Title: PSI 2018/2019 - Math for QFT - Lecture 4

Date: Aug 22, 2018 10:30 AM

URL: http://pirsa.org/18080020

Abstract:

Saddle-point Approximation

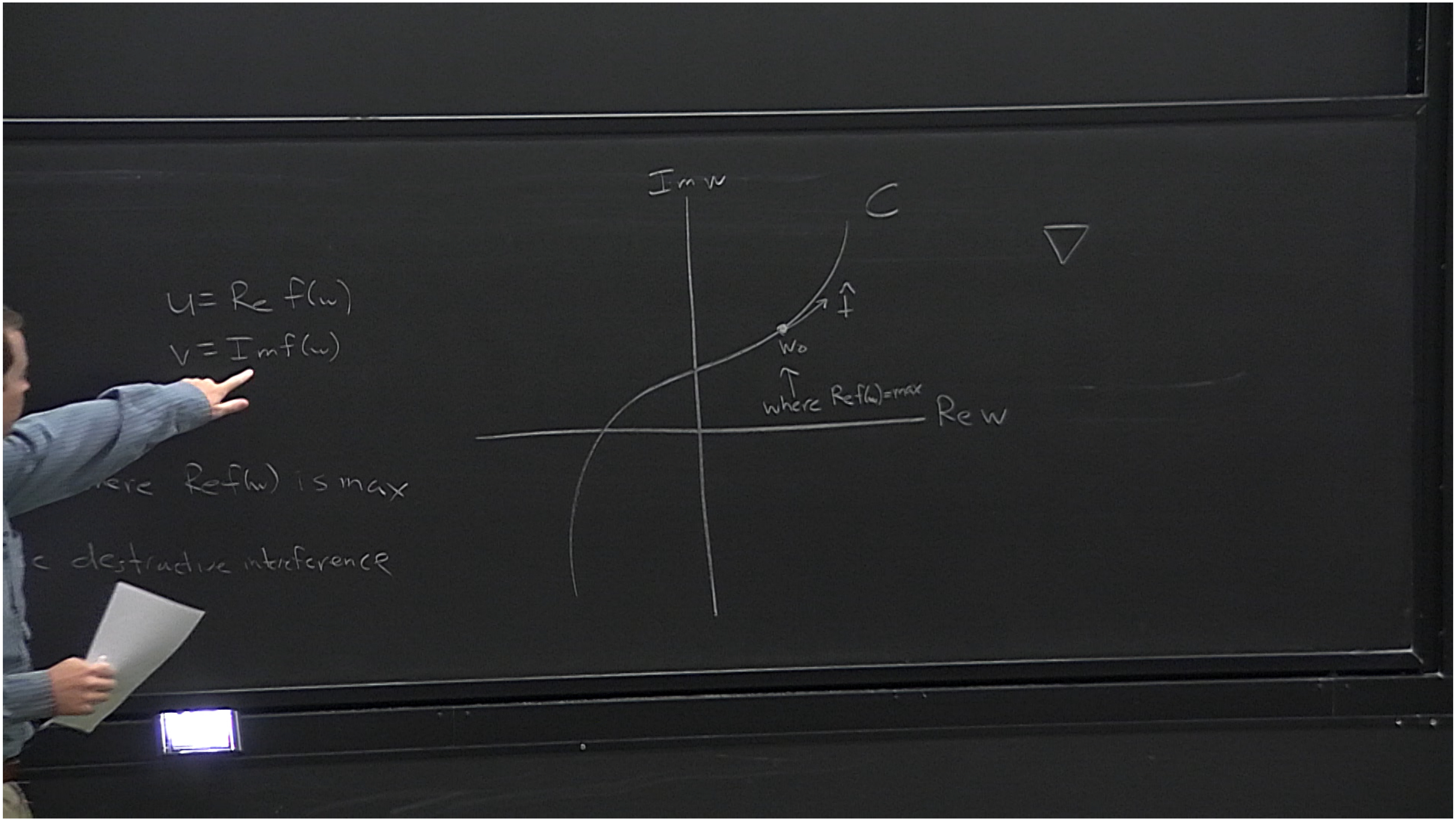
$$I(z) = \int_C g(w)\, e^{z f(w)}\, dw$$

$z \to \infty$

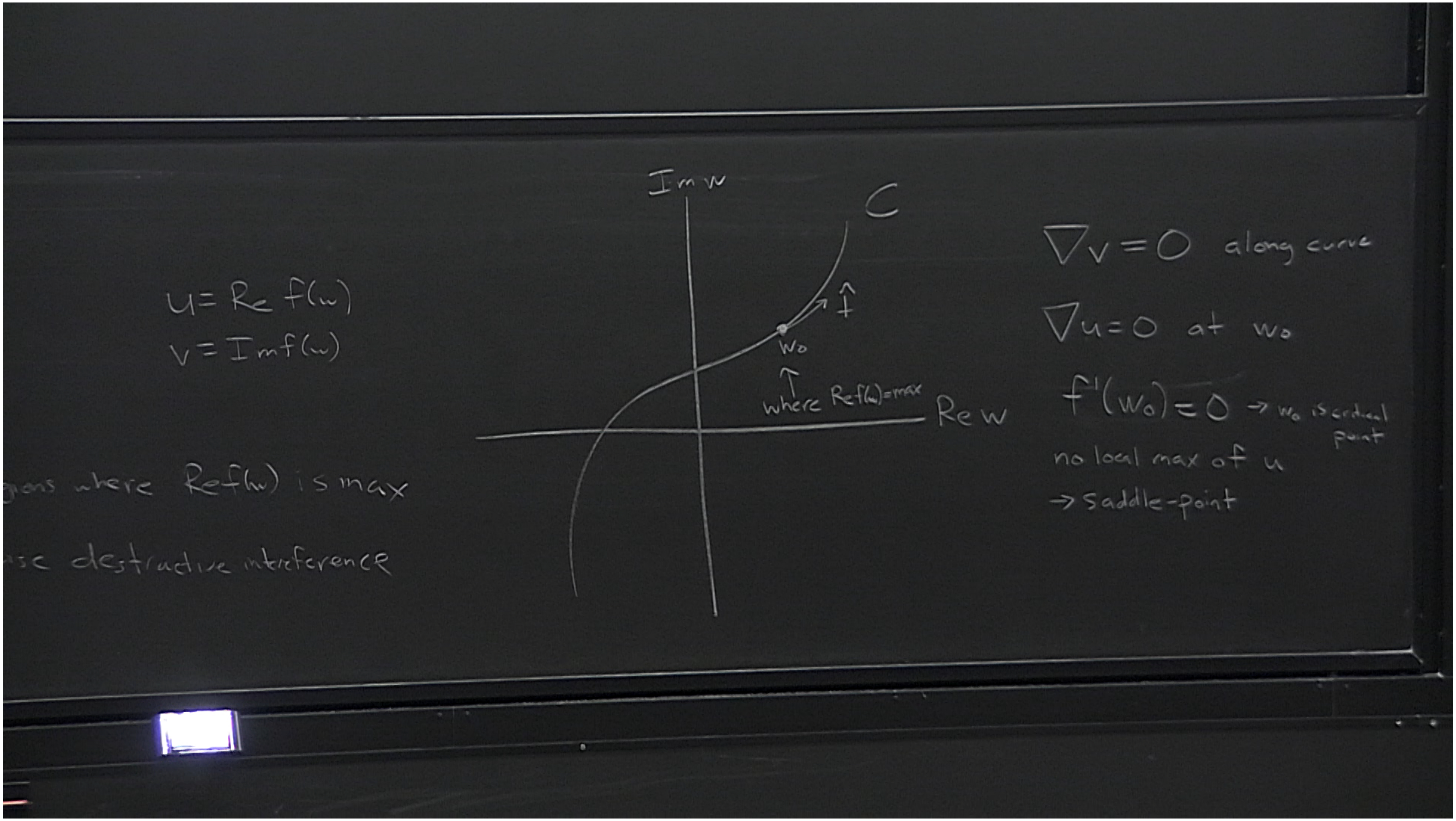
$z = \frac{1}{\lambda}$

might expect dominant contributions to come from regions where $\operatorname{Re} f(w)$ is max

