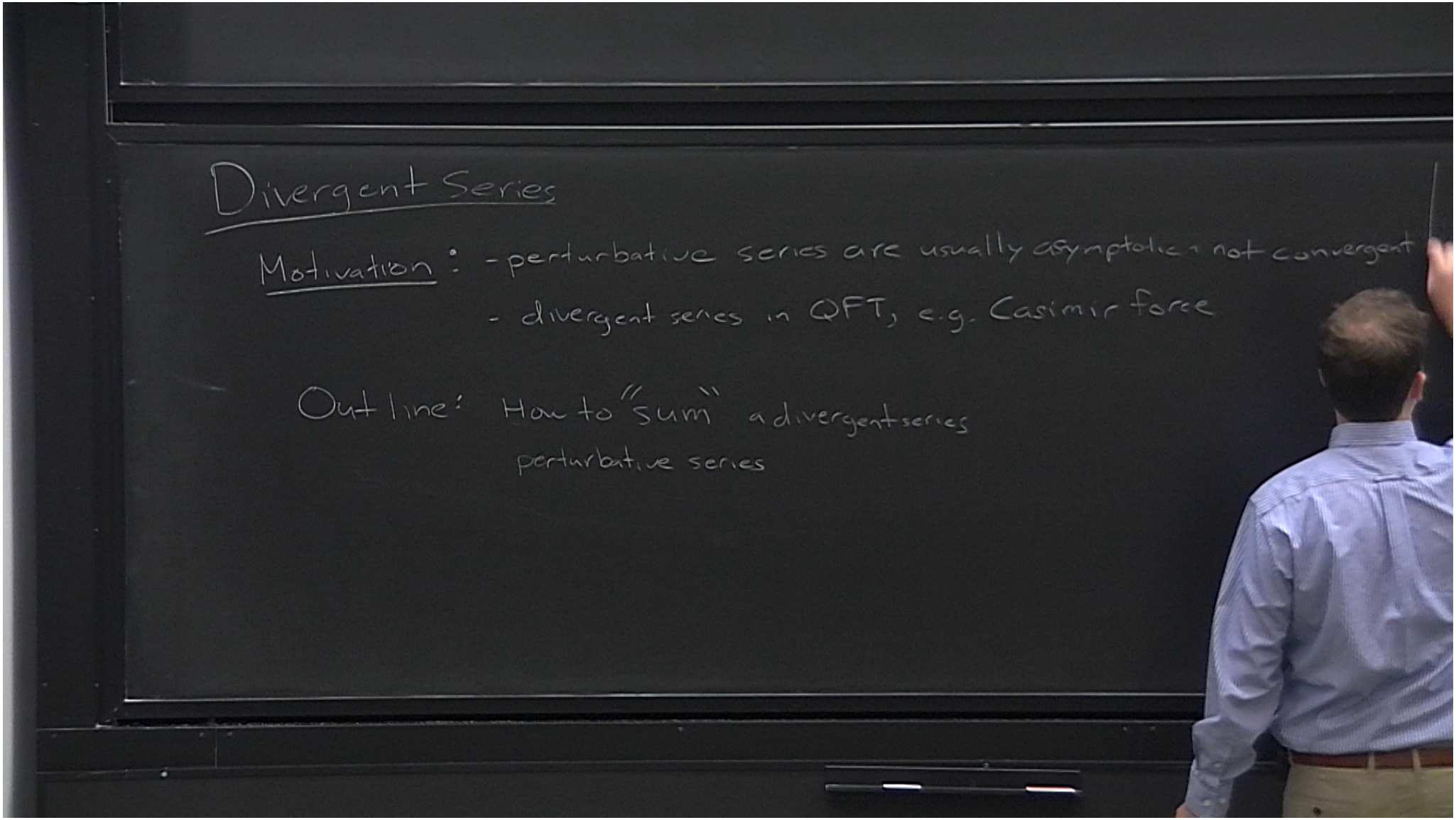
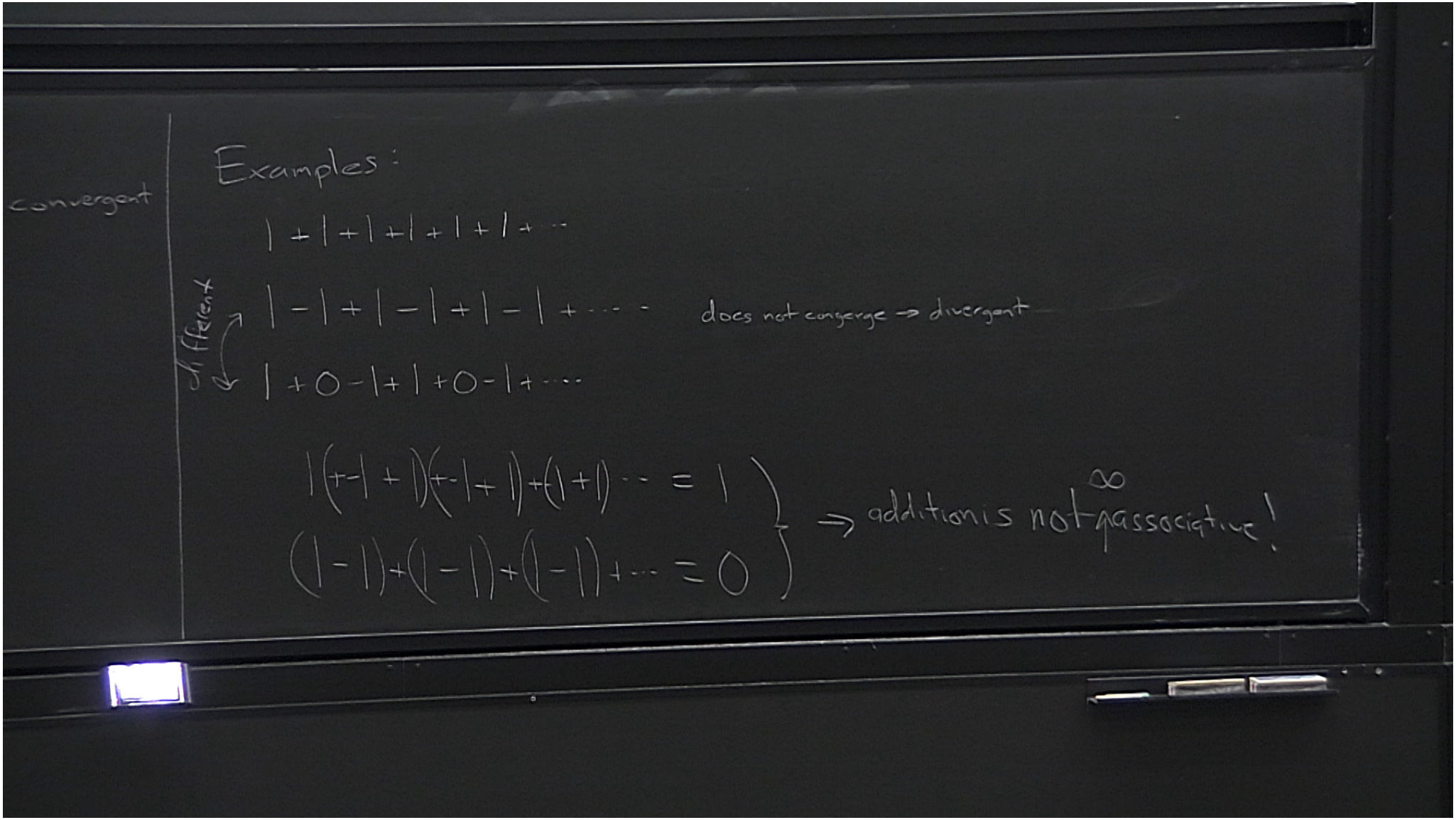
Title: PSI 2018/2019 - Math for QFT - Lecture 5

