Title: A compositional approach to quantum functions, and the Morita theory of quantum graph isomorphisms

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Abstract: Certain nonlocal games exhibiting quantum advantage, such as the quantum graph homomorphism and isomorphism games, have composable quantum strategies which are naturally interpreted as structure-preserving functions between finite sets. We propose a natural compositional framework for noncommutative finite set theory in which these quantum strategies appear naturally, and which connects nonlocal games with recent work on compact quantum groups. We apply Morita-theoretical machinery within this framework to characterise, classify, and construct quantum strategies for the graph isomorphism game. This is joint work with Benjamin Musto and David Reutter, based on the papers 1711.07945 and 1801.09705.

A compositional approach to quantum functions, and the Morita theory of quantum graph isomorphisms

Dominic Verdon[1,2]
(with Benjamin Musto[1] and David Reutter[1])

University of Oxford[1] / University of Bristol[2]

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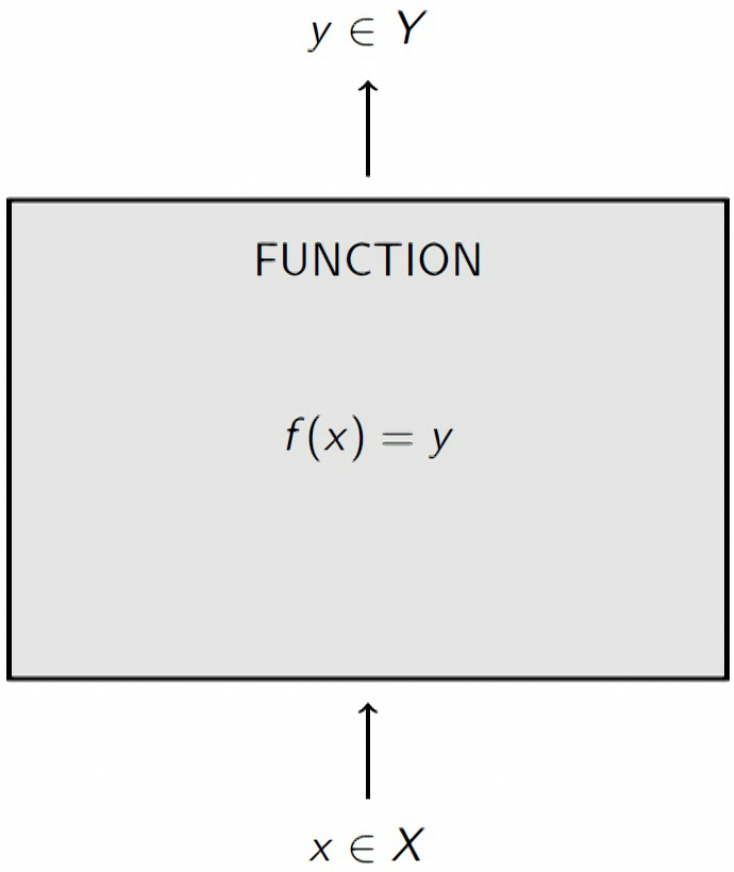
Papers

1. Benjamin Musto, David Reutter & DV, "A compositional theory of quantum functions". To appear in *J. Math. Phys.*, 2018. (arxiv:1711.07945).
2. Benjamin Musto, David Reutter & DV, "The Morita theory of quantum graph isomorphisms". To appear in *Comm. Math. Phys.*, 2018. (arxiv:1801.09705)

Foundational interest

- New mathematical tools.
- A connection between 'quantisation' and 'categorification'.
- New 'non-abelian' contextuality constructions.
- Clarifies compositional aspects of quantum functions.

What is a quantum function?



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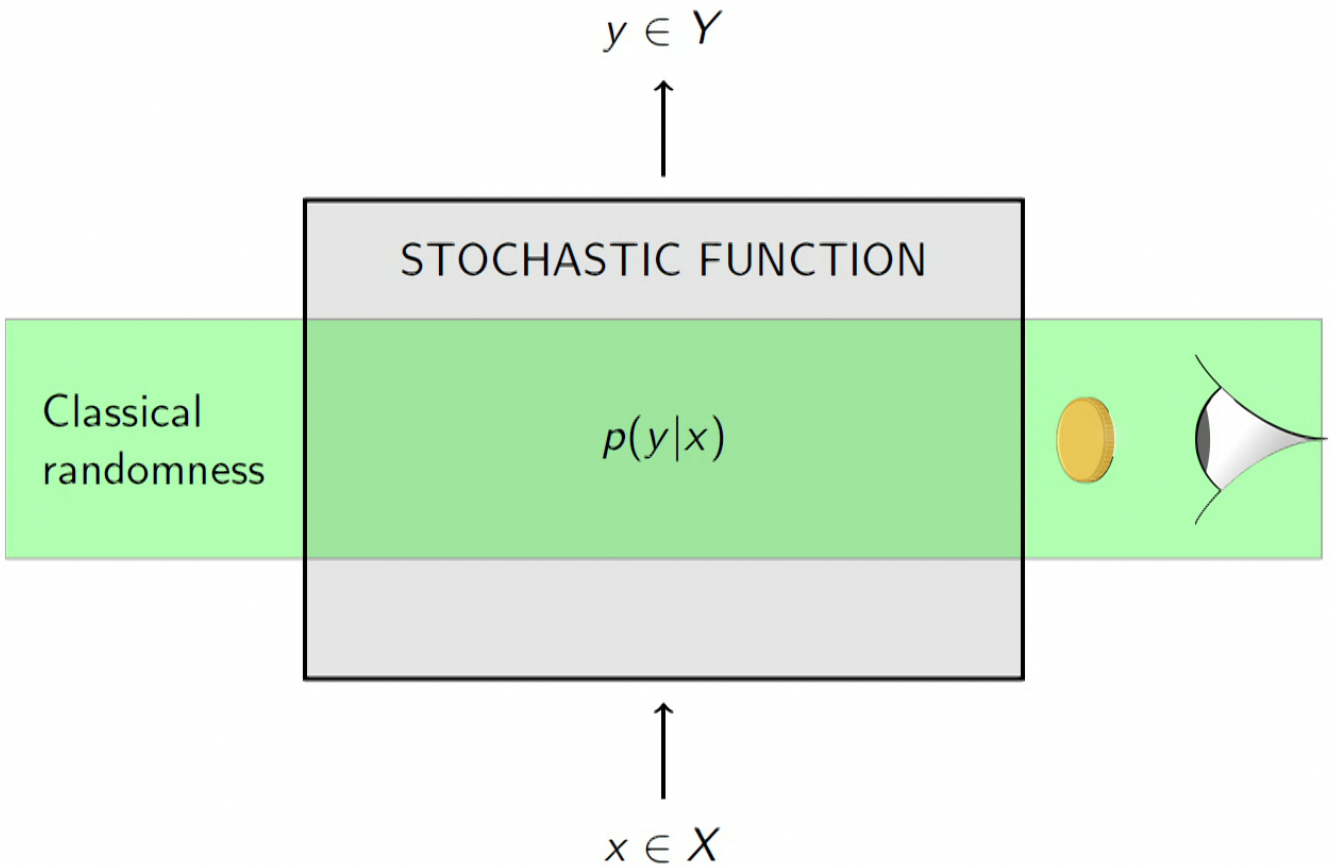

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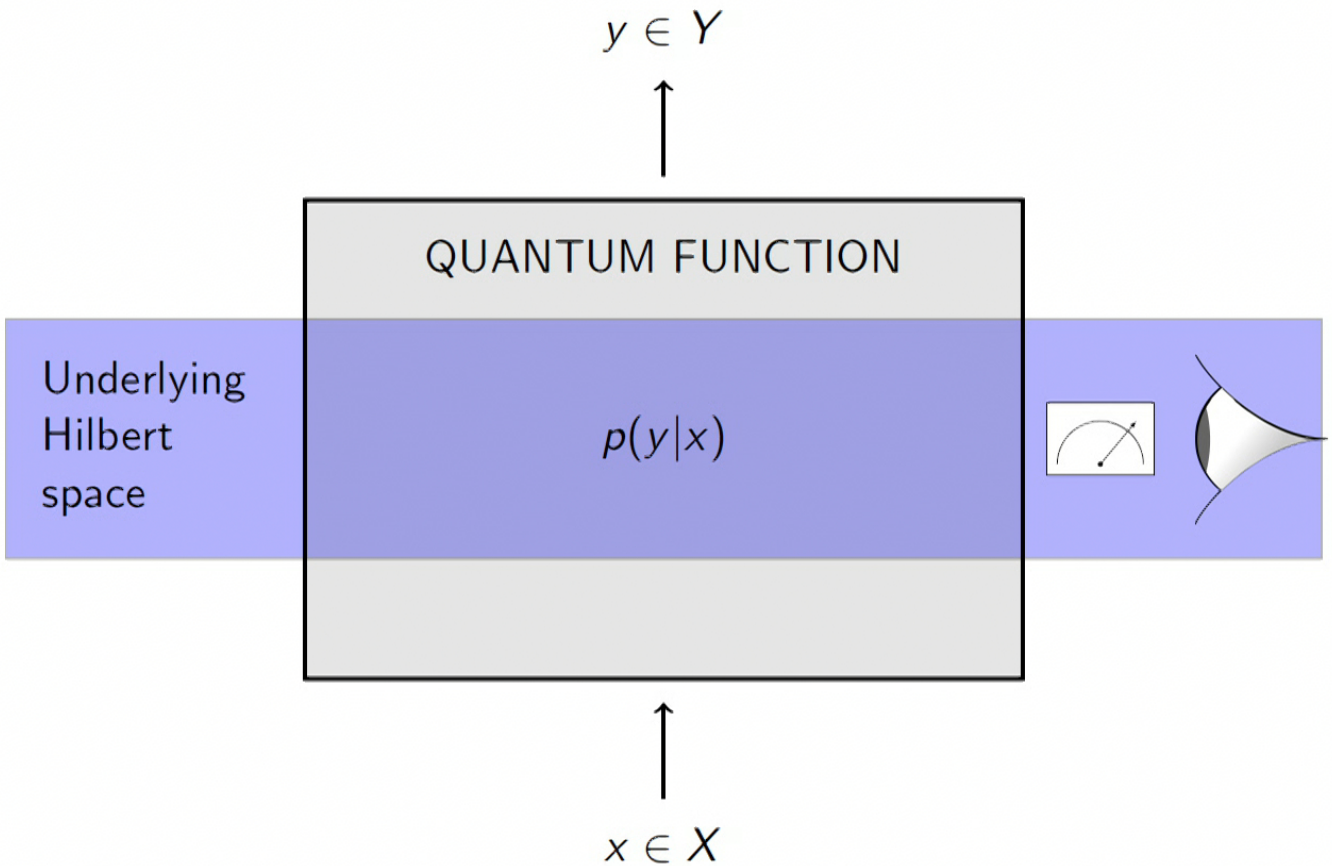


Gelfand duality

- Problem: sets and Hilbert spaces live in different categories.
- Solution: embed theory of sets and functions in the category of Hilbert spaces and linear maps.

