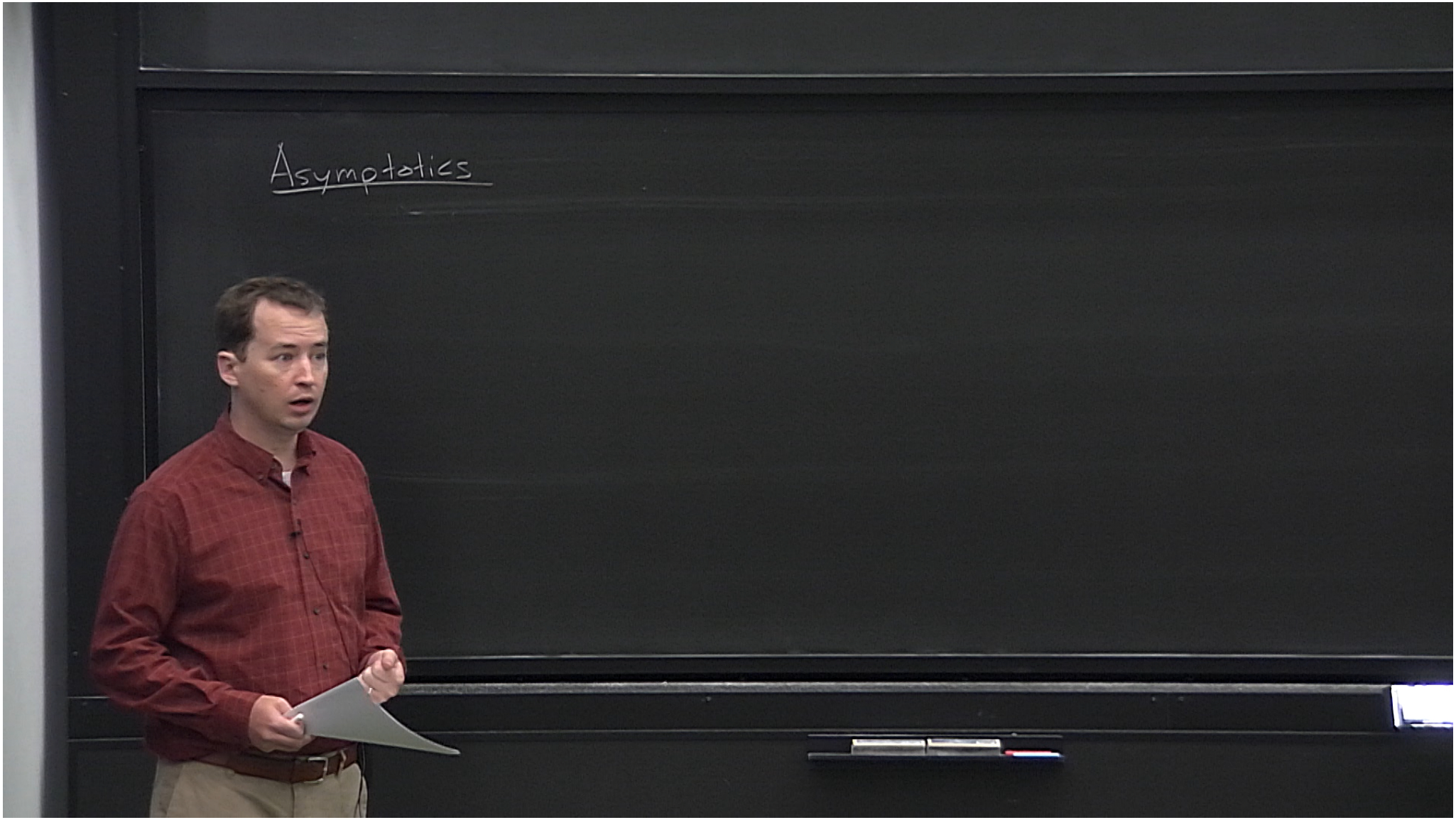
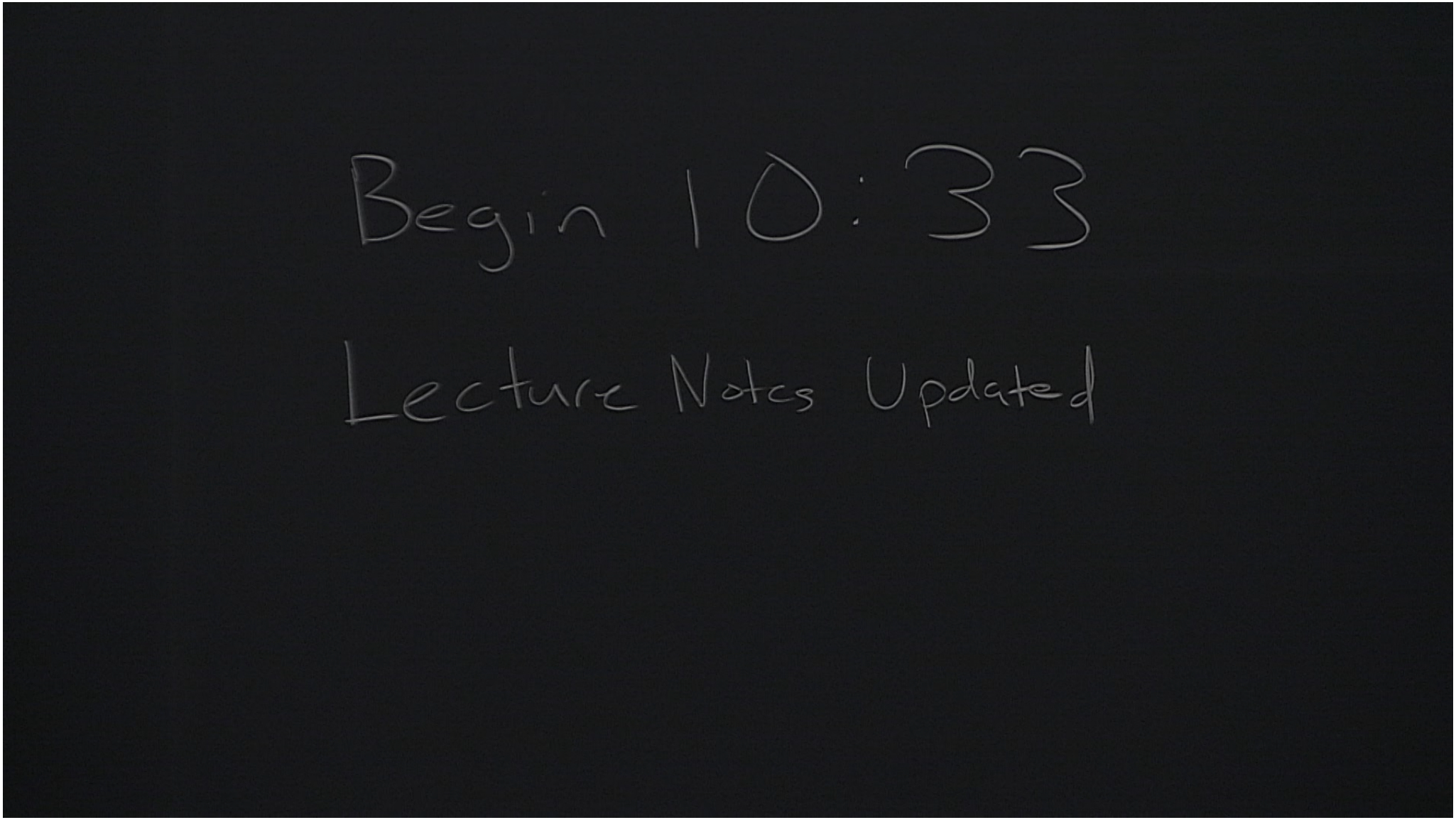
Title: PSI 2018/2019 - Math for QFT - Lecture 3

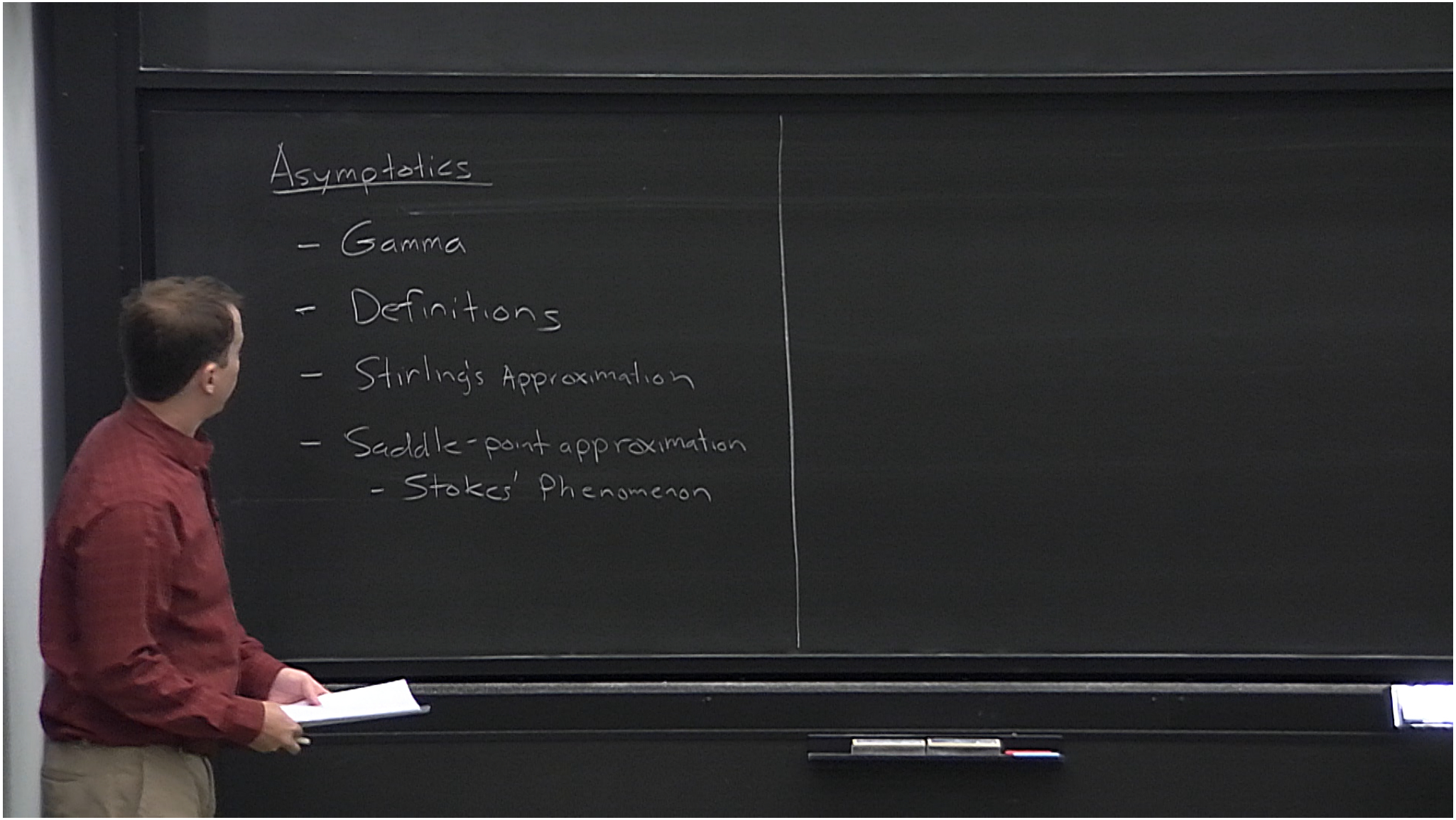
Date: Aug 21, 2018 10:30 AM

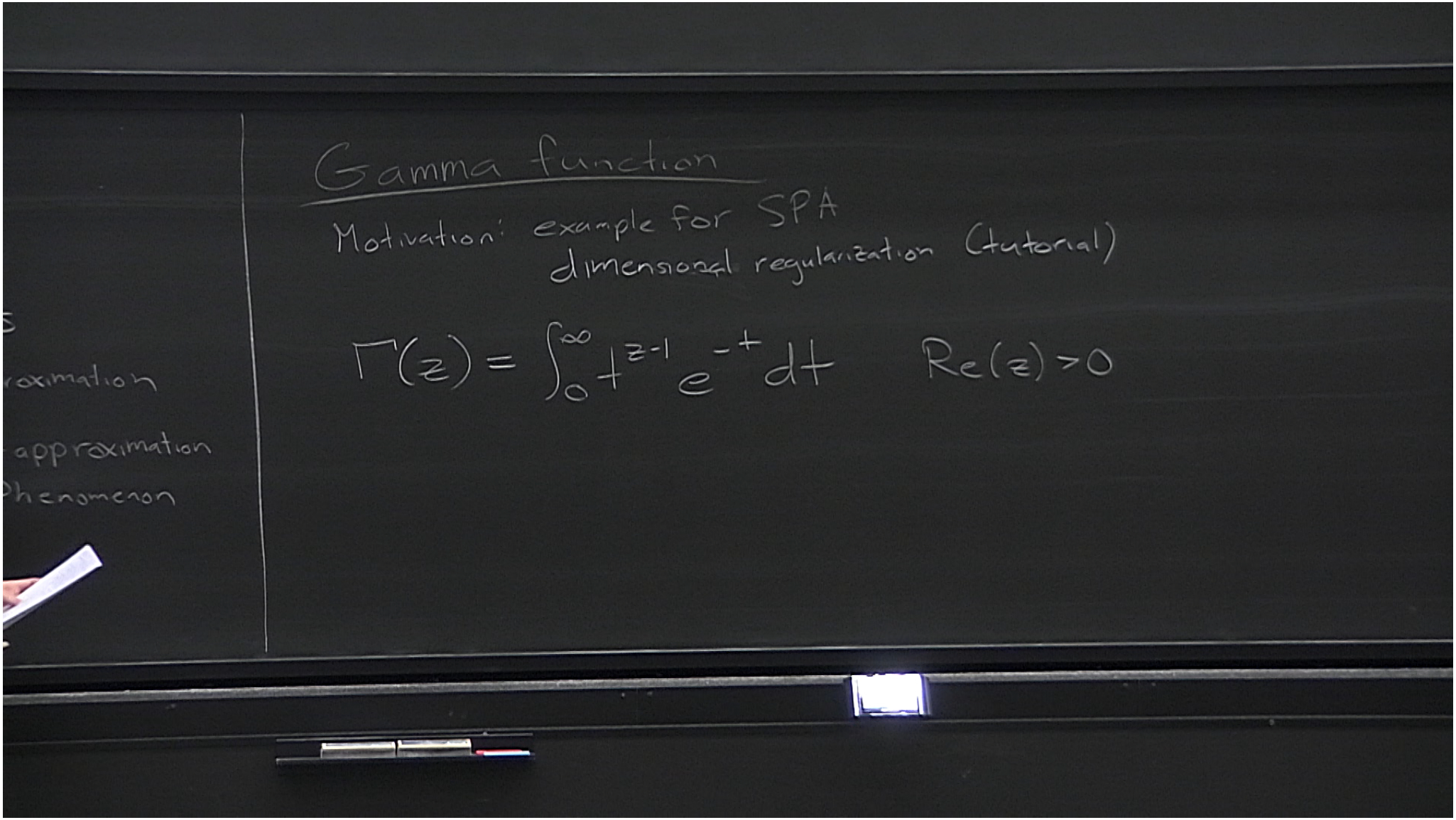
URL: http://pirsa.org/18080019

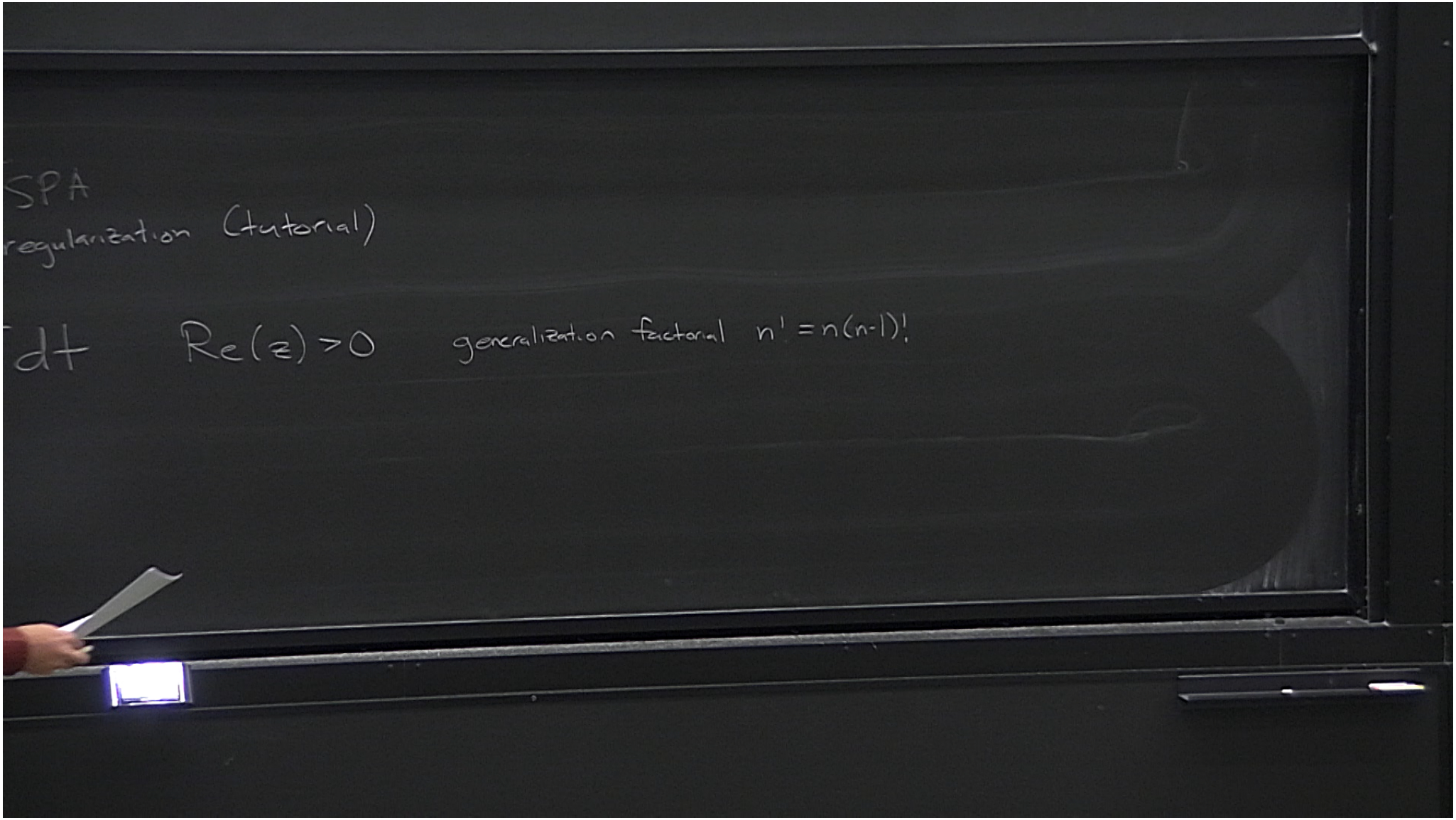
Abstract:

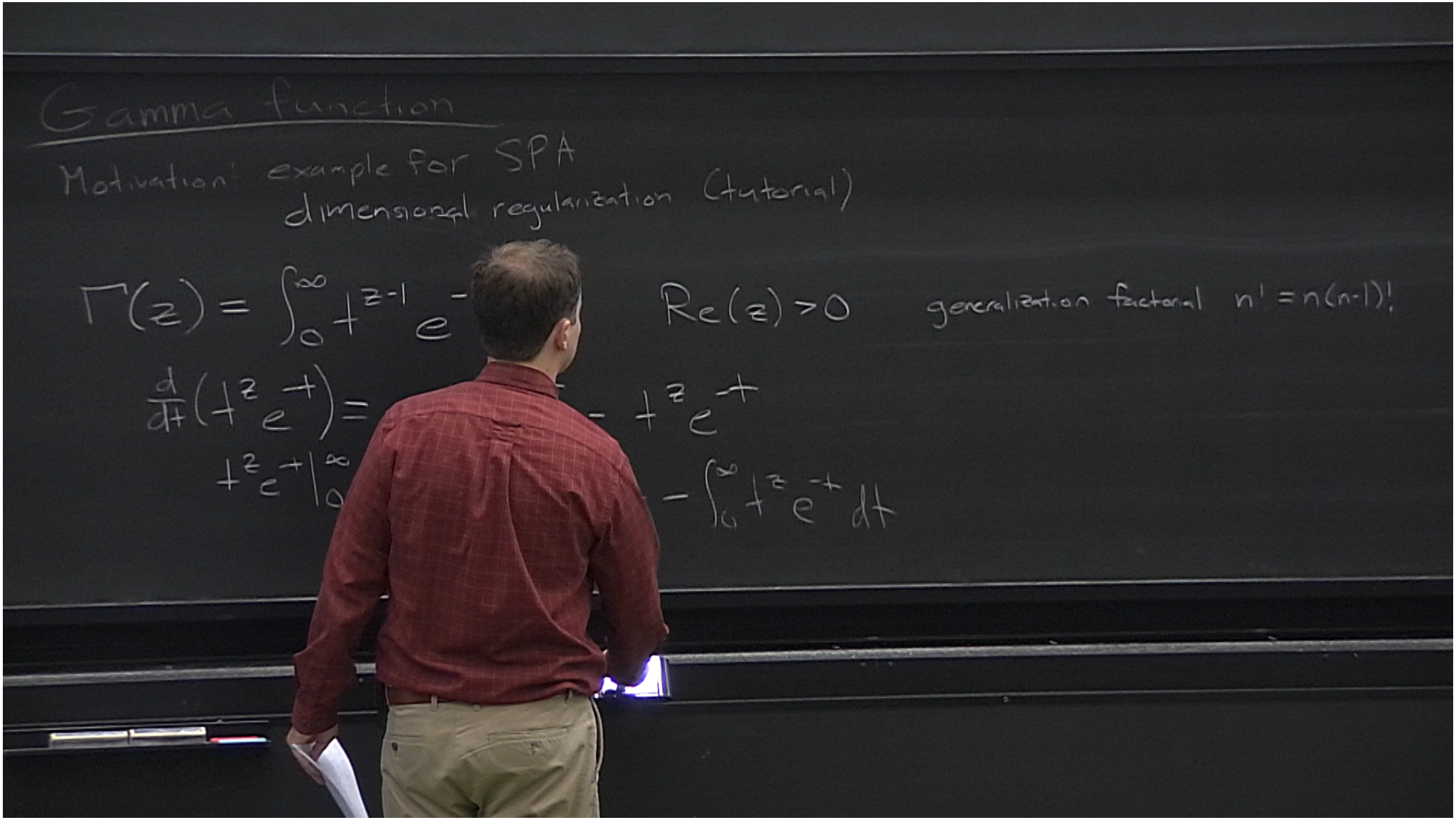

Asymptotics

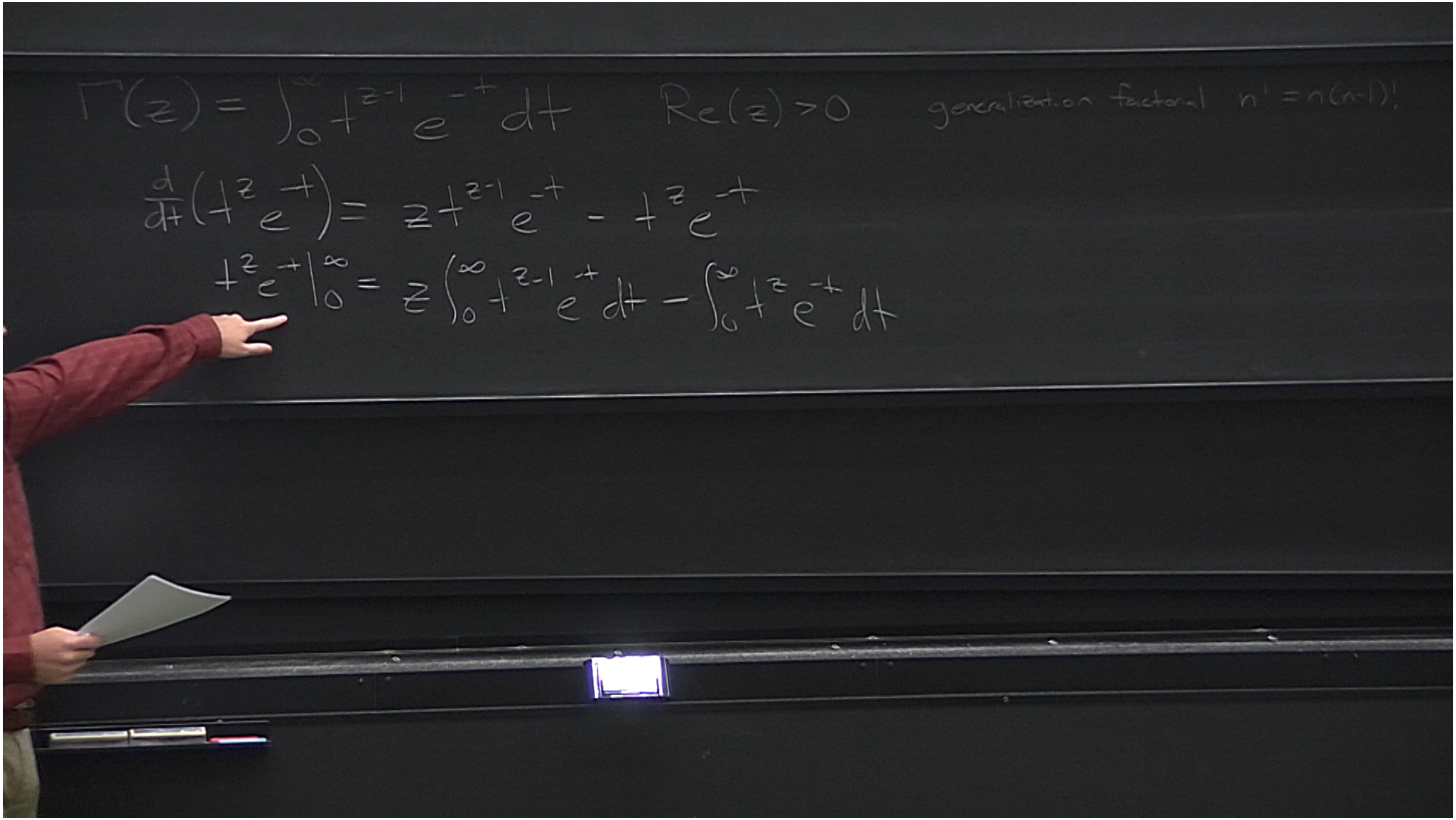

Begin 10:33
Lecture Notes Updated

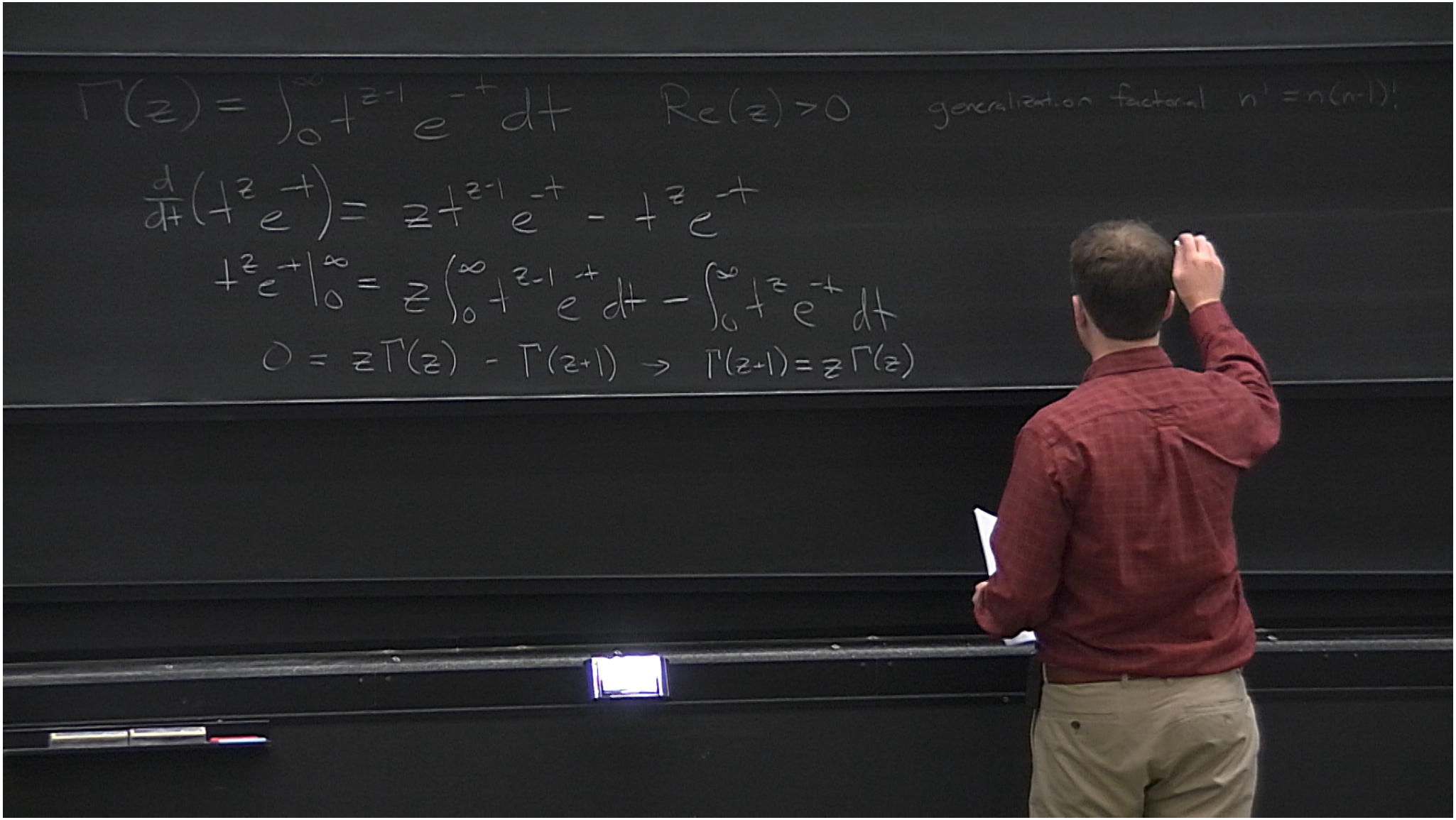

Asymptotics
- Gamma
- Definitions
- Stirling's Approximation
- Saddle-point approximation
- Stokes' Phenomenon

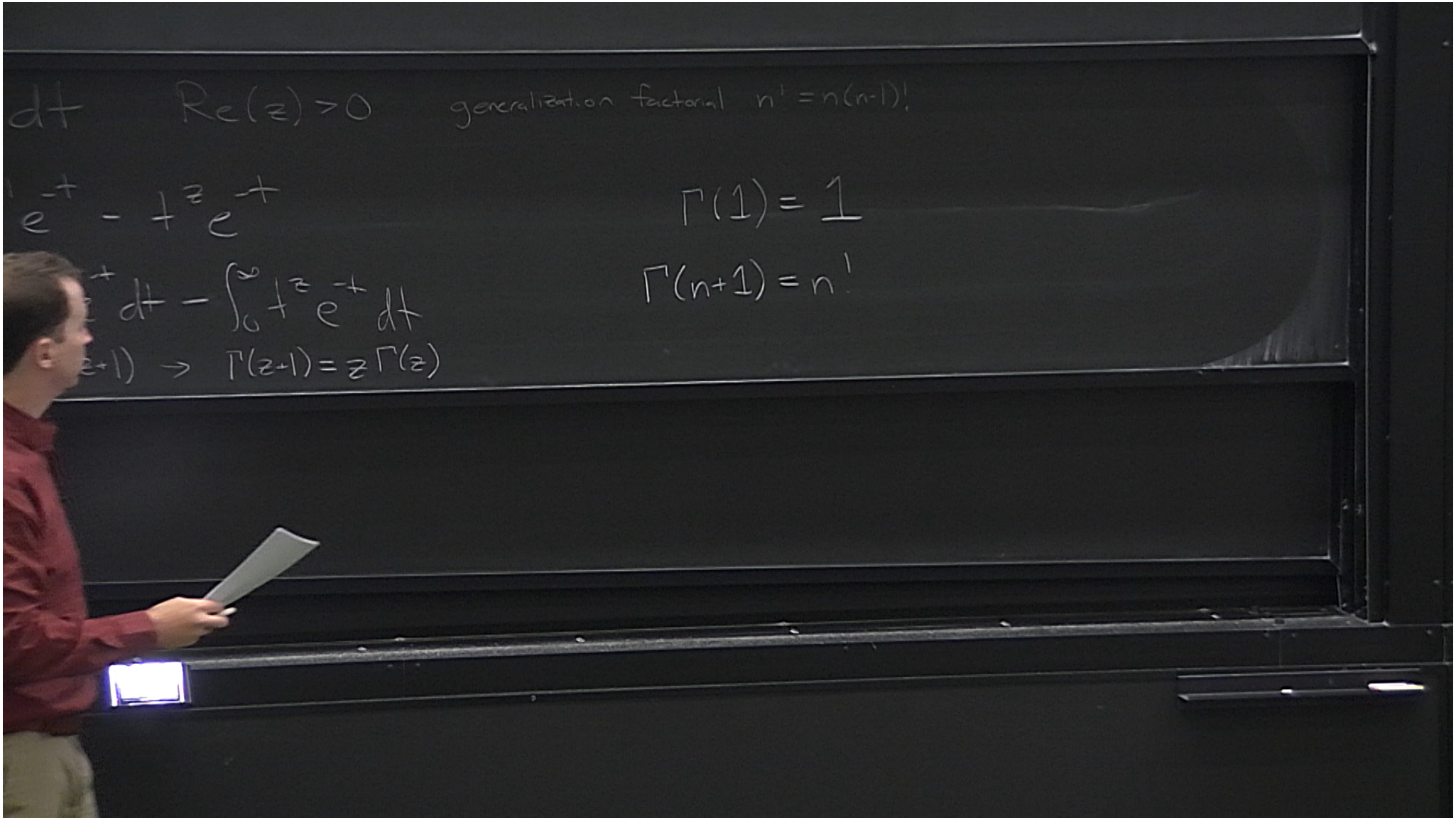

Gamma function
Motivation: example for SPA
dimensional regularization (tutorial)
$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad Re(z) > 0$
roximation
approximation
Phenomenon

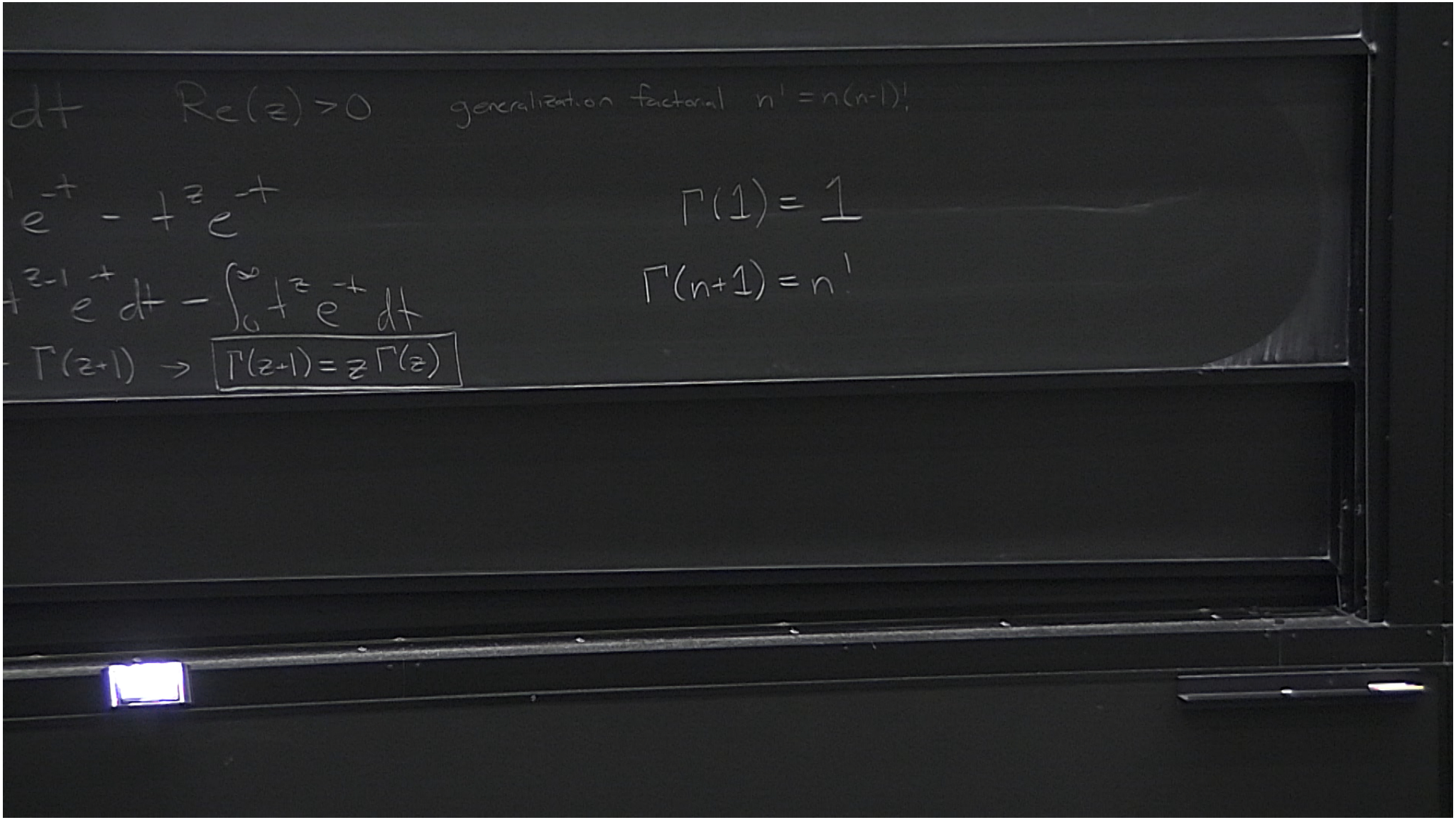

SPA
regularization (tutorial)
dt
Re(z) > 0
generalization factorial n! = n(n-1)!