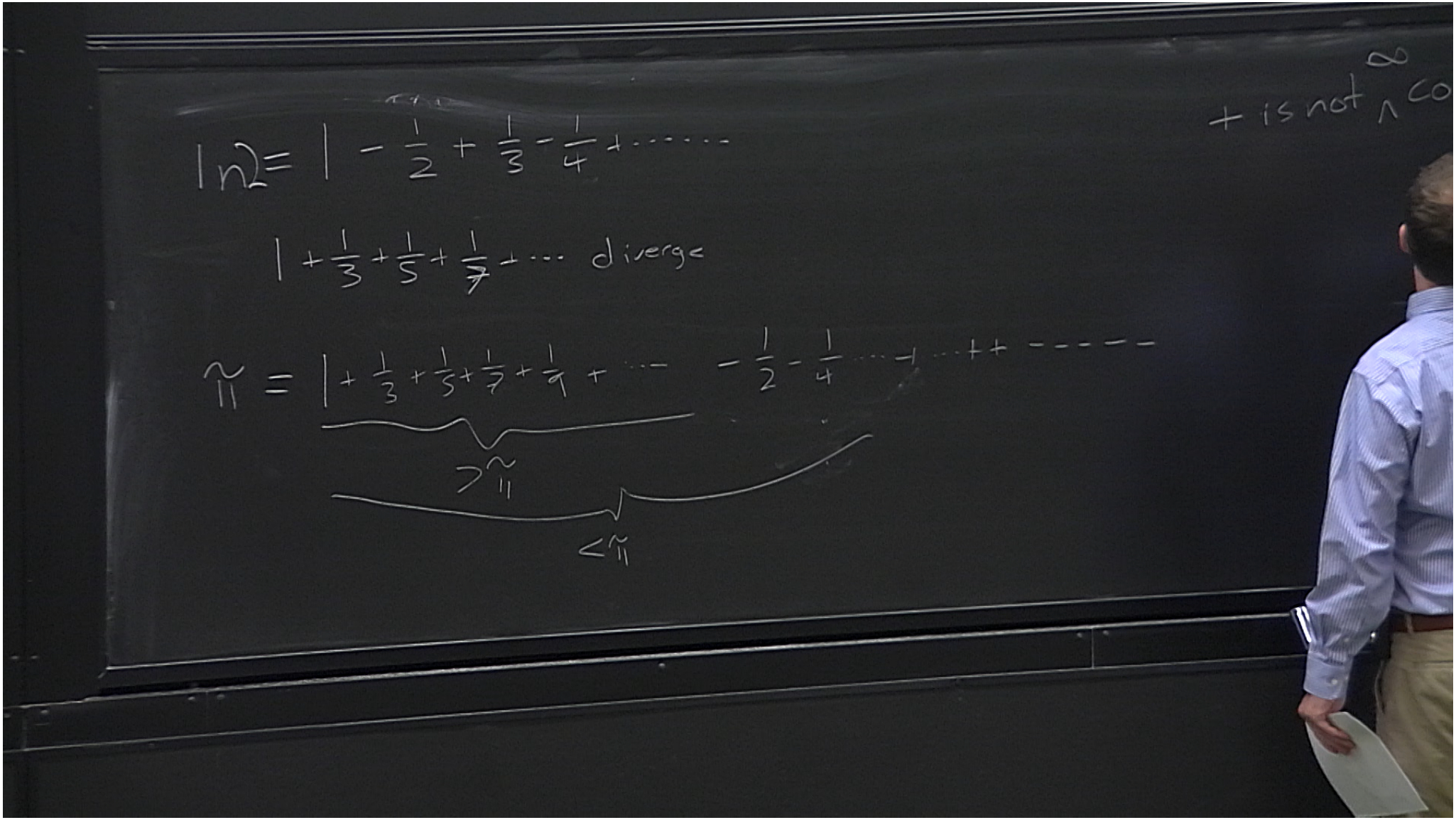
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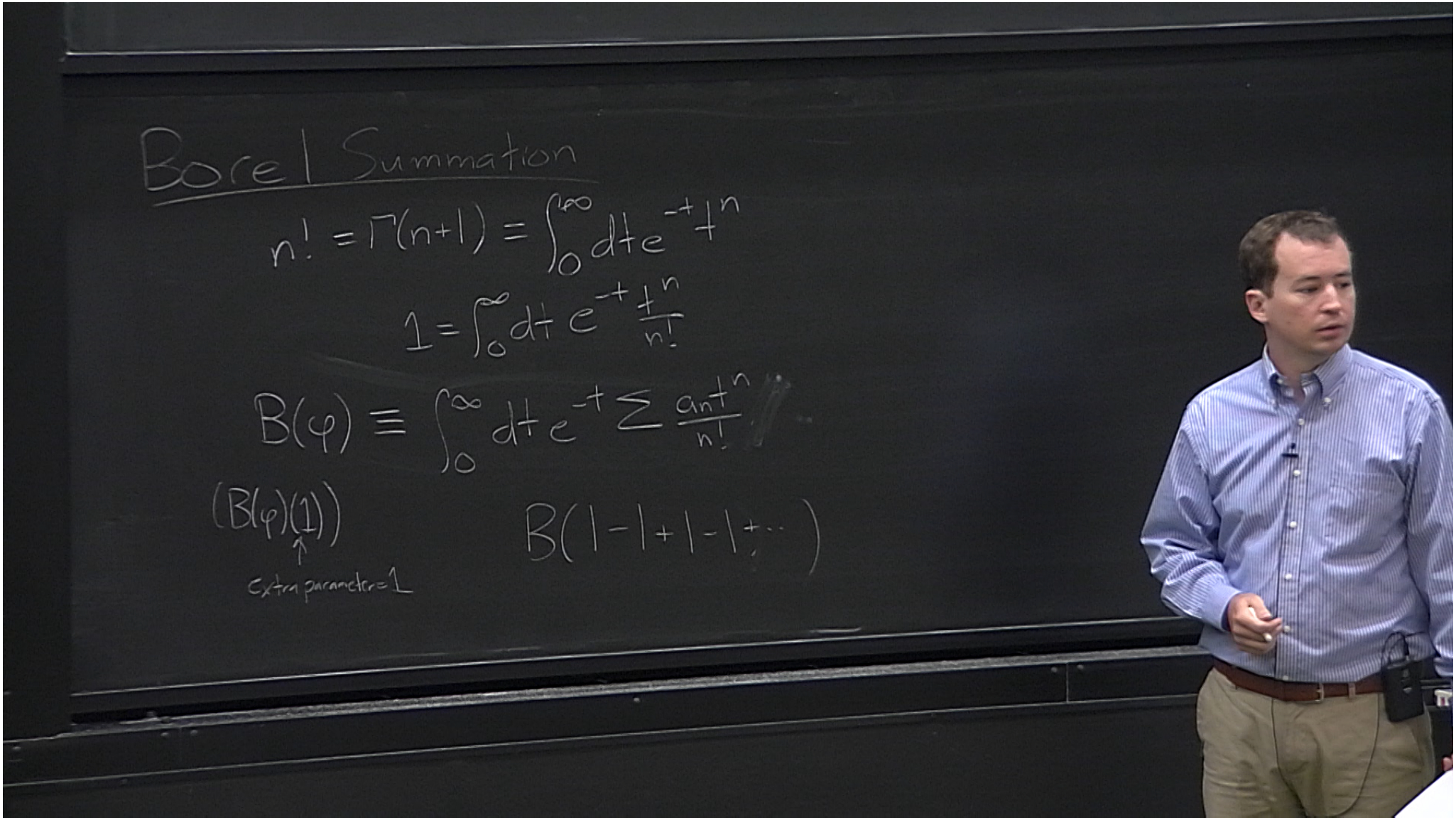
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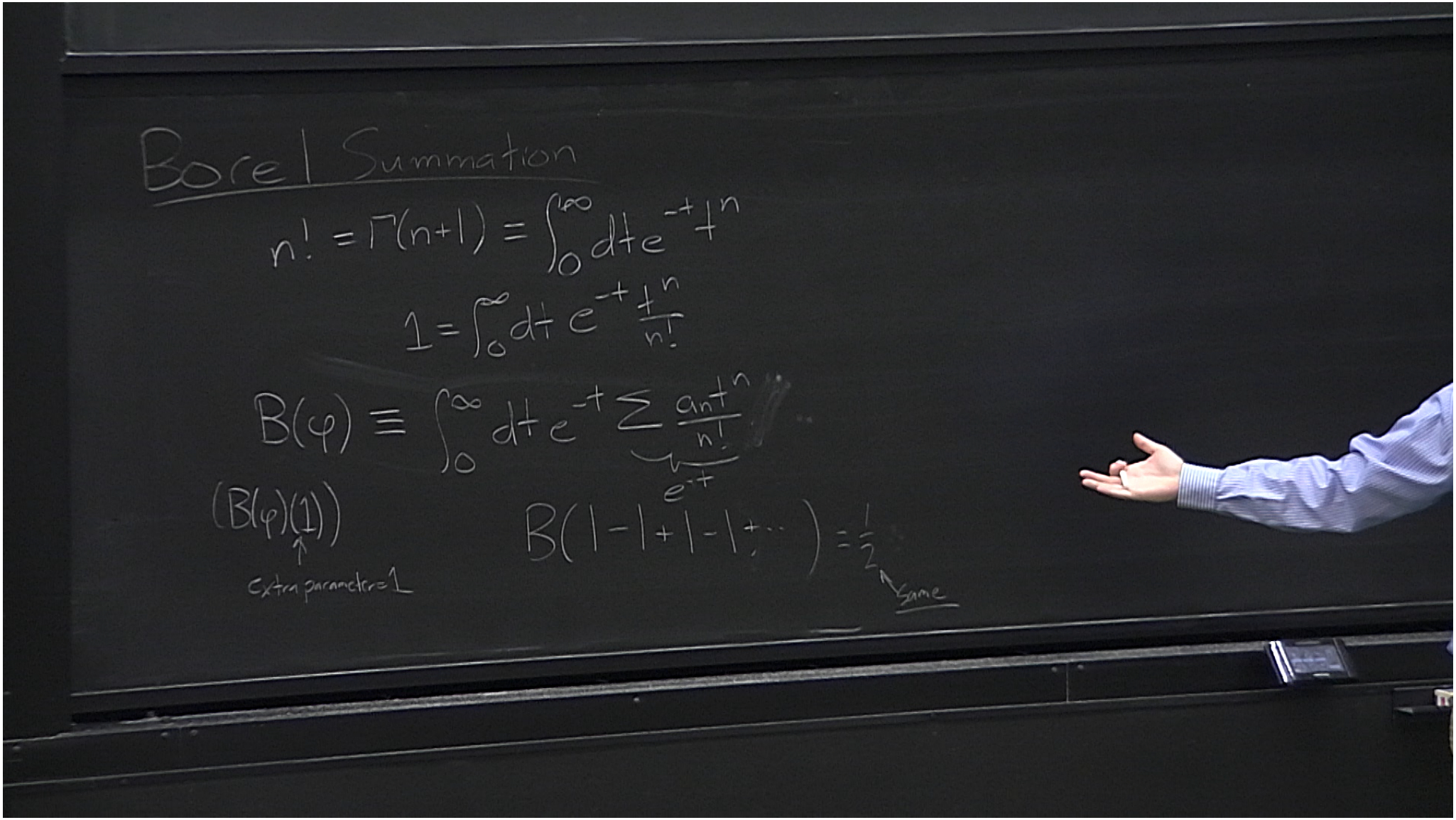
Abstract:

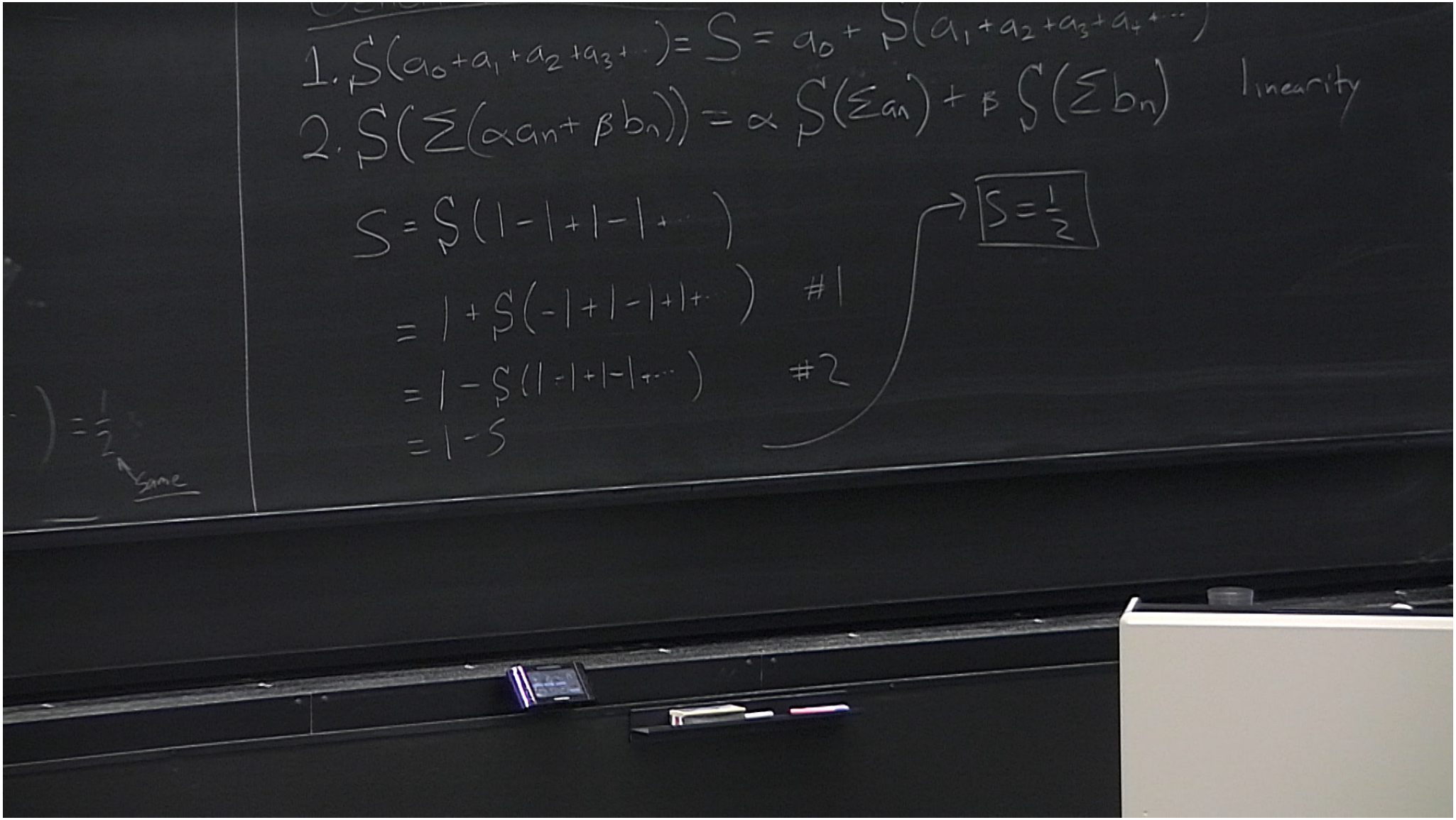

Divergent Series
Motivation: - perturbative series are usually asymptotic + not convergent
- divergent series in QFT, e.g. Casimir force
Outline: How to "sum" a divergent series
perturbative series