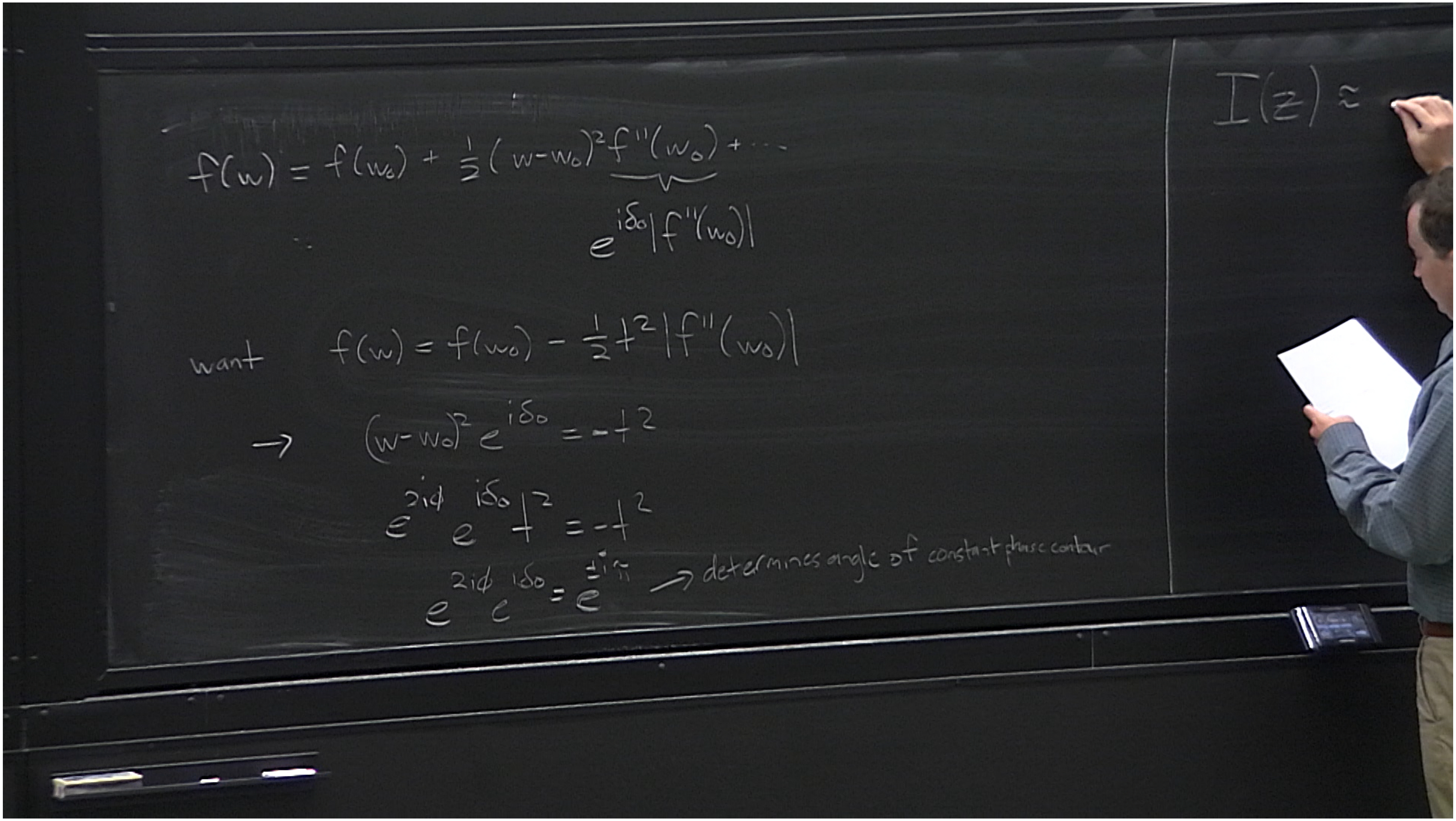
- works if C is a constant phase contour - otherwise destructive interference

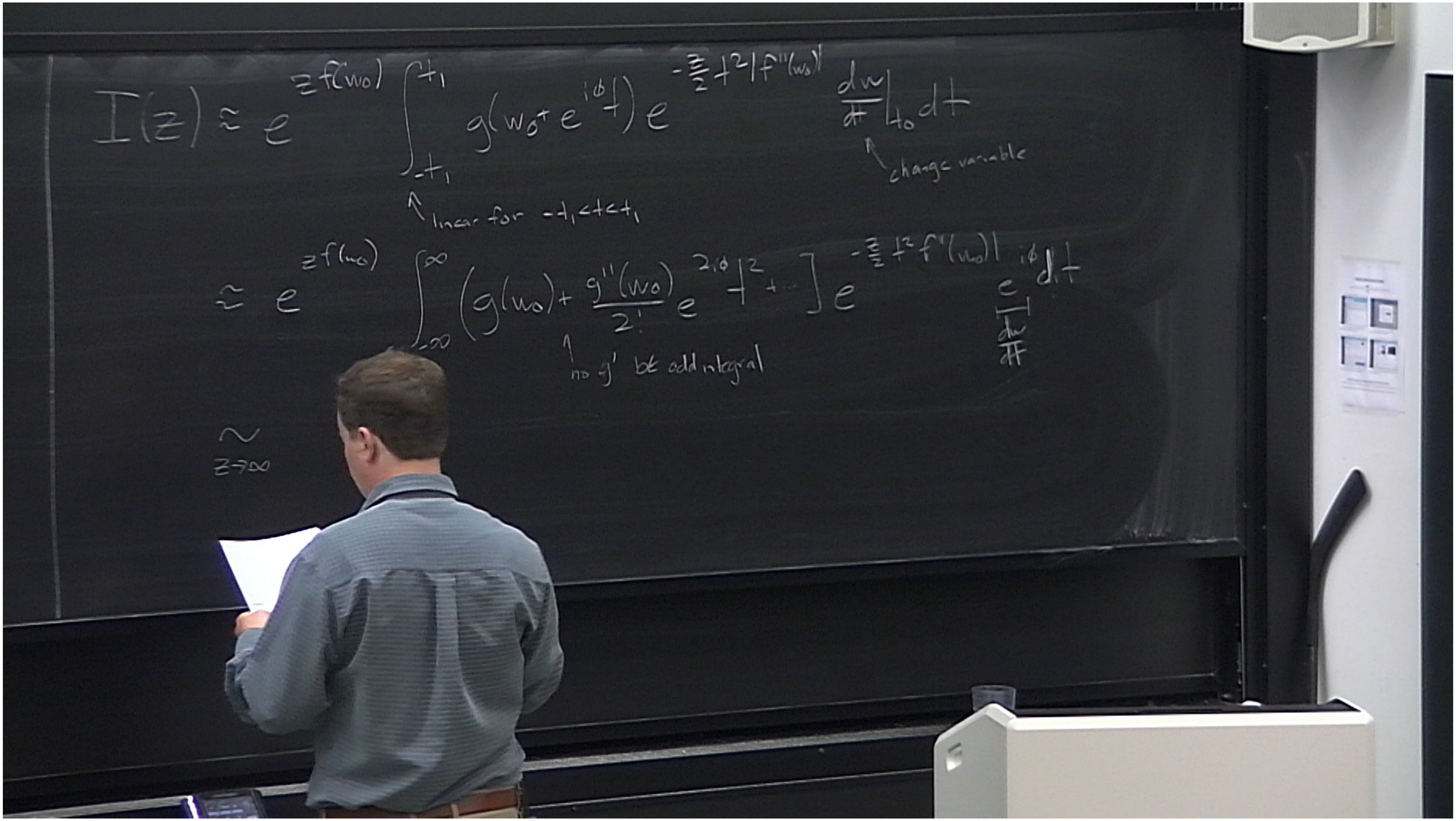


u = Re f(w)
v = Im f(w)
Ref(w) is max
destructive interference
Im w
C
w0
where Ref(w)=max
Re w