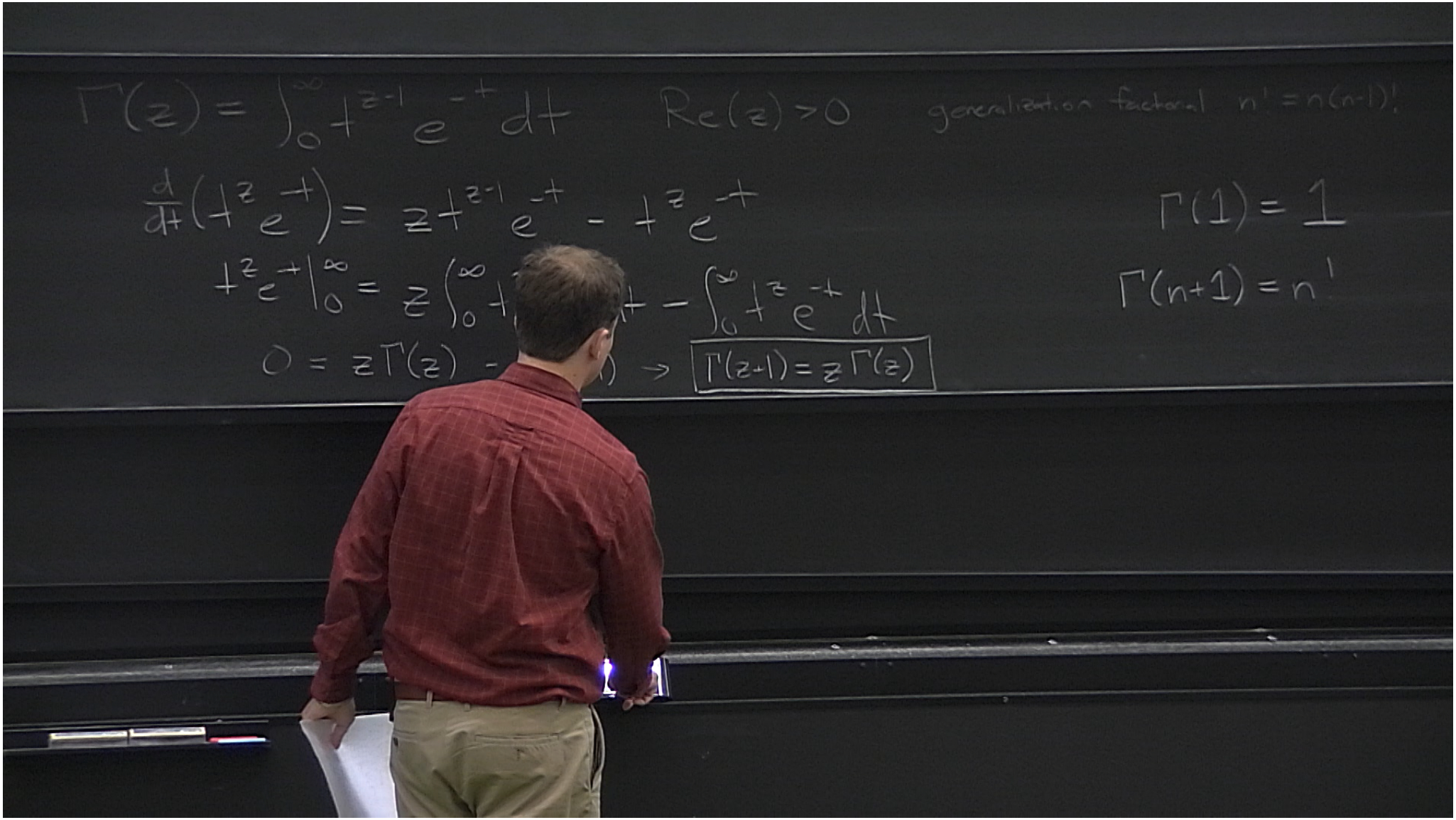

Gamma function
Motivation: example for SPA
dimensional regularization (tutorial)
$\Gamma(z) = \int_0^\infty t^{z-1} e^{-}$
$\mathrm{Re}(z) > 0$
generalization factorial $n! = n(n-1)!$
$\frac{d}{dt}(t^z e^{-t}) =$
$- t^z e^{-t}$
$t^z e^{-t}\Big|_0^\infty$
$- \int_0^\infty t^z e^{-t} dt$


$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ $\mathrm{Re}(z) > 0$ generalization factorial $n! = n(n-1)!$
$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$
$t^z e^{-t}\Big|_0^\infty = z\int_0^\infty t^{z-1} e^{-t}\, dt - \int_0^\infty t^z e^{-t}\, dt$


$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ $\mathrm{Re}(z) > 0$ generalization factorial $n! = n(n-1)!$
$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$
$t^z e^{-t}\Big|_0^\infty = z\int_0^\infty t^{z-1} e^{-t} dt - \int_0^\infty t^z e^{-t} dt$
$0 = z\Gamma(z) - \Gamma(z+1) \rightarrow \Gamma(z+1) = z\Gamma(z)$


dt Re(z) > 0 generalization factorial n! = n(n-1)!
$e^{-t} - t^z e^{-t}$
$\Gamma(1) = 1$
$\Gamma(n+1) = n!$
$\int_0^\infty t^z e^{-t} dt$
$\Gamma(z+1) \rightarrow \Gamma(z+1) = z\Gamma(z)$


dt Re(z) > 0 generalization factorial n! = n(n-1)!
$e^{-t} - t^z e^{-t}$
$t^{z-1} e^{-t} dt - \int_0^\infty t^z e^{-t} dt$
$\Gamma(z+1) \rightarrow \boxed{\Gamma(z+1) = z\Gamma(z)}$
$\Gamma(1) = 1$
$\Gamma(n+1) = n!$


$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ $\mathrm{Re}(z) > 0$ generalization factorial $n! = n(n-1)!$
$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$
$\Gamma(1) = 1$
$t^z e^{-t}\Big|_0^\infty = z \int_0^\infty t$ $t - \int_0^\infty t^z e^{-t} dt$
$\Gamma(n+1) = n!$
$0 = z\Gamma(z)$ $\rightarrow$ $\Gamma(z+1) = z\Gamma(z)$

# Gamma function

Motivation: example for SPA
dimensional regularization (tutorial)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad \mathrm{Re}(z) > 0$$

generalization factorial $n! = n(n-1)!$

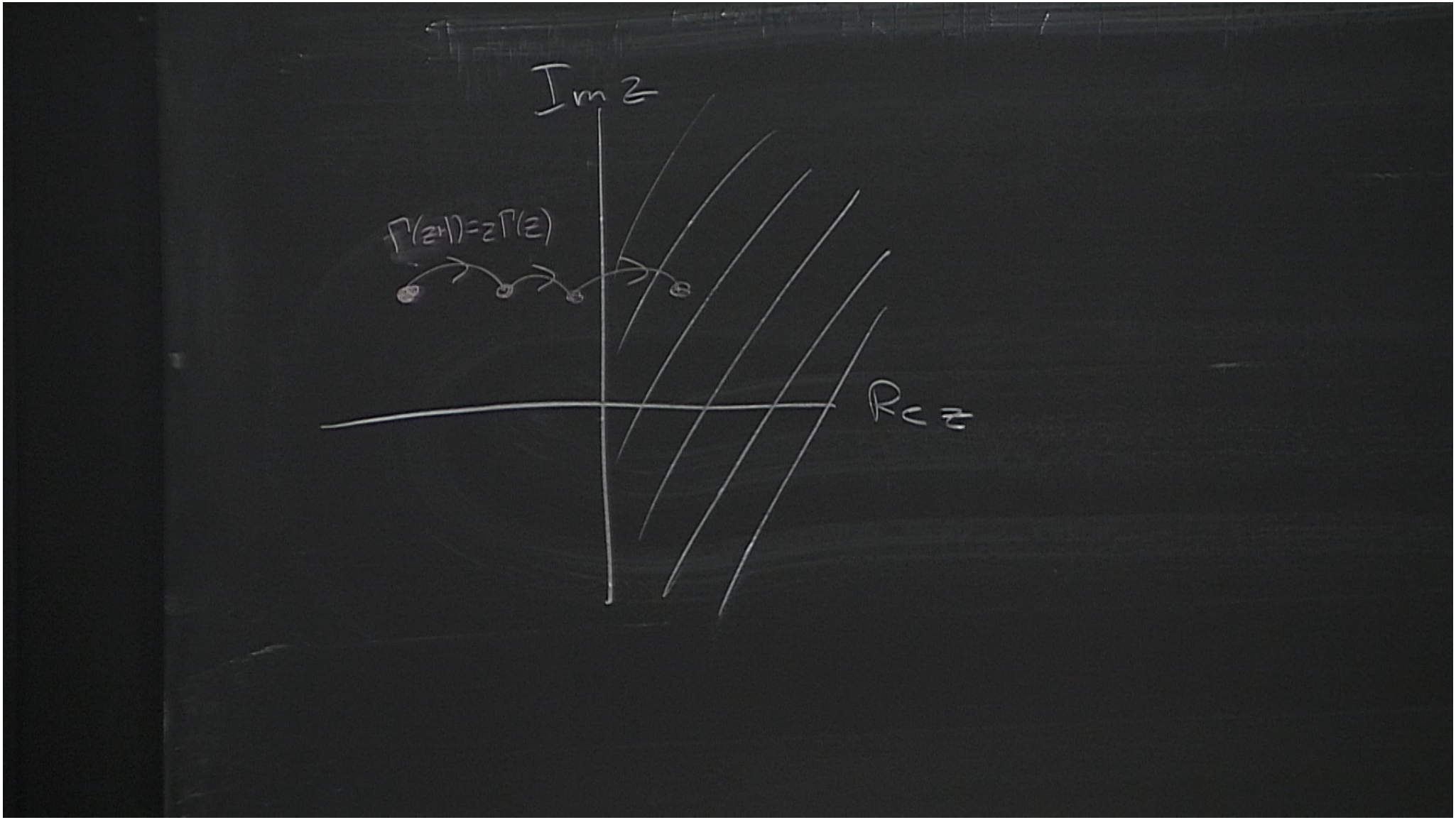
$$\frac{d}{dt}\left(t^z e^{-t}\right) = z t^{z-1} e^{-t} - t^z e^{-t}$$

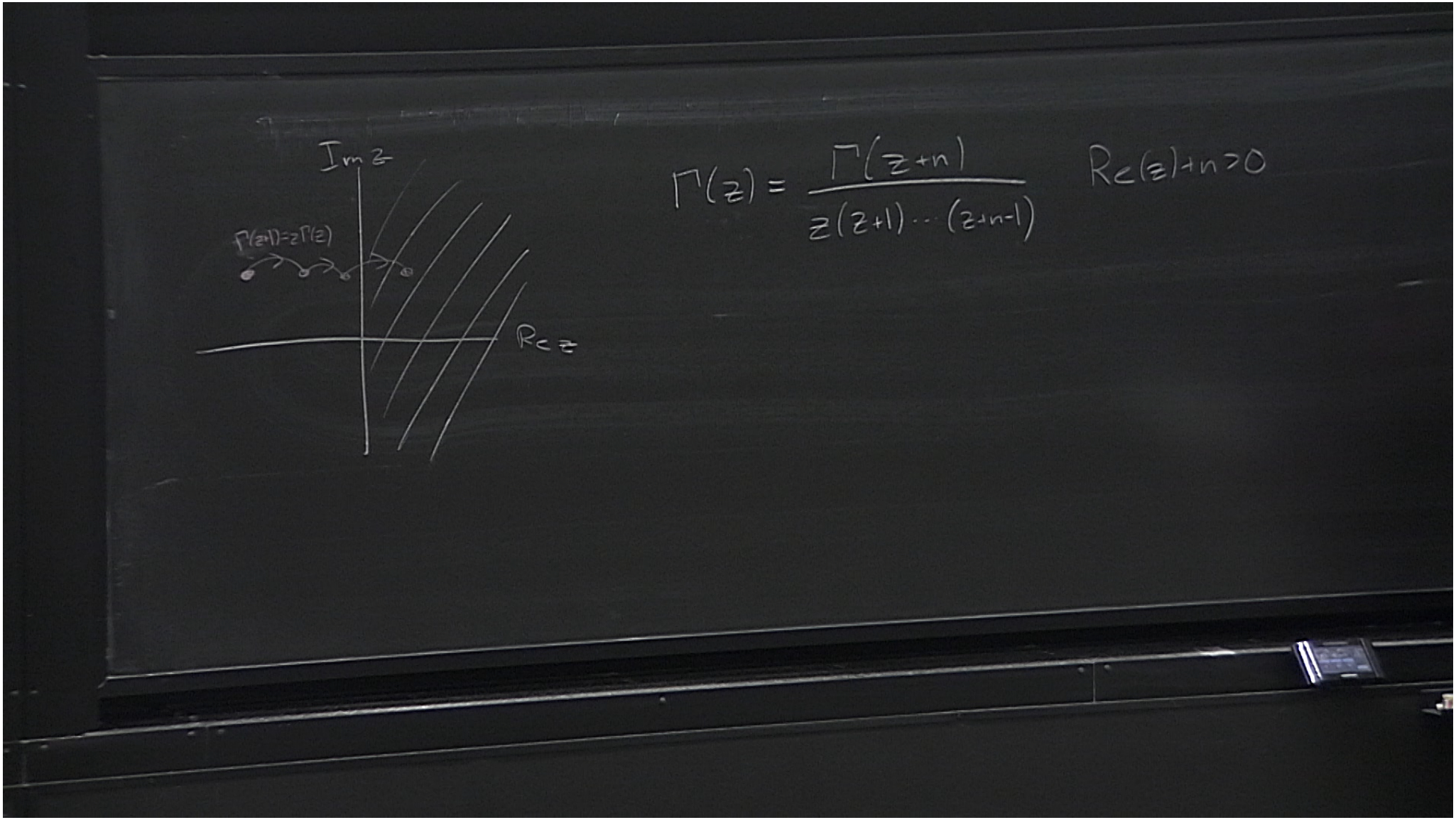
$$\Gamma(1) = 1$$

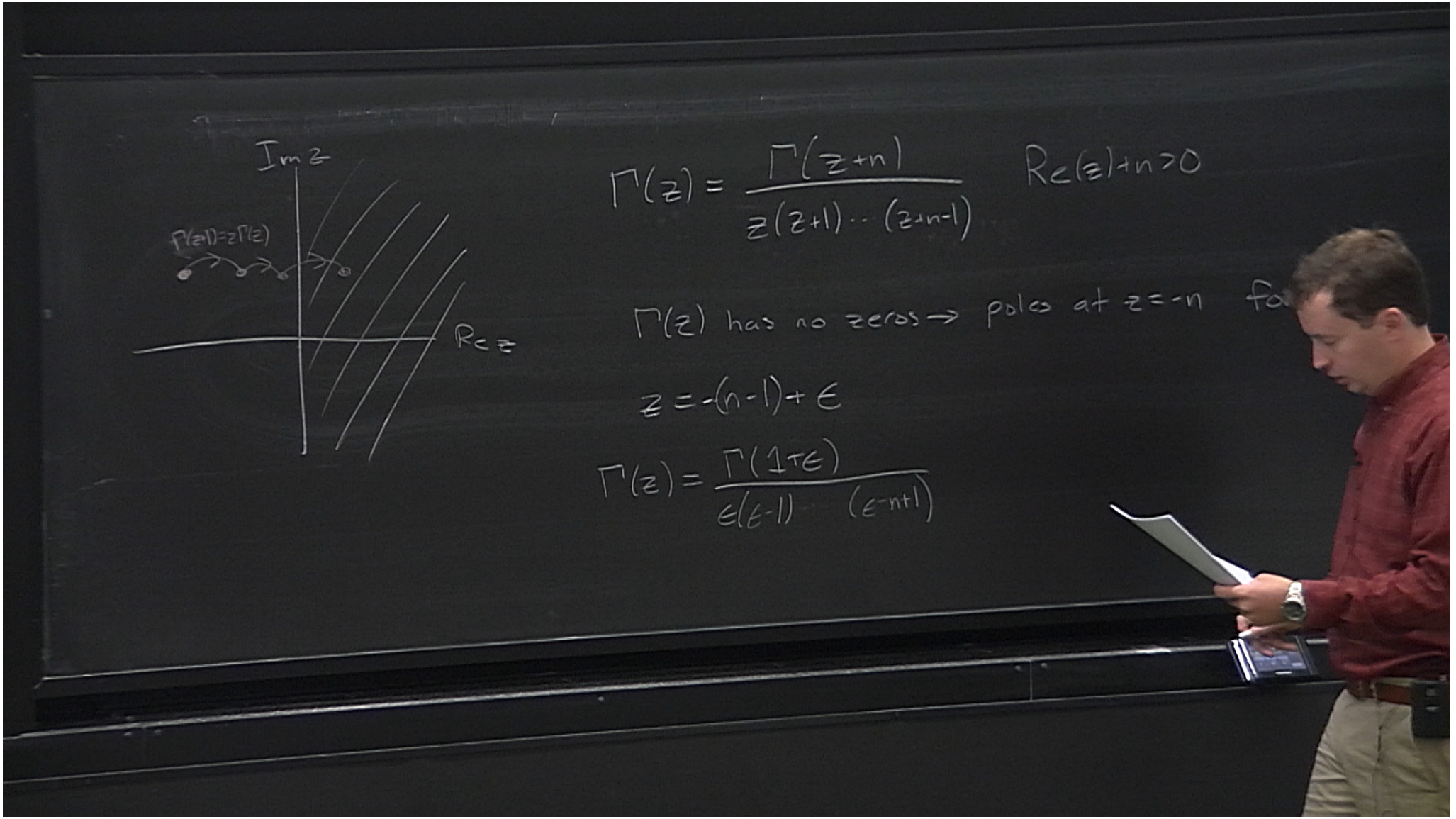
$$t^z e^{-t}\Big|_0^\infty = z\int_0^\infty t^{z-1} e^{-t} dt - \int_0^\infty t^z e^{-t} dt$$

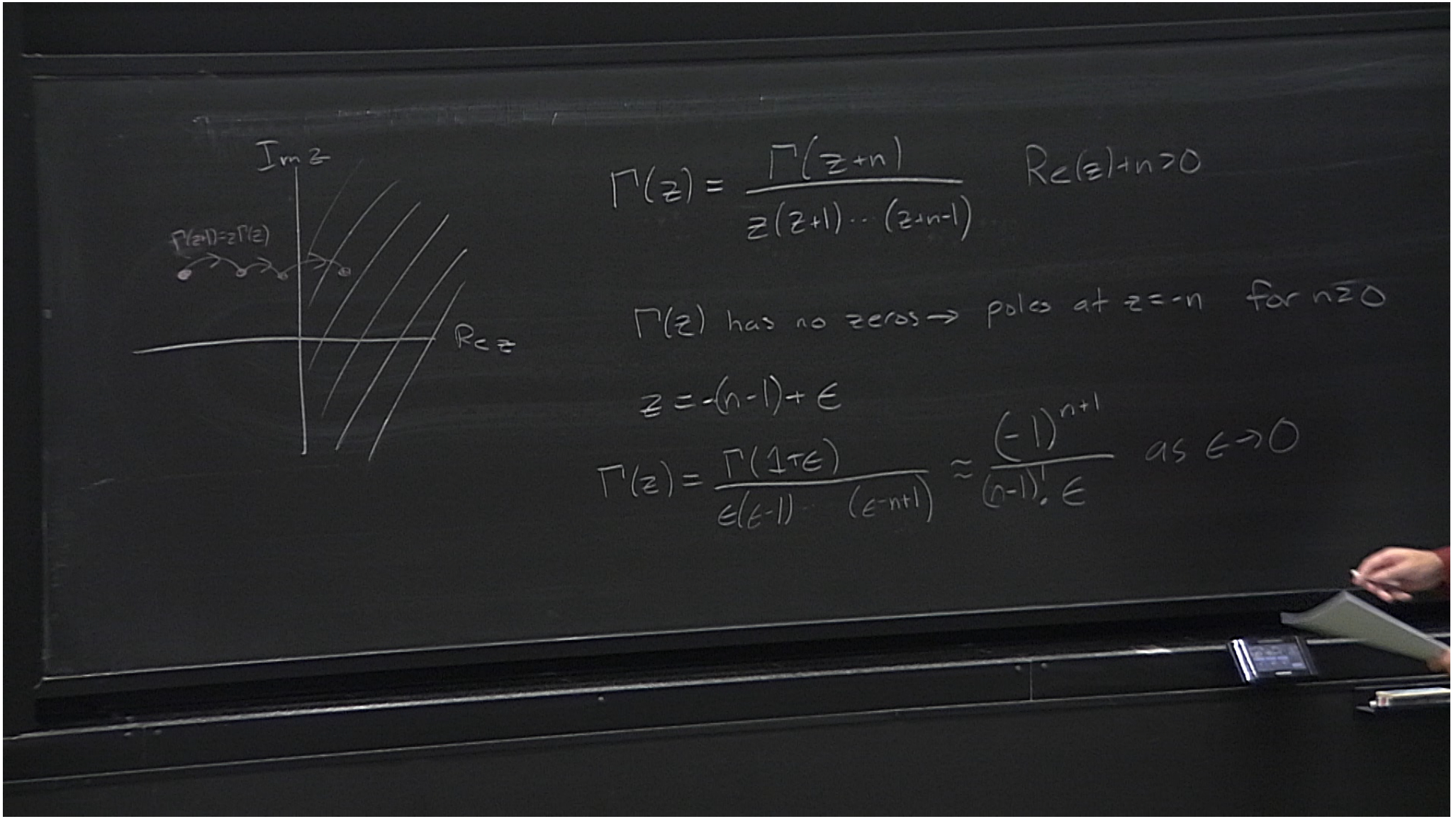
$$\Gamma(n+1) = n!$$

$$0 = z\Gamma(z) - \Gamma(z+1) \rightarrow \boxed{\Gamma(z+1) = z\Gamma(z)}$$


Im z
Γ(z+1)=zΓ(z)
Re z


Im z
Γ(z+1)=zΓ(z)
Re z
Γ(z) = Γ(z+n) / z(z+1)···(z+n-1)
Re(z)+n>0


Im z
Γ(z+1)=zΓ(z)
Re z
Γ(z) = Γ(z+n) / z(z+1)···(z+n-1)
Re(z)+n>0
Γ(z) has no zeros → poles at z=-n fo
z=-(n-1)+ϵ
Γ(z) = Γ(1+ϵ) / ϵ(ϵ-1)···(ϵ-n+1)