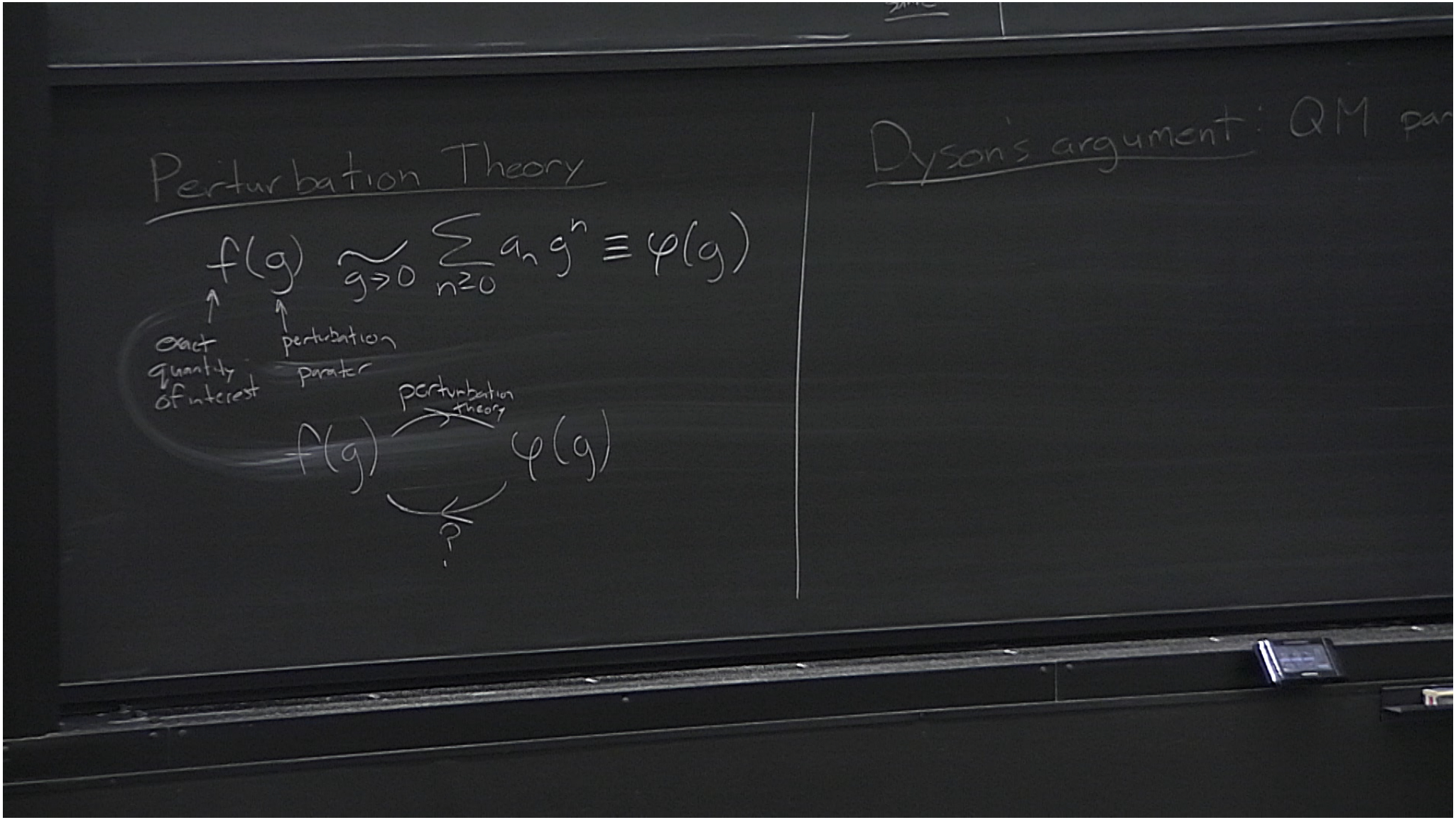

convergent
Examples:
1 + 1 + 1 + 1 + 1 + 1 + 1 + ···
different
1 − 1 + 1 − 1 + 1 − 1 + ··· does not converge ⇒ divergent
1 + 0 − 1 + 1 + 0 − 1 + ···
1 + (−1 + 1) + (−1 + 1) + (1 + 1) ··· = 1
(1 − 1) + (1 − 1) + (1 − 1) + ··· = 0
⇒ addition is not associative!

