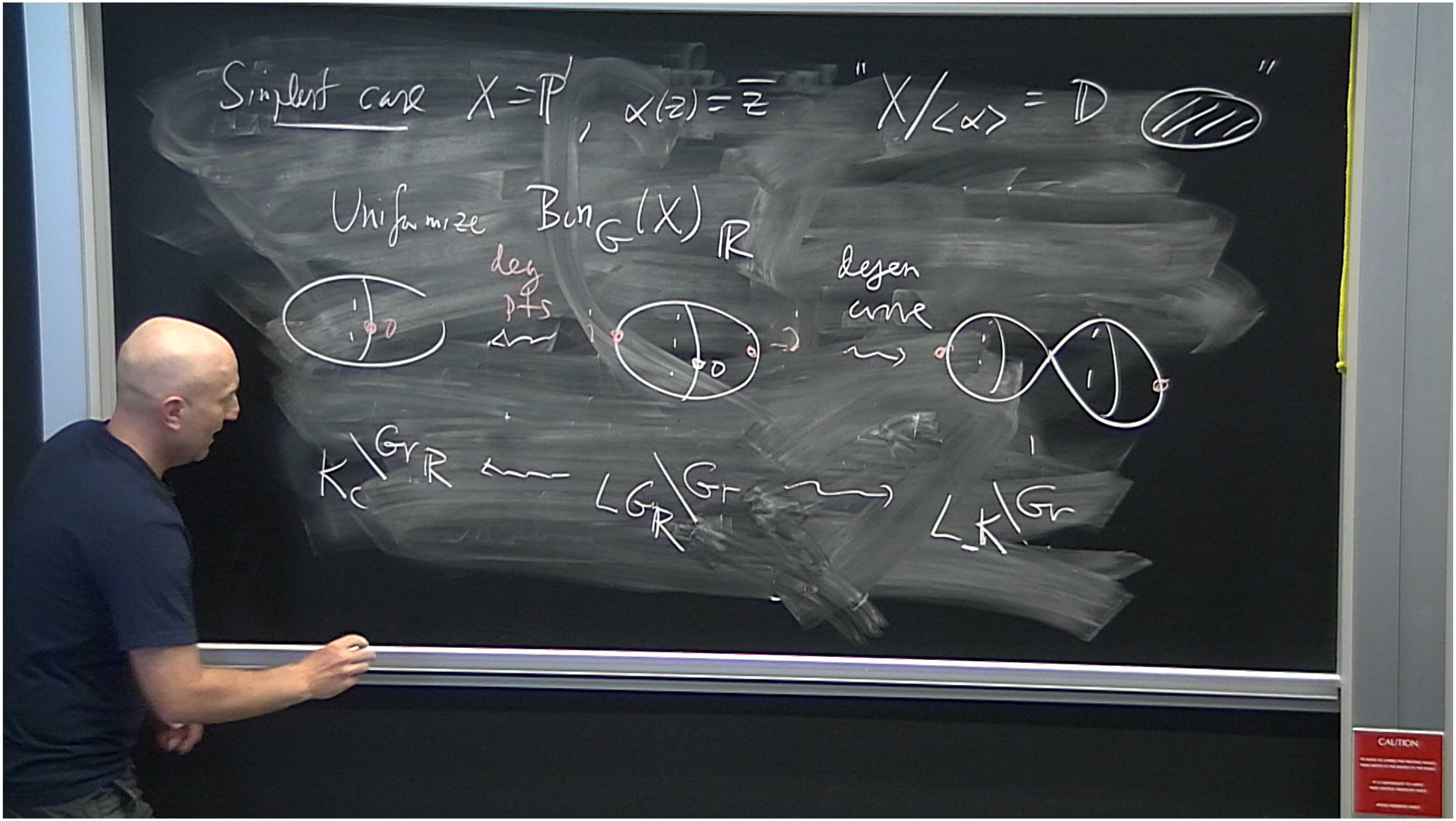
0
CAUTION


Simplest case $X = \mathbb{P}^1$, $\alpha(z) = \bar{z}$
" $X/\langle\alpha\rangle = \mathbb{D}$ "
Uniformize $\mathrm{Bun}_G(X)_{\mathbb{R}}$
0
$LG_{\mathbb{R}} \backslash Gr$
CAUTION

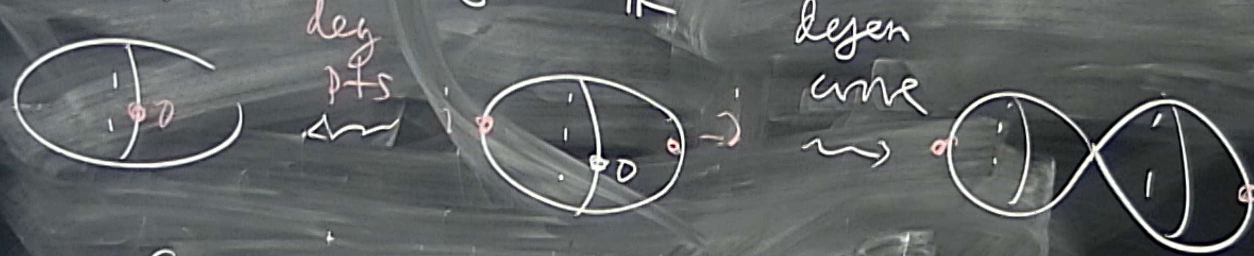

Simplest case $X = \mathbb{P}^1$, $\alpha(z) = \bar{z}$
" $X/\langle\alpha\rangle = \mathbb{D}$ "
Uniformize $\mathrm{Bun}_G(X)_{\mathbb{R}}$
deg
pts
$LG_{\mathbb{R}} \backslash Gr$


Simplest case X = P^1, α(z) = z̄
"X/⟨α⟩ = D"
Uniformize Bun_G(X)_R
deg pts
K_c\Gr_R
LG_R\Gr


Simplest case $X = \mathbb{P}^1$, $\alpha(z) = \bar{z}$
" $X/\langle\alpha\rangle = \mathbb{D}$ "
Uniformize $Bun_G(X)_{\mathbb{R}}$
deg pts
degen curve
$K_{\mathbb{C}} \backslash Gr_{\mathbb{R}}$
$LG_{\mathbb{R}} \backslash Gr$
$L_K \backslash Gr$

Simplest case $X = \mathbb{P}^1$, $\alpha(z) = \bar{z}$ "$X/\langle\alpha\rangle = \mathbb{D}$"

Uniformize $\mathrm{Bun}_G(X)_{\mathbb{R}}$



$$\begin{array}{ccccc} K_{\mathbb{C}}\backslash Gr_{\mathbb{R}} & \leftsquigarrow & LG_{\mathbb{R}}\backslash Gr & \rightsquigarrow & L_{-}K\backslash Gr \\ \cup & & & & \cup \\ N_{\mathbb{R}} & & & & N_{p} \end{array}$$