$u = \mathrm{Re}\, f(w)$
$v = \mathrm{Im}\, f(w)$
gions where Re f(w) is max
se destructive interference
Im w
C
$\hat{t}$
w_0
where Re f(w)=max
Re w
$\nabla v = 0$ along curve
$\nabla u = 0$ at w_0
$f'(w_0) = 0$ → w_0 is critical point
no local max of u
→ saddle-point


$f(w) = f(w_0) + \frac{1}{2}(w-w_0)^2 \underbrace{f''(w_0)}_{e^{i\delta_0}|f''(w_0)|} + \cdots$
want $f(w) = f(w_0) - \frac{1}{2}t^2|f''(w_0)|$
$\rightarrow (w-w_0)^2 e^{i\delta_0} = -t^2$
$e^{2i\phi} e^{i\delta_0} t^2 = -t^2$
$e^{2i\phi} e^{i\delta_0} = e^{\pm i\pi}$ → determines angle of constant phase contour
$I(z) \simeq$


$I(z) \simeq e^{zf(w_0)} \int_{-t_1}^{t_1} g(w_0 + e^{i\phi} t)\, e^{-\frac{z}{2} t^2 |f''(w_0)|} \frac{dw}{dt}\Big|_{t_0} dt$
linear for $-t_1 < t < t_1$
change variable
$\simeq e^{zf(w_0)} \int_{-\infty}^{\infty} \left(g(w_0) + \frac{g''(w_0)}{2!} e^{2i\phi} t^2 + \dots\right] e^{-\frac{z}{2} t^2 |f''(w_0)|} \, e^{i\phi} dt$
no g' b/c odd integral
$\sim_{z \to \infty}$

$$I(z) \simeq e^{z f(w_0)} \int_{-t_1}^{t_1} g(w_0 + e^{i\phi} t)\, e^{-\frac{z}{2} t^2 |f''(w_0)|} \left.\frac{dw}{dt}\right|_{t_0} dt$$

linear for $-t_1 < t < t_1$

change variable

$$\simeq e^{z f(w_0)} \int_{-\infty}^{\infty} \left(g(w_0) + \frac{g''(w_0)}{2!} e^{2i\phi} t^2 + \dots\right] e^{-\frac{z}{2} t^2 |f''(w_0)|}\, e^{i\phi}\, dt$$

$e^{i\phi} = \frac{dw}{dt}$

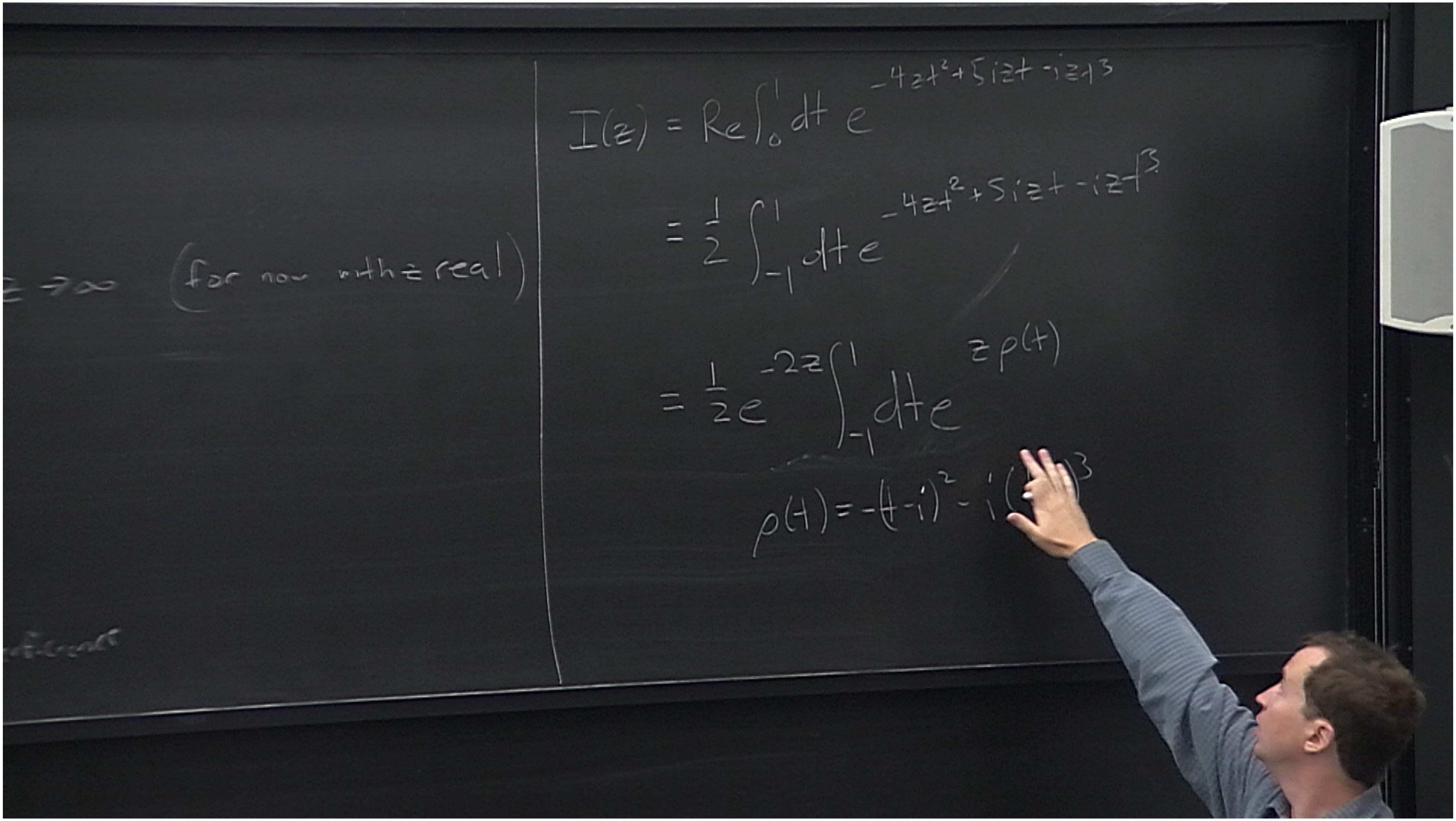
no g' b/c odd integral

$$\underset{z\to\infty}{\sim} \sqrt{\frac{2\pi}{z}}\, \frac{e^{z f(w_0)}}{\sqrt{-f''(w_0)}} \left[g(w_0) - \frac{1}{z} \frac{g''(w_0)}{1!\, 2 f''(w_0)} + \dots \right]$$

Stokes' Phenomenon

- abrupt change in asymptotic relations as phase of z changes

Example: $$I(z) = \int_0^1 dt\, e^{-4zt^2} \cos(5zt - zt^3) \qquad z \to \infty$$ (for now with z r


$z \to \infty$ (for now with z real)
$I(z) = \mathrm{Re} \int_0^1 dt\, e^{-4zt^2+5izt-izt^3}$
$= \frac{1}{2} \int_{-1}^{1} dt\, e^{-4zt^2+5izt-izt^3}$
$= \frac{1}{2} e^{-2z} \int_{-1}^{1} dt\, e^{z\rho(t)}$
$\rho(t) = -(t-i)^2 - i($