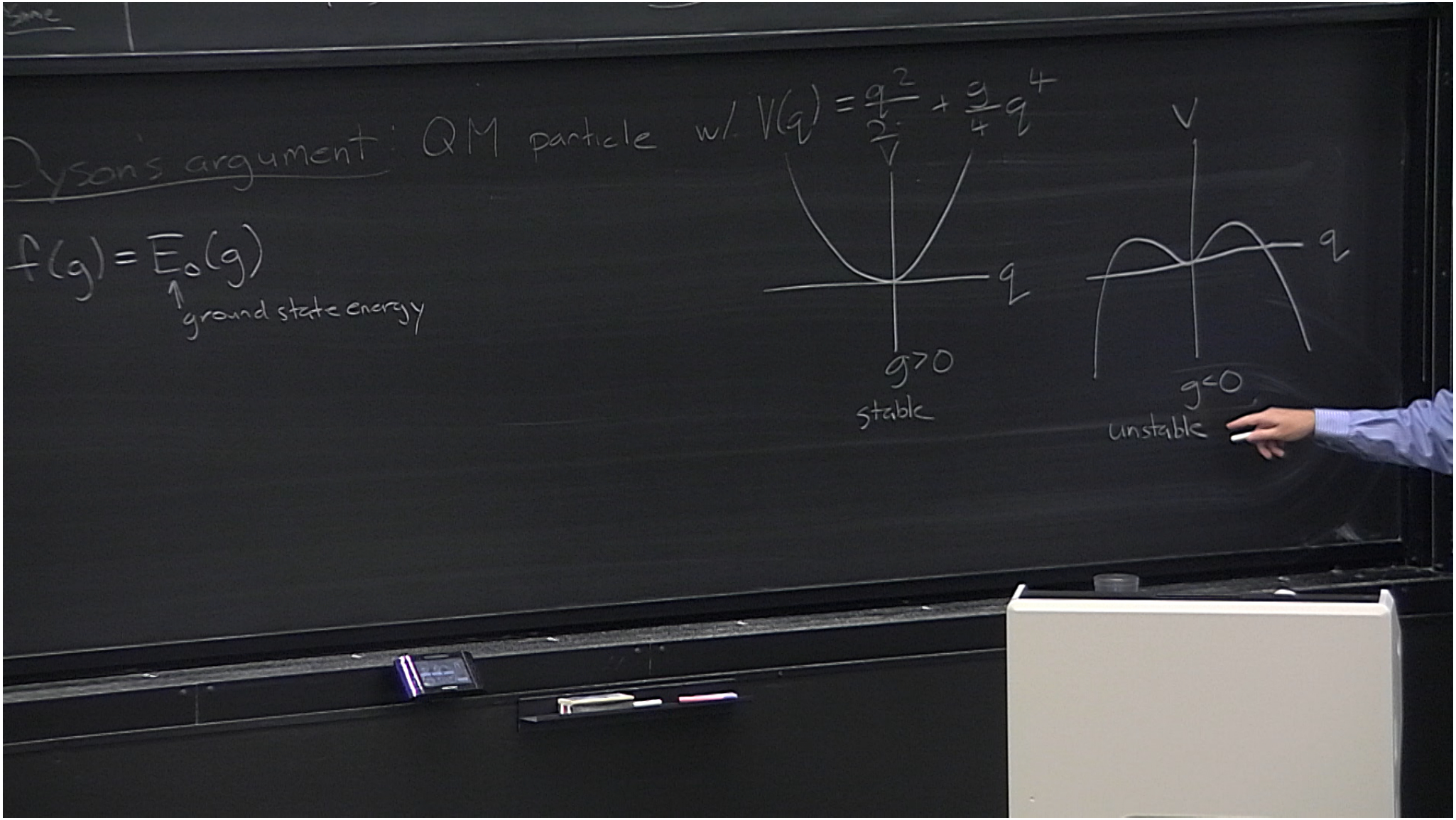

ln2 = 1 - 1/2 + 1/3 - 1/4 + ...
+ is not ∞ ∧ co
1 + 1/3 + 1/5 + 1/7 + ... diverge
π̃ = 1 + 1/3 + 1/5 + 1/7 + 1/9 + ... - 1/2 - 1/4 ... + + - - - - -
> π̃
< π̃


Borel Summation
$n! = \Gamma(n+1) = \int_0^\infty dt\, e^{-t} t^n$
$1 = \int_0^\infty dt\, e^{-t} \frac{t^n}{n!}$
$B(\varphi) \equiv \int_0^\infty dt\, e^{-t} \sum \frac{a_n t^n}{n!}$
$(B(\varphi)(1))$
Extra parameter = 1
$B(1-1+1-1+\cdots)$


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e^{-t}
$(B(\varphi)(1))$
Extra parameter = 1
$B(1-1+1-1+\cdots) = \frac{1}{2}$
Same


1. $S(a_0+a_1+a_2+a_3+\cdots) = S = a_0 + S(a_1+a_2+a_3+a_4+\cdots)$
2. $S(\Sigma(\alpha a_n+\beta b_n)) = \alpha S(\Sigma a_n) + \beta S(\Sigma b_n)$ linearity
$S = S(1-1+1-1+\cdots)$
$= 1 + S(-1+1-1+1+\cdots)$ #1
$= 1 - S(1-1+1-1+\cdots)$ #2
$= 1 - S$
$S = \frac{1}{2}$
same

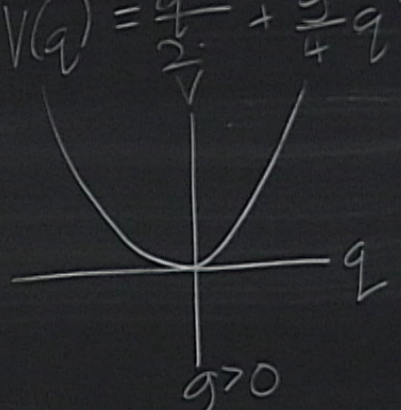

Perturbation Theory
$f(g) \underset{g>0}{\sim} \sum_{n\geq 0} a_n g^n \equiv \varphi(g)$
exact quantity of interest
perturbation parater
perturbation theory
f(g)
φ(g)
?
Dyson's argument: QM

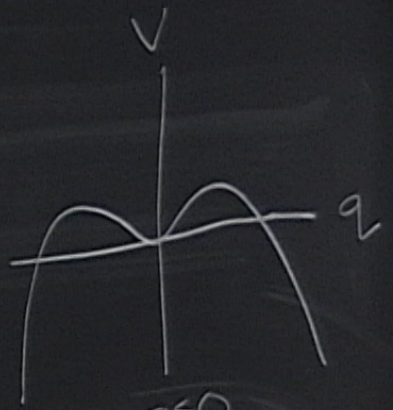

Dyson's argument: QM particle w/ $V(q) = \frac{q^2}{2} + \frac{g}{4}q^4$
$f(g) = E_0(g)$
↑ ground state energy
V
q
g>0
stable
V
q
g<0
unstable