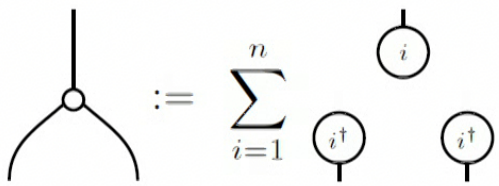
$$\begin{array}{ll} \text{commutative f.d. } C*\text{-algebras and homomorphisms} & \\ \leftrightarrow^* & \text{finite sets and functions} \end{array}$$

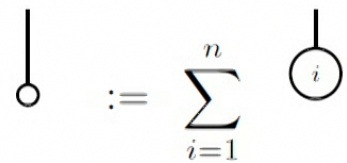
$$\begin{array}{ll} \text{f.d. } C*\text{-algebras and homomorphisms} & \leftrightarrow^* \\ \text{'finite quantum sets' and 'functions'} & \end{array}$$

- For finite sets we can formulate all this using Frobenius algebras.

Diagrammatic Gelfand duality

- A set X is identified with an orthonormal basis of a vector space of dimension $|X|$.
- The algebra structure compares the basis elements:





$$m : |i\rangle \otimes |j\rangle \mapsto \delta_{i,j} |i\rangle$$

$$u : 1 \mapsto \sum_{i=1}^{n} |i\rangle$$

[1] Bob Coecke, Duško Pavlović, and Jamie Vicary. "A new description of orthogonal bases". In: *Mathematical Structures in Computer Science* (2009). DOI: 10.1017/s0960129512000047. eprint: 0810.0812.

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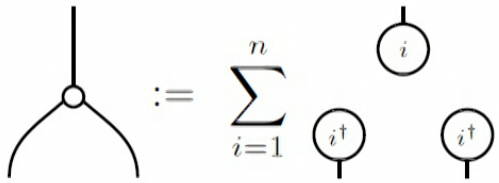

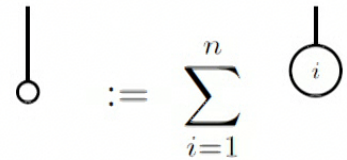

$$m : |i\rangle \otimes |j\rangle \mapsto \delta_{i,j} |i\rangle \qquad u : 1 \mapsto \sum_{i=1}^{n} |i\rangle$$

- Together with their adjoints, these maps give V_X the structure of a *commutative special dagger Frobenius algebra* (CSFA).[1]

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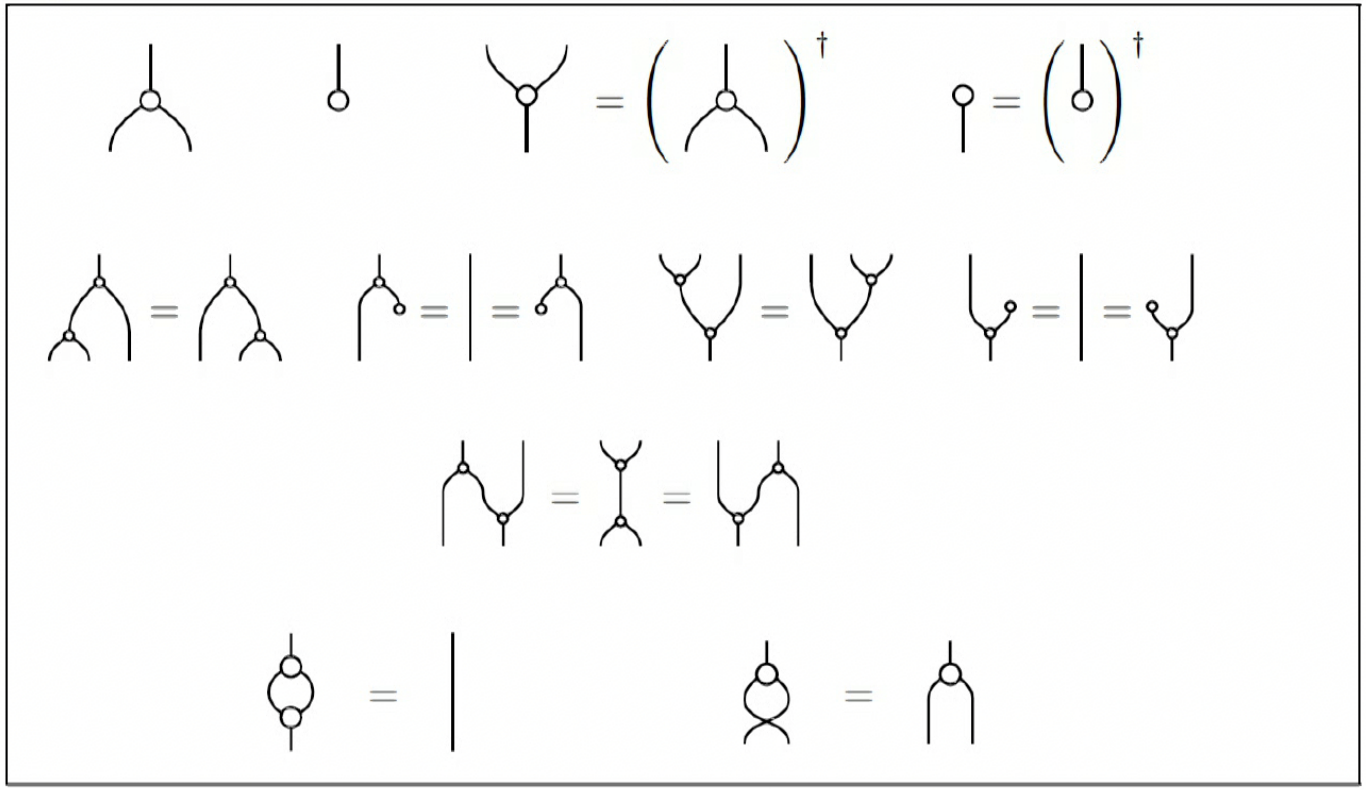

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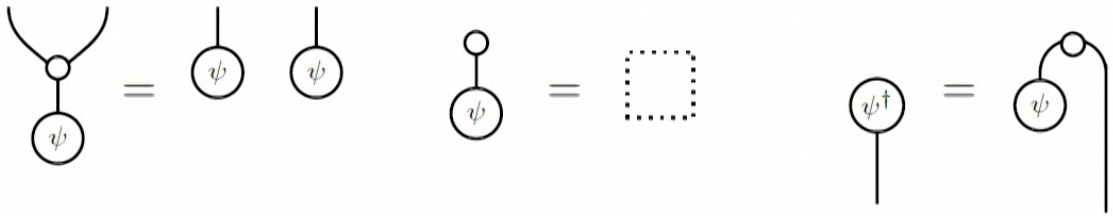
Diagrammatic Gelfand duality

- The elements of the set can be recovered as the *copyable states* $\psi : \mathbb{C} \to A$ of the SCFA:



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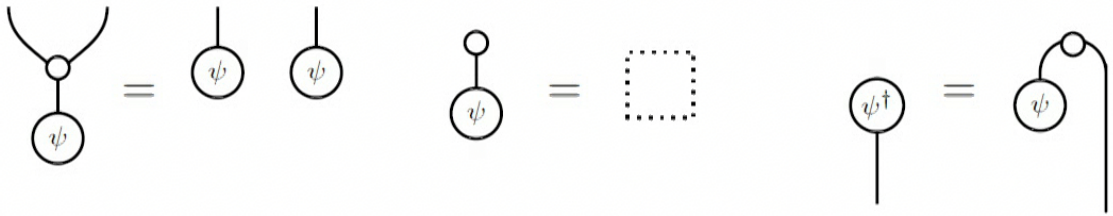
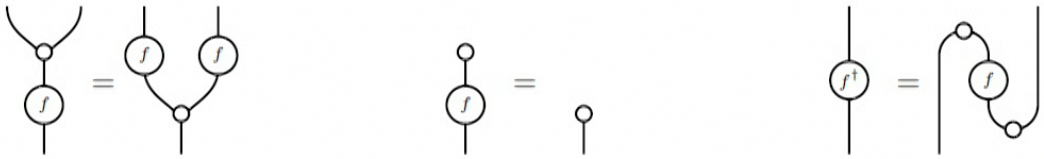


- Functions correspond to $*$-*cohomomorphisms*, which map elements to elements:



Diagrammatic Gelfand duality

- Generalise to *quantum sets* by moving from *commutative* to *symmetric* special dagger Frobenius algebras:

[2] Jamie Vicary. "Categorical Formulation of Finite-Dimensional Quantum Algebras". In: *Communications in Mathematical Physics* 304.3 (2010), pp. 765–796. DOI: 10.1007/s00220-010-1138-0. eprint: 0805.0432.