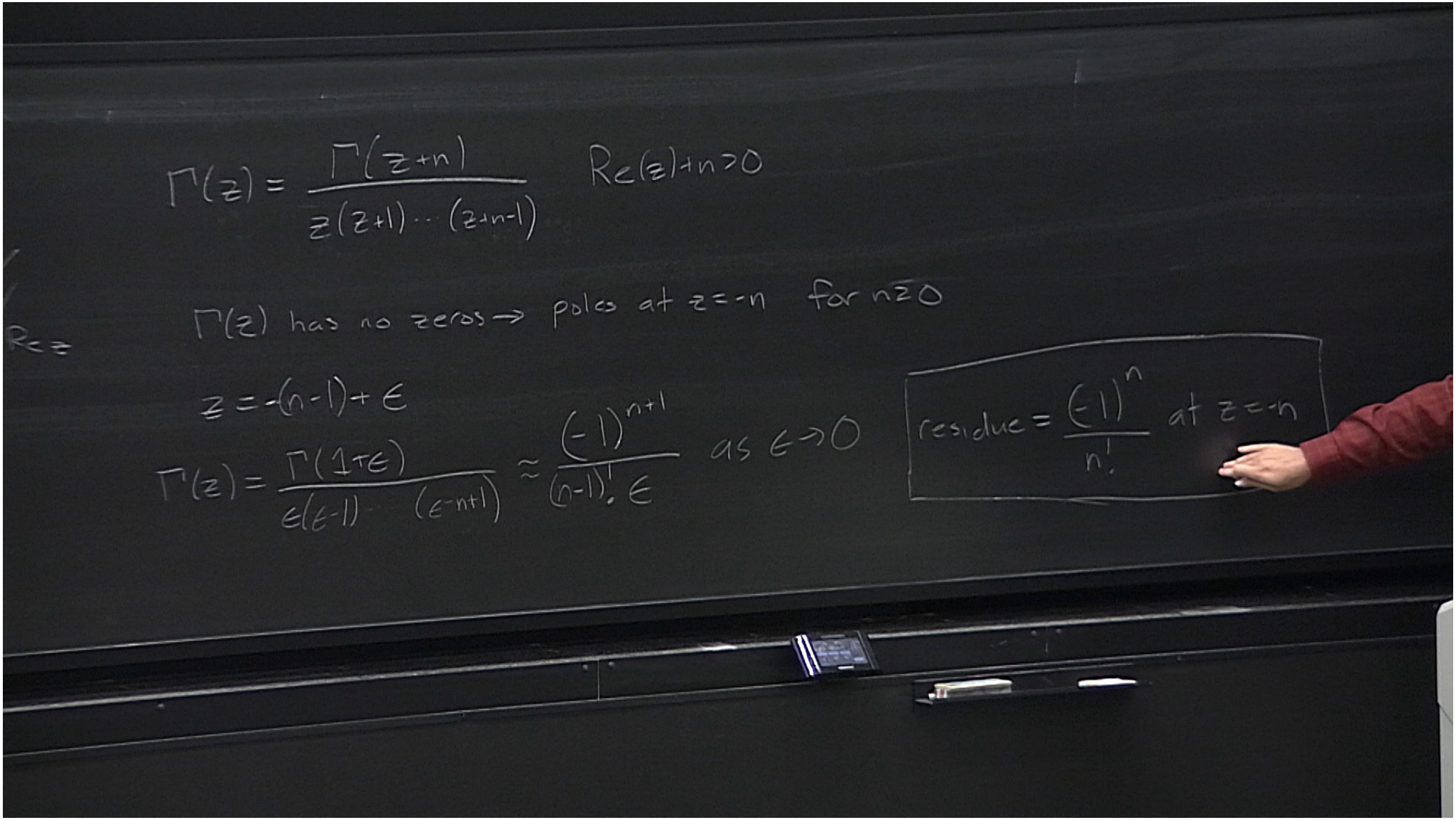


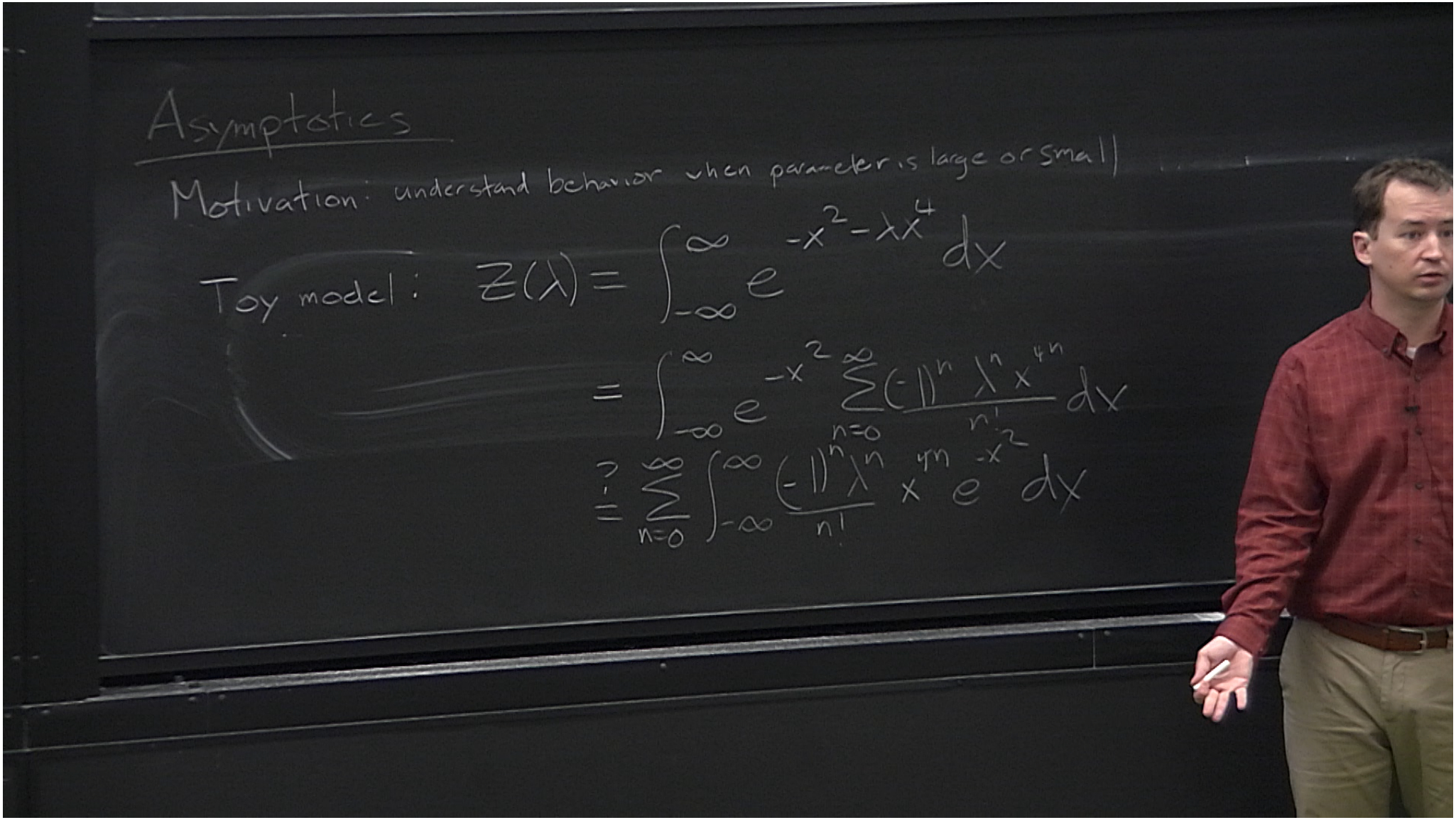

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \qquad \mathrm{Re}(z)+n>0$$

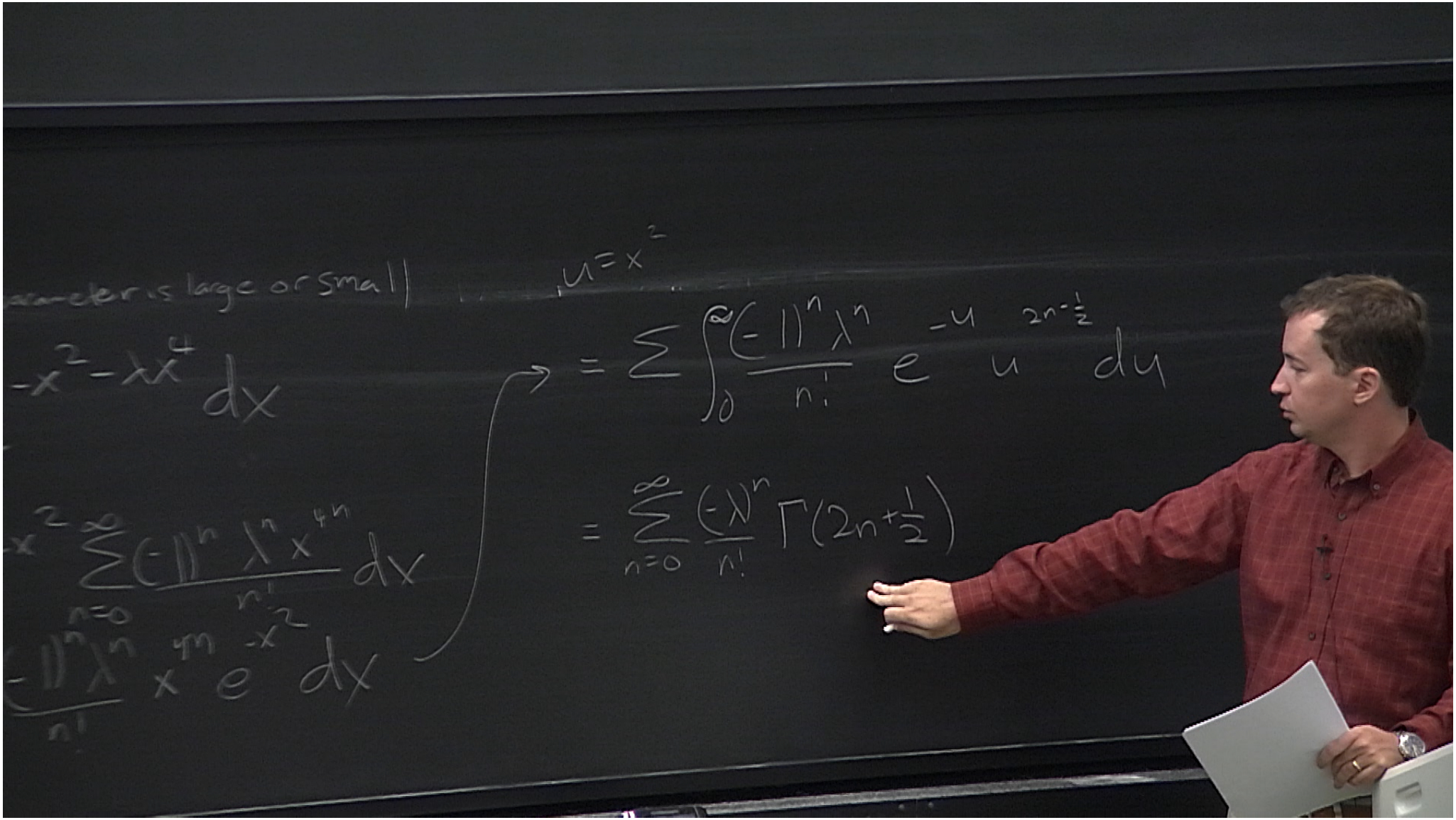
$\Gamma(z)$ has no zeros → poles at $z=-n$ for $n \geq 0$

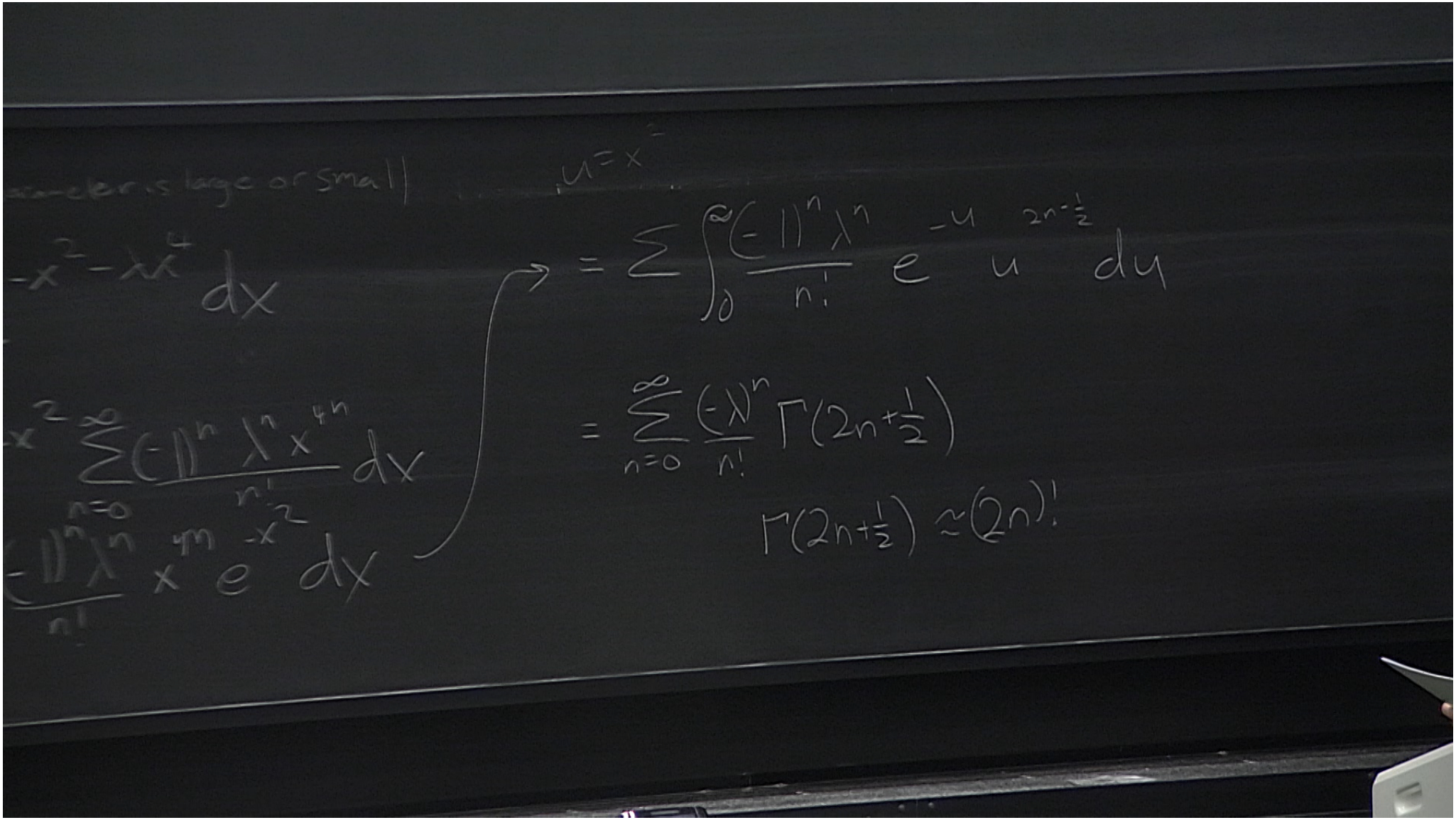
$$z = -(n-1)+\epsilon$$

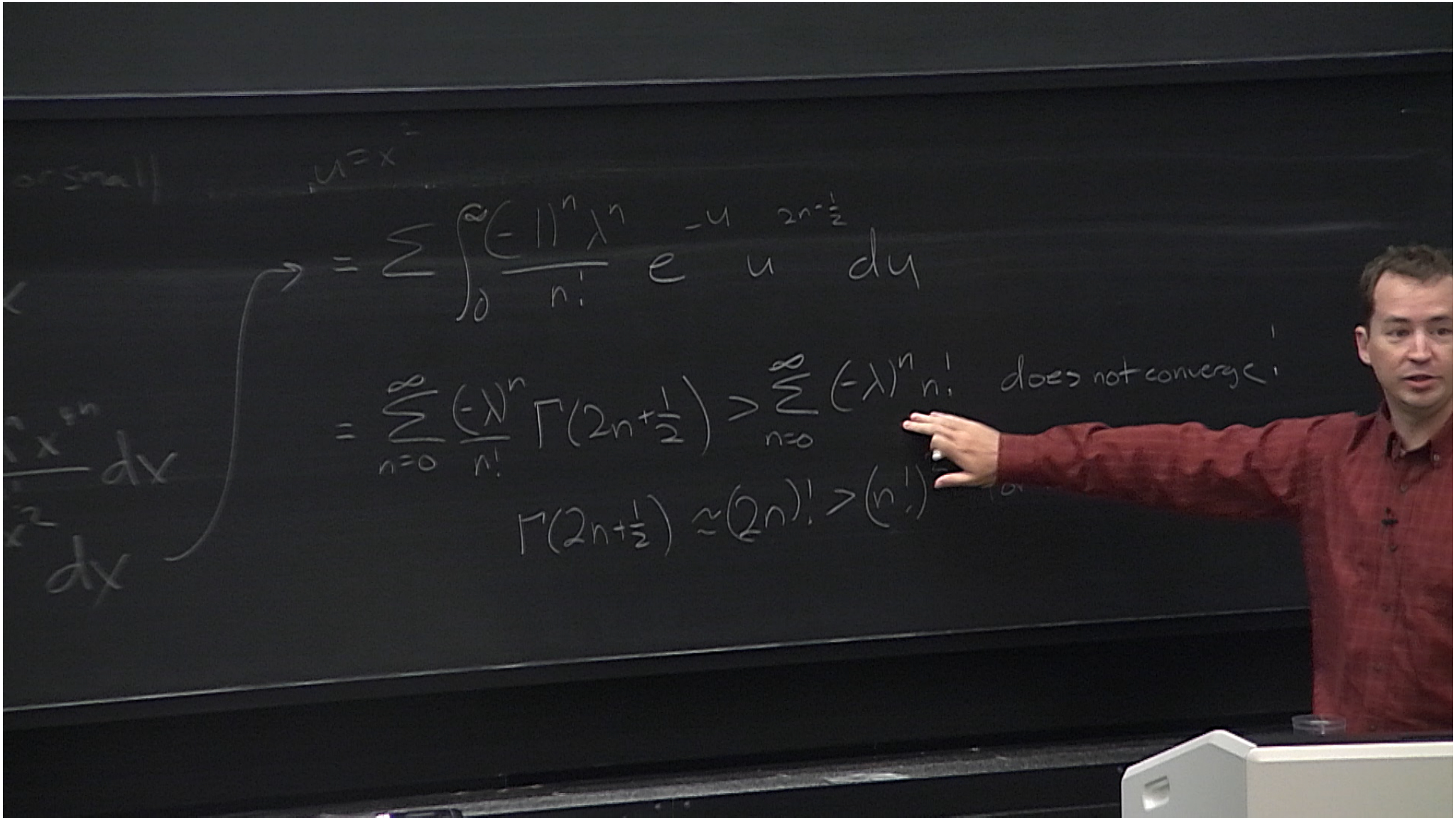
$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n+1)} \approx \frac{(-1)^{n+1}}{(n-1)!\,\epsilon} \quad \text{as } \epsilon \to 0$$

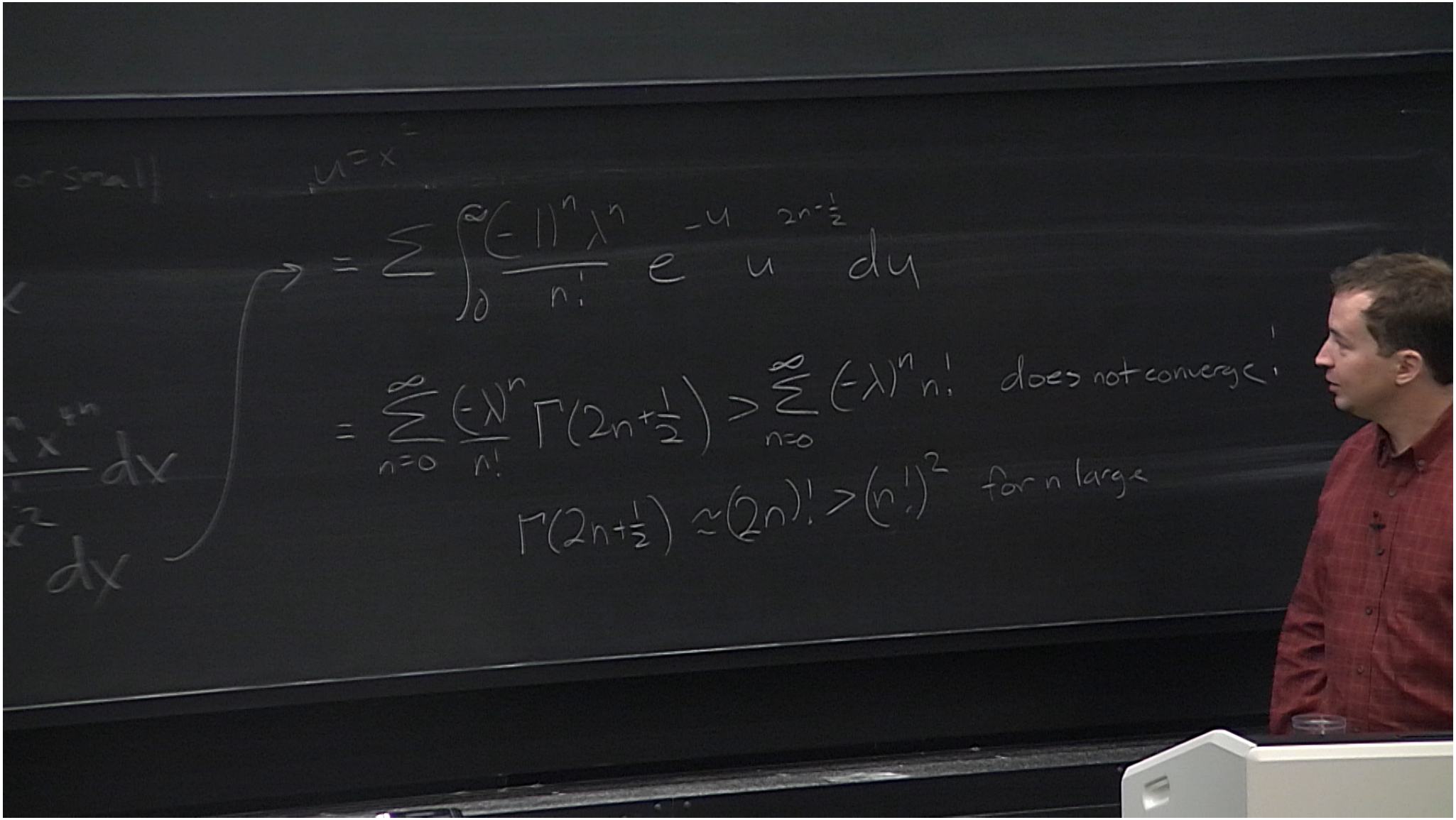

$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$ $\mathrm{Re}(z)+n>0$
$\Gamma(z)$ has no zeros → poles at $z=-n$ for $n\geq 0$
$z=-(n-1)+\epsilon$
$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n+1)} \approx \frac{(-1)^{n+1}}{(n-1)!\cdot\epsilon}$ as $\epsilon\to 0$
residue $= \frac{(-1)^n}{n!}$ at $z=-n$