Dyson's argument: QM particle w/ $V(q) = \frac{1}{2}q^2 + \frac{g}{4}q^4$

$f(g) = E_0(g)$

↑ ground state energy





If $\varphi(g)$ has finite radius of convergence

→ describes $g>0$ and $g<0$ NP physics → divergence of φ

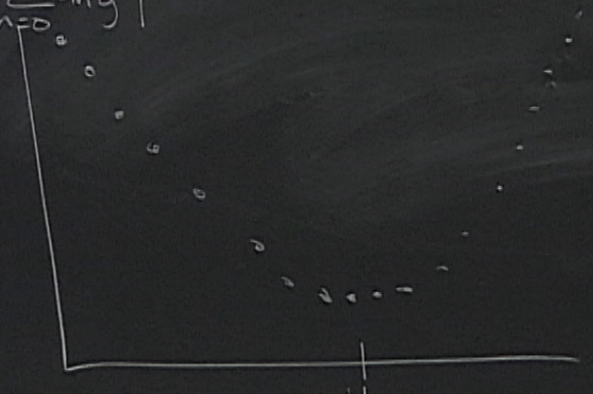
→ contradiction

$\varphi(g)$ must have zero radius of convergence

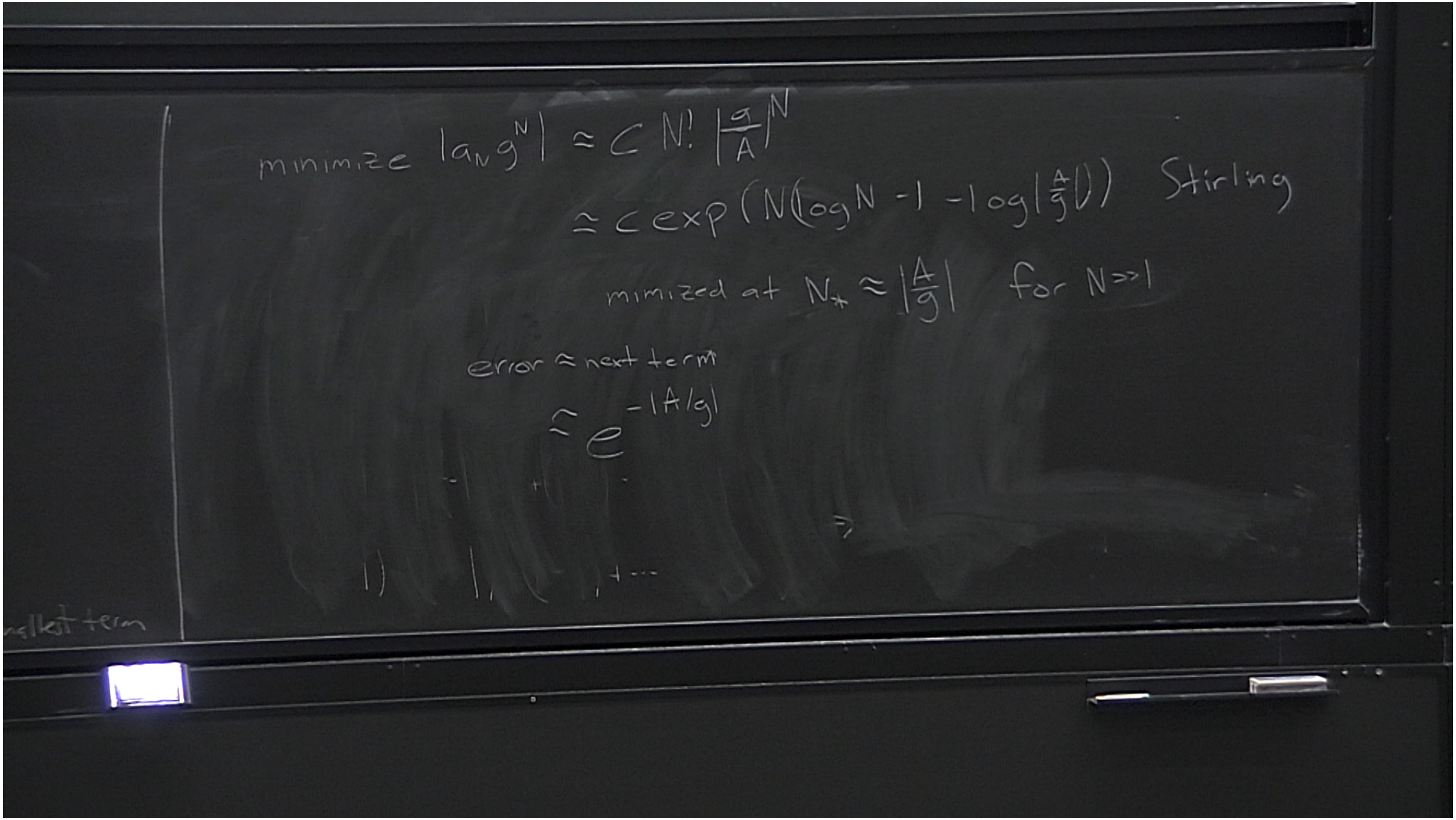
Optimal truncation

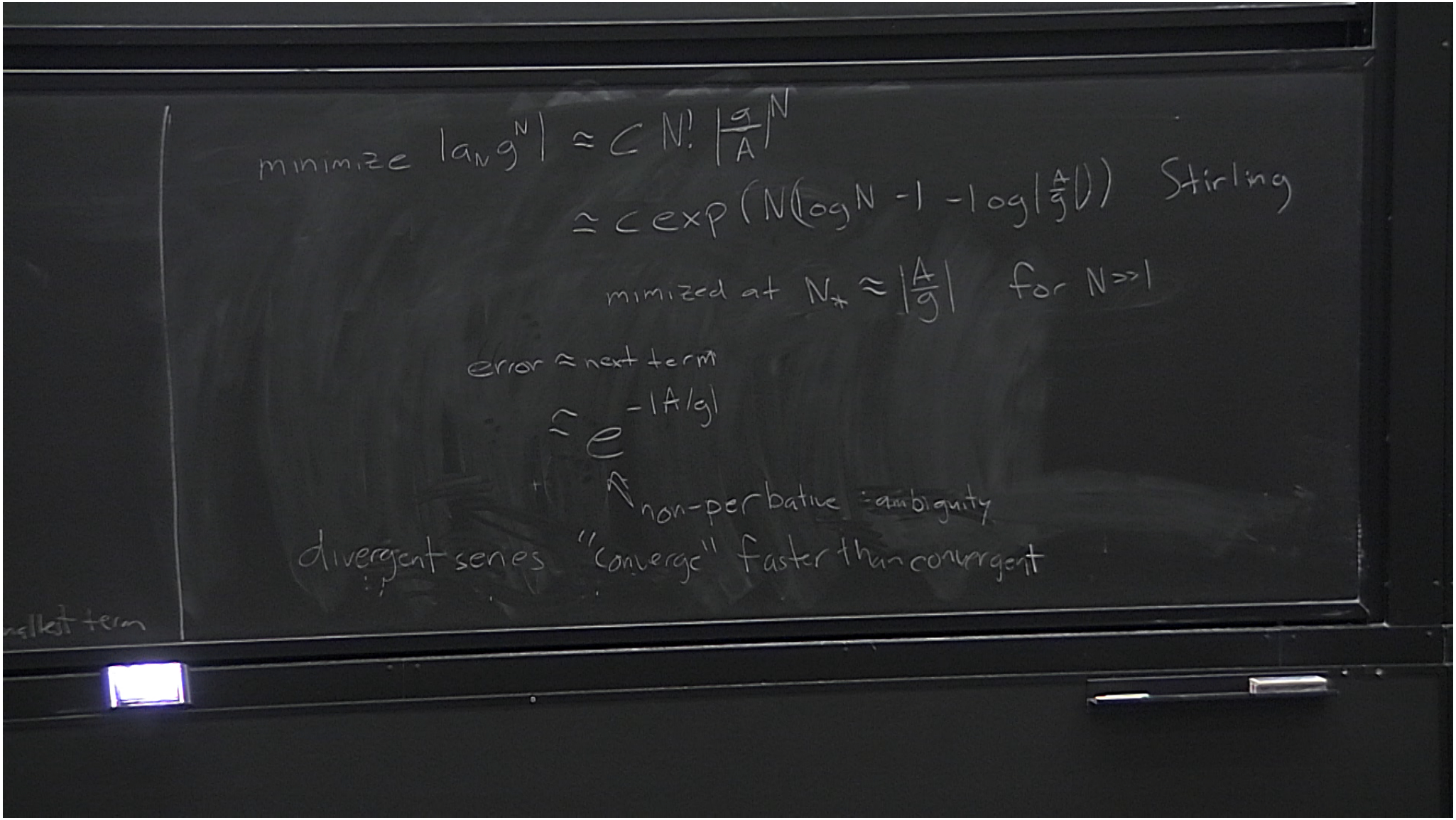
typically in QM or QFT

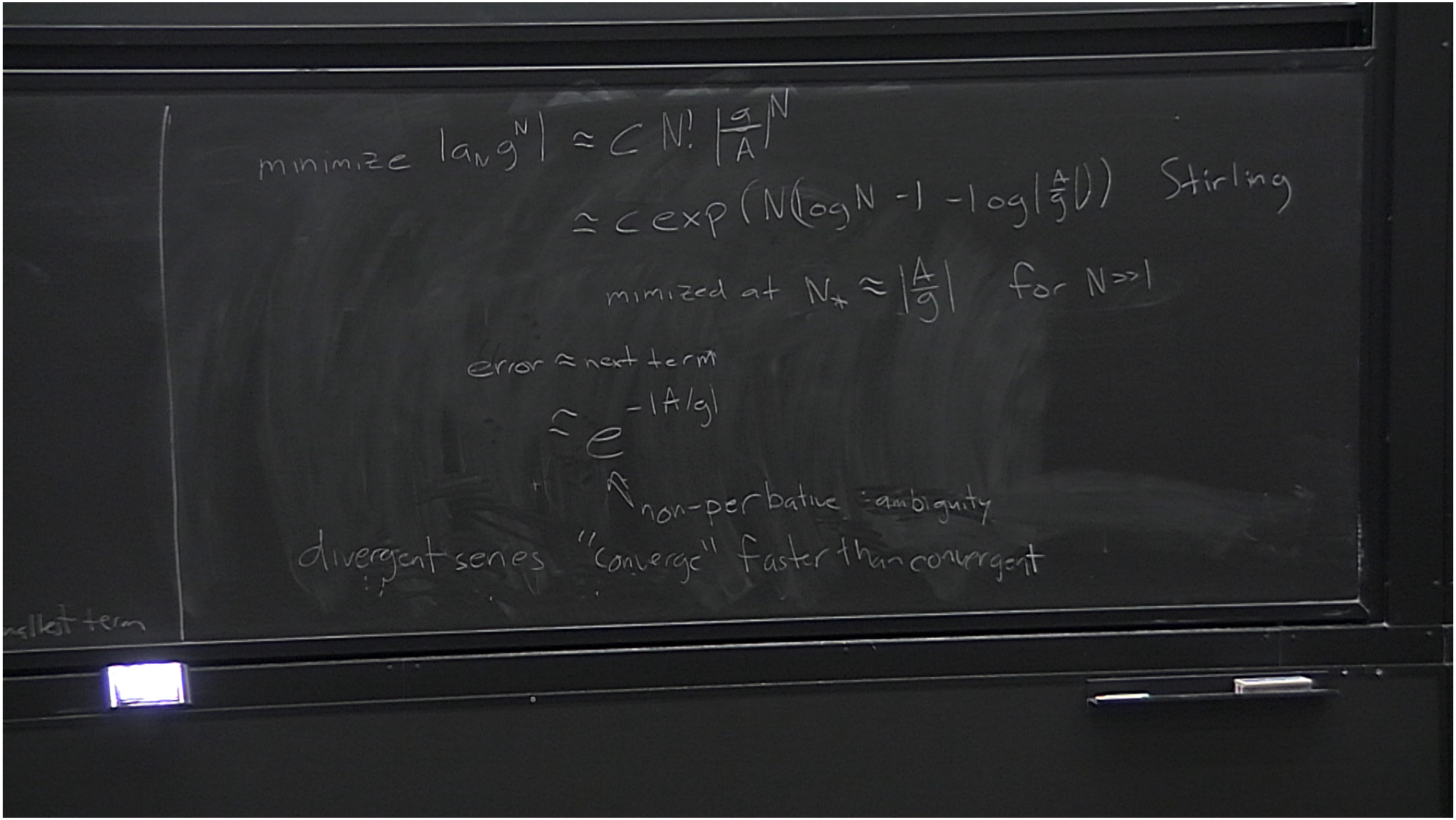
$$\text{If } -\sum_{n=0}^{N} a_n g^n \Big| \quad a_n \tilde{\underset{n\to\infty}{=}} A^{-n} n!$$

