Title: Measures of Preparation Contextuality

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Abstract: In a large medical trial, if one obtained a ridiculously small p-value like 10^-12, one would typically move from a plain hypothesis test to trying to estimate the parameters of the effect. For example, one might try to estimate the optimal dosage of a drug or the optimal length of a course of treatment. Tests of Bell and noncontextuality inequalities are hypotheses tests, and typical p-values are much lower than this, e.g. 12-sigma effects are not unheard of and a 7-sigma violation already corresponds to a p-value of about 10^-12. Why then, in quantum foundations, are we still obsessed with proposing and testing new inequalities rather than trying to estimate the parameters of the effect from the experimental data? Here, we will try to do this for preparation contextuality, but will also make some related comments on recent loophole-free Bell inequality tests.

We introduce two measures of preparation contextuality: the maximal overlap and the preparation contextuality fraction. The latter is linearly related to the degree of violation of a preparation noncontextuality inequality, so can be estimated from experimental data. Although the measures are different in general, they can be equal for proofs of preparation contextuality that have sufficient symmetry, such as the timelike analogue of the CHSH scenario. We give the value of these measures for this scenario. Using our result, we can consider pairty-epsilon multiplexing, Alice must try to communicate two bits to Bob so that he can choose to determine either of them with high probability, but where Alice must ensure that Bob cannot guess the parity of the bits with probability greater than 1/2 + epsilon, and determine the range of epislon for which there is still an advantage in preparation contextual theories. If time permits, I will make some brief comments on how to robustify experimental tests of this result.

joint work with Eric Freda and David Schmid

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Measures of Preparation Contextuality

Matthew Leifer

with E. Freda (Chapman) and D. Schmid (Perimeter)

Foundations of Quantum Mechanics - Perimeter Institute - August 1, 2018

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The Golden Rule of Resource Measures

- Paraphrasing Spekkens:
  - If you want to measure a physical resource, then you ought to first define a resource theory, i.e. a set of state spaces and free operations. Meaningful measures are monotones under the free operations. The most meaningful measures are the operational conversion rates between states under the free operations.
  - If you are proposing arbitrary measures without a resource theory behind them then you are a clown.

Outline

1. Loophole Free Bell Tests
2. Preparation Contextuality
3. Measures of Preparation Contextuality
4. Time-like CHSH Setup
5. Parity ϵ Multiplexing
6. Time-like Chained Setup
7. Conclusions

1. Loophole Free Bell Tests

Loophole-Free Bell Experiments as Hypothesis Tests

- Reported p-values of loophole free Bell tests under the most stringent analysis (accounting for memory loophole and partial predictability of measurement settings).
- Delft B. Hensen et. al., Nature **526** 682–686 (2015). *Scientific Reports* **6** 30289 (2016)
  - 2015: p=0.039
  - 2016: p=0.061
- NIST 2015 L. K. Shalm et. al., *Phys. Rev. Lett.* **115**, 250402 (2015)
  - p $\leq 2.3 \times 10^{-7}$
- Vienna: 2015 M. Giustina et. al., *Phys. Rev. Lett.* **115** 250401 (2015)
  - p $\leq 3.74 \times 10^{-31}$

Fraction Measures

- Suppose you have some observed probabilities $\mathrm{Prob}_{\mathrm{obs}}(a|x)$ and you want to measure to what extent they can be reproduced by a model with property Y (e.g. $Y =$ locality, noncontextuality, etc.)
- Write

$$\mathrm{Prob}_{\mathrm{obs}}(a|x) = p\mathrm{Prob}_{Y}(a|x) + (1-p)\mathrm{Prob}'(a|x)$$

where $\mathrm{Prob}_{Y}(a|x)$ can be reproduced by a model with property Y and $\mathrm{Prob}'(a|x)$ is an arbitrary distribution.

- Define the *Y-fraction* as

$$p_{\mathrm{Y}} = \max p$$

where the maximum is taken over all such decompositions.

- The *non Y-fraction* is defined as $p_{\mathrm{NY}} = 1 - p_{\mathrm{Y}}$.

Fraction Measures

- Suppose also that there is a linear inequality $I(\mathrm{Prob}_Y) \leq I_Y$ that is bounded by I_Y for models with property Y, but violated by your observed probabilities.
- Then,

$$\begin{aligned} I_{\mathrm{obs}} = I(\mathrm{Prob}_{\mathrm{obs}}) &= pI(\mathrm{Prob}_Y) + (1-p)I(\mathrm{Prob}') \\ &\leq pI_Y + (1-p)I_{\max} \end{aligned}$$

- Rearranging

$$p_Y \leq \frac{I_{\max} - I_{\mathrm{obs}}}{I_{\max} - I_Y}$$

with equality if $I(\mathrm{Prob}_{\mathrm{Y}}) \leq I_Y$ is a tight inequality.