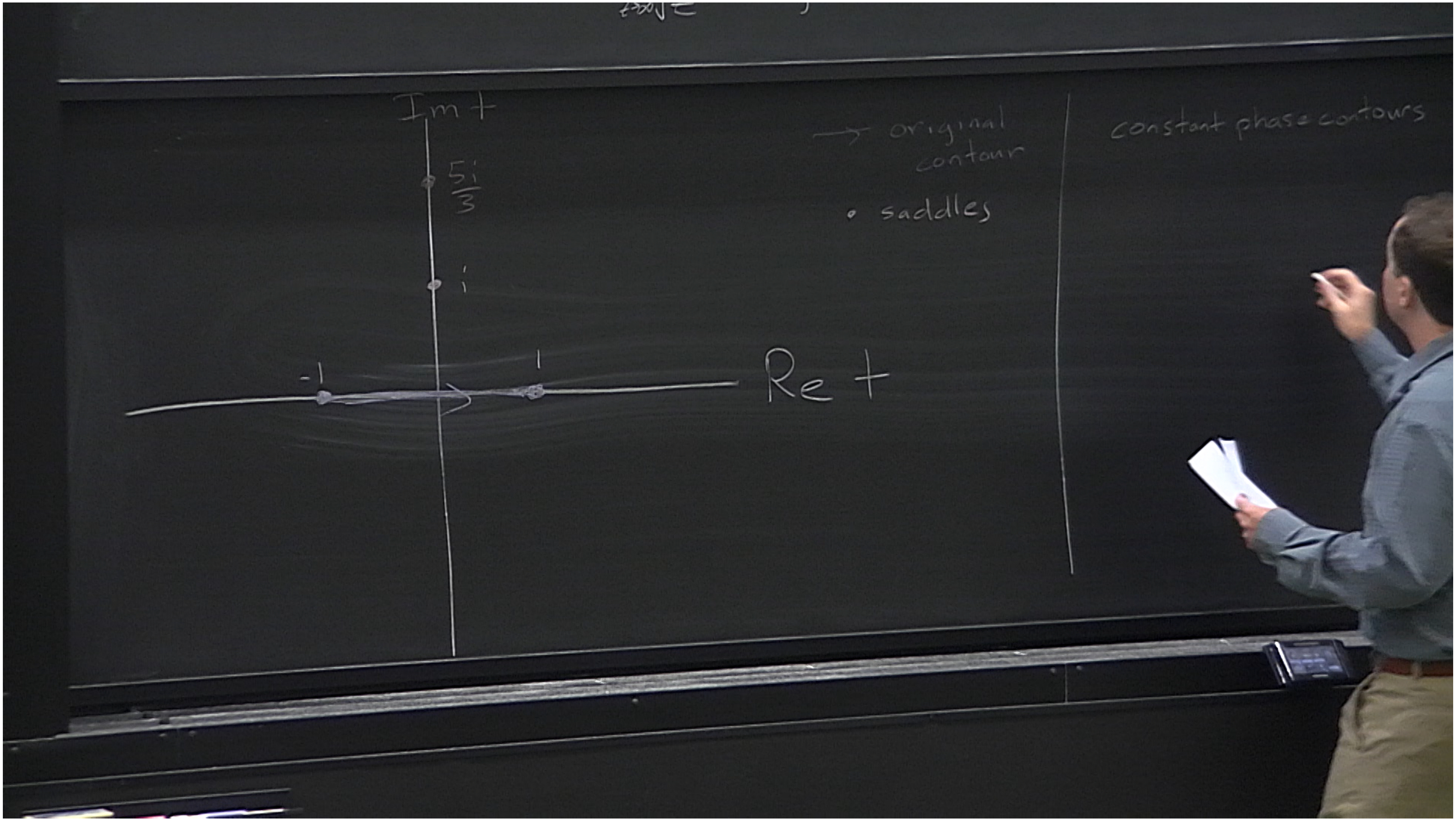
$z \to \infty$ (for now with z real)

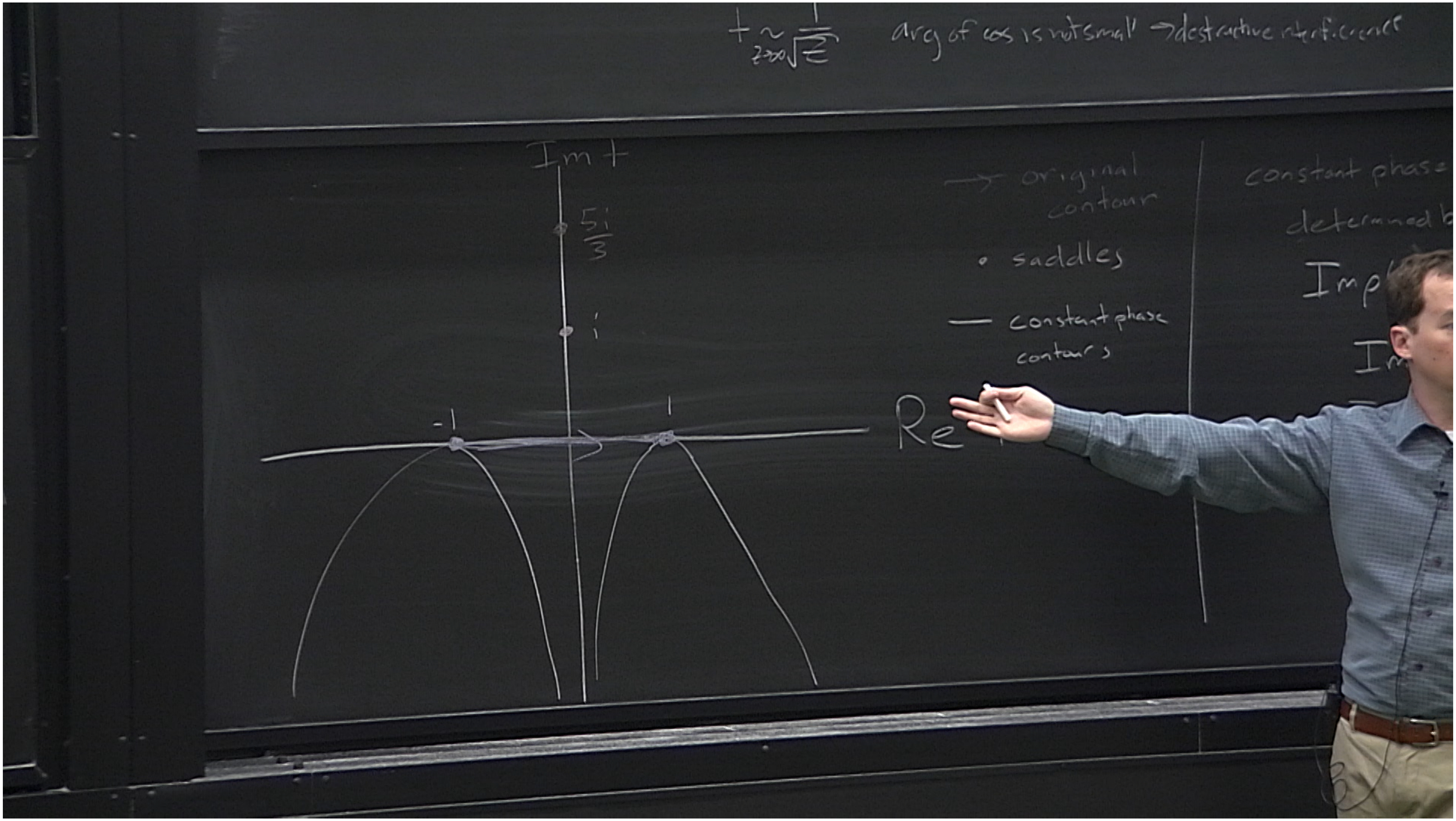
$$= \frac{1}{2}\int_{-1}^{1} dt\, e^{-4z\ldots}$$