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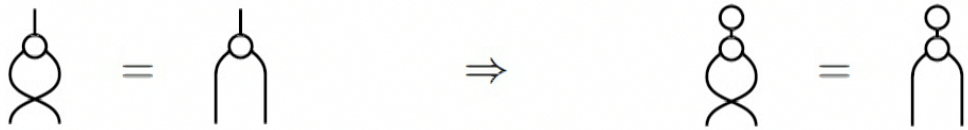

- This is all just a reformulation of the C^*-algebraic version.[2]

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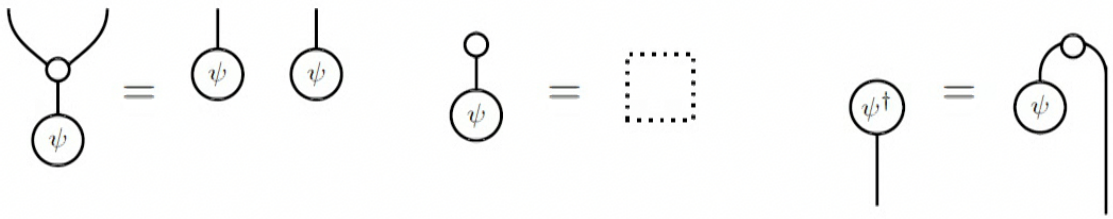
Quantum elements

- Identify elements with copyable states $\mathbb{C} \to A$:



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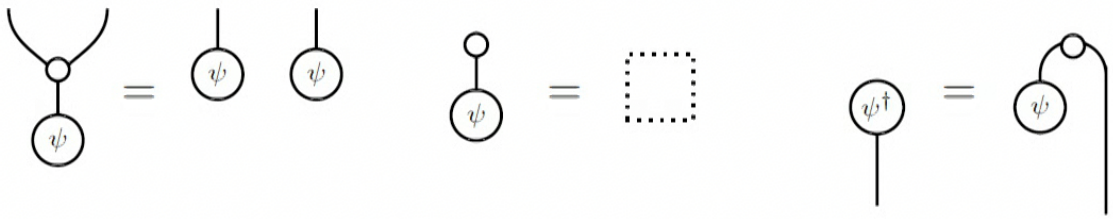
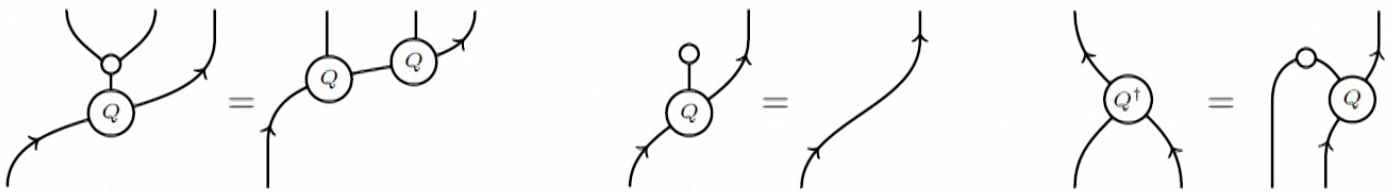
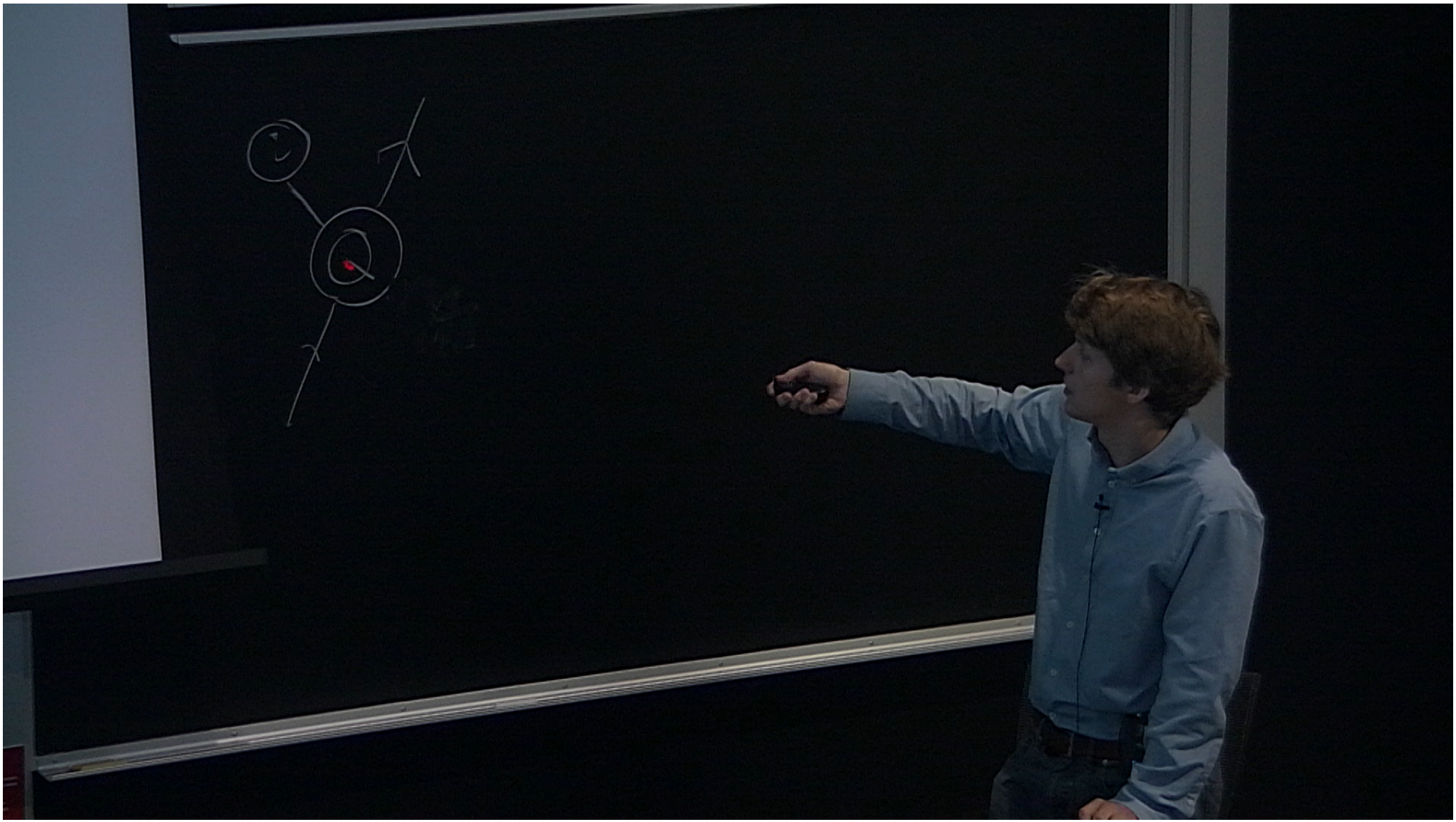


- Define a *quantum element* as a map $H \to A$ for a Hilbert space H:





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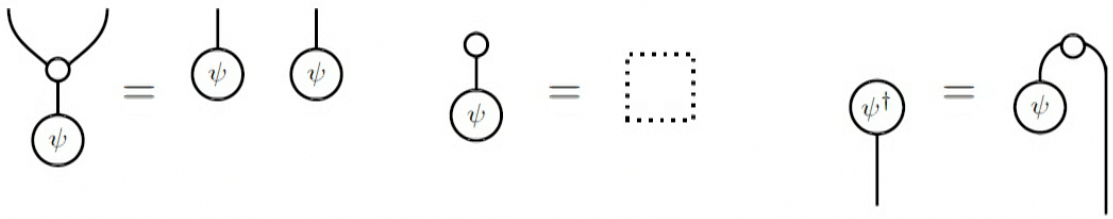

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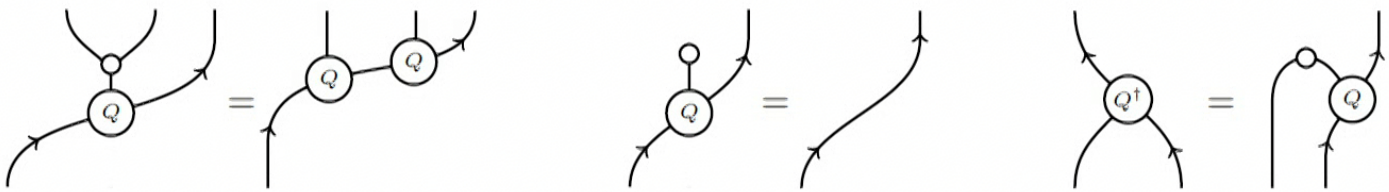


Quantum elements of classical sets

Theorem

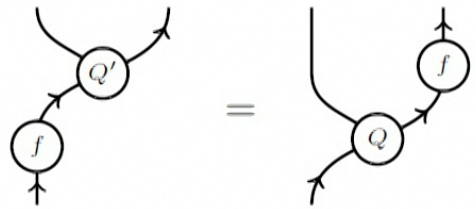
Quantum elements of a classical set X are precisely projective measurements with outcomes in X.

Proof.

Consider the maps $Q_i : H \to H$ corresponding to output i. The first equation gives idempotency and orthogonality of the $\{Q_i\}$, the second completeness, and the third that the projectors Q_i are Hermitian. □

A category of quantum elements

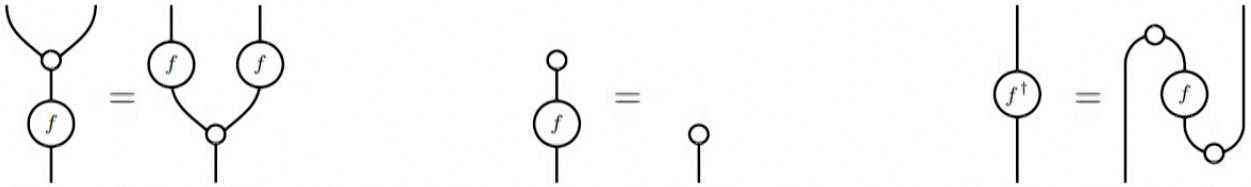
- More generally, quantum elements of A are comodules of A.
- Quantum elements have *intertwiners*:



- The *set* of classical elements becomes a *category* $\mathrm{QEl}(A)$ of quantum elements:
  - Objects are quantum elements.
  - Morphisms are intertwiners.
- The classical elements live in $\mathrm{QEl}(A)$ as the one-dimensional quantum elements.
- The quantum set A can be recovered from $\mathrm{QEl}(A)$ and the forgetful functor $F : \mathrm{QEl}(A) \to$ **Hilb**.

Quantum functions

- Identify functions with $*$-cohomomorphisms $f : A \to B$:



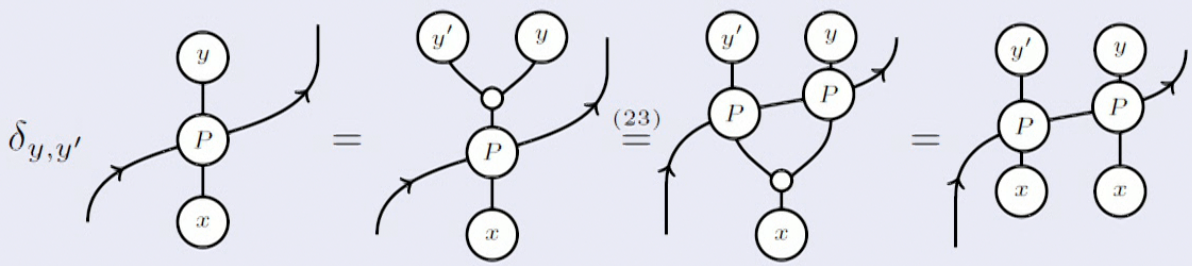
Quantum functions between classical sets

Theorem

A quantum function $X \to Y$ between classical sets X and Y is precisely a controlled projective measurement.