- Locality fraction[1] and noncontextuality fraction[2] are examples of this.

[1] R. Colbeck & R. Renner, *Phys. Rev. Lett.* **101**, 050403 (2008)

[2] S. Abramsky et. al., *Phys. Rev. Lett.* **119**, 050504 (2017)

Example: Local Fraction for the CHSH Inequality

$$I(\text{Prob}) = \langle ab \rangle_{x=0,y=0} + \langle ab \rangle_{x=0,y=1} + \langle ab \rangle_{x=1,y=0} - \langle ab \rangle_{x=1,y=1}$$

- $I_{\text{max}} = 4$, obtained using the PR box
- $I_{\text{LHV}} = 2$, from the CHSH inequality
- $I_{\text{obs}} \leq 2\sqrt{2}$ from the Tsirelson bound, assuming our experiment obeys quantum mechanics.

$$p_{\text{L}} \geq \frac{4 - 2\sqrt{2}}{4 - 2} \approx 0.586 \qquad p_{\text{NL}} \lesssim 0.414$$

Loophole-Free Bell Experiments as Parameter Estimation

- Delft:
  - 2015: $p_{\mathrm{NL}} = 0.21 \pm 0.1$
  - 2016: $p_{\mathrm{NL}} = 0.175 \pm 0.09$
  - Note: Errors are standard deviations without closing memory loophole.

- Vienna:
  - $p_{\mathrm{NL}} = 7.27 \times 10^{-6}$ (4×10^{-5} is maximum possible for CH/Eberhard inequality)
  - Note: Errors cannot be estimated from the data in the paper.

Rate of nonlocal bit production

- We can define the *rate of nonlocal bit production* as

$$R_{\mathrm{NL}} = \frac{p_{\mathrm{NL}}}{\text{time per run of the experiment}}$$

- Delft:
  - 2015: $R_{\mathrm{NL}} = (6.5 \pm 0.3) \times 10^{-5}\ \mathrm{s}^{-1}$
  - 2016: $R_{\mathrm{NL}} = (6.9 \pm 0.4) \times 10^{-5}\ \mathrm{s}^{-1}$
  - Note: Errors are standard deviations without closing memory loophole.
- Vienna:
  - $R_{\mathrm{NL}} = 2.54 \times 10^{-2}\ \mathrm{s}^{-1}$

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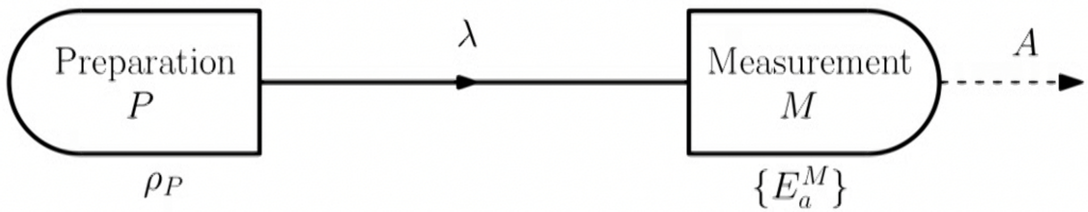
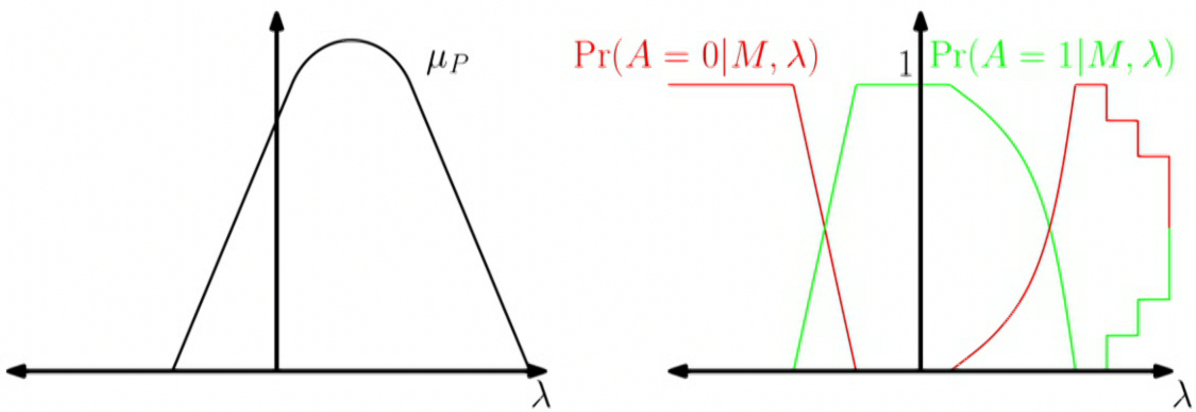
2. Preparation Contextuality