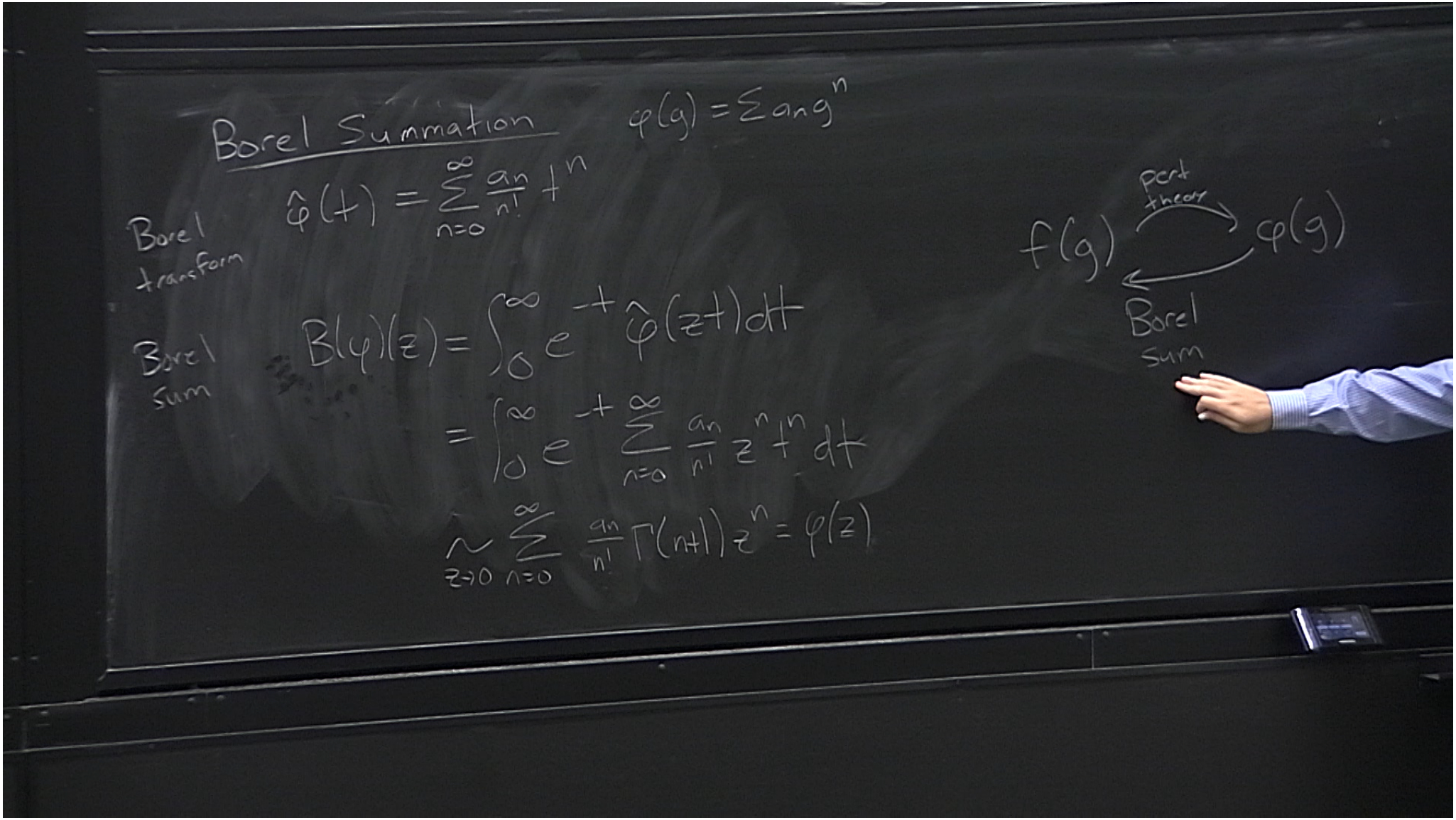


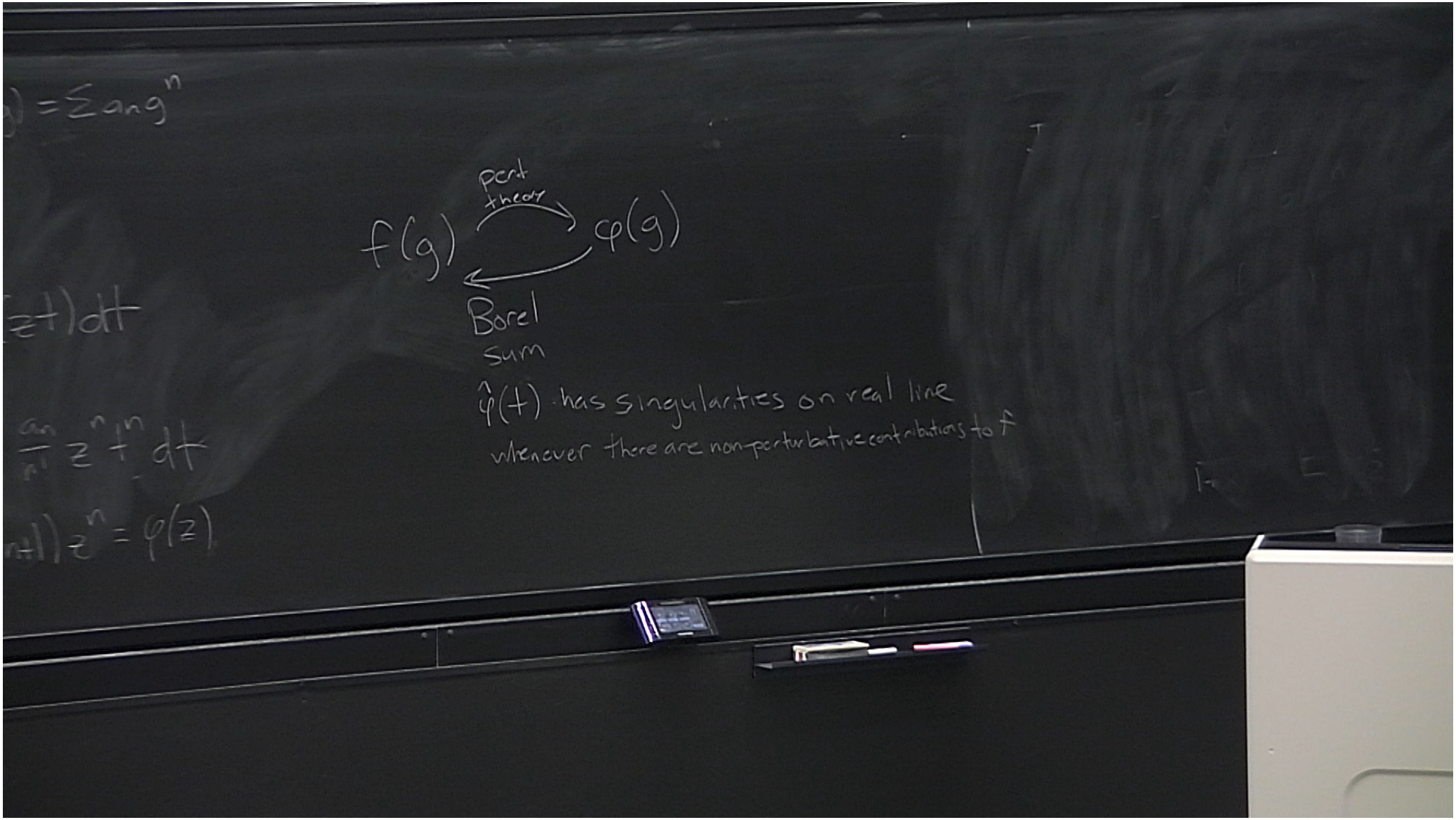
N_* ← optimal truncation is usually to keep terms up to the smallest

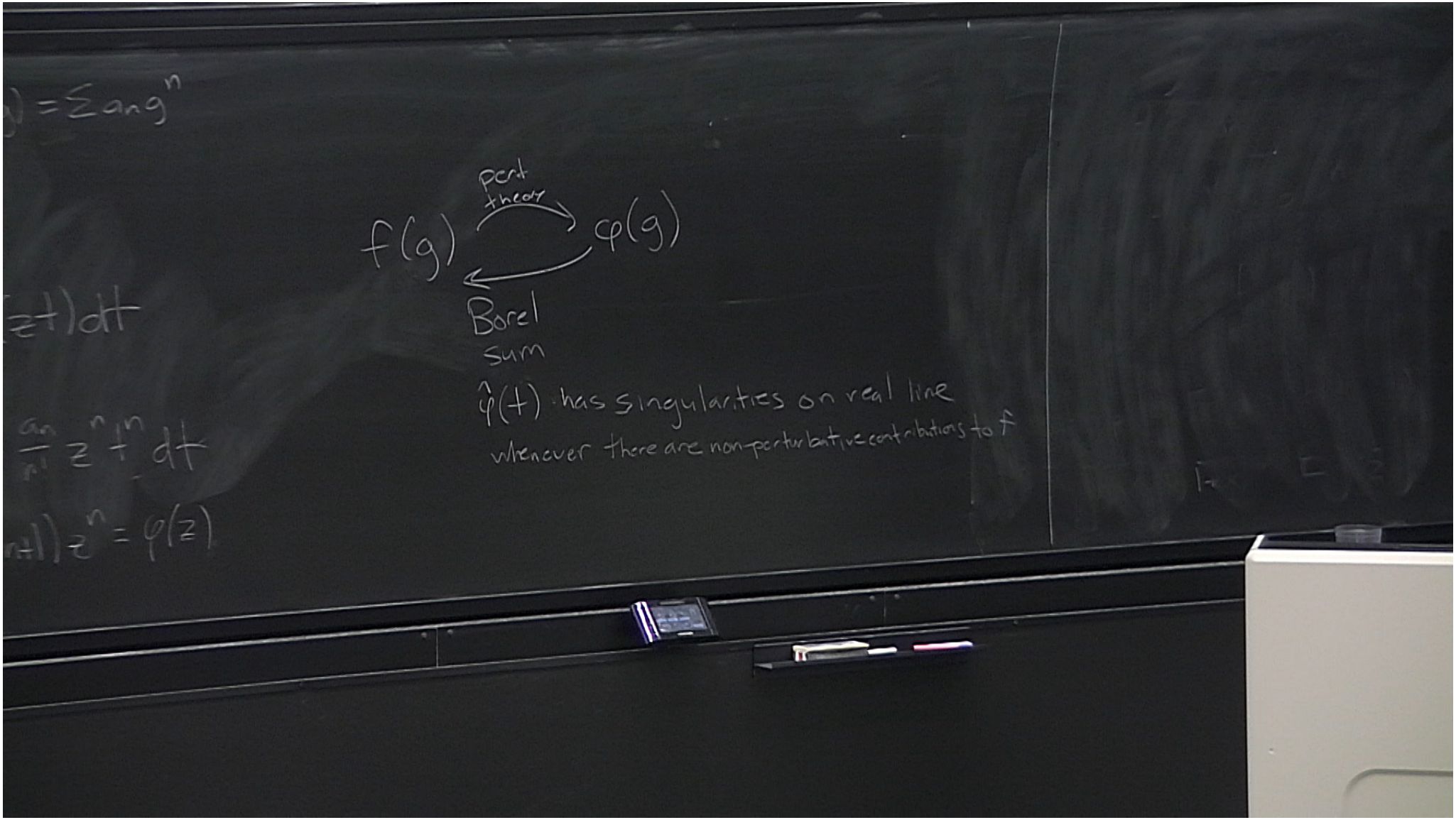

minimize $|a_N g^N| \approx C\, N!\, \left|\frac{g}{A}\right|^N$
$\simeq C \exp\left(N(\log N - 1 - \log\left|\frac{A}{g}\right|)\right)$ Stirling
minimized at $N_* \approx \left|\frac{A}{g}\right|$ for $N \gg 1$
error ≈ next term
$\approx e^{-|A/g|}$


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error ≈ next term
$\sim e^{-|A/g|}$
non-perbative : ambiguity
divergent series "converge" faster than convergent
smallest term


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$\simeq e^{-|A/g|}$
non-perbative : ambiguity
divergent series "converge" faster than convergent
smallest term


Borel Summation
$\varphi(g) = \sum a_n g^n$
Borel transform
$\hat{\varphi}(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$
Borel sum
$B(\varphi)(z) = \int_0^{\infty} e^{-t} \hat{\varphi}(zt)\,dt$
$= \int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n t^n\,dt$
$\sim_{z \to 0} \sum_{n=0}^{\infty} \frac{a_n}{n!} \Gamma(n+1) z^n = \varphi(z)$
pert theory
$f(g)$
$\varphi(g)$
Borel sum


$= \sum a_n g^n$
$f(g)$
pert theory
$\varphi(g)$
Borel sum
$\hat{\varphi}(t)$ has singularities on real line
whenever there are non-perturbative contributions to f
$zt)dt$
$\frac{a_n}{n!} z^n t^n dt$
$n+1) z^n = \varphi(z)$


$= \Sigma a_n g^n$
$f(g)$
pert theory
$\varphi(g)$
Borel sum
$\hat{\varphi}(t)$ has singularities on real line
whenever there are non-perturbative contributions to f