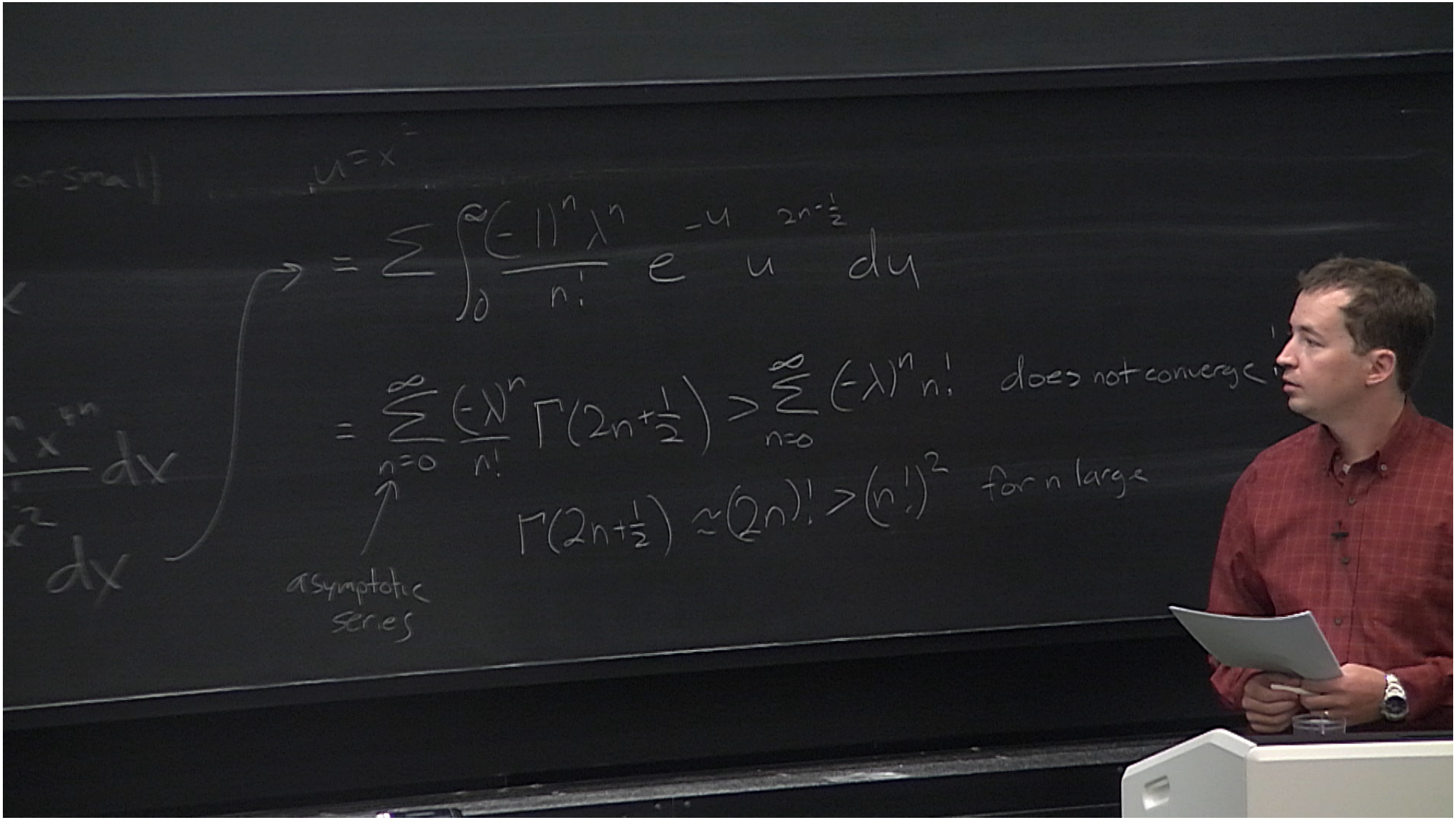

Asymptotics
Motivation: understand behavior when parameter is large or small
Toy model: $Z(\lambda) = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4} dx$
$= \int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{4n}}{n!} dx$
$\stackrel{?}{=} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{(-1)^n \lambda^n}{n!} x^{4n} e^{-x^2} dx$

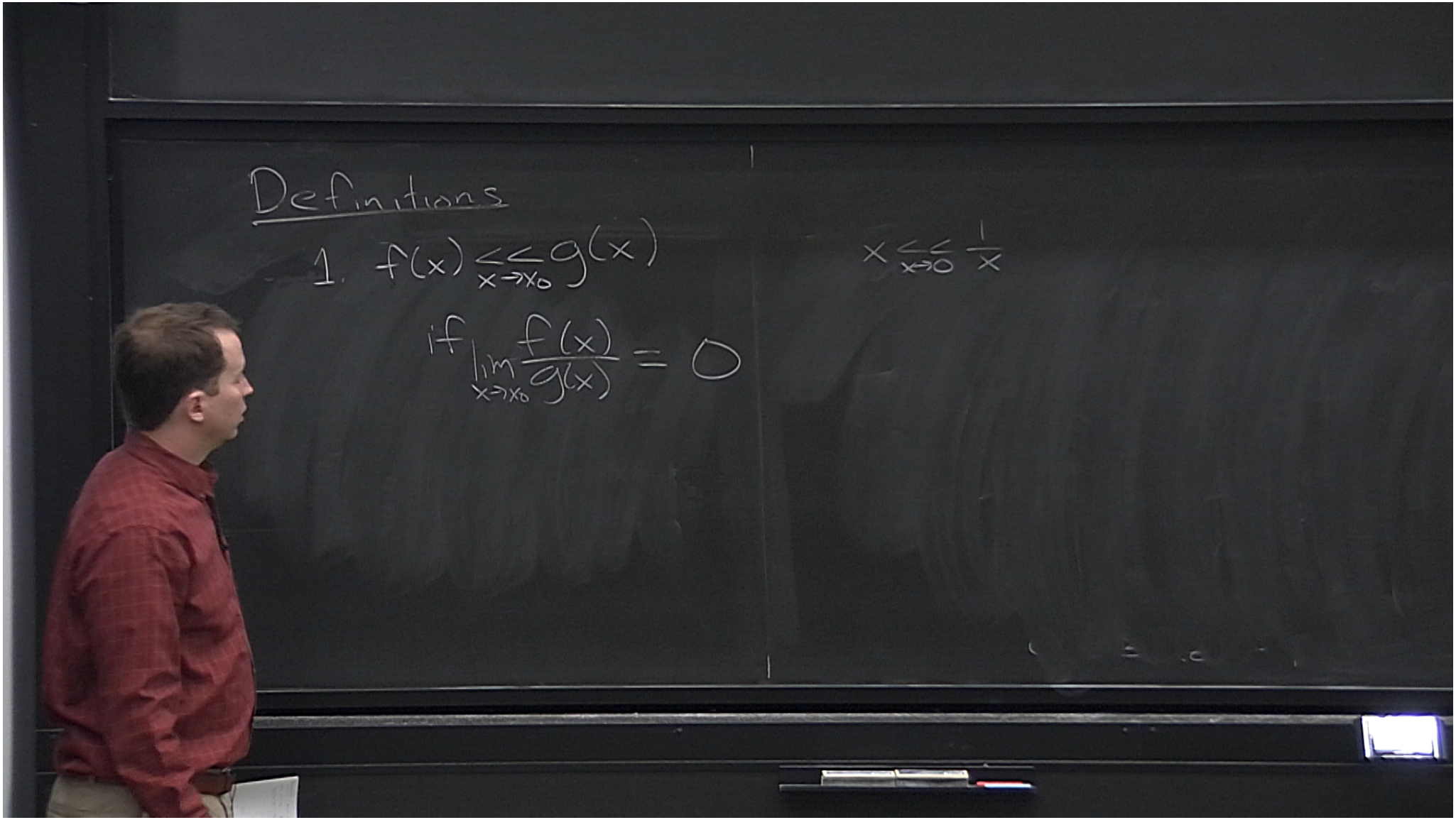

parameter is large or small
$u = x^2$
$-x^2 - \lambda x^4 \, dx$
$= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} x^{4n} \, dx$
$= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{(-1)^n \lambda^n}{n!} e^{-u} u^{2n-\frac{1}{2}} \, du$
$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma(2n+\frac{1}{2})$
$\frac{(-1)^n \lambda^n}{n!} x^m e^{-x^2} \, dx$

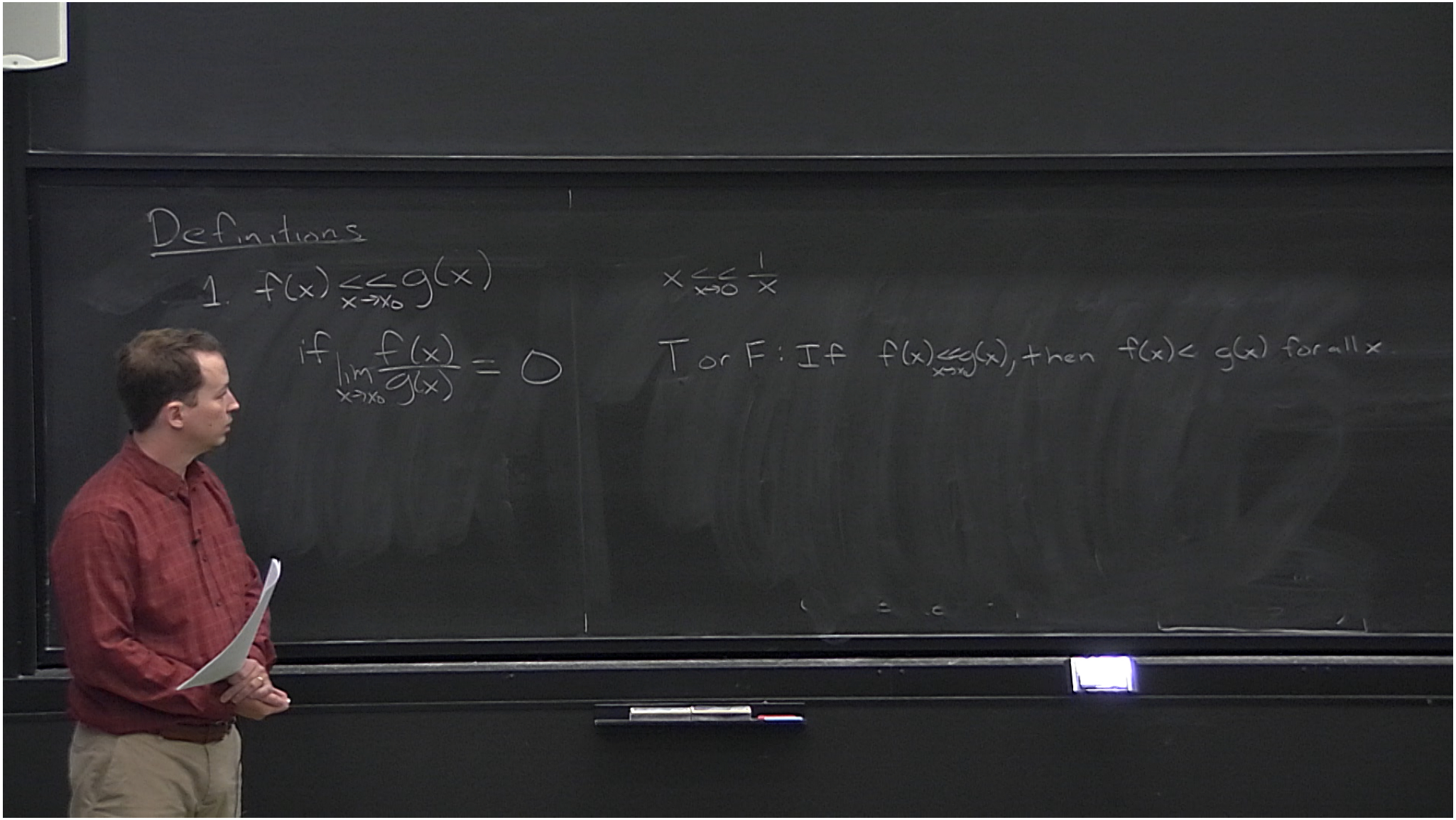

$u = x^2$
$= \sum \int_0^\infty \frac{(-1)^n \lambda^n}{n!} e^{-u} u^{2n-\frac{1}{2}} du$
$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma(2n+\frac{1}{2})$
$\Gamma(2n+\frac{1}{2}) \simeq (2n)!$
$\int e^{-x^2} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{4n}}{n!} dx$
$\sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} x^m e^{-x^2} dx$
$-x^2 - \lambda x^4 dx$
meter is large or small

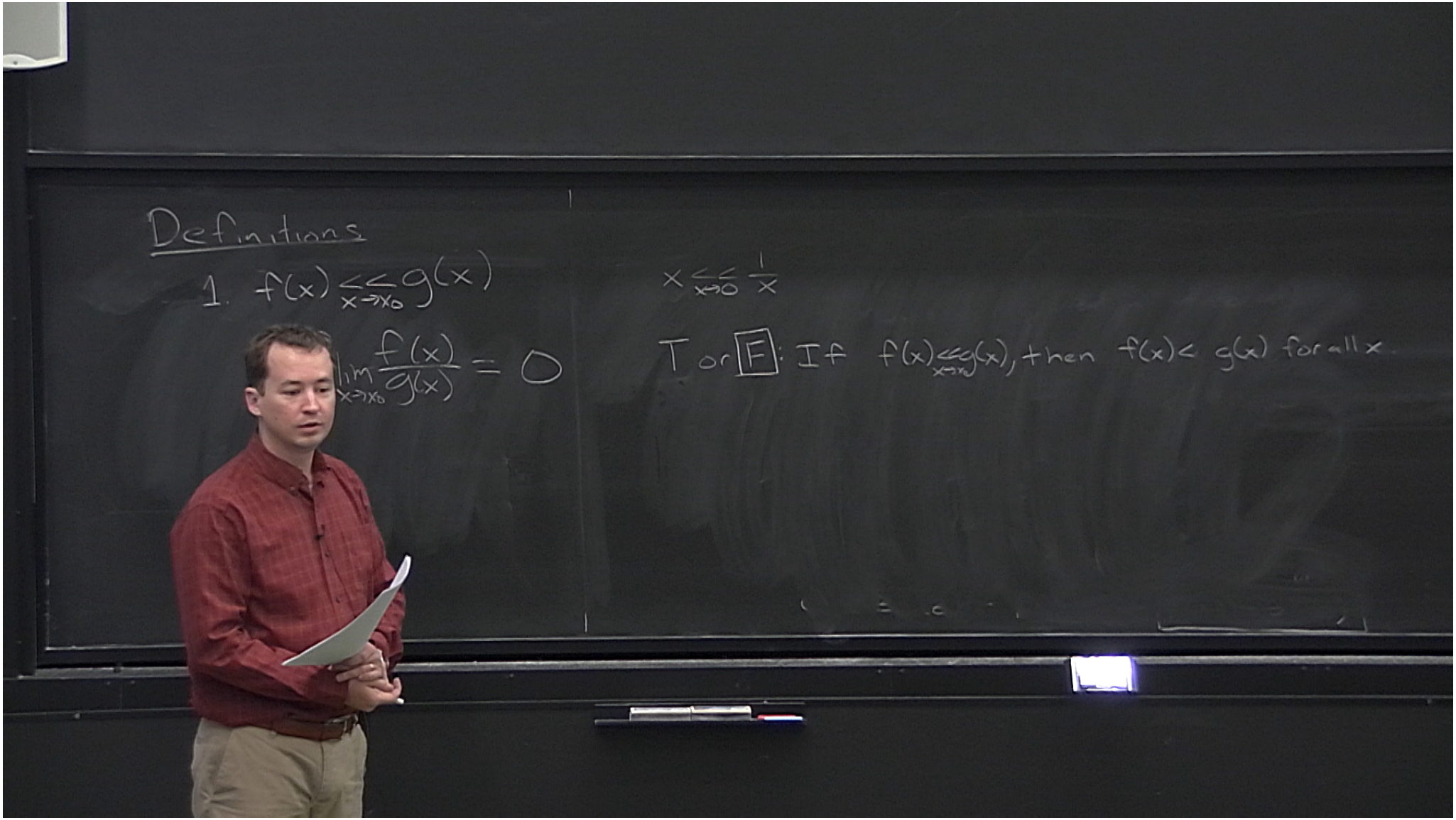

small
$u = x^{\frac{1}{2}}$
$= \sum \int_0^\infty \frac{(-1)^n \lambda^n}{n!} e^{-u} u^{2n-\frac{1}{2}} du$
$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma(2n+\frac{1}{2}) > \sum_{n=0}^{\infty} (-\lambda)^n n!$ does not converge!
$\Gamma(2n+\frac{1}{2}) \approx (2n)! > (n!)$
$\frac{x^{\frac{1}{2}n}}{x^2} dx$
$dx$

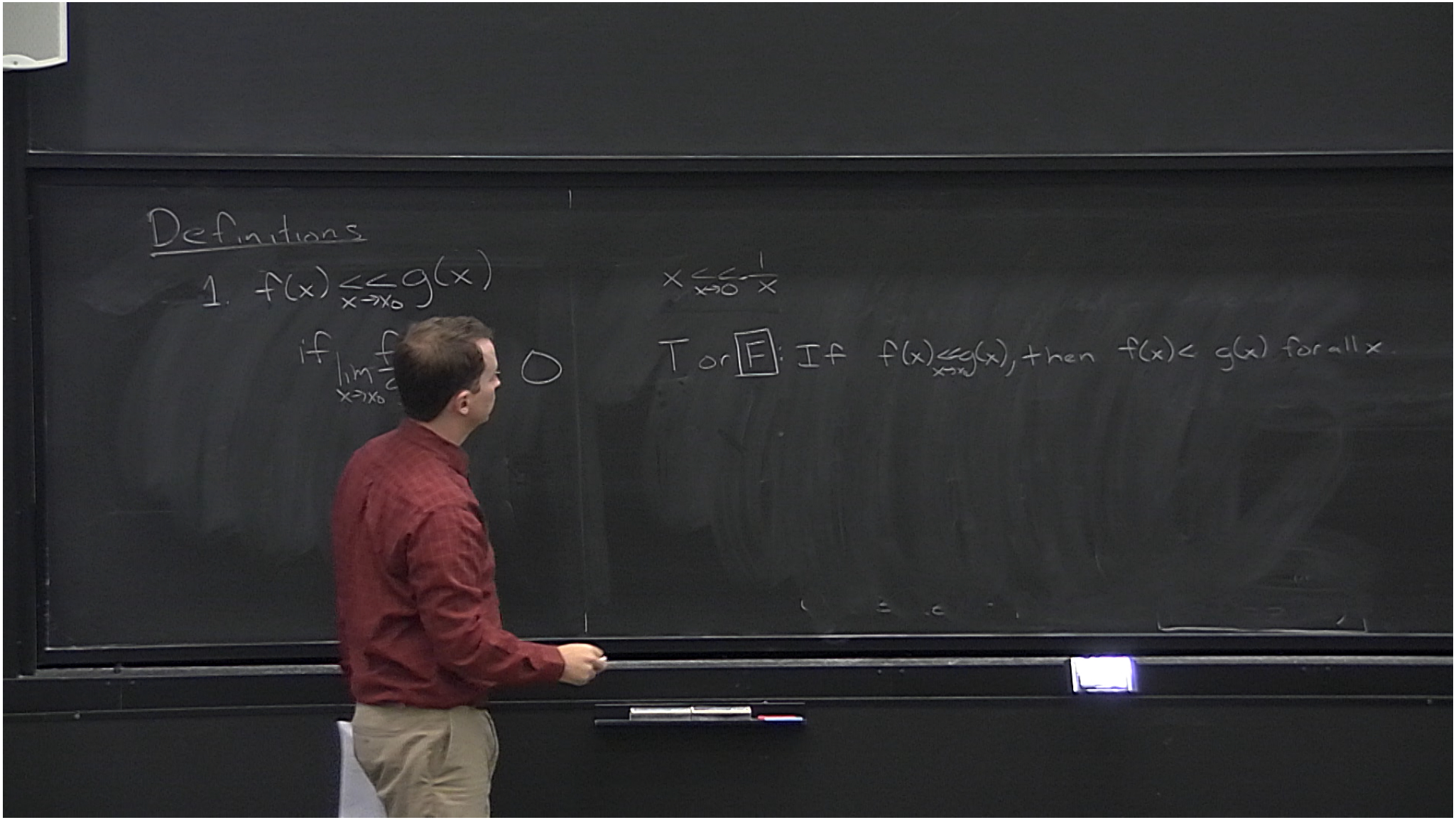

or small
$u = x^2$
$= \sum \int_0^\infty \frac{(-1)^n \lambda^n}{n!} e^{-u} u^{2n-\frac{1}{2}} du$
$= \sum_{n=0}^\infty \frac{(-\lambda)^n}{n!} \Gamma(2n+\frac{1}{2}) > \sum_{n=0}^\infty (-\lambda)^n n!$ does not converge!
$\Gamma(2n+\frac{1}{2}) \approx (2n)! > (n!)^2$ for n large
$\frac{x^{4n}}{} dx$
$x^2 dx$


small
$u = x^2$
$= \sum \int_0^\infty \frac{(-1)^n \lambda^n}{n!} e^{-u} u^{2n-\frac{1}{2}} du$
$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma(2n+\frac{1}{2}) > \sum_{n=0}^{\infty} (-\lambda)^n n!$ does not converge
$\Gamma(2n+\frac{1}{2}) \approx (2n)! > (n!)^2$ for n large
asymptotic series
$\frac{x^{4n}}{} dx$
$x^2$
$dx$