Ontological Models



$$\mathrm{Prob}(A = a|P, M) = \mathrm{Tr}(E_a^M \rho_P)$$



$$\mathrm{Prob}(A = a|P, M) = \int_\Lambda \mathrm{Pr}(A = a|M, \lambda) d\mu_P(\lambda)$$

Ontological Models

- An ontological model consists of
  - A measurable space (Λ, Σ)
  - For each preparation P, a probability measure $\mu_P \colon \Sigma \to [0,1]$.
  - For each measurement M a conditional probability function $\Pr(A = a|M,\cdot)\colon \Lambda \to [0,1]$ satisfying
$$\forall \lambda, \qquad \sum_a \Pr(A = a|M, \lambda) = 1$$
  - The observed probabilities predicted by the model are
$$\mathrm{Prob}(A = a|P, M) = \int_\Lambda \Pr(A = a|M, \lambda)\, \mathrm{d}\mu_P$$

Preparation Contextuality

- Two preparations P and Q are *operationally equivalent* if, for all (M, a)

$$\mathrm{Prob}(A = a|P, M) = \mathrm{Prob}(A = a|Q, M)$$

- Note, if $\rho_P = \rho_Q$ then P and Q are operationally equivalent.

- A model is *preparation noncontextual* if, whenever P and Q are operationally equivalent then

$$\mu_P = \mu_Q$$

- A model is *preparation contextual* this fails to hold.

Preparation Noncontextual Fraction

- Spekkens showed that even a model of a qubit must be preparation contextual[1], but now we want to measure how contextual.
- A simple way of doing this is to use the *Preparation Noncontextual fraction*.

$$p_{\mathrm{NC}} = \max\{p \mid \mathrm{Prob}(A|P,M) = p\mathrm{Prob}_{\mathrm{NC}}(A|P,M) + (1-p)\mathrm{Prob}'(A|P,M)\}$$

and the associated *Preparation Contextual (PC) fraction*

$$p_{\mathrm{C}} = 1 - p_{\mathrm{NC}}$$

[1] R. Spekkens, Phys. Rev. A **71**, 052108 (2005)

Preparation Noncontextual Fraction

- As before, we can use any preparation noncontextuality inequality

$$I(\mathrm{Prob}_{\mathrm{NC}}) \leq I_{\mathrm{NC}}$$

to calculate

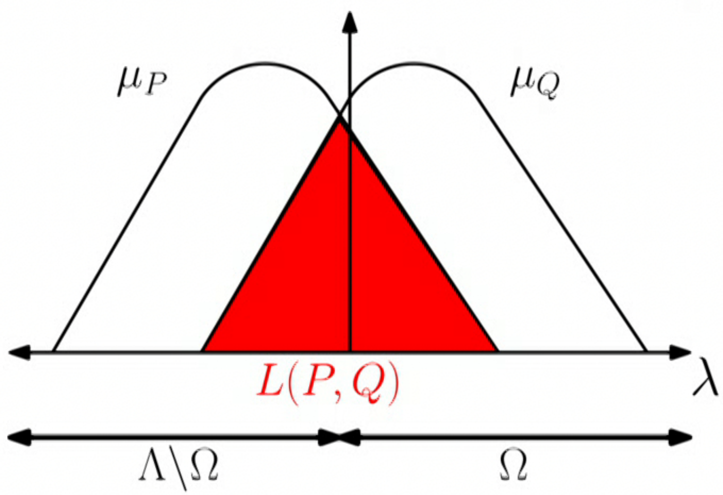
$$p_{\mathrm{NC}} \leq \frac{I_{\max} - I_{\mathrm{obs}}}{I_{\max} - I_{\mathrm{NC}}}$$

- This is a rather crude measure, so we will also consider more operationally meaningful measures that tell us about the contextuality of specific operationally equivalent preparations.

Overlap Measures

- Consider two operationally equivalent preparations, P and Q. We can define their *ontological overlap*

$$L(P,Q) = \inf_{\Omega\in\Sigma}\big(\mu_P(\Omega) + \mu_Q(\Lambda\setminus\Omega)\big)$$



- $L(P,Q) = 1$ in a preparation noncontextual model.
- $\frac{1}{2}(2 - L(P,Q))$ is the optimal probability of guessing whether P or Q was prepared given λ.