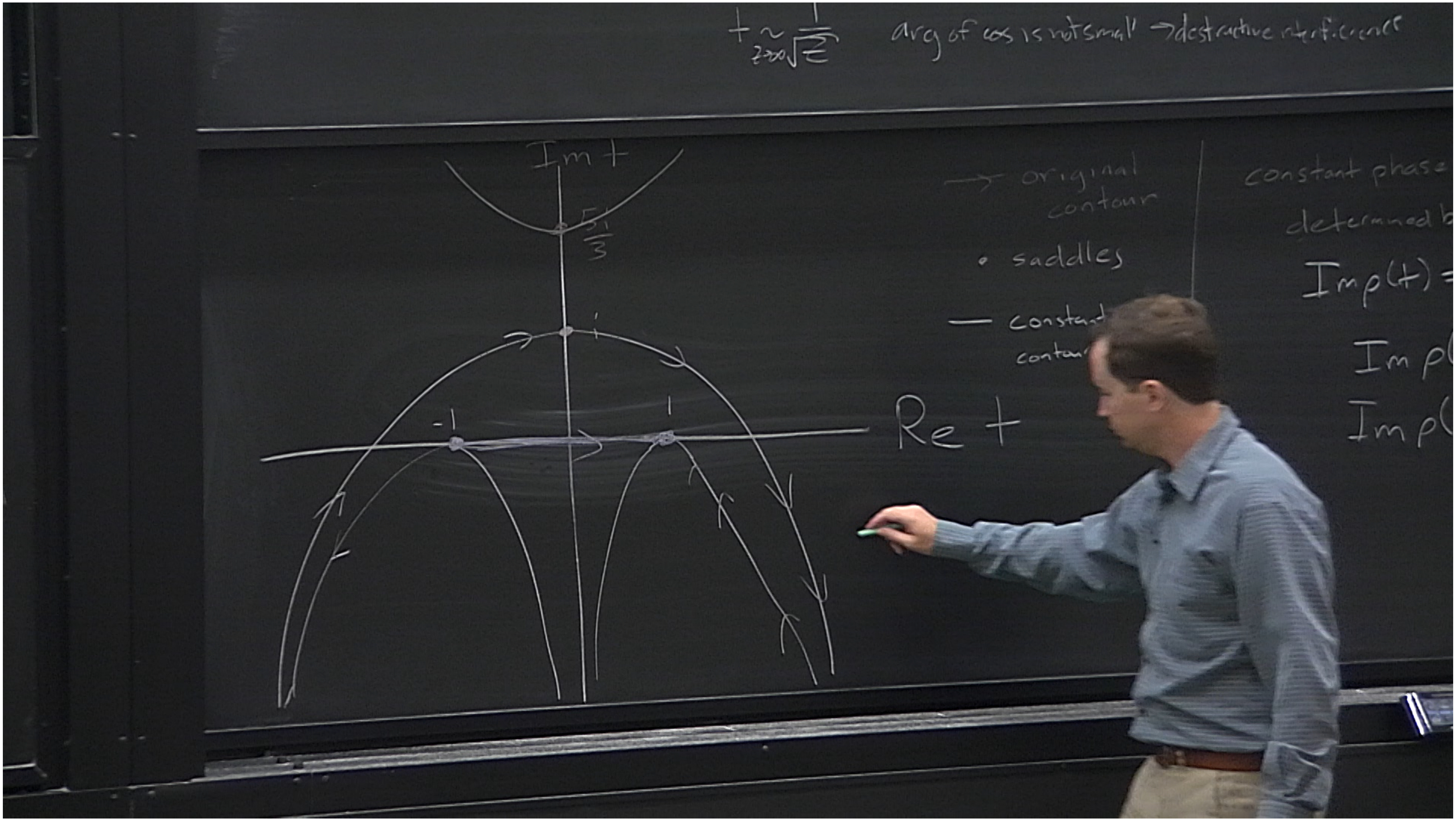
$$= \frac{1}{2} e^{-2z}\int_{-1}^{1} dt\, e^{z\rho(t)}$$

$$\rho(t) = -(t-i)^2 - i(t-i)^3$$

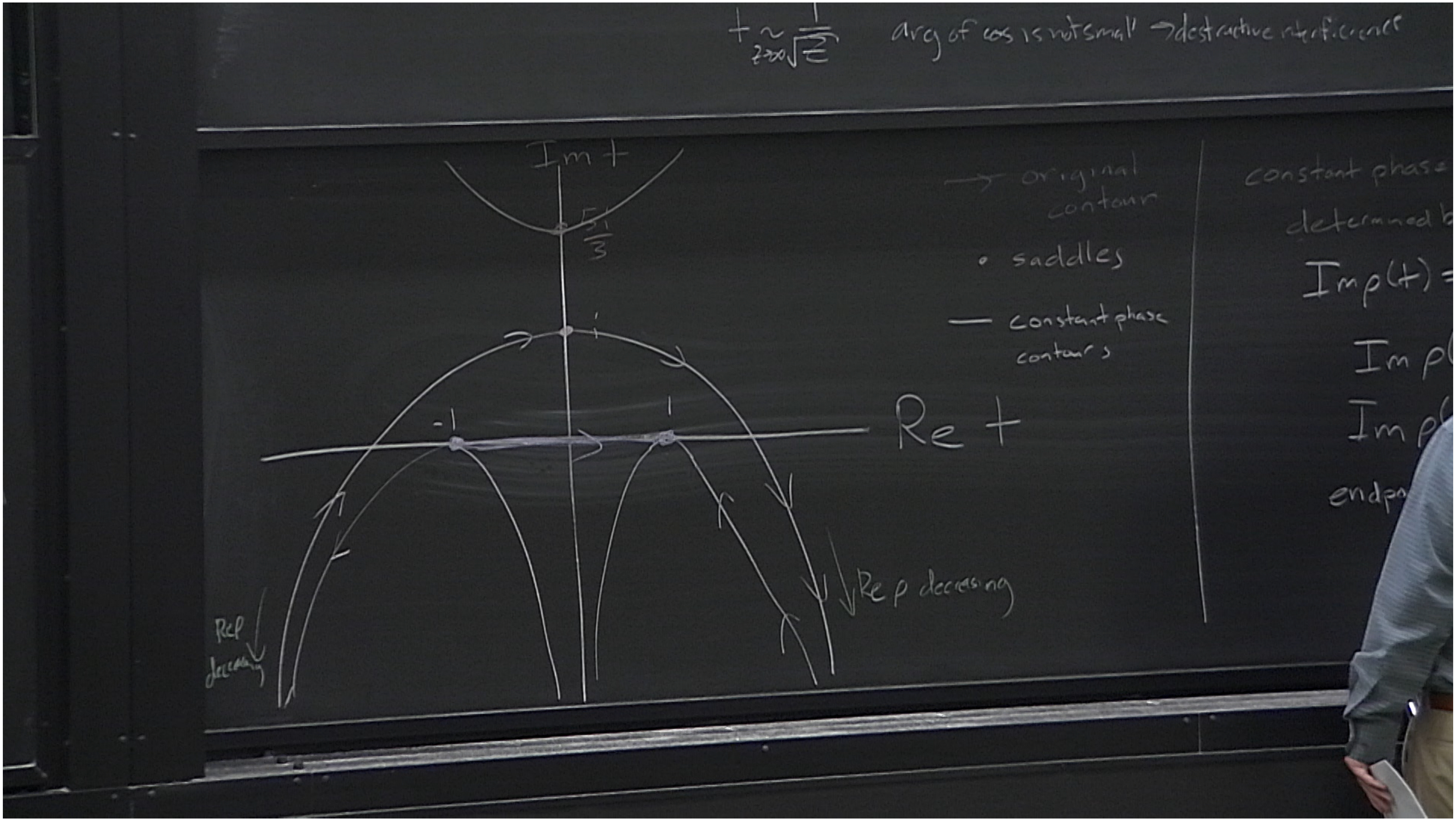
$$\rho'(t) = 0 \text{ if } \left.\begin{array}{l} t = i \\ t = \frac{5i}{3} \end{array}\right\} \text{saddle points}$$