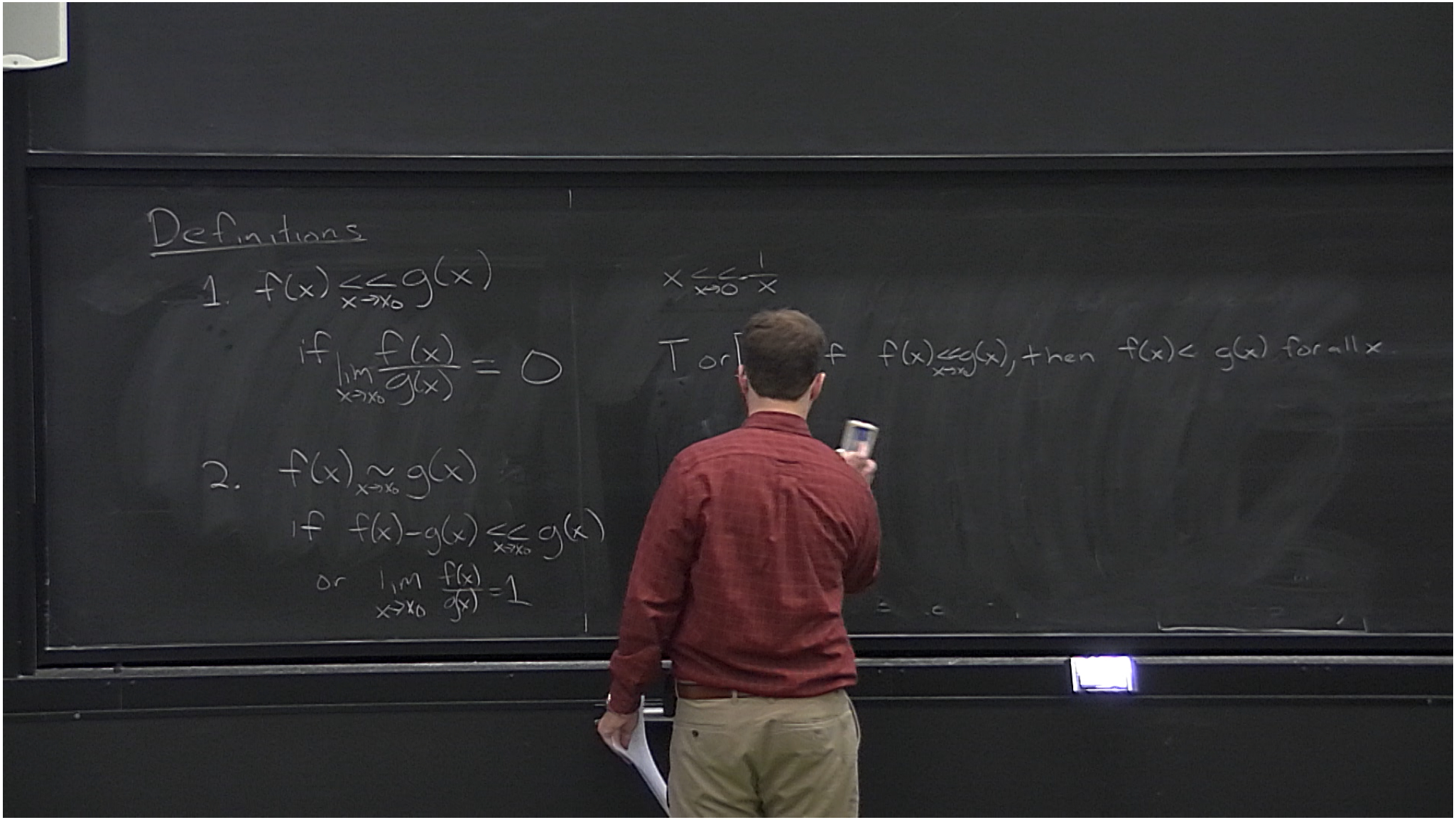

Definitions
1. $f(x) \underset{x \to x_0}{\ll} g(x)$
if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$
$x \underset{x \to 0}{\ll} \frac{1}{x}$

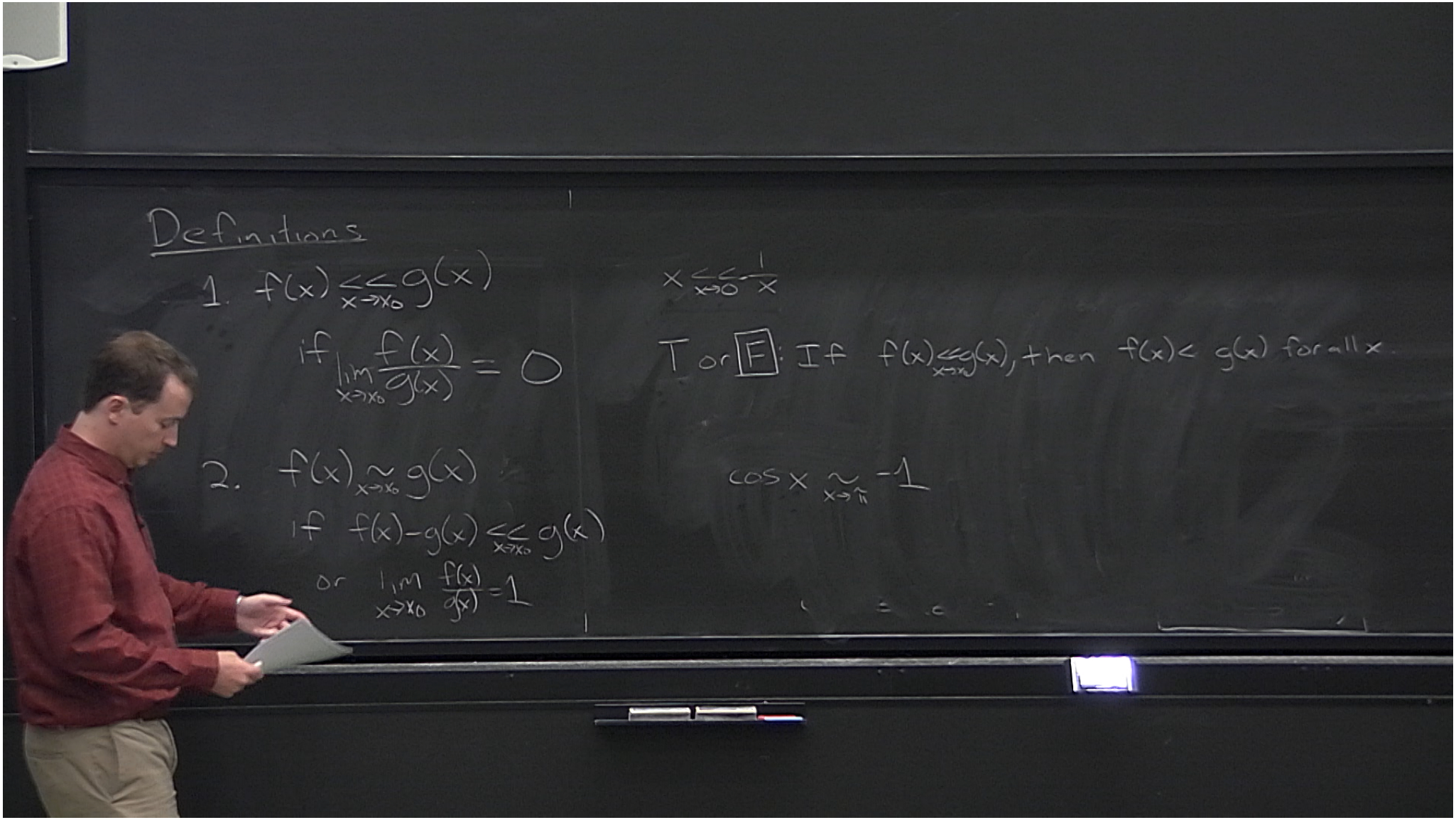

Definitions
1. $f(x) \ll_{x \to x_0} g(x)$
if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$
$x \ll_{x \to 0} \frac{1}{x}$
T or F: If $f(x) \ll_{x \to x_0} g(x)$, then $f(x) < g(x)$ for all $x$


Definitions
1. $f(x) \underset{x \to x_0}{\ll} g(x)$
$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$
$x \underset{x \to 0}{\ll} \frac{1}{x}$
T or F: If $f(x) \underset{x \to x_0}{\ll} g(x)$, then $f(x) < g(x)$ for all $x$


Definitions
1. $f(x) \ll g(x)$
$x \ll \frac{1}{x}$
T or F: If $f(x) \ll g(x)$, then $f(x) < g(x)$ for all $x$


Definitions
1. $f(x) \ll_{x \to x_0} g(x)$
if $\lim_{x \to x_0} \frac{f}{g} = 0$
$x \ll_{x \to 0} \frac{1}{x}$
T or F: If $f(x) \ll_{x \to x_0} g(x)$, then $f(x) < g(x)$ for all $x$.
$x^2 \ll_{x \to 0} -1$


Definitions
1. $f(x) \ll_{x \to x_0} g(x)$
if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$
$x \ll_{x \to 0} \frac{1}{x}$
2. $f(x) \sim_{x \to x_0} g(x)$
if $f(x) - g(x) \ll_{x \to x_0} g(x)$
or $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$


Definitions
1. $f(x) \ll_{x \to x_0} g(x)$
if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$
$x \ll_{x \to 0} \frac{1}{x}$
T or F: If $f(x) \ll_{x \to x_0} g(x)$, then $f(x) < g(x)$ for all $x$.
2. $f(x) \sim_{x \to x_0} g(x)$
if $f(x) - g(x) \ll_{x \to x_0} g(x)$
or $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$
$\cos x \sim_{x \to \pi} -1$

$x \ll \frac{1}{x}$ as $x \to 0$

T or F: If $f(x) \ll g(x)$ as $x \to x_0$, then $f(x) < g(x)$ for all $x$.

$\cos x \sim -1$ as $x \to \pi$

T or F. If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$, then $f(x) \sim g(x)$ as $x \to x_0$.

T or F. If $\lim_{x \to x_0} f(x) = Y$, then $f(x) \sim Y$ as $x \to x_0$

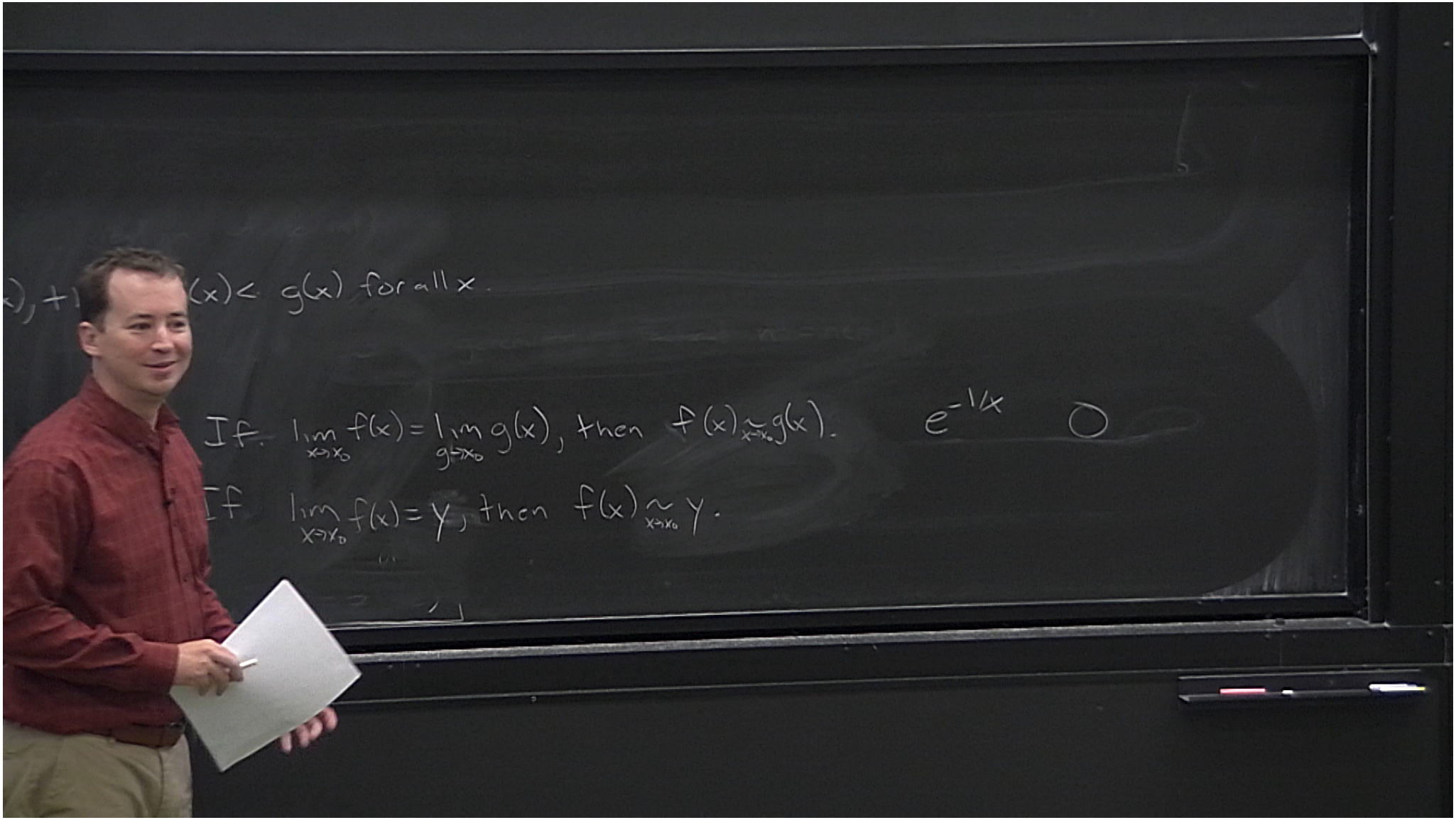
$x \ll \frac{1}{x}$ as $x \to 0$

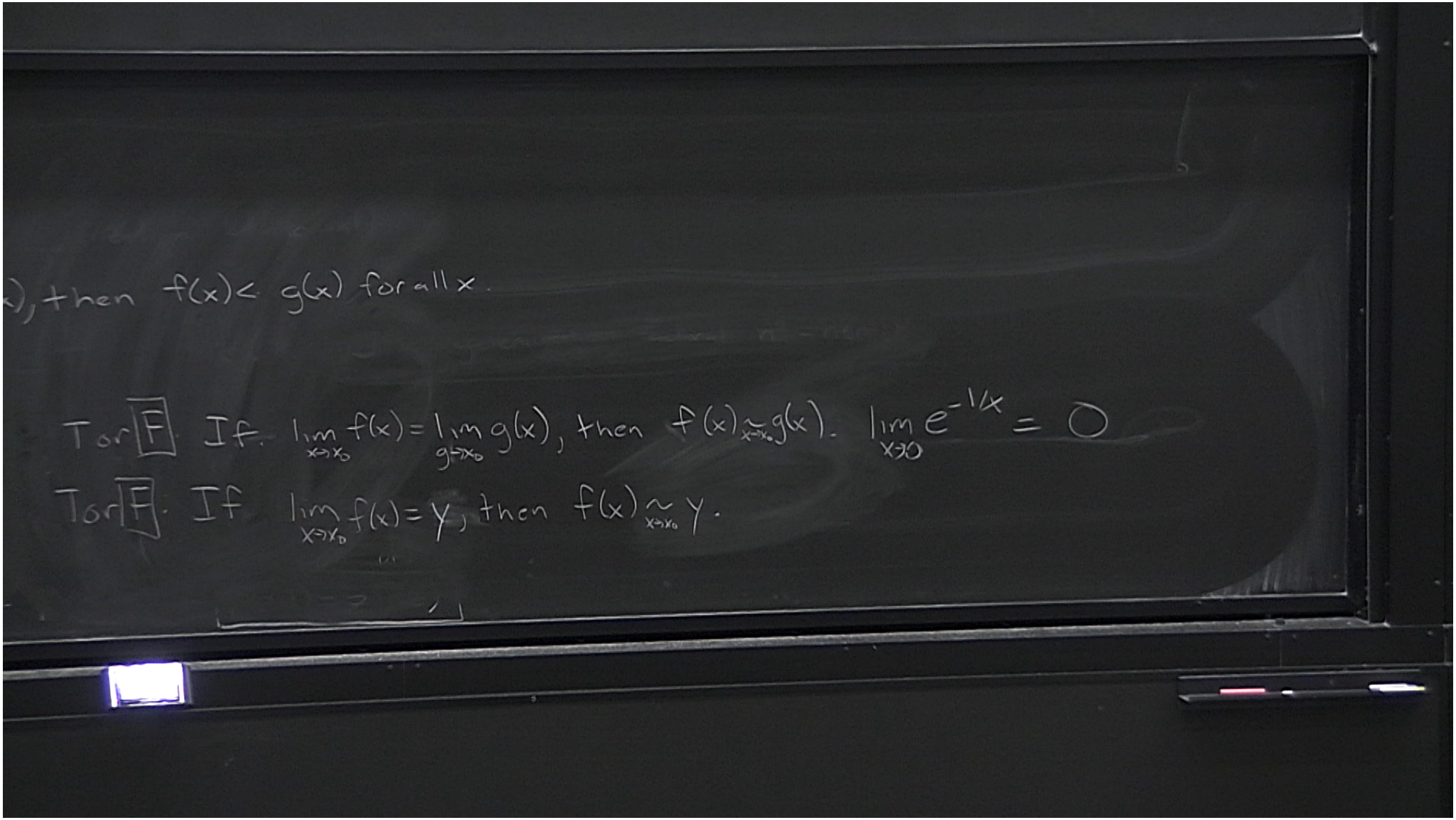
T or F: If $f(x) \ll g(x)$ as $x \to x_0$, then $f(x) < g(x)$ for all $x$.