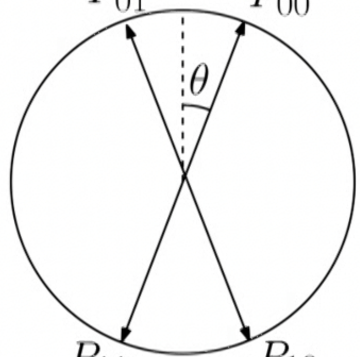
Overlap Measures

- Let $\mathcal{P}$ be the set of n-element partitions of Λ,

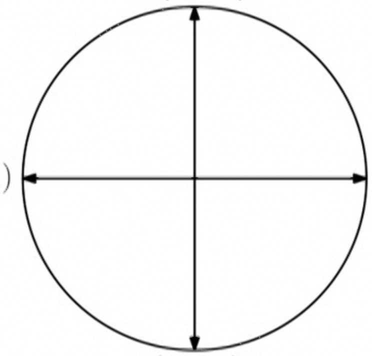
  i.e. $\{\Omega_1, \Omega_2, \cdots, \Omega_n\} \in \mathcal{P}$ if
  - $\Omega_j \in \Sigma$
  - $\Omega_j \cap \Omega_k = \emptyset$ for $j \neq k$
  - $\cup_{j=1}^{n} \Omega_j = \Lambda$
- If $P_1, P_2, \cdots, P_n$ are all operationally equivalent, then we can generalize the overlap to

$$L(P_1, \cdots, P_n) = \inf_{\{\Omega_1, \Omega_2, \cdots, \Omega_n\} \in \mathcal{P}} \left(\sum_{j=1}^{n} \mu_{P_j}(\Omega_j) \right)$$

Time-like CHSH Setup



Preparations



Measurements

- Define the two mixed preparations

$$P_+ = \tfrac{1}{2}P_{00} + \tfrac{1}{2}P_{11}$$
$$P_- = \tfrac{1}{2}P_{01} + \tfrac{1}{2}P_{10}$$

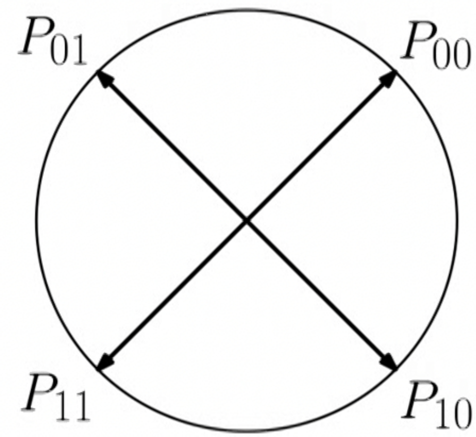
- P_+ and P_- are operationally equivalent.
- Spekkens et. al.[1] proved that a PNC model must satisfy

$$\mathrm{Prob}(0|P_{00}, M_0) + \mathrm{Prob}(0|P_{01}, M_0) + \mathrm{Prob}(1|P_{10}, M_0) + \mathrm{Prob}(1|P_{11}, M_0)$$
$$+ \mathrm{Prob}(0|P_{00}, M_1) + \mathrm{Prob}(0|P_{10}, M_1) + \mathrm{Prob}(1|P_{01}, M_1) + \mathrm{Prob}(0|P_{11}, M_1) \leq 6$$

- The algebraic maximum is 8.
- In quantum mechanics, we get $2(2 + \cos\theta + \sin\theta)$
- So $p_{\mathrm{NC}} \leq 2 - \cos\theta - \sin\theta$ or $p_{\mathrm{C}} \geq \cos\theta + \sin\theta - 1$

[1] R. Spekkens et. al., *Phys. Rev. Lett.* **102** 010401 (2009)

Time-like CHSH Setup



- The optimum occurs at $\theta = \frac{\pi}{4}$, where we get

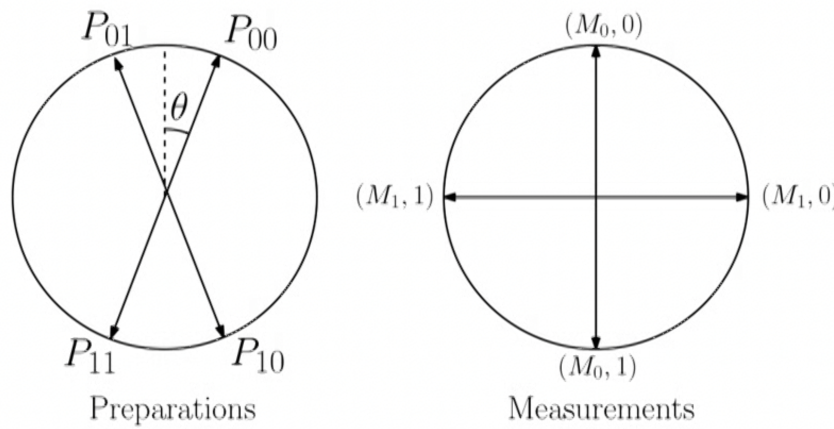
$$p_{\mathrm{NC}} \leq 2 - \sqrt{2} \approx 0.586$$
$$p_{\mathrm{C}} \gtrsim 0.414$$