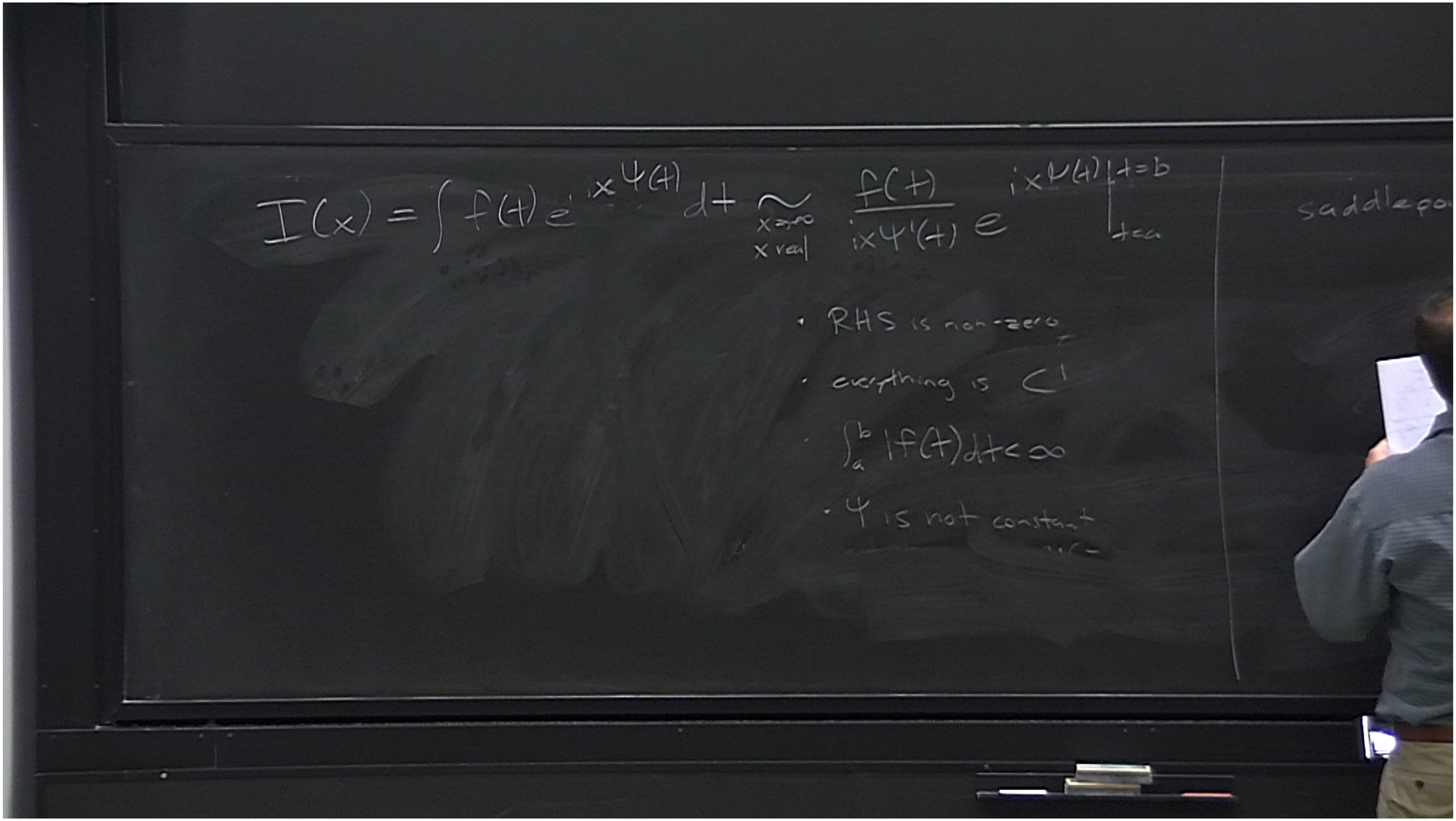

Im t
5i/3
i
-1
1
Re t
→ original contour
• saddles
constant phase contours

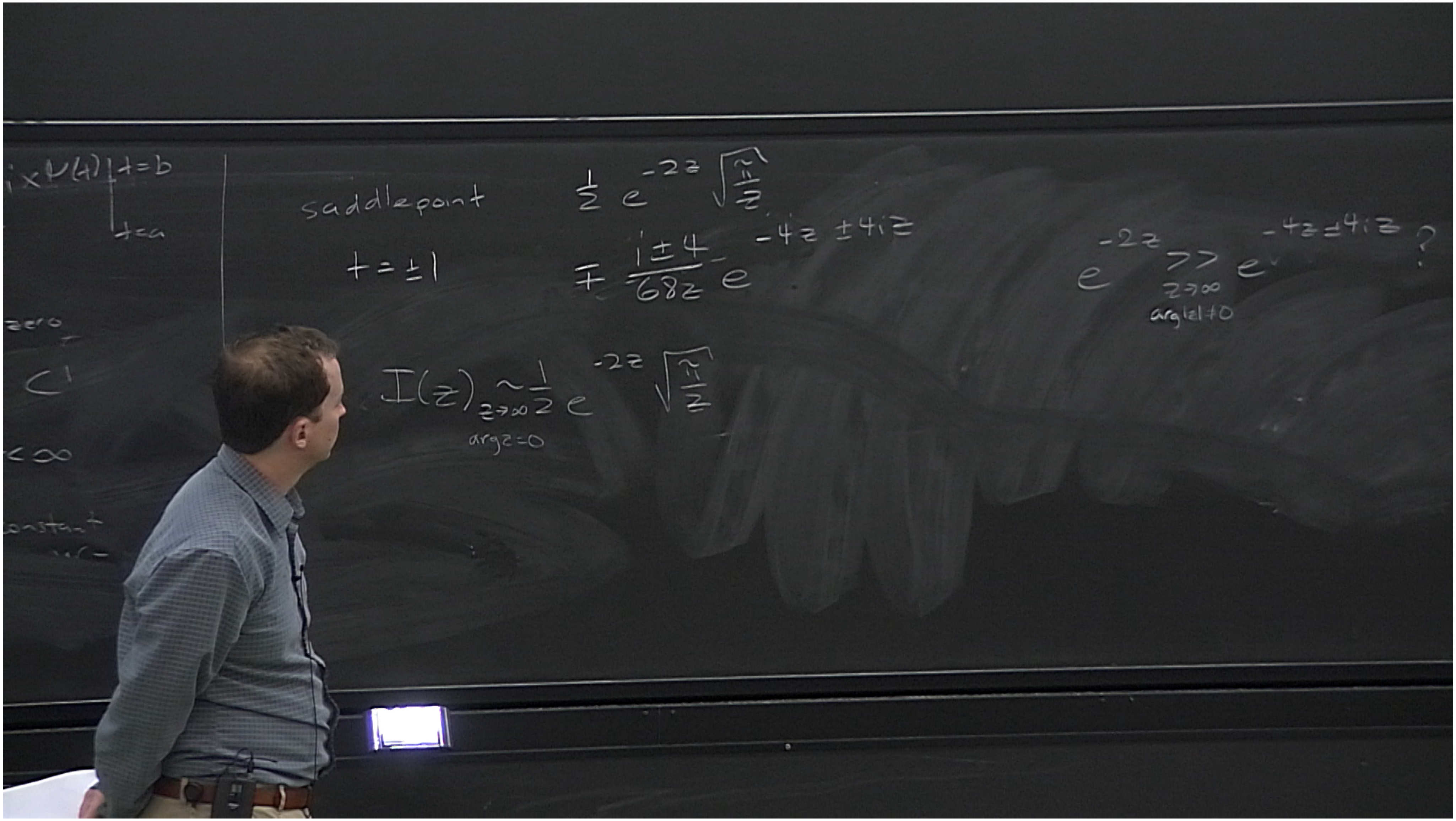

arg of cos is not small → destructive interference
Im t
original contour
saddles
constant phase contours
Re
constant phase
determined b



arg of cos is not small → destructive interference
Im t
Re t
original contour
saddles
constant phase
determined b
Im ρ(t) =


arg of cos is not small ⇒ destructive interference
Im t
Re t
original contour
saddles
constant phase contours
Re ρ decreasing
Re ρ decreasing
constant phase
Im ρ(t) =