Proof.

For example, the first equation, expanded in the classical basis, specifies orthogonality and idempotency of the projectors for each control x:



□

Quantum functions between classical sets

- Every quantum function $P : X \to Y$ can be considered as a matrix of projectors $\{P_{x,y}\}_{x \in X, y \in Y}$ on a Hilbert space H, where the following holds for all $x \in X$ and $y_1, y_2 \in Y$:

$$P_{x,y_1} P_{x,y_2} = \delta_{y_1,y_2} P_{x,y_1} \qquad \sum_{y \in Y} P_{x,y} = \mathbb{1}_H$$

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- Composition is normal matrix multiplication where the product of entries is tensor product.

Quantum functions between quantum sets

- Sołtan: quantum set of quantum functions $[A, B]$ is the internal Hom $[A, B]$ in the category C^*-Alg^{op}.[3]

Theorem

The category of quantum functions $A \to B$ is the opposite of the category of finite dimensional C^*-representations of $[A, B]$.

- Quantum elements are the case $[\mathbb{C}, B]$.
- In general, can't recover $[A, B]$ from the category of quantum functions.

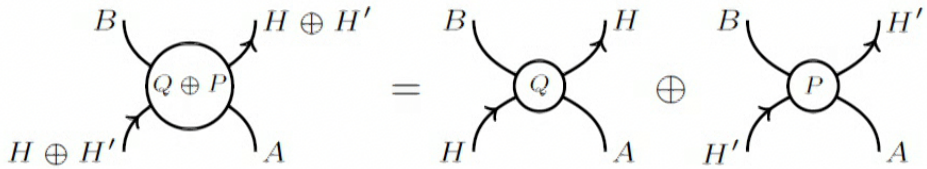
[3] Piotr M. Sołtan. "Quantum families of maps and quantum semigroups on finite quantum spaces". In: *Journal of Geometry and Physics* 59.3 (2009), pp. 354–368. DOI: 10.1016/j.geomphys.2008.11.007. eprint: 0610922.

Interval

- We've defined quantum elements and quantum functions using Gelfand duality and an operational principle.
- We've shown that they form *categories* rather than sets.
- Concretely
  - Quantum functions between classical sets are controlled projective measurements.
  - Quantum functions between quantum sets are representations of the internal Hom in the category of quantum sets.
- Next:
  - Investigate the *semisimple* structure of the categories of quantum functions $A \to B$.
  - Show how they fit into a 2-category of quantum sets, quantum functions and intertwiners.

The direct sum of quantum functions

- There is a *direct sum* of quantum functions:



- For a quantum function between classical sets, the direct sum is equivalent to:

$$(P \oplus Q)_{a,b} = P_{a,b} \oplus Q_{a,b}$$

- Operationally, the direct sum is a 'mixing' of quantum functions. TFAE:
  - Perform the quantum measurement $(H \oplus H', [P \oplus Q]_{a,-})$ depending on a received input a.

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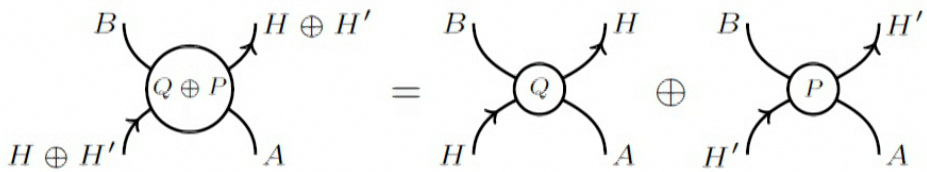

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  - Before receiving any input, perform a projective measurement onto H and H', and then, depending on the outcome, perform the quantum measurement $(H, P_{a,-})$ or $(H', Q_{a,-})$ upon receiving the input a.

Semisimplicity

- *Simple* quantum functions P cannot be decomposed as a direct sum of two nonzero quantum functions. Equivalently, $\mathrm{QSet}(P, P) \cong \mathbb{C}$.
- For classical sets, this reduces to:

$$\forall X \in \mathrm{End}(H): \qquad P_{a,b}X = XP_{a,b} \;\; \forall a \in A, b \in B \;\; \Rightarrow \;\; X \propto \mathbb{1}_H$$

Theorem (Semisimplicity of $\mathrm{QSet}(A, B)$)

Every quantum function in $\mathrm{QSet}(A, B)$ can be decomposed as a finite direct sum of simple quantum functions.

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- This completely characterises the intertwiners in $\mathrm{QSet}(A, B)$.
- In general, there are *infinitely* (uncountably!) many simple quantum functions $A \to B$.

Essential classicality

- Semisimplicity allows us to derive a notion of classicality of a quantum function between classical sets.

Definition

A quantum function $X \to Y$ is *essentially classical* if it is a direct sum of classical (i.e. one-dimensional) functions.

- This is equivalent to commutation of all the projectors $P_{x,y}$.
- In an essentially classical quantum function, the underlying Hilbert space is just being used to generate *classical* randomness.
- A projective measurement is essentially classical.

Example

$$\begin{pmatrix} |0\rangle\langle 0| & |1\rangle\langle 1| & |2\rangle\langle 2| \\ |1\rangle\langle 1| & |2\rangle\langle 2| & |0\rangle\langle 0| \\ |2\rangle\langle 2| & |0\rangle\langle 0| & |1\rangle\langle 1| \end{pmatrix}$$

Under the decomposition $\mathbb{C}^3 = \mathbb{C}\,|0\rangle \oplus \mathbb{C}\,|1\rangle \oplus \mathbb{C}\,|2\rangle$, this is a sum of permutations:

$$\begin{pmatrix} |0\rangle\langle 0| & |1\rangle\langle 1| & |2\rangle\langle 2| \\ |1\rangle\langle 1| & |2\rangle\langle 2| & |0\rangle\langle 0| \\ |2\rangle\langle 2| & |0\rangle\langle 0| & |1\rangle\langle 1| \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Non-example

$$\begin{pmatrix} |0\rangle\langle 0| & |1\rangle\langle 1| & |2\rangle\langle 2| & |3\rangle\langle 3| \\ |\psi_-\rangle\langle\psi_-| & |\phi_+\rangle\langle\phi_+| & |\phi_-\rangle\langle\phi_-| & |\psi_+\rangle\langle\psi_+| \\ |\psi_+\rangle\langle\psi_+| & |\phi_-\rangle\langle\phi_-| & |\phi_+\rangle\langle\phi_+| & |\psi_-\rangle\langle\psi_-| \\ |3\rangle\langle 3| & |2\rangle\langle 2| & |1\rangle\langle 1| & |0\rangle\langle 0| \end{pmatrix}$$

where

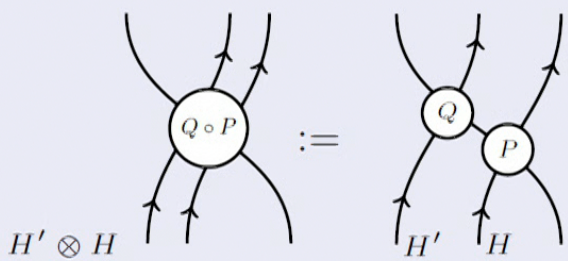
$$\begin{aligned} |\psi_+\rangle &= \tfrac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \\ |\psi_-\rangle &= \tfrac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right) \\ |\phi_+\rangle &= \tfrac{1}{\sqrt{5}}\left(i\,|0\rangle + 2\,|3\rangle\right) \\ |\phi_-\rangle &= \tfrac{1}{\sqrt{5}}\left(2\,|0\rangle + i\,|3\rangle\right) \end{aligned}$$

QSet

Definition (The 2-category QSet)

- **objects** are quantum sets $A, B, \ldots$;
- **1-morphisms** $A \to B$ are quantum functions $(H, P) : A \to B$;
- **2-morphisms** $(H, P) \to (H', P')$ are intertwiners of quantum functions.

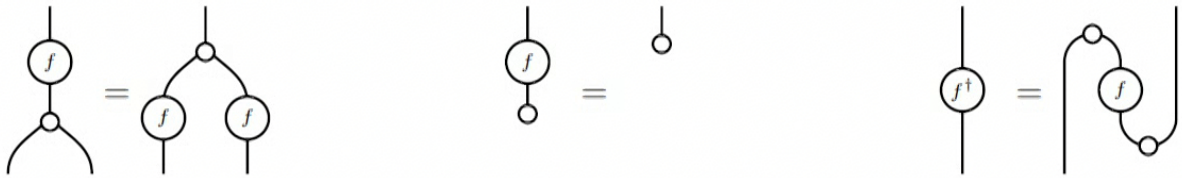
Two quantum functions $(H, P) : A \to B$ and $(H', Q) : B \to C$ compose to give $(H' \otimes H, Q \circ P)$:



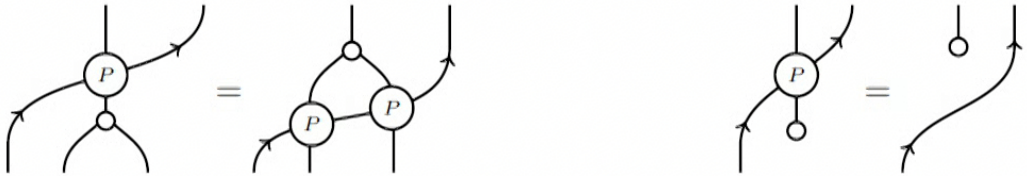
Can compose 2-morphisms by ordinary composition, and by tensor product of linear maps.

Quantum bijections

- What's the quantisation of a bijection?
- A bijection is a $*$-cohomomorphism obeying the following additional equations:



- Add a Hilbert space wire to obtain additional equations making a quantum function into a *quantum bijection*:

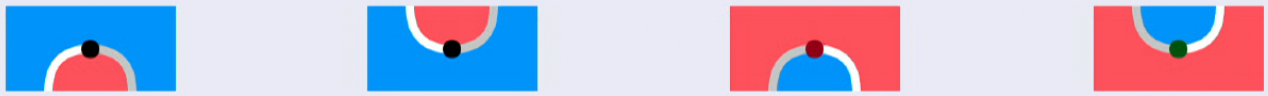


- In what sense is this invertible?