- This is all very well, but can we say anything about the overlap $L(P_+, P_-)$, which is more operationally meaningful.
- It turns out that, due to the symmetry of this setup, we get

$$L(P_+, P_-) = p_{\mathrm{NC}}$$

for this case.

Overlap From Symmetry



- Notation: $\mu_{jk} = \mu_{P_{jk}}$ and $\mu_{\pm} = \mu_{P_{\pm}}$
- The infimum in

  $$L(P_+, P_-) = \inf_{\Omega \in \Sigma} (\mu_+(\Omega) + \mu_-(\Lambda \setminus \Omega))$$

  is given by p in a decomposition

  $$\mu_+ = p\mu^{\mathrm{NC}} + (1-p)\mu'_+ \qquad \mu_- = p\mu^{\mathrm{NC}} + (1-p)\mu'_-$$

  with maximal p.
- But this does not mean that $p = p_{\mathrm{NC}}$ because we cannot necessarily write

  $$\mu_{jk} = p\mu_{jk}^{\mathrm{NC}} + (1-p)\mu'_{P_{jk}}$$

  with

  $$\mu^{\mathrm{NC}} = \tfrac{1}{2}\mu_{00}^{\mathrm{NC}} + \tfrac{1}{2}\mu_{11}^{\mathrm{NC}} = \tfrac{1}{2}\mu_{01}^{\mathrm{NC}} + \tfrac{1}{2}\mu_{10}^{\mathrm{NC}}$$

Overlap From Symmetry

- All we can conclude is that

$$\mu_{jk} = p_{jk}\mu_{jk}^{\mathrm{NC}} + (1 - p_{jk})\mu'_{P_{jk}}$$

with

$$p\mu^{\mathrm{NC}} = \tfrac{1}{2}p_{00}\mu_{00}^{\mathrm{NC}} + \tfrac{1}{2}p_{11}\mu_{11}^{\mathrm{NC}} = \tfrac{1}{2}p_{01}\mu_{01}^{\mathrm{NC}} + \tfrac{1}{2}p_{10}\mu_{10}^{\mathrm{NC}}$$

- If, for any allowed p, there always exists a model with $p_{jk} = p$ we would be done, and the symmetry of the setup allows us to construct such a model.

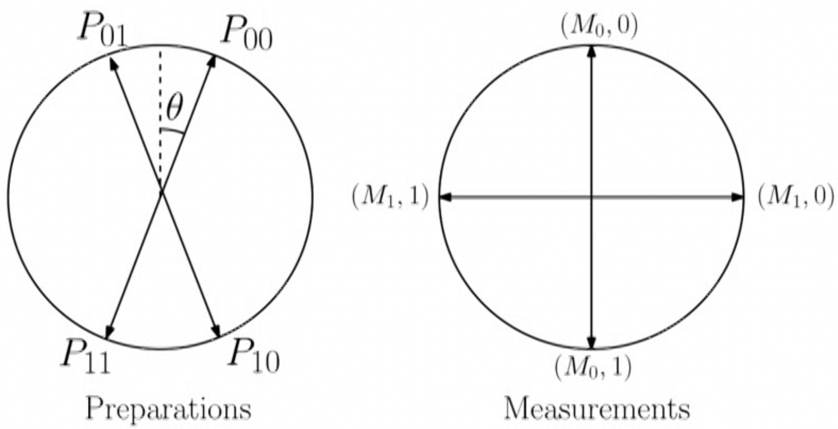
Overlap From Symmetry

- We now construct a third model by taking the "direct sum" of the two:
  - The ontic state space is $\tilde{\tilde{\Lambda}} = \Lambda \sqcup \tilde{\Lambda}$
  - Let $\tilde{\tilde{\mu}}_{jk}(\Omega) = \frac{1}{2}\mu_{jk}(\Omega \setminus \Lambda) + \frac{1}{2}\tilde{\mu}(\Omega \setminus \tilde{\Lambda})$
  - Let $\widetilde{\widetilde{\Pr}}(k|M_j,\lambda) = \begin{cases} \Pr(k|M_j,\lambda), & \lambda \in \Lambda \\ \widetilde{\Pr}(k|M_j,\lambda), & \lambda \in \tilde{\Lambda} \end{cases}$
- In this model $\tilde{\tilde{p}}_{00} = \tilde{\tilde{p}}_{11} = \frac{1}{2}(p_{00} + p_{11})$ and $\tilde{\tilde{p}}_{01} = \tilde{\tilde{p}}_{10} = \frac{1}{2}(p_{01} + p_{10})$ and we also have

$$\tilde{\tilde{p}} = p = \tilde{\tilde{p}}_{00} = \tilde{\tilde{p}}_{11} = \tilde{\tilde{p}}_{01} = \tilde{\tilde{p}}_{10}$$