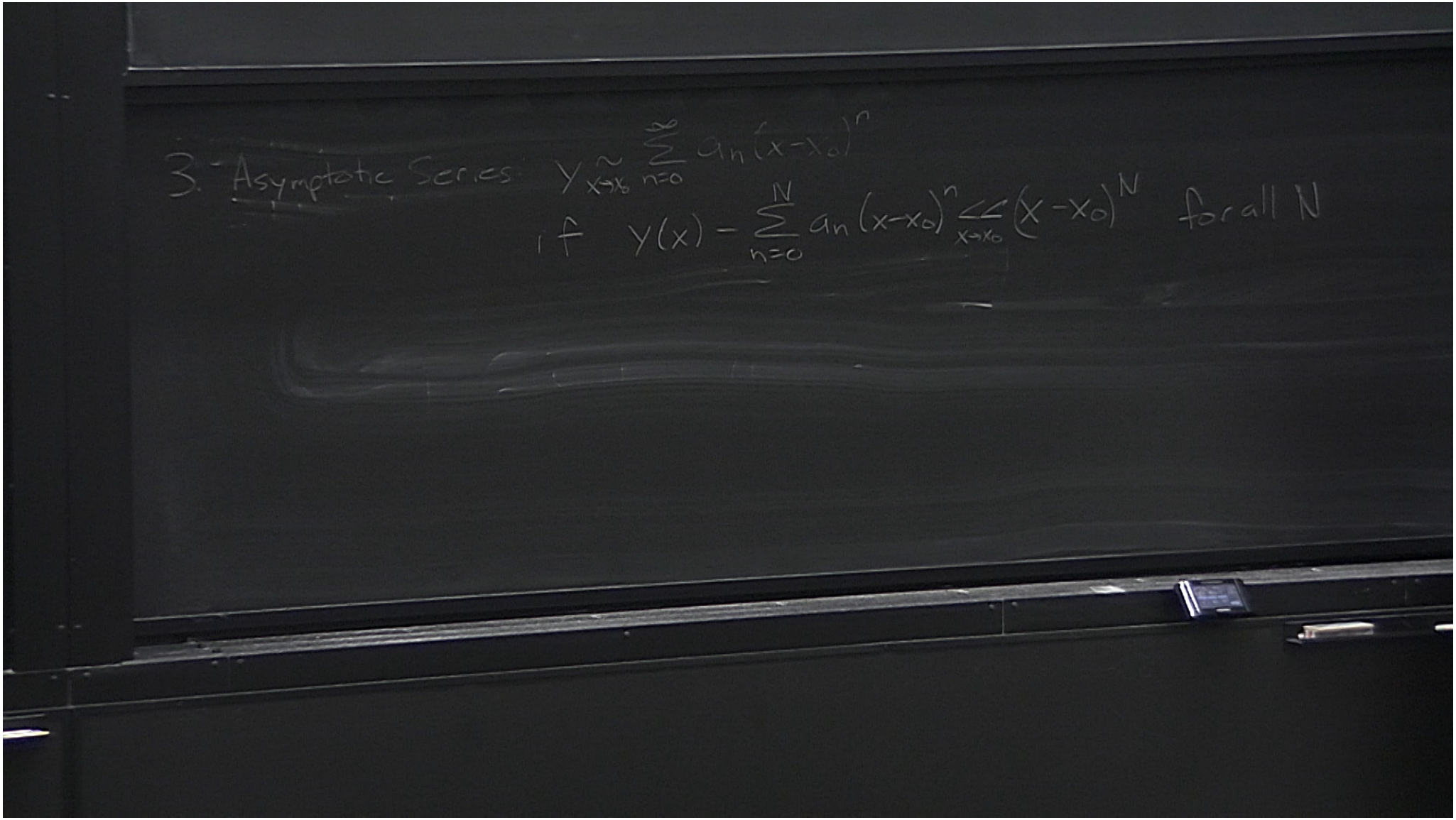
$\cos x \sim -1$ as $x \to \pi$

T or F: If $\lim_{x \to x_0} f(x) = \lim_{g \to x_0} g(x)$, then $f(x) \sim g(x)$ as $x \to x_0$.

T or F: If $\lim_{x \to x_0} f(x) = Y$, then $f(x) \sim Y$ as $x \to x_0$.


$f(x) < g(x)$ for all $x$.
If $\lim_{x \to x_0} f(x) = \lim_{g \to x_0} g(x)$, then $f(x) \underset{x \to x_0}{\simeq} g(x)$.
$e^{-1/x}$ 0
If $\lim_{x \to x_0} f(x) = Y$, then $f(x) \underset{x \to x_0}{\simeq} Y$.

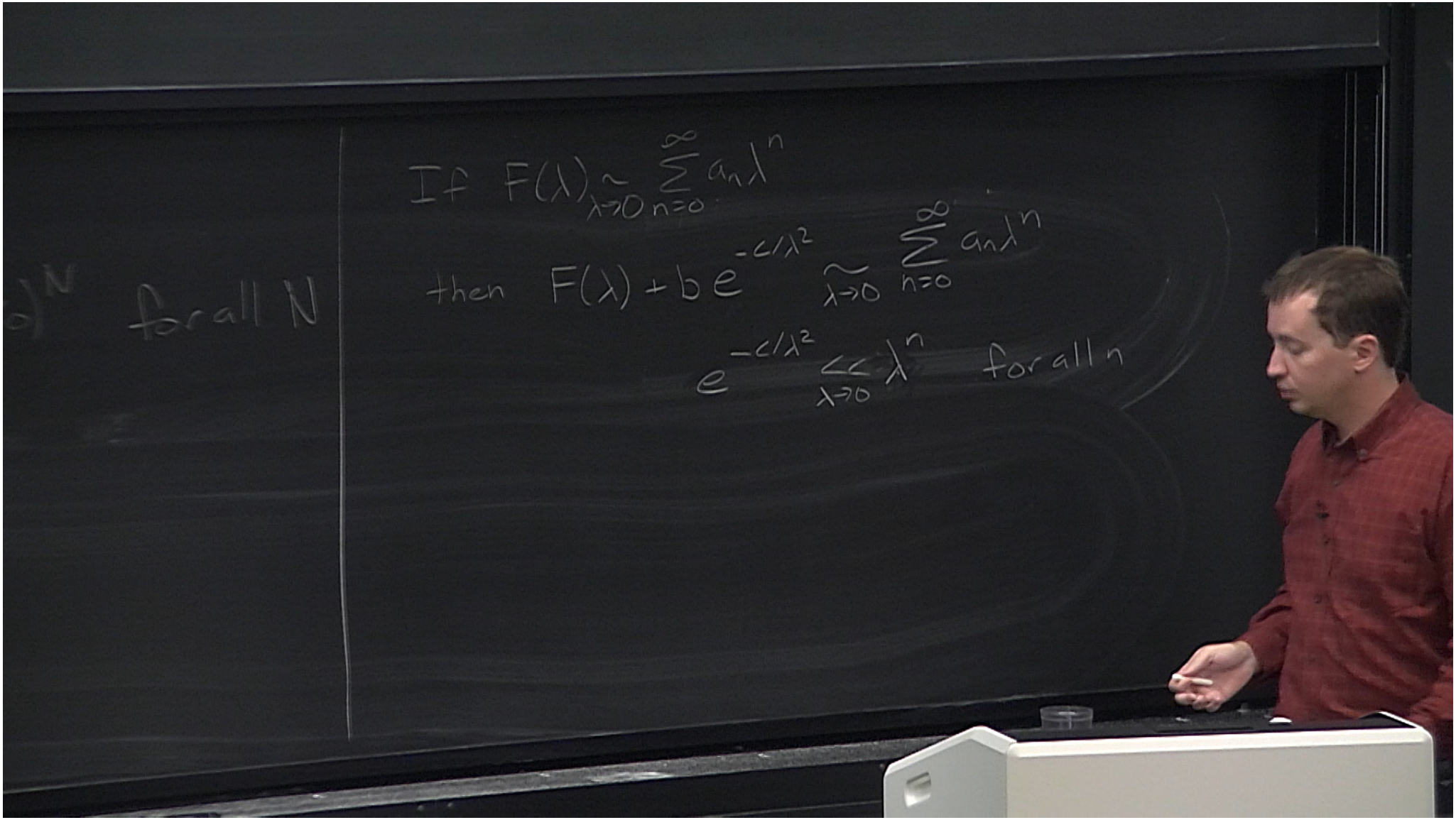

), then $f(x) < g(x)$ for all $x$.
T or F If $\lim_{x \to x_0} f(x) = \lim_{g \to x_0} g(x)$, then $f(x) \underset{x \to x_0}{\simeq} g(x)$. $\lim_{x \to 0} e^{-1/x} = 0$
T or F If $\lim_{x \to x_0} f(x) = Y$, then $f(x) \underset{x \to x_0}{\simeq} Y$.

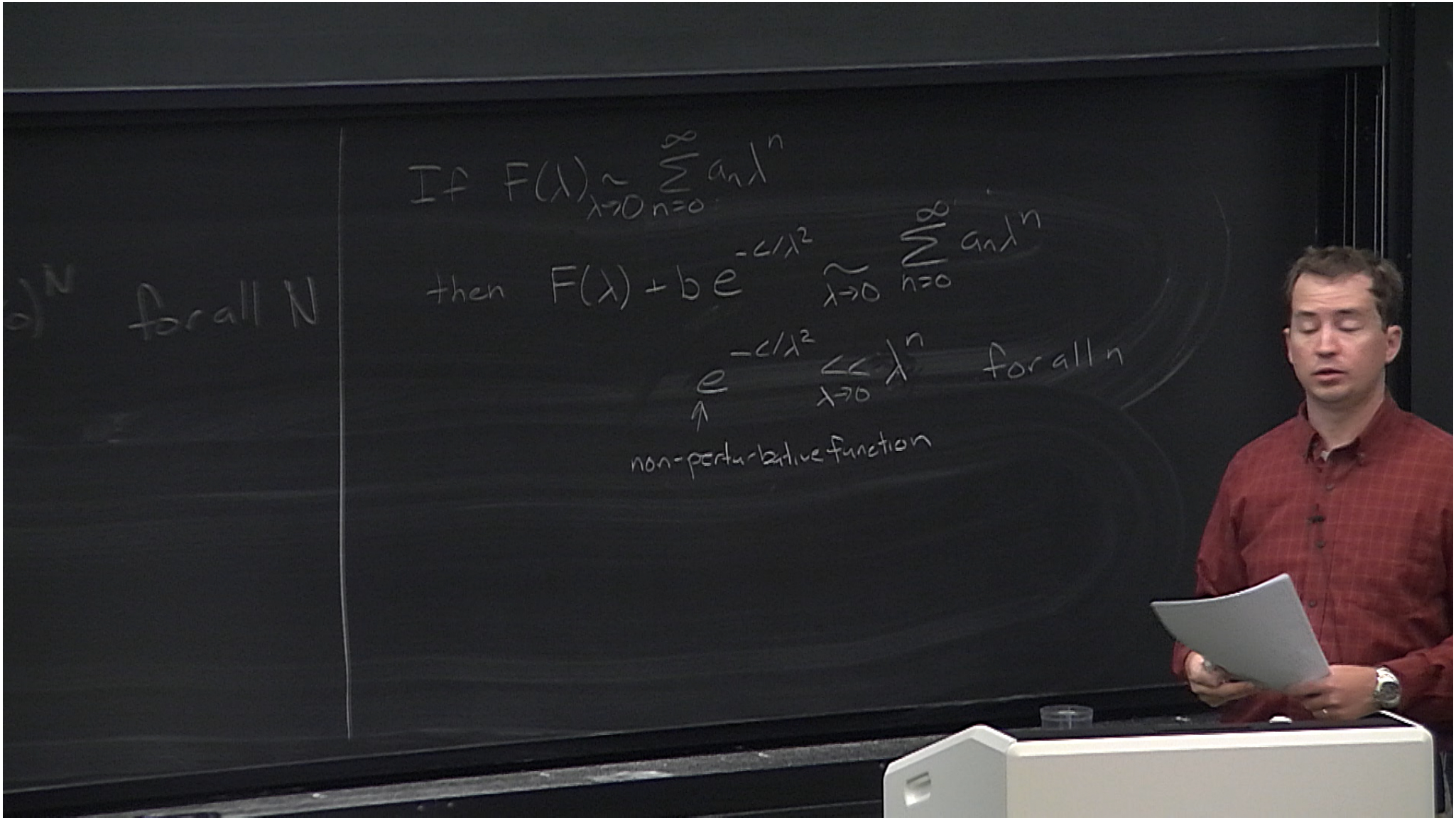

3. Asymptotic Series: $y \underset{x \to x_0}{\sim} \sum_{n=0}^{\infty} a_n (x-x_0)^n$
if $y(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \underset{x \to x_0}{\ll} (x-x_0)^N$ for all $N$

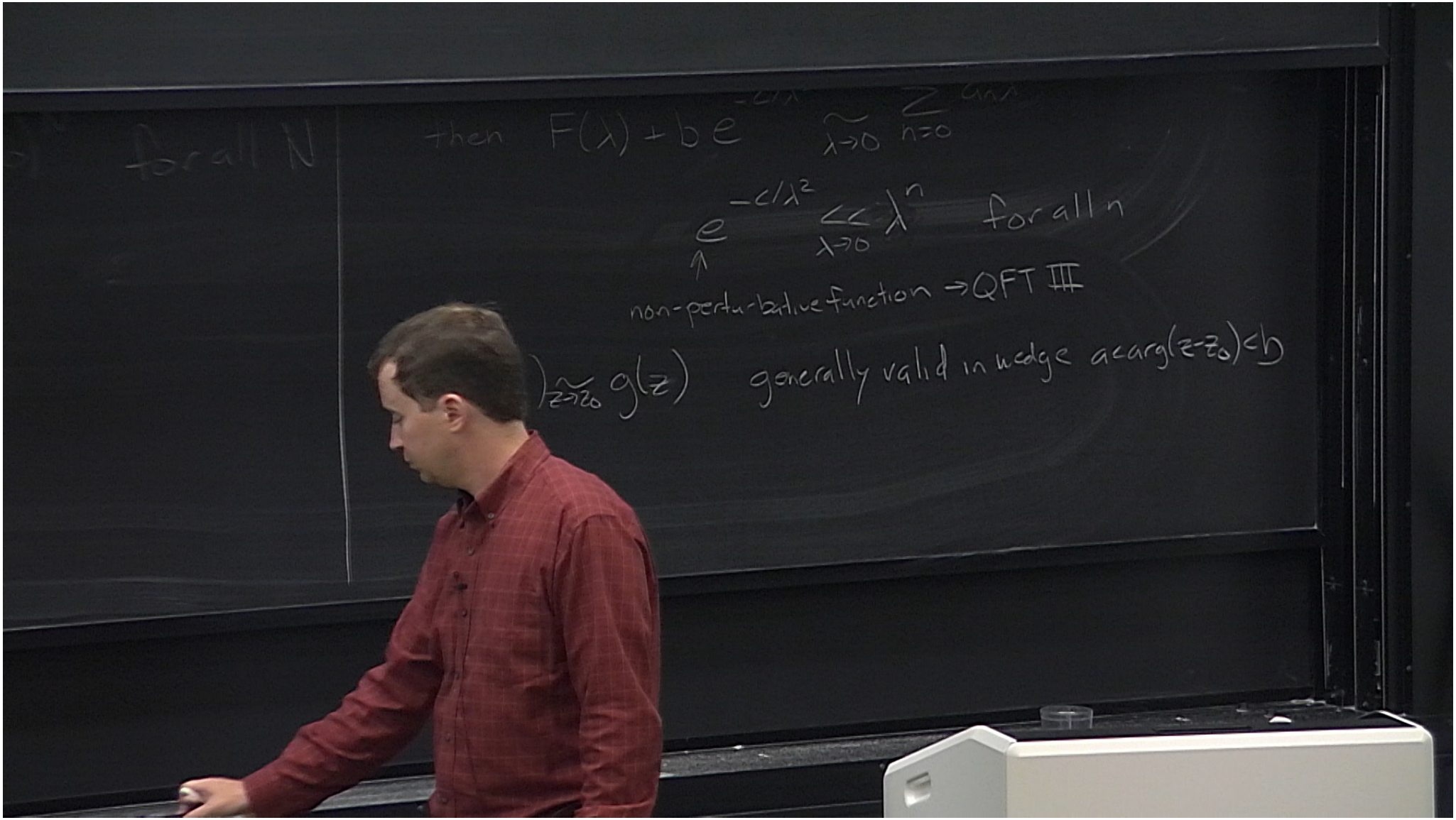
3. Asymptotic Series: $y \underset{x \to x_0}{\sim} \sum_{n=0}^{\infty} a_n (x-x_0)^n$

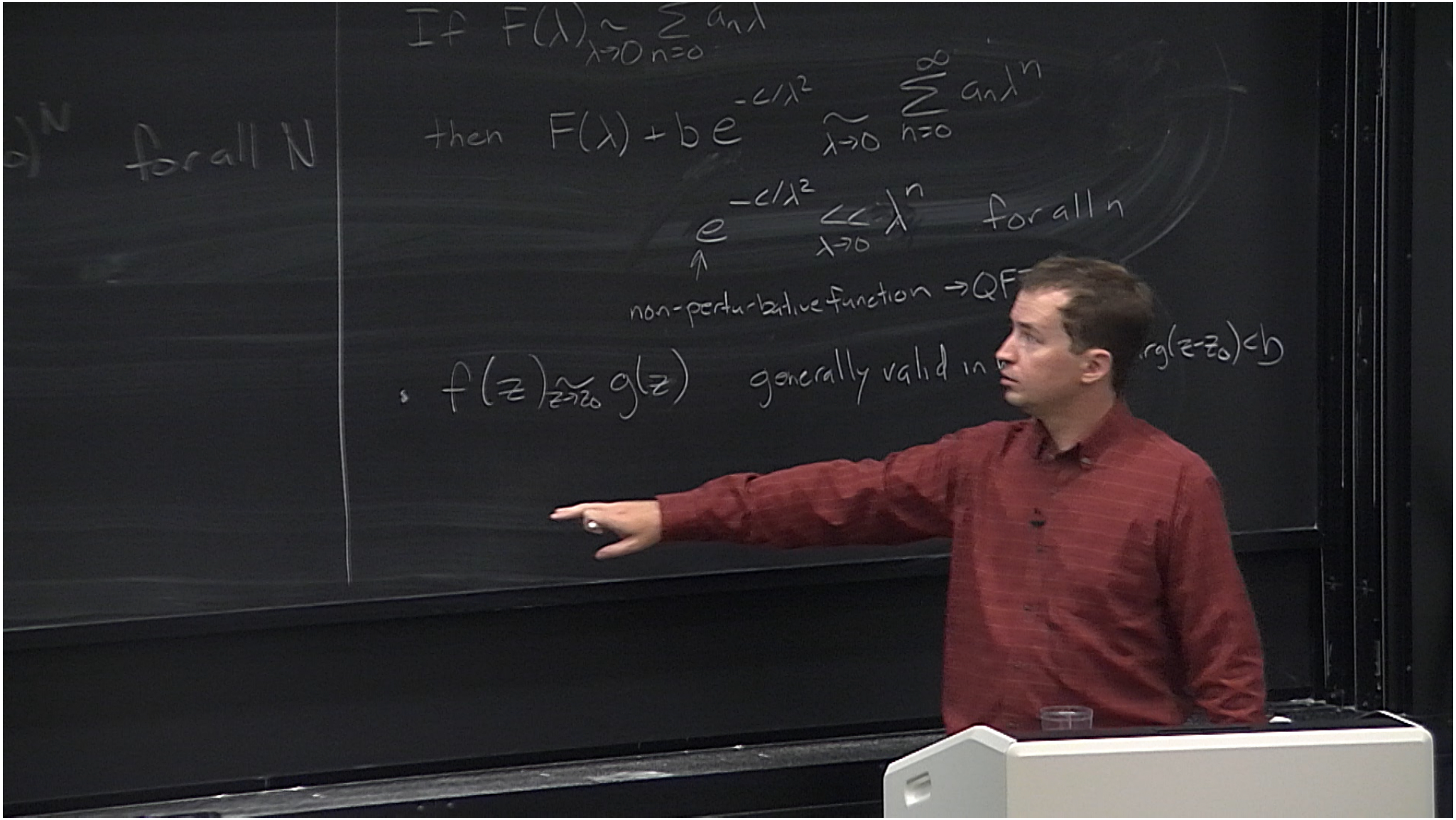
if $y(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \underset{x \to x_0}{\ll} (x-x_0)^N$ for all $N$