Invertibility: Equivalences

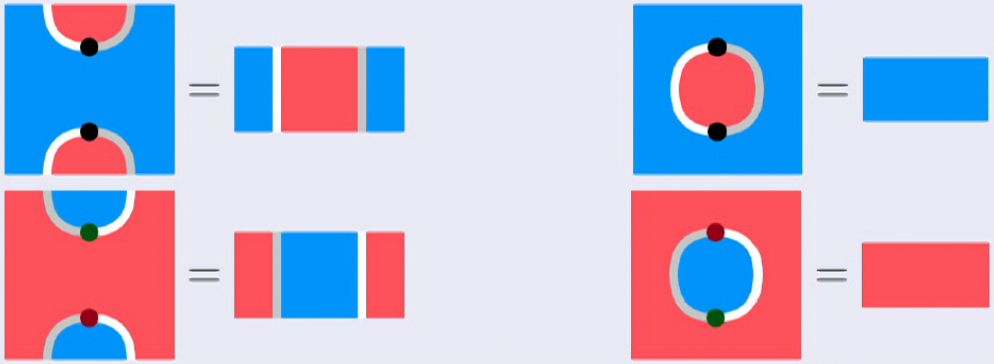
Definition (Equivalence)

A 1-morphism $f : A \to B$ is an *equivalence* if there exist $g : B \to A$ and



$\alpha : g \circ f \to \mathrm{id}_A$ $\quad \alpha^{-1} : \mathrm{id}_A \to g \circ f$ $\quad \beta : f \circ g \to \mathrm{id}_B$ $\quad \beta^{-1} : \mathrm{id}_B \to f \circ g$

such that α, α^{-1} and β, β^{-1} are inverses:



Invertibility: Equivalences

Theorem

Equivalences between quantum sets in QSet *are* $*$*-isomorphisms.*

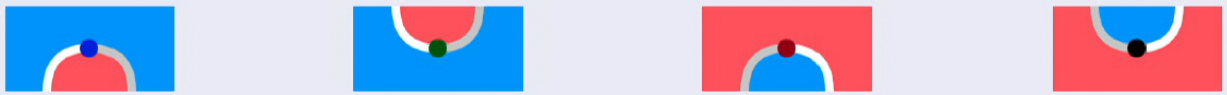
Proof.

Let $Q : H \otimes A \to B \otimes H$ be a quantum function. Then for equivalence there must exist $Q^{-1} : H' \otimes B \to A \otimes H'$ and an invertible 2-morphism from $Q^{-1} \circ Q : (H \otimes H') \otimes A \to A \otimes (H \otimes H')$ to $\mathrm{id}_A : A \to A$. But the 2-morphisms are just intertwining linear maps $H \times H' \to \mathbb{C}$. For such a map to be invertible implies that $H \times H'$ is one-dimensional, which implies that H and H' are also one-dimensional. $\square$

Invertibility: Dualisable morphisms

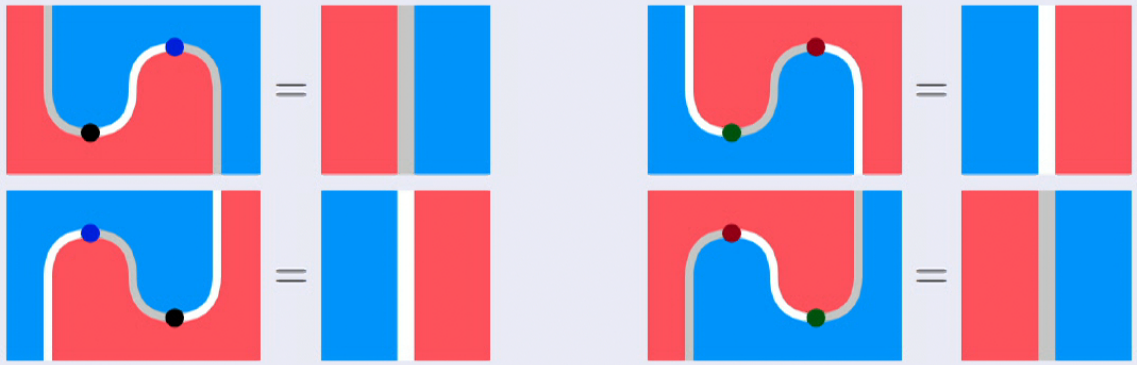
Definition (Dualisability)

A 1-morphism $f : A \to B$ is *dualisable* (aka an *adjunction*) if there exist $g : B \to A$ and



$\alpha : g \circ f \to \mathrm{id}_A \quad \alpha^* : \mathrm{id}_A \to g \circ f \quad \beta : f \circ g \to \mathrm{id}_B \quad \beta^* : \mathrm{id}_B \to f \circ g$

such that $\alpha, \alpha^*, \beta, \beta^*$ obey the *snake equations*:

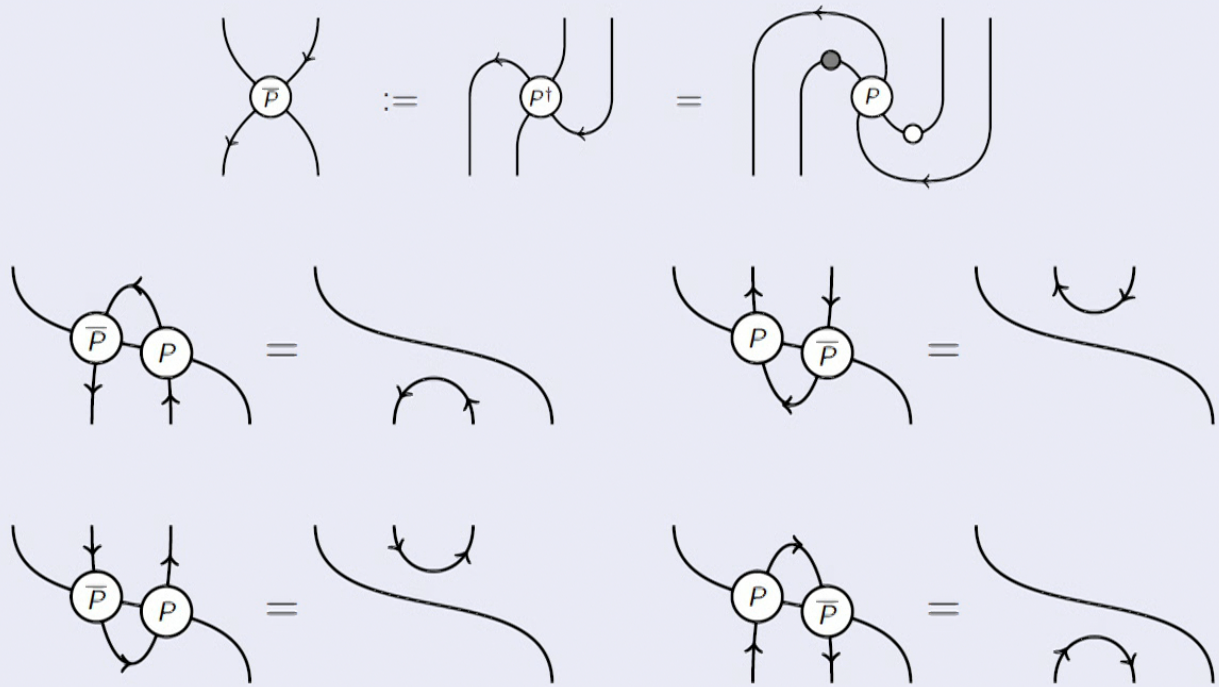


Invertibility: Dualisable morphisms

Theorem

Quantum bijections are precisely dagger dualisable morphisms in QSet.

Proof.



Quantum bijections between classical sets

- Quantum bijections $H \otimes X \to Y \otimes H$ only exist if $|X| = |Y|$.
- Expanding the definition in the classical basis, we obtain orthogonality and completeness equations for the columns of the matrix of projectors:

$$P_{x_1,y} P_{x_2,y} = \delta_{x_1,x_2} P_{x_1,y} \qquad \sum_{x \in X} P_{x,y} = \mathbb{1}_H$$

- Quantum bijections are square matrices of projections whose rows and columns form partitions of unity.
- These are known as *magic unitaries* or *projective permutation matrices*.

Quantum permutation groups

- Wang[4] defined the quantum automorphism group $A_{\mathrm{Aut}}(X)$ of a finite set X to be the *final object* in the category of *quantum transformation groups* of the commutative C^*-algebra of functions on X.
- Generators $a_{i,j}$ $(i, j = 1, \ldots n)$ and relations:

$$a_{i,j}^2 = a_{i,j} = a_{i,j}^* \quad \sum_{i=1}^{n} a_{i,j} = 1, \quad \forall 1 \leq j \leq n \quad \sum_{j=1}^{n} a_{i,j} = 1, \quad \forall 1 \leq i \leq n$$

- Quantum bijections are thus precisely f.d. representations $A_{\mathrm{Aut}}(X) \to B(H)$.

[4] Shuzhou Wang. "Quantum Symmetry Groups of Finite Spaces". In: *Communications in Mathematical Physics* 195.1 (1998), pp. 195–211. DOI: 10.1007/s002200050385. eprint: math/9807091.

Some facts about quantum bijections

- All quantum bijections on an n-element set with $n \leq 3$ are essentially classical.[5]

[5] Teodor Banica, Julien Bichon, and Benoît Collins. "Quantum permutation groups: A survey". In: *Noncommutative Harmonic Analysis with Applications to Probability*. Institute of Mathematics Polish Academy of Sciences, 2007. DOI: 10.4064/bc78-0-1. eprint: math/0612724.

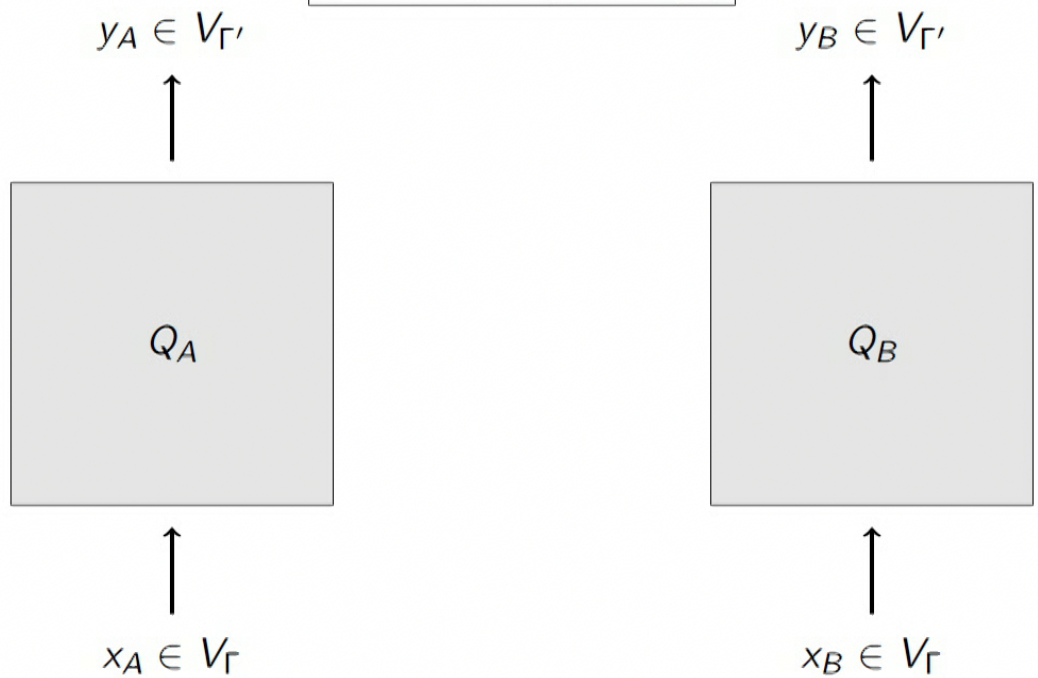
[6] Teodor Banica and Julien Bichon. "Quantum groups acting on 4 points". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2009.626 (2009), pp. 75–114. DOI: 10.1515/crelle.2009.003. eprint: math/0703118.