$I(x) = \int f(t) e^{ix\psi(t)} dt \underset{\substack{x\to\infty \\ x \text{ real}}}{\sim} \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)} \Big|_{t=a}^{t=b}$
• RHS is non-zero
• everything is C^1
• $\int_a^b |f(t)| dt < \infty$
• ψ is not constant
saddlepo


saddlepoint
t = ±1
$\frac{1}{2} e^{-2z} \sqrt{\frac{\pi}{z}}$
$\mp \frac{1 \pm 4}{68z} e^{-4z \pm 4iz}$
$e^{-2z} \underset{z\to\infty}{>>} e^{-4z \pm 4iz}$?
arg z ≠ 0
$I(z) \underset{z\to\infty}{\sim} \frac{1}{2} e^{-2z} \sqrt{\frac{\pi}{z}}$
arg z = 0

$$\frac{1}{2} e^{-2z}\sqrt{\frac{\pi}{z}}$$

$$\mp \frac{i\pm 4}{68z} e^{-4z \pm 4iz}$$

$$\underset{\substack{z\to\infty \\ \arg z = 0}}{\sim} \frac{1}{2} e^{-2z}\sqrt{\frac{\pi}{z}}$$

dominates if $|\arg z| < \arctan\frac{1}{2}$

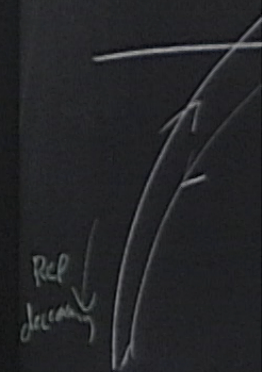
otherwise one of these will dominate

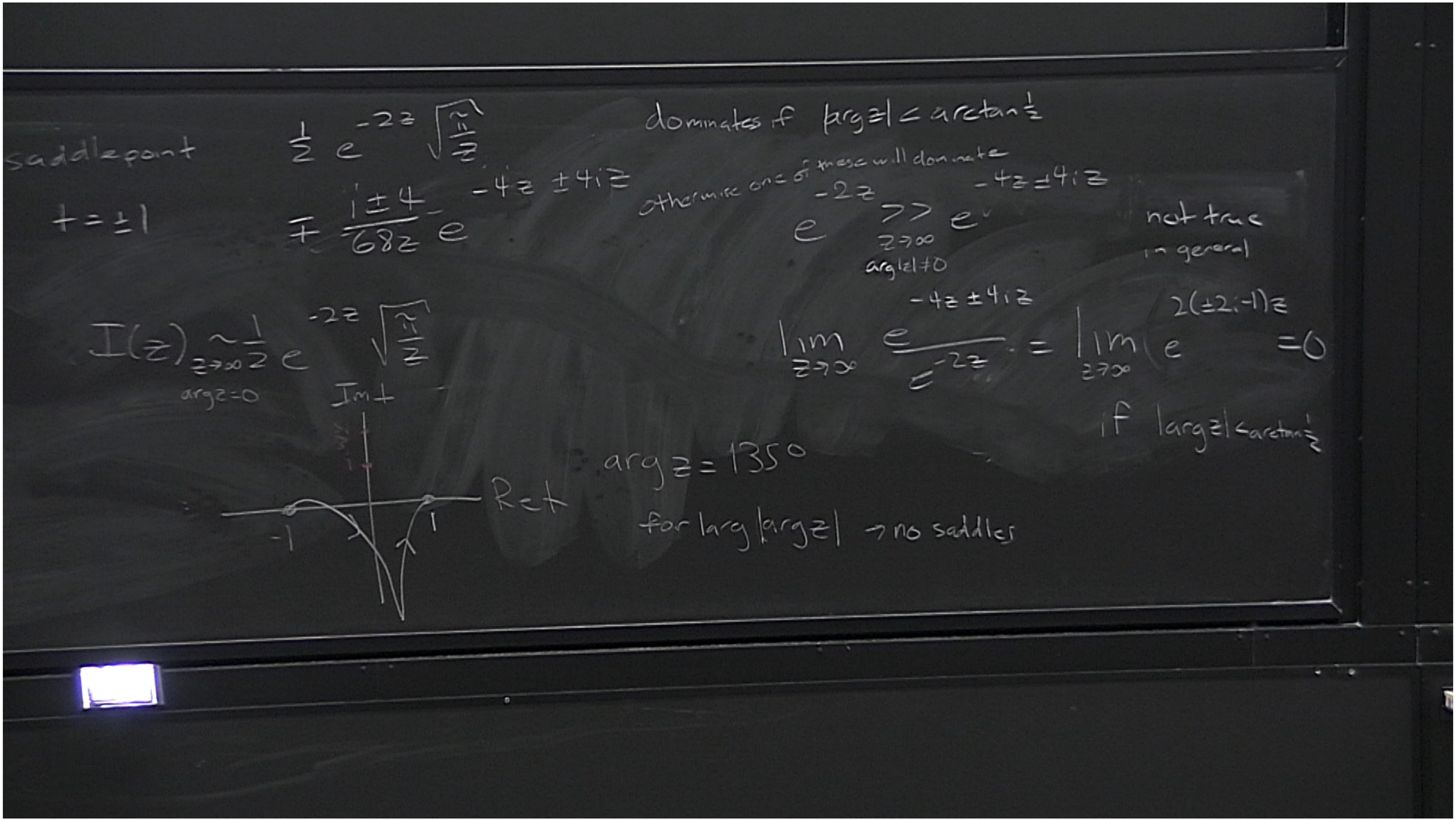
$$e^{-2z} \underset{\substack{z\to\infty \\ \arg|z|\neq 0}}{\gg} e^{-4z\pm 4iz}$$