- Since this model exists, we can conclude that $L(P_+, P_-) = p_{\mathrm{NC}}$ for this setup.

Overlap From Symmetry



- We can construct a new model by reflecting everything through the origin of the Bloch sphere.

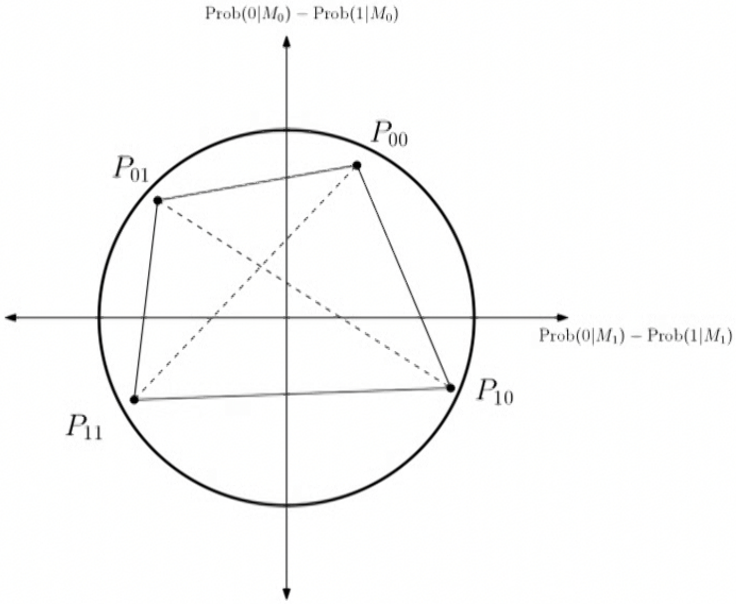
$$\tilde{\mu}_{00} = \mu_{11}, \qquad \tilde{\mu}_{11} = \mu_{00}, \qquad \tilde{\mu}_{01} = \mu_{10}, \qquad \tilde{\mu}_{10} = \mu_{01}$$

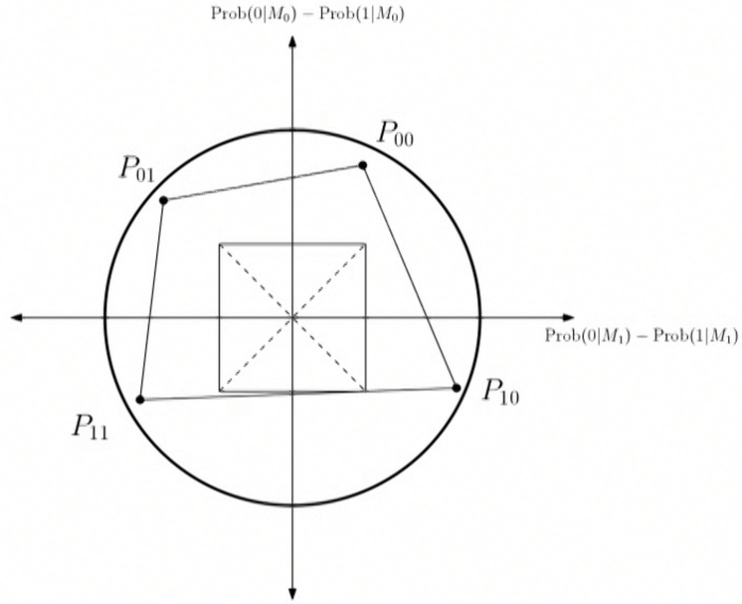
$$\widetilde{\Pr}\,(0|M_0,\lambda) = \Pr(1|\mathrm{M}_0,\lambda), \qquad \widetilde{\Pr}\,(1|M_0,\lambda) = \Pr(0|\mathrm{M}_0,\lambda)$$
$$\widetilde{\Pr}\,(0|M_1,\lambda) = \Pr(1|\mathrm{M}_1,\lambda), \qquad \widetilde{\Pr}\,(1|M_1,\lambda) = \Pr(0|\mathrm{M}_1,\lambda)$$

- Because the quantum predictions are invariant, the new model will also reproduce them.