Interval

- We've defined a 2-categorical compositional theory of quantum functions incorporating notions from noncommutative geometry.
- We're now going to apply this to classify quantum pseudo-telepathy in the *graph isomorphism game*.

Quantum graph homomorphisms[7]

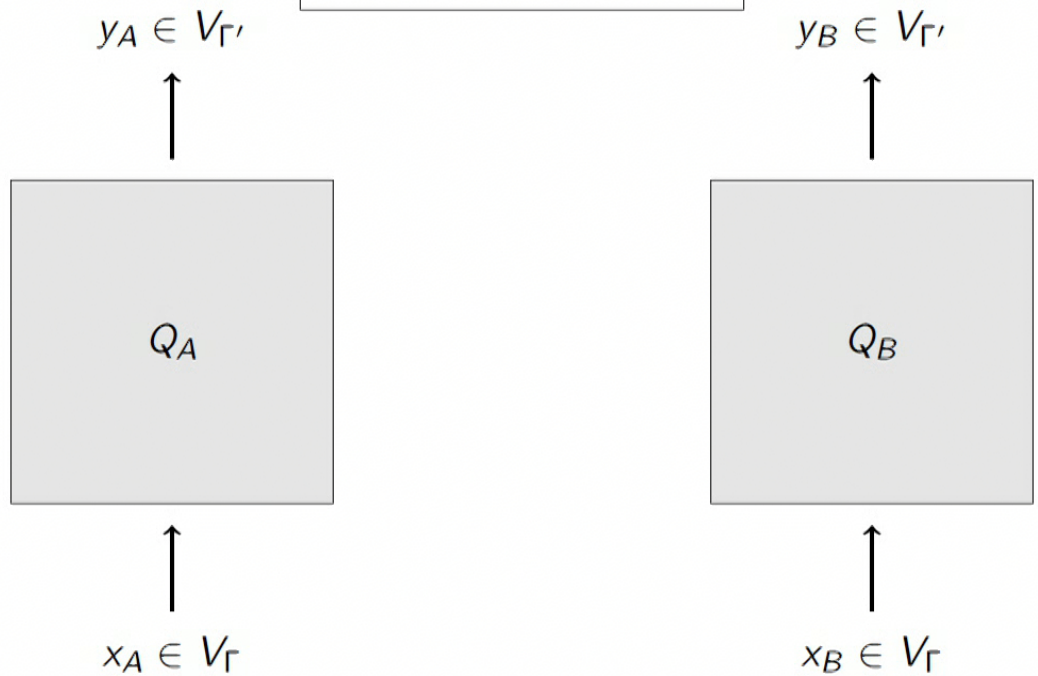
$$\boxed{x_A \sim_\Gamma x_B \Rightarrow y_A \sim_{\Gamma'} y_B}$$



[7] Laura Mančinska and David E Roberson. "Quantum homomorphisms". In: *Journal of Combinatorial Theory, Series B* 118 (2016), pp. 228–267. DOI: 10.1016/j.jctb.2015.12.009. eprint: 1212.1724.

Quantum graph isomorphisms[8]

$$\boxed{\mathrm{rel}(x_A, x_B) = \mathrm{rel}(y_A, y_B)}$$



[8] Albert Atserias et al. "Quantum and non-signalling graph isomorphisms". In: (2016). eprint: 1611.09837.

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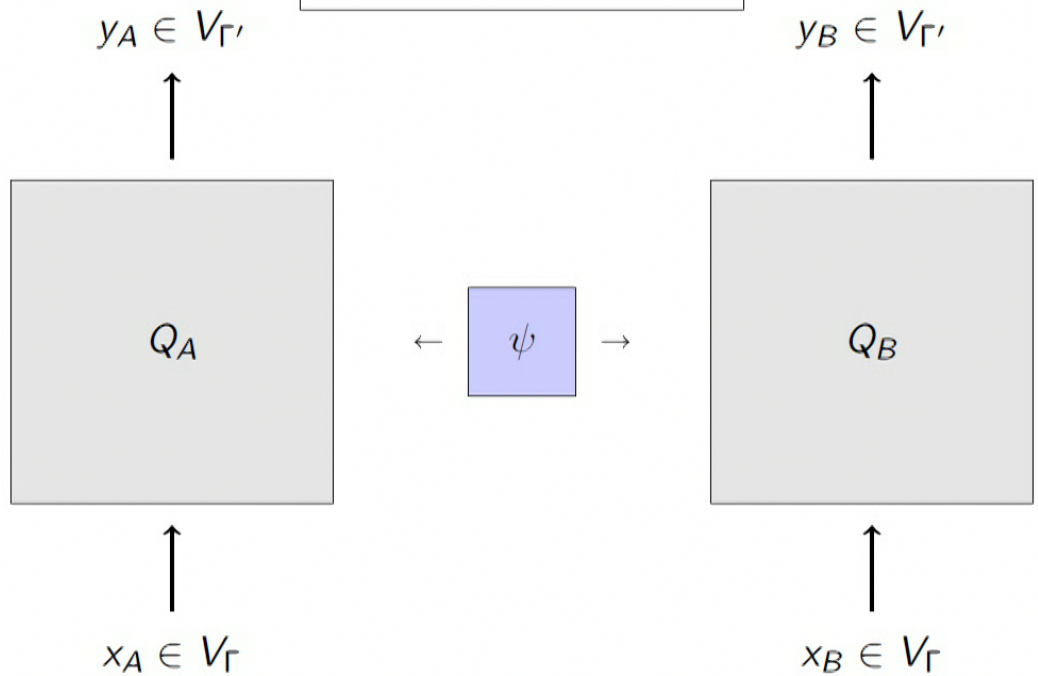
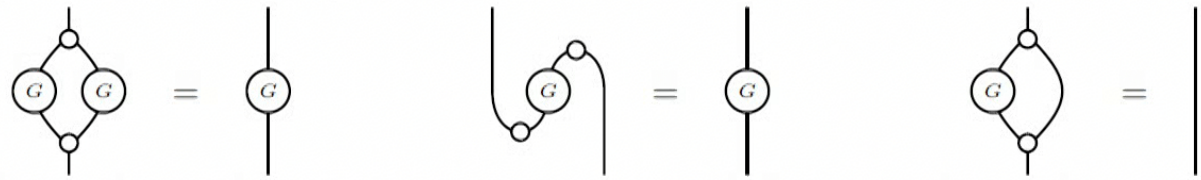


[8] Albert Atserias et al. "Quantum and non-signalling graph isomorphisms". In: (2016). eprint: 1611.09837.

Quantum graphs

- We define a *quantum graph* to be a quantum set equipped with a *quantum adjacency matrix*:



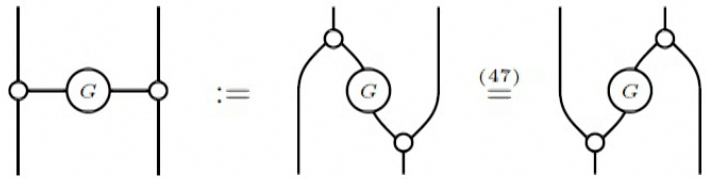
- On a classical set this is the standard definition of an adjacency matrix: $G^2_{v,w} = G_{v,w}$, $G_{v,w} = G_{w,v}$, and $G_{v,v} = 1$.
- On a quantum set these are the *symmetric quantum relations* of Kuperberg and Weaver[9], generalising the *quantum non-confusability graphs* of Duan, Severini and Winter[10].

[9] Greg Kuperberg and Nik Weaver. *A von Neumann algebra approach to quantum metrics/quantum relations*. Vol. 215. 1010. American Mathematical Society, 2012. DOI: 10.1090/S0065-9266-2011-00637-4. eprint: 1005.0353.

[10] Runyao Duan, Simone Severini, and Andreas Winter. "Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number". In: *IEEE Transactions on Information Theory* 59.2 (2013), pp. 1164–1174. DOI: 10.1109/TIT.2012.2221677. eprint: 1002.2514.

Graph homomorphisms

- We define the following projector:



- For a classical graph, this projects onto $\{(v, w) | v \sim_G w\} \subseteq V_G \times V_G$.