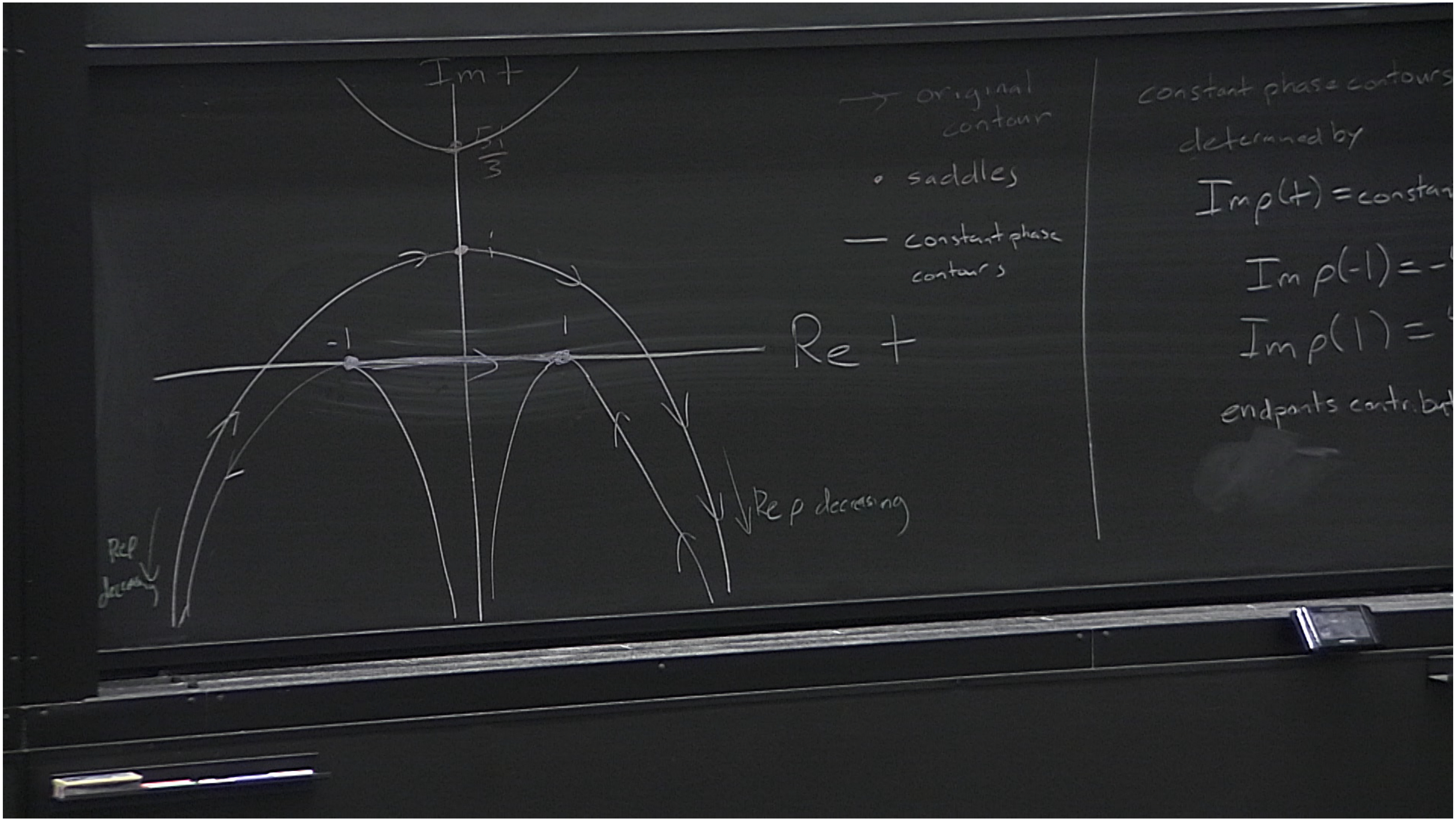
not true in general

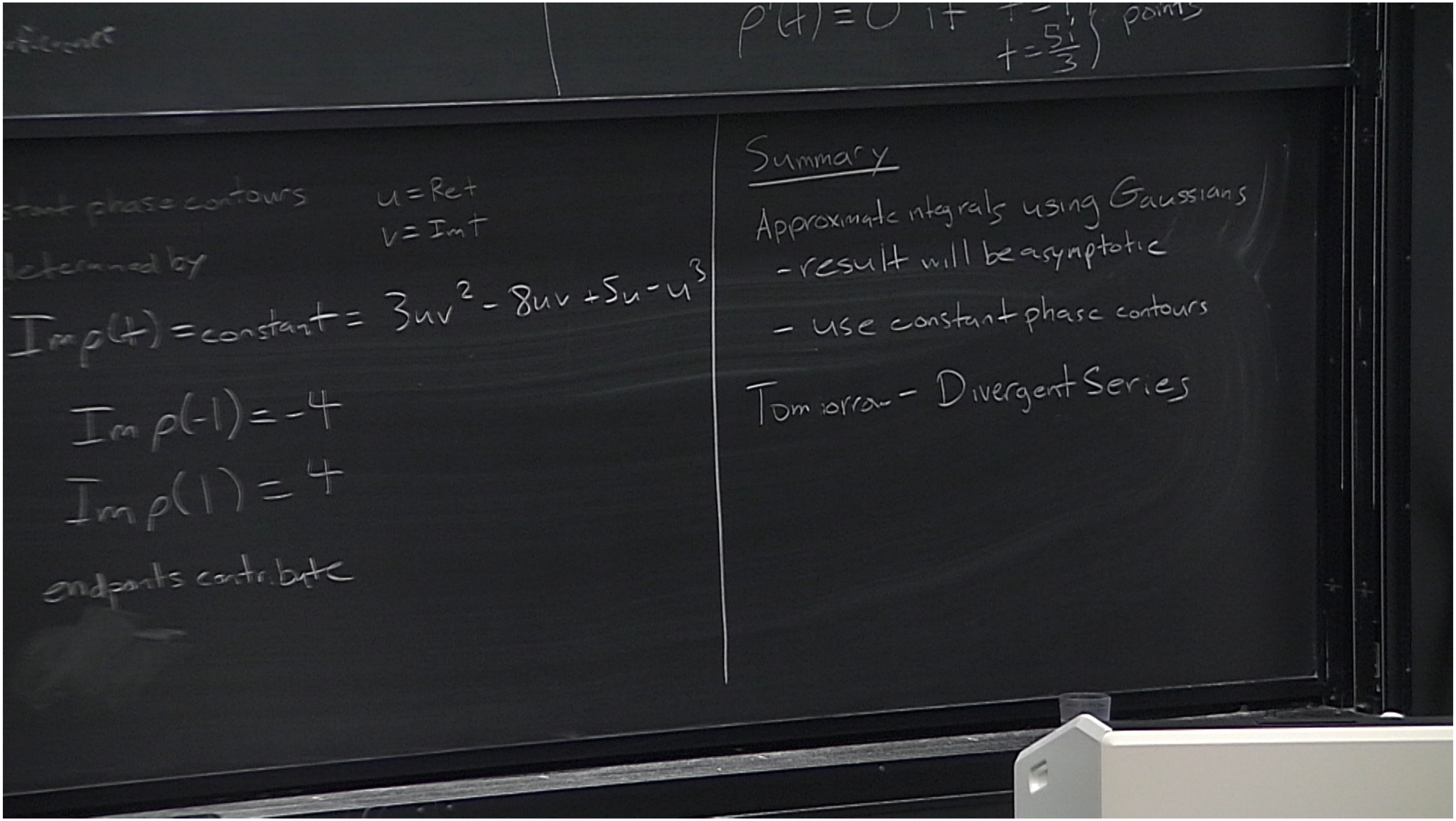
$$\lim_{z\to\infty} \frac{e^{-4z\pm 4iz}}{e^{-2z}} = \lim_{z\to\infty} e^{2(\pm 2i-1)z} = 0$$

if $|\arg z| < \arctan\frac{1}{2}$




saddlepoint
$\frac{1}{2} e^{-2z}\sqrt{\frac{\pi}{z}}$
dominates if $|\arg z| < \arctan\frac{1}{2}$
$t = \pm 1$
$\mp \frac{i \pm 4}{68z} e^{-4z \pm 4iz}$
otherwise one of these will dominate
$e^{-2z} \gg_{z\to\infty,\ \arg|z|\neq 0} e^{-4z\pm 4iz}$
not true in general
$I(z) \sim_{z\to\infty} \frac{1}{z} e^{-2z}\sqrt{\frac{\pi}{z}}$
$\arg z = 0$
$\lim_{z\to\infty} \frac{e^{-4z\pm4iz}}{e^{-2z}} = \lim_{z\to\infty} e^{2(\pm 2i-1)z} = 0$
if $|\arg z| < \arctan\frac{1}{2}$
Im t
Re t
-1
1
$\arg z = 135°$
for larg $|\arg z|$ → no saddles


Im t
original contour
saddles
constant phase contours
Re t
Re ρ decreasing
5i/3
i
-1
1
constant phase contours
determined by
Im ρ(t) = constan
Im ρ(-1) = -
Im ρ(1) =
endpoints contribut


u = Re t
v = Im t
Im ρ(t) = constant = 3uv² - 8uv + 5u - u³
Im ρ(-1) = -4
Im ρ(1) = 4
endpoints contribute
Summary
Approximate integrals using Gaussians
- result will be asymptotic
- use constant phase contours
Tomorrow - Divergent Series