Robustness

- In a real experiment, operational equivalences will not hold exactly. Pusey[1] showed how to deal with this for the CHSH case.





[1] M. Pusey, arXiv:1506.04178 (2015).

5. Parity ϵ Multiplexing

Parity Oblivious Multiplexing

- Alice has two bits a_0 and a_1 chosen uniformly at random.
- She sends a message m to Bob, such that m contains no information about the parity $a_0 \bigoplus a_1$.
- Bob is asked to guess a_j with j chosen uniformly at random. Call his guess b.
- He succeeds if $b = a_j$

$$p_{\text{succ}} = \frac{1}{4}[\text{Prob}(b = 0|j = 0, a_0 = 0) + \text{Prob}(b = 0|j = 1, a_1 = 0)$$
$$+\text{Prob}(b = 1|j = 0, a_0 = 1) + \text{Prob}(b = 1|j = 1, a_1 = 1)]$$

- Classical success probability is $\frac{3}{4}$, quantum is $\frac{2+\sqrt{2}}{4} \approx 0.85$ by PNC inequality[1]

[1] R. Spekkens et. al., *Phys. Rev. Lett.* **102** 010401 (2009)

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Parity ϵ Multiplexing

- Alice has two bits a_0 and a_1 chosen uniformly at random.
- She sends a message m to Bob, such that m allows Bob to guess $a_0 \oplus a_1$ with probability at most ϵ.
- Bob is asked to guess a_j with j chosen uniformly at random. Call his guess b.
- He succeeds if $b = a_j$

$$p_{\text{succ}} = \frac{1}{4}[\text{Prob}(b = 0|j = 0, a_0 = 0) + \text{Prob}(b = 0|j = 1, a_1 = 0)$$
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Overlap Measures

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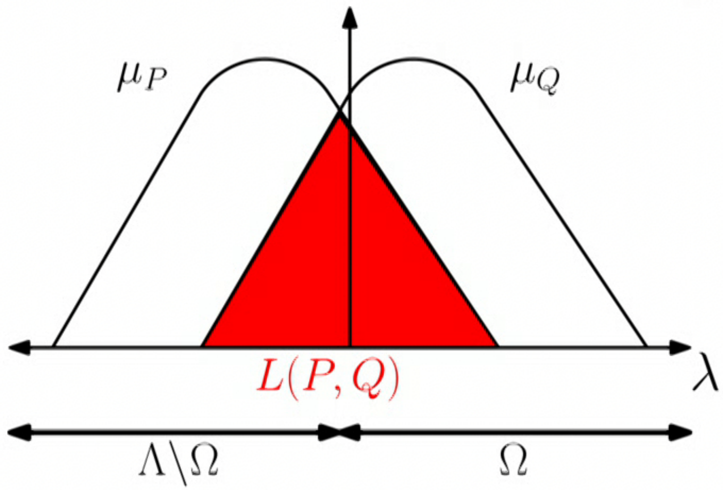


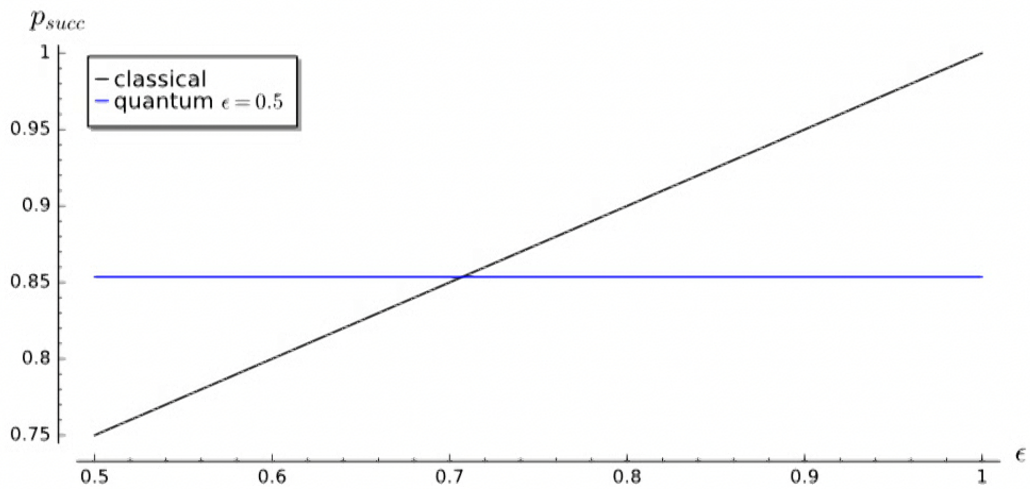
- $L(P,Q) = 1$ in a preparation noncontextual model.
- $\frac{1}{2}(2-L(P,Q))$ is the optimal probability of guessing whether P or Q was prepared given λ.