Graph homomorphisms

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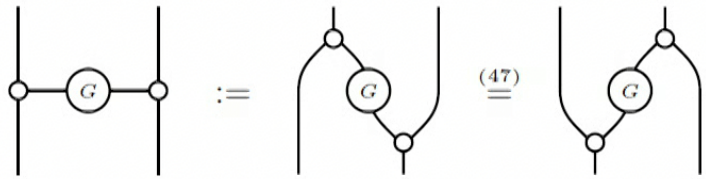
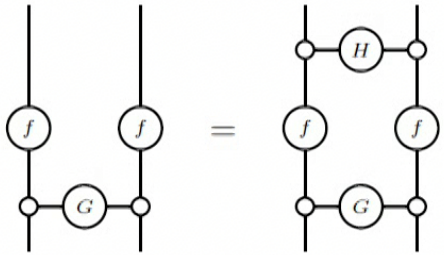
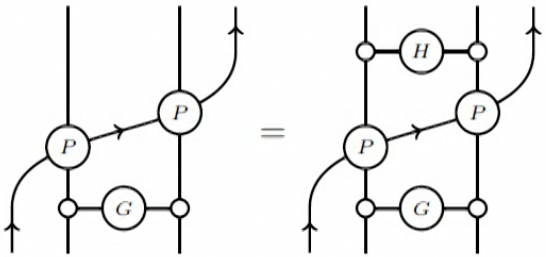


- For a classical graph, this projects onto $\{(v, w) | v \sim_G w\} \subseteq V_G \times V_G$.
- A classical graph homomorphism $G \to H$ is a $*$-cohomomorphism $f : G \to H$ fulfilling:



Quantum graph homomorphisms

- Define a *quantum graph homomorphism* as a quantum function satisfying:



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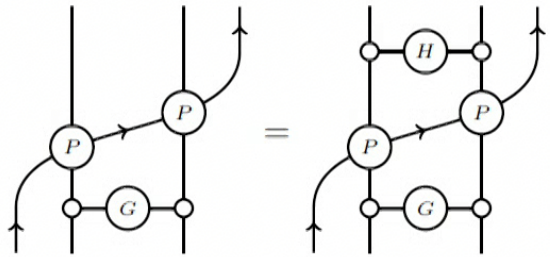
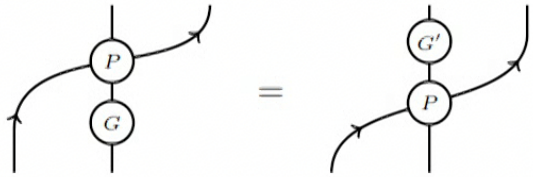


- A *quantum graph isomorphism* is a quantum graph homomorphism which is also a bijection; or equivalently, a quantum bijection commuting with the adjacency matrix:



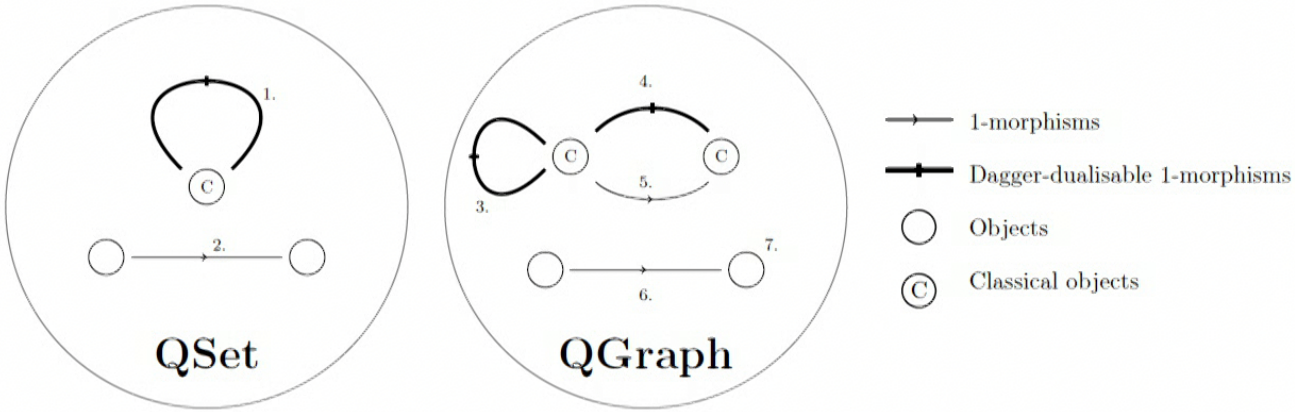
Recovering notions

- Given graphs G and H, there is a perfect quantum strategy for the graph homomorphism game if and only if there is a quantum graph homomorphism $(H, P) : (V_G, G) \to (V_H, H)$.
- Given graphs G and H, there is a perfect quantum strategy for the graph isomorphism game if and only if there is a quantum graph isomorphism $(H, P) : (V_G, G) \to (V_H, H)$.
- Quantum graph isomorphisms $(H, P) : (V_G, G) \to (V_G, G)$ are precisely finite dimensional representations of Banica's *quantum automorphism group* algebra.[11]

[11] Teodor Banica. "Quantum automorphism groups of homogeneous graphs". In: *Journal of Functional Analysis* 224.2 (2005), pp. 243–280. DOI: 10.1016/j.jfa.2004.11.002. eprint: math/0311402.

QGraph

- We can define a 2-category QGraph whose objects are quantum graphs, whose morphisms are quantum graph homomorphisms, and whose 2-morphisms are intertwiners.



1. Finite-dimensional (f.d.) representations of Wang's quantum permutation group algebra [?]. Projective permutation matrices of Atserias et al. [?].
2. F.d. representations of Sołtan's quantum space of all maps between finite quantum spaces [?].
3. F.d. representations of Banica's quantum automorphism group algebra [?].
4. Quantum graph isomorphisms of Atserias et al. [?].
5. Quantum graph homomorphisms of Mančinska and Roberson [?].
6. 'Pure' entanglement-assisted graph homomorphisms of Stahlke [?].
7. Quantum graphs of Weaver [?], generalising noncommutative graphs of Duan et al [?].

Objective

- We will classify quantum graphs Γ' with an isomorphism $\Gamma' \to \Gamma$ based on structures in the category $\mathrm{QAut}(\Gamma)$.
- We will derive a construction of quantum isomorphic graphs from *central type subgroups* of the automorphism group.

Morita equivalence

- Dualisable 1-morphisms $f : A \to B$ give rise to Frobenius monoids in the category of endomorphisms $A \to A$.

Morita equivalence

- Dualisable 1-morphisms $f : A \to B$ give rise to Frobenius monoids in the category of endomorphisms $A \to A$.
- The carrier object is $O = \bar{f} \circ f$, and the algebra is defined using the cups and caps of the duality:

$m : O \circ O \to O$ $\quad c : O \to O \circ O$ $\quad u : \mathrm{id}_A \to O$ $\quad cou : O \to \mathrm{id}_A$

Morita equivalence

- In sufficiently nice 2-categories, we can go in the other direction.
- We say that two Frobenius monoids are *Morita equivalent* if there is an invertible bimodule between them.
- We say that a Frobenius monoid is *split* if it can be obtained a dualisable 1-morphism as in the previous slide.

Corollary

Let $\mathbb{B}$ be a sufficiently nice 2-category, and let A be an object of $\mathbb{B}$. There is a bijection between the following sets:

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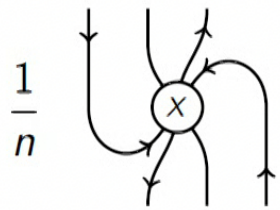
Corollary

Let $\mathbb{B}$ be a sufficiently nice 2-category, and let A be an object of $\mathbb{B}$. There is a bijection between the following sets:

- *Dagger equivalence classes of objects X such that there exists a special 1-morphism $S : X \to A$.*
- *Morita equivalence classes of split special dagger Frobenius monoids $F \in \mathbb{B}(A, A)$.*

Splitting in QGraph

- QGraph is sufficiently nice!
- A Frobenius monoid in $\mathrm{QAut}(\Gamma)$ is split if its underlying algebra is isomorphic to an endomorphism algebra.
- The splitting is defined by the following idempotent:



  Here n is the dimension of the H in $(\mathrm{End}(H), X)$.
- One can define a quantum graph structure Γ on the resulting vector space A, and a quantum graph isomorphism $\Gamma \to \Gamma'$.

Classification of quantum graphs

Theorem

Let Γ be a quantum graph. There is a bijective constructive correspondence between:

- *Isomorphism classes of quantum graphs Γ' such that there exists a quantum isomorphism $\Gamma' \to \Gamma$.*
- *Morita equivalence classes of simple dagger Frobenius monoids in $\mathrm{QAut}(\Gamma)$.*

Classical subcategory

- There is no hope of finding all Morita equivalence classes of simple dagger Frobenius monoids in $\mathrm{QAut}(\Gamma)$ for general Γ.
- We can do this in a simpler setting, however.

Definition

The *classical subcategory* of $\mathrm{QAut}(\Gamma)$ is the full semisimple monoidal subcategory of quantum automorphisms which are decomposable into a direct sum of ordinary automorphisms.

Classical subcategory

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- We can do this in a simpler setting, however.

Definition

The *classical subcategory* of $\mathrm{QAut}(\Gamma)$ is the full semisimple monoidal subcategory of quantum automorphisms which are decomposable into a direct sum of ordinary automorphisms.

- For some graphs which have *no quantum symmetries*, the classical subcategory is the whole of $\mathrm{QAut}(\Gamma)$.

Frobenius monoids in the classical subcategory

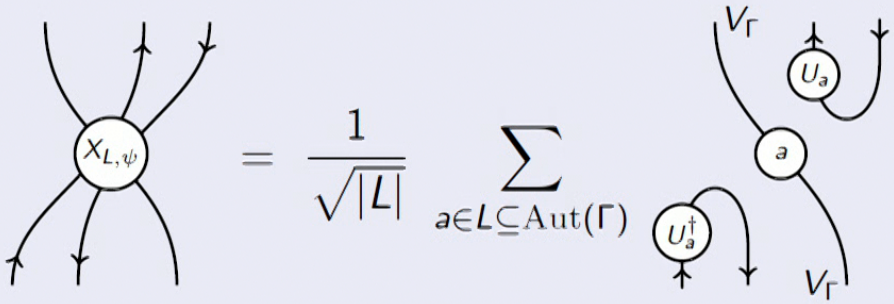
Definition (Groups of central type)

A group of central type (G, ψ) is a group G and a 2-cocycle ψ such that there is a single irreducible projective ψ-representation.
A *nice UEB* is a unitary irreducible projective ψ-representation of a group (G, ψ) of central type.

There is an analogue of an alternating form, $\rho : G \times G \to U(1)$, on a group of central type which marks certain subgroups out as *coisotropic*.

Theorem

Let Γ be a graph. Every simple dagger Frobenius monoid in the classical subcategory is Morita equivalent to a simple dagger Frobenius monoid $(H \otimes H^*, X_{L,\psi})$ for some Hilbert space H, where the underlying linear map $X_{L,\psi} : (H \otimes H^*) \otimes V_\Gamma \to V_\Gamma \otimes (H \otimes H^*)$ is defined as follows:



Here, (L, ψ) is a subgroup of central type of $\mathrm{Aut}(\Gamma)$ and $\{U_a \mid a \in L\}$ is a corresponding nice UEB.

Classification of quantum isomorphic graphs

Theorem

Let Γ be a classical graph. Then, every subgroup of central type (L, ψ) of $\mathrm{Aut}(\Gamma)$ with coisotropic stabilisers induces a classical graph $\Gamma_{L,\psi}$ and a quantum isomorphism $\Gamma_{L,\psi} \to \Gamma$. Moreover, if Γ has no quantum symmetries, this gives rise to a bijective correspondence between the following sets:

- *Isomorphism classes of classical graphs Γ' such that there exists a quantum isomorphism $\Gamma' \to \Gamma$.*
- *Subgroups of central type (L, ψ) of $\mathrm{Aut}(\Gamma)$ with coisotropic stabilisers up to the equivalence relation:*

$$(L, \psi) \sim (L', \psi') \;\Leftrightarrow\; L' = gLg^{-1} \text{ and } \psi' \text{ is cohomologous to } \psi^g(x, y) := \psi(gxg^{-1}, gyg^{-1})$$

Graphs with no quantum symmetries

The following is a complete list of all vertex-transitive graphs of order ≤ 11 with no quantum symmetries.[12]

Graph	Automorphism group
C_{11}, $C_{11}(2)$, $C_{11}(3)$	D_{11}
Petersen	S_5
C_{10}, $C_{10}(2)$, C_{10}^{+}, $\mathrm{Pr}(C_5)$	D_{10}
Torus	$S_3 \wr \mathbb{Z}_2$
C_9, $C_9(3)$	D_9
C_8, C_8^{+}	D_8
C_7	D_7
C_6	D_6
C_5	D_5
K_3	S_3
K_2	$\mathbb{Z}_2$

[12] Teodor Banica and Julien Bichon. "Quantum automorphism groups of vertex-transitive graphs of order ≤ 11". In: *Journal of Algebraic Combinatorics* 26.1 (2007), pp. 83–105. DOI: 10.1007/s10801-006-0049-9. eprint: math/0601758, Simon Schmidt. "The Petersen graph has no quantum symmetry". In: (2018). eprint: 1801.02942.

Graphs with no quantum symmetries

Theorem

None of the graphs in the above table exhibit pseudo-telepathy in the graph isomorphism game.

Proof.

Recipe:

1. Find all central type subgroups (L, ψ) of $\mathrm{Aut}(\Gamma)$ up to conjugacy.

Graphs with no quantum symmetries

Theorem

None of the graphs in the above table exhibit pseudo-telepathy in the graph isomorphism game.

Proof.

Recipe:

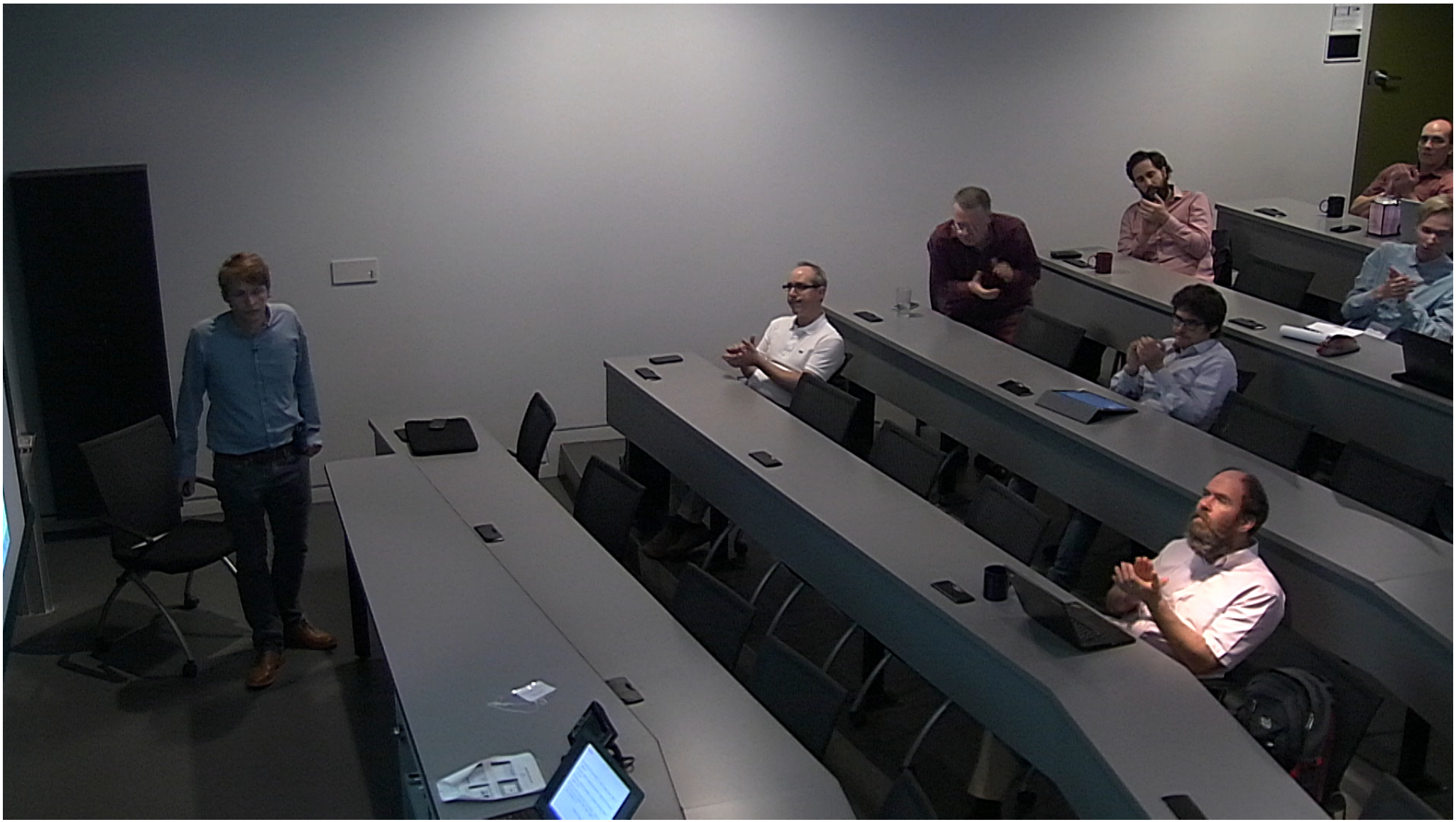
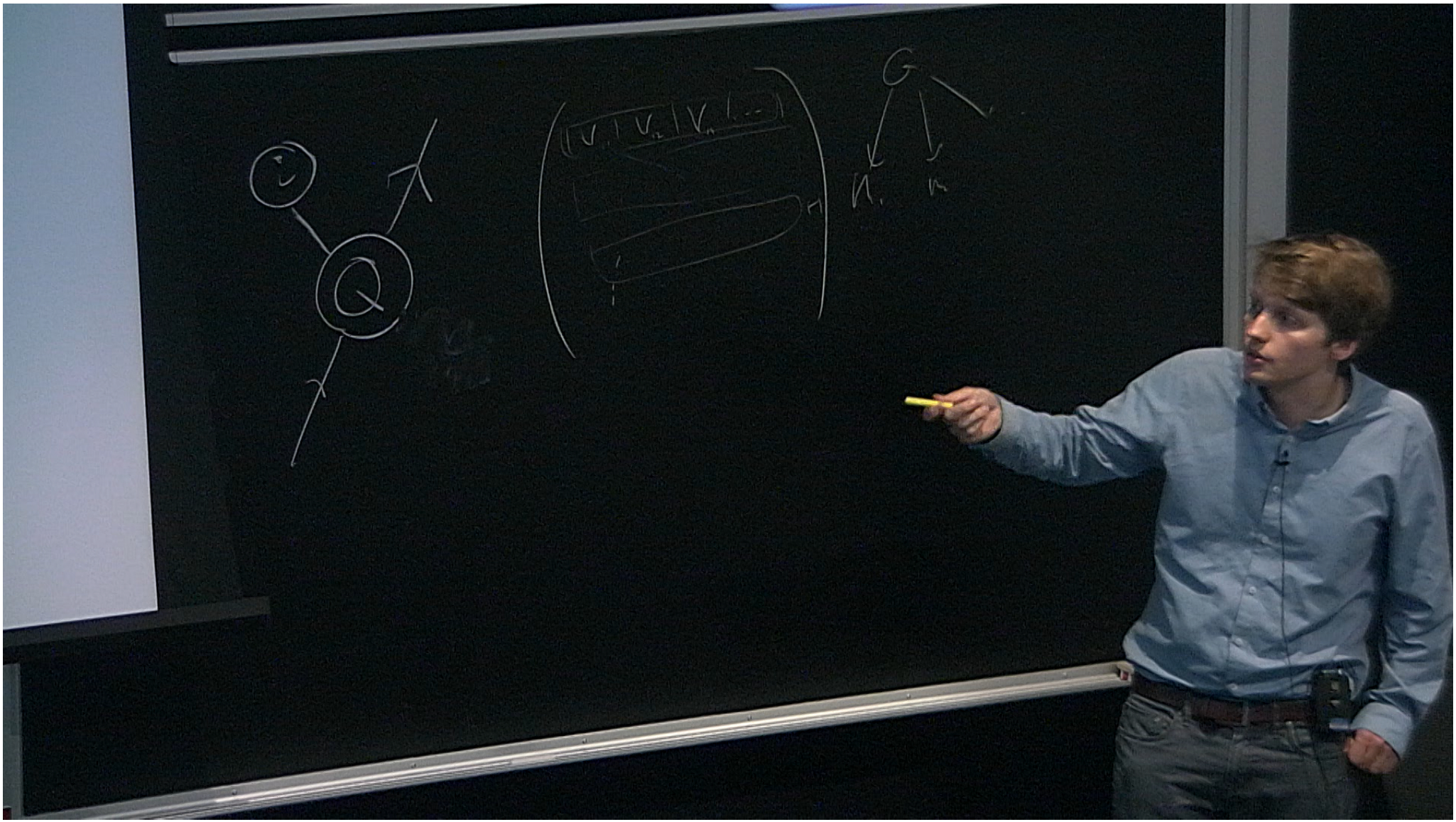
1. Find all central type subgroups (L, ψ) of $\mathrm{Aut}(\Gamma)$ up to conjugacy.
2. Show that the stabilisers of the action for each group are not all coisotropic.

□

This also classifies isomorphic quantum graphs.

Some things that weren't covered

- Quantum relations.
- Classification of isomorphisms $\Gamma' \to \Gamma$.
- Arkhipov's construction.
- The details of the group-theoretical construction.

Outlook

- Understand the construction in a more conceptual way. (Work in progress.)
- Develop broader representation-theoretic constructions of quantum functions.
- Develop a physical interpretation of quantum functions between quantum sets.
- Explore quantum function theory more broadly: quantum monotone functions between quantum posets, higher algebraic structures in QSet and QGraph, etc.