Parity ϵ Multiplexing

- Classical success probability is

$$p_{\text{succ}} = \frac{1}{2}(1 + \epsilon)$$

- There is a quantum advantage provided $\epsilon \leq \sqrt{2}/2 \approx 0.707$



Note: We have not optimized the quantum protocol for $\epsilon > \frac{1}{2}$.

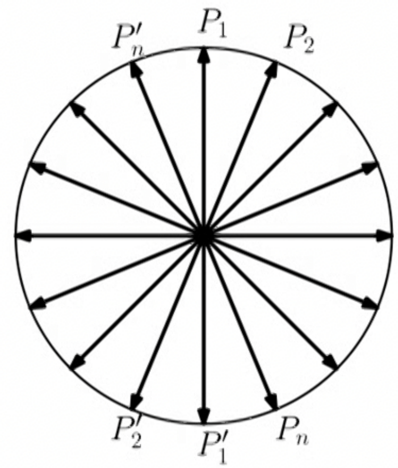
6. Time-like Chained Setup

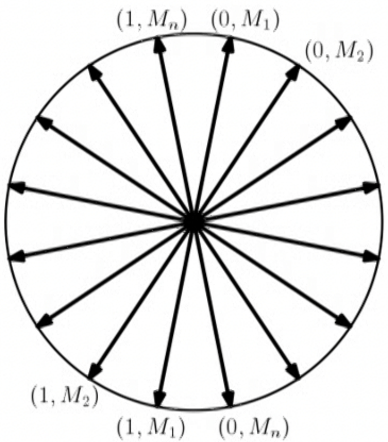
Time-like Chained Setup

- $Q_j = \frac{1}{2}P_j + \frac{1}{2}P_j'$ are operationally equivalent for all j.

- Converting Colbeck-Renner result[1] into preparation contextuality, gives

$$p_{\mathrm{NC}} \to 0 \text{ as } n \to \infty$$



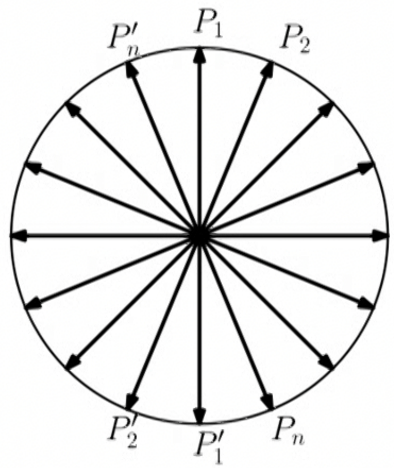


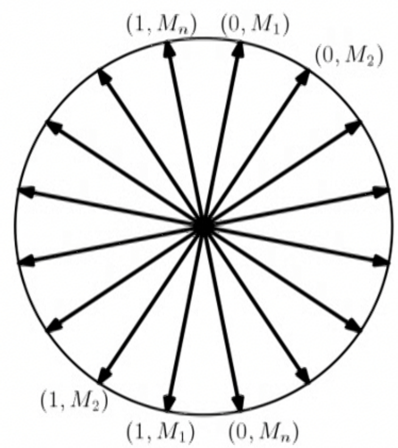
[1] R. Colbeck & R. Renner, *Phys. Rev. Lett.* **101**, 050403 (2008)

Time-like Chained Setup

- Because we still have symmetry under reflection through the origin, we can still get an overlap bound from this, but it is

$$L(Q_1, Q_2, \cdots, Q_n) \to 0$$
$$\text{as } n \to 0$$





Conclusions

- When your hypothesis tests are extremely compelling, you should move to parameter estimation.
- In Bell and Contextuality tests, there are some simple parameters you can estimate from existing data.
  - For loophole free Bell tests, this leads to a different comparison of the relative merits of the different experiments, and shows how far we still have to go.
- In preparation contextuality tests, we can estimate more operationally meaningful parameters provided the setup is sufficiently symmetric. This can be used to robustify information processing tasks that are powered by preparation contextuality.

Open Questions

- Do the overlap bounds get smaller in more complicated preparation contextuality proofs?
  - Any Kochen-Specker proof can be converted into a preparation contextuality proof. Can we relate the contextuality fraction of Abramsky et. al. to the prepartion contextuality fraction and overlap bounds?
- Can we develop similar measures for contextuality as a whole (preparation, transformation, measurement)?
- Can other information processing tasks that are powered by contextuality be robustified using overlap measures?
- Are our measures monotones in a well-motivated resource theory of contextuality?