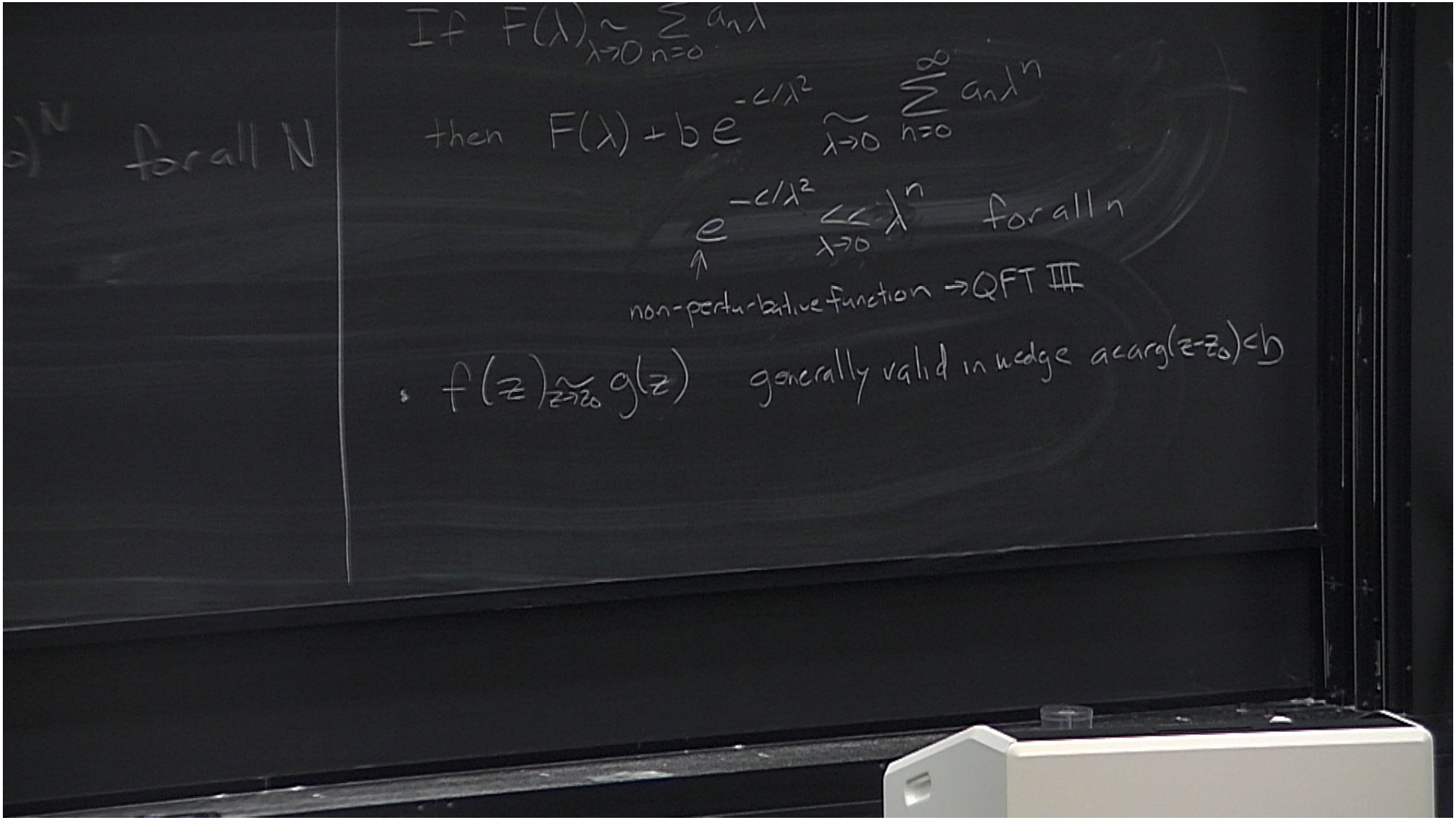
- comments:
  - convergent series get better with more terms
  - asymptotic series - better as $x \to x_0$ with a fixed # of terms

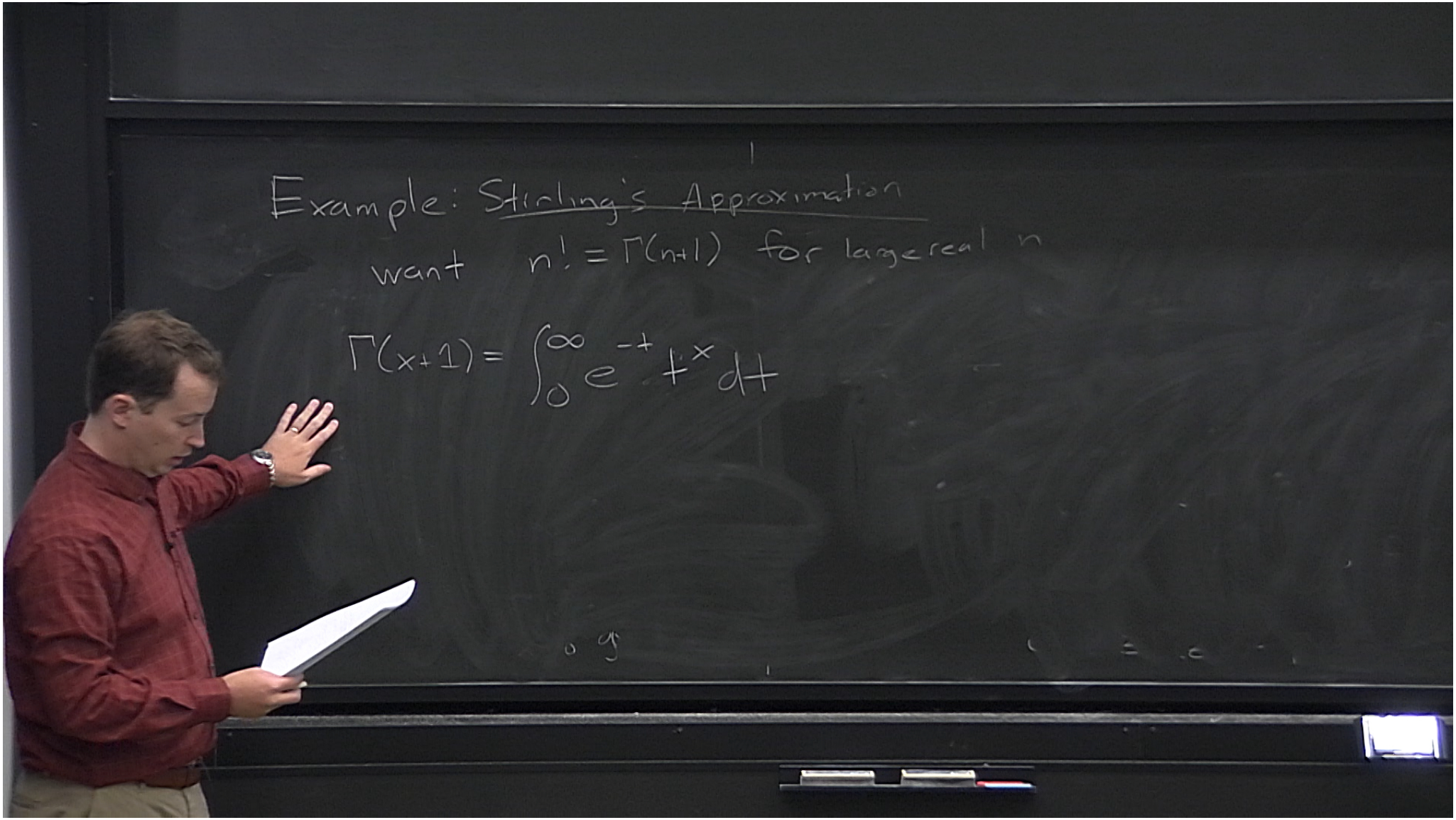

If $F(\lambda) \underset{\lambda \to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$
then $F(\lambda) + b e^{-c/\lambda^2} \underset{\lambda \to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$
$e^{-c/\lambda^2} \underset{\lambda \to 0}{<<} \lambda^n$ for all $n$
for all $N$

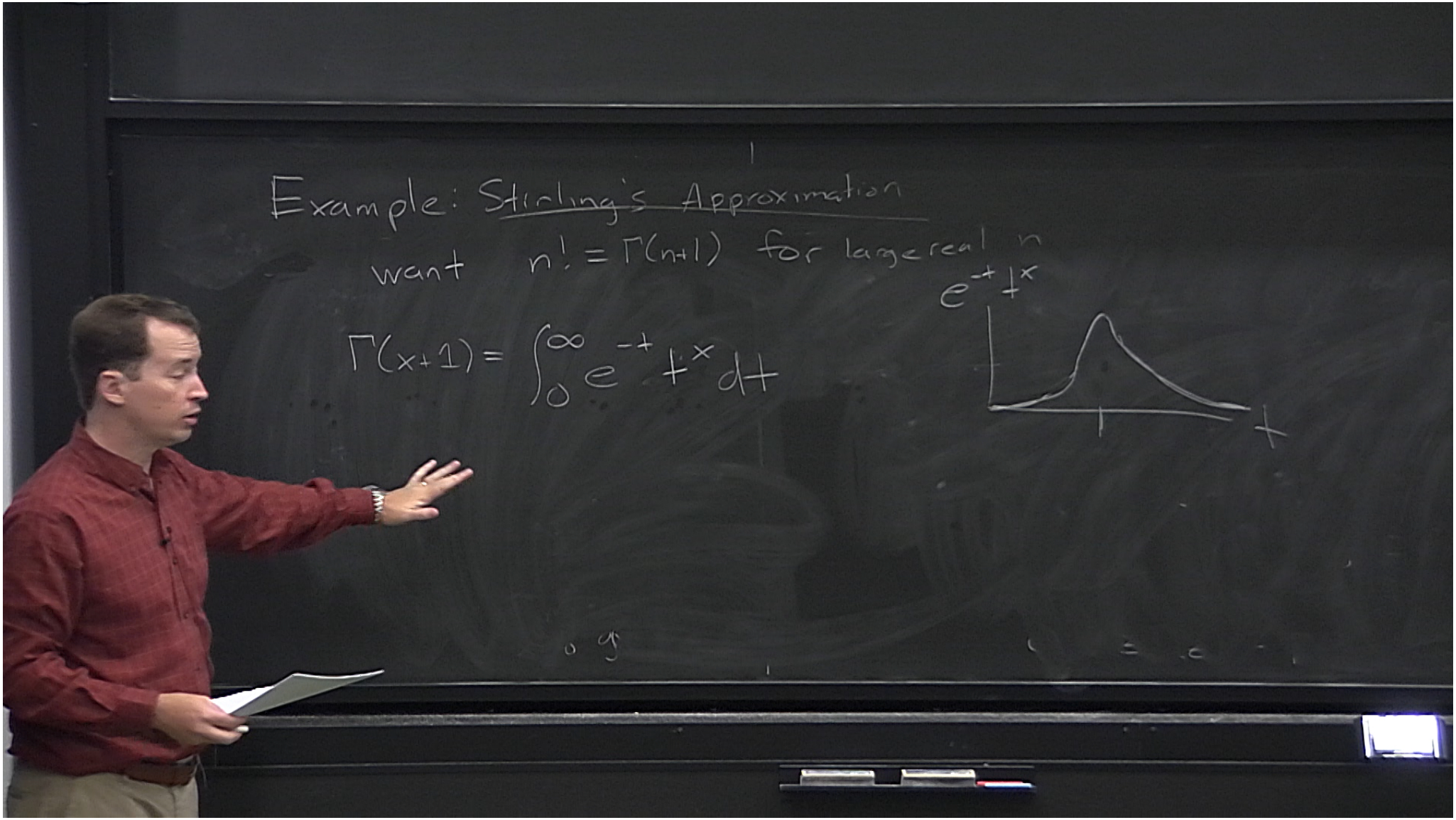

forall N
If $F(\lambda) \underset{\lambda \to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$
then $F(\lambda) + b e^{-c/\lambda^2} \underset{\lambda \to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$
$e^{-c/\lambda^2} \underset{\lambda \to 0}{<<} \lambda^n$ forall n
non-perturbative function

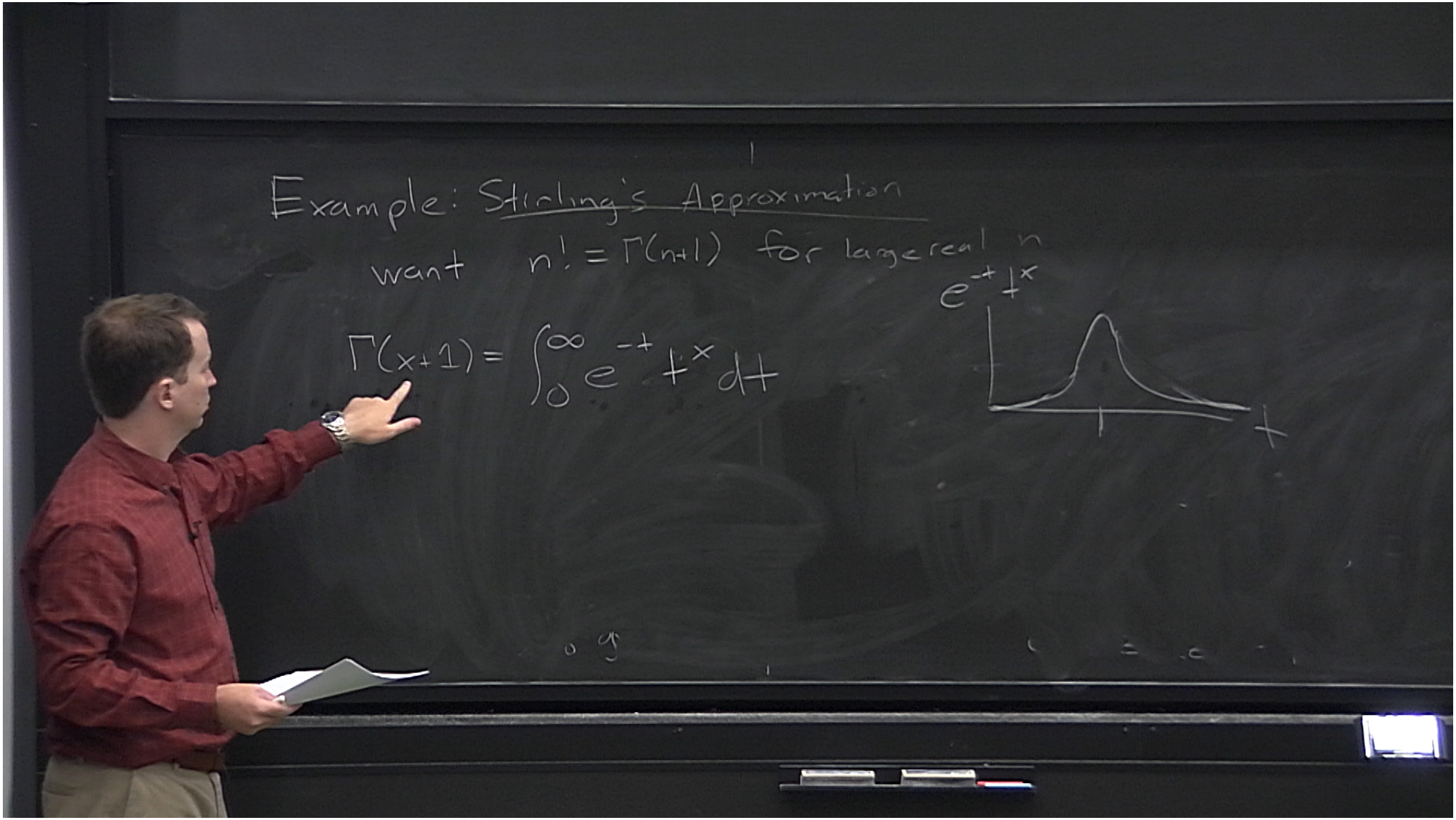

for all N
then $F(\lambda) + b e^{-c/\lambda} \underset{\lambda \to 0}{\sim} \sum_{n=0} a_n \lambda^n$
$e^{-c/\lambda^2} \underset{\lambda \to 0}{<<} \lambda^n$ for all n
non-perturbative function → QFT III
$\underset{z \to z_0}{\sim} g(z)$ generally valid in wedge $a < \arg(z - z_0) < b$


If $F(\lambda) \underset{\lambda\to 0}{\sim} \sum_{n=0} a_n \lambda$
forall N
then $F(\lambda) + b e^{-c/\lambda^2} \underset{\lambda\to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$
$e^{-c/\lambda^2} \underset{\lambda\to 0}{<<} \lambda^n$ forall n
non-perturbative function
$f(z) \underset{z\to z_0}{\sim} g(z)$ generally valid in


If $F(\lambda) \underset{\lambda\to 0}{\sim} \sum_{n=0} a_n \lambda^n$
then $F(\lambda) + b e^{-c/\lambda^2} \underset{\lambda\to 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$
$e^{-c/\lambda^2} \underset{\lambda\to 0}{<<} \lambda^n$ for all $n$
non-perturbative function → QFT III
$f(z) \underset{z\to z_0}{\sim} g(z)$ generally valid in wedge $a < \arg(z - z_0) < b$


Example: Stirling's Approximation
want $n! = \Gamma(n+1)$ for large real $n$
$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt$


Example: Stirling's Approximation
want $n! = \Gamma(n+1)$ for large real $n$
$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt$
$e^{-t} t^x$
t


Example: Stirling's Approximation
want $n! = \Gamma(n+1)$ for large real $n$
$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt$
$e^{-t} t^x$
t

Example: Stirling's Approximation

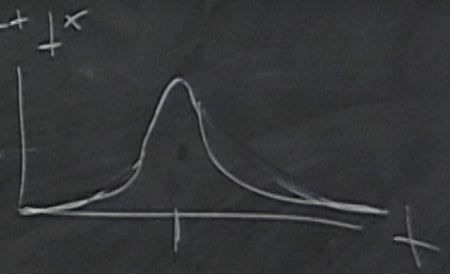
want $n! = \Gamma(n+1)$ for large real $n$

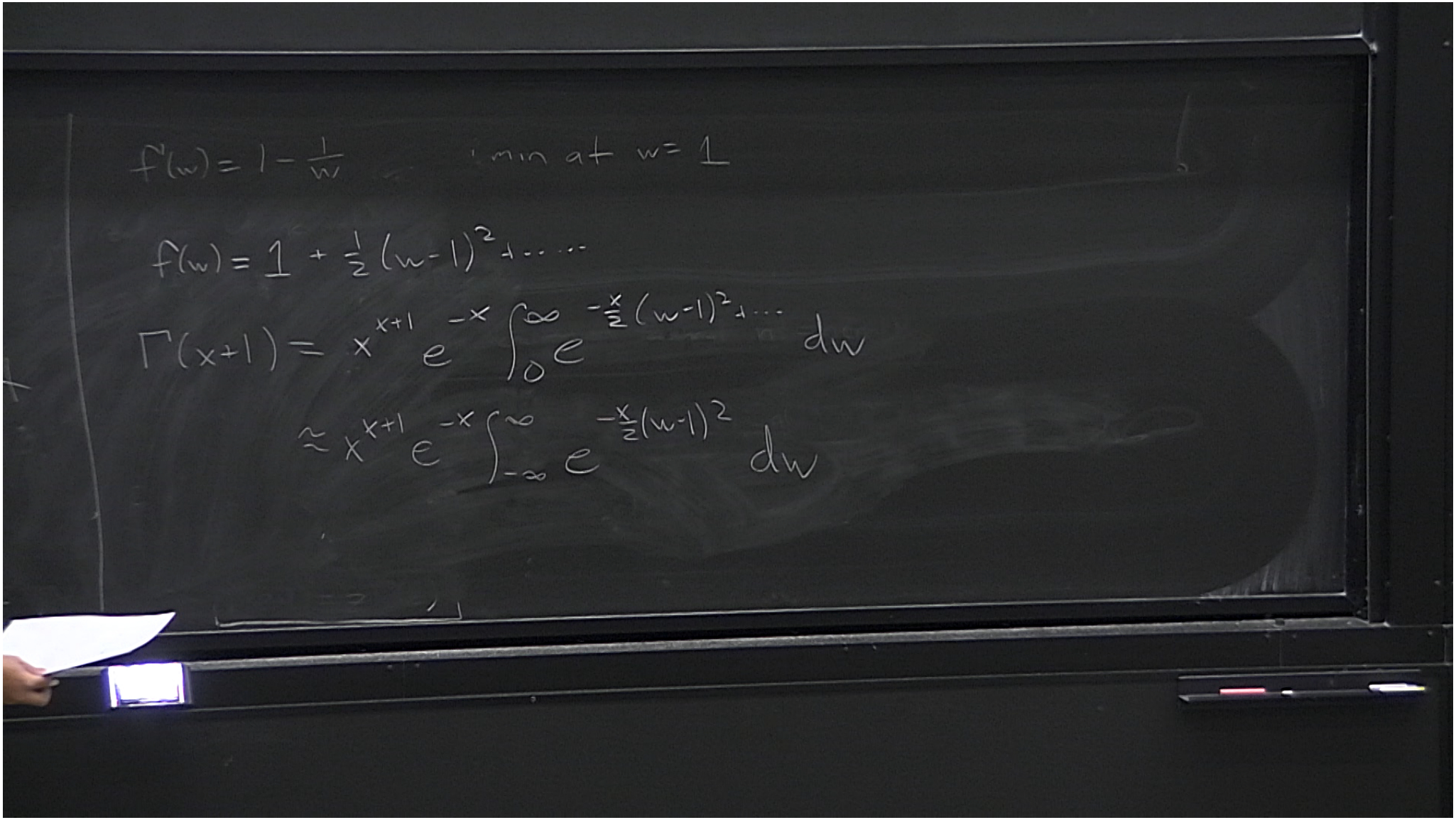
$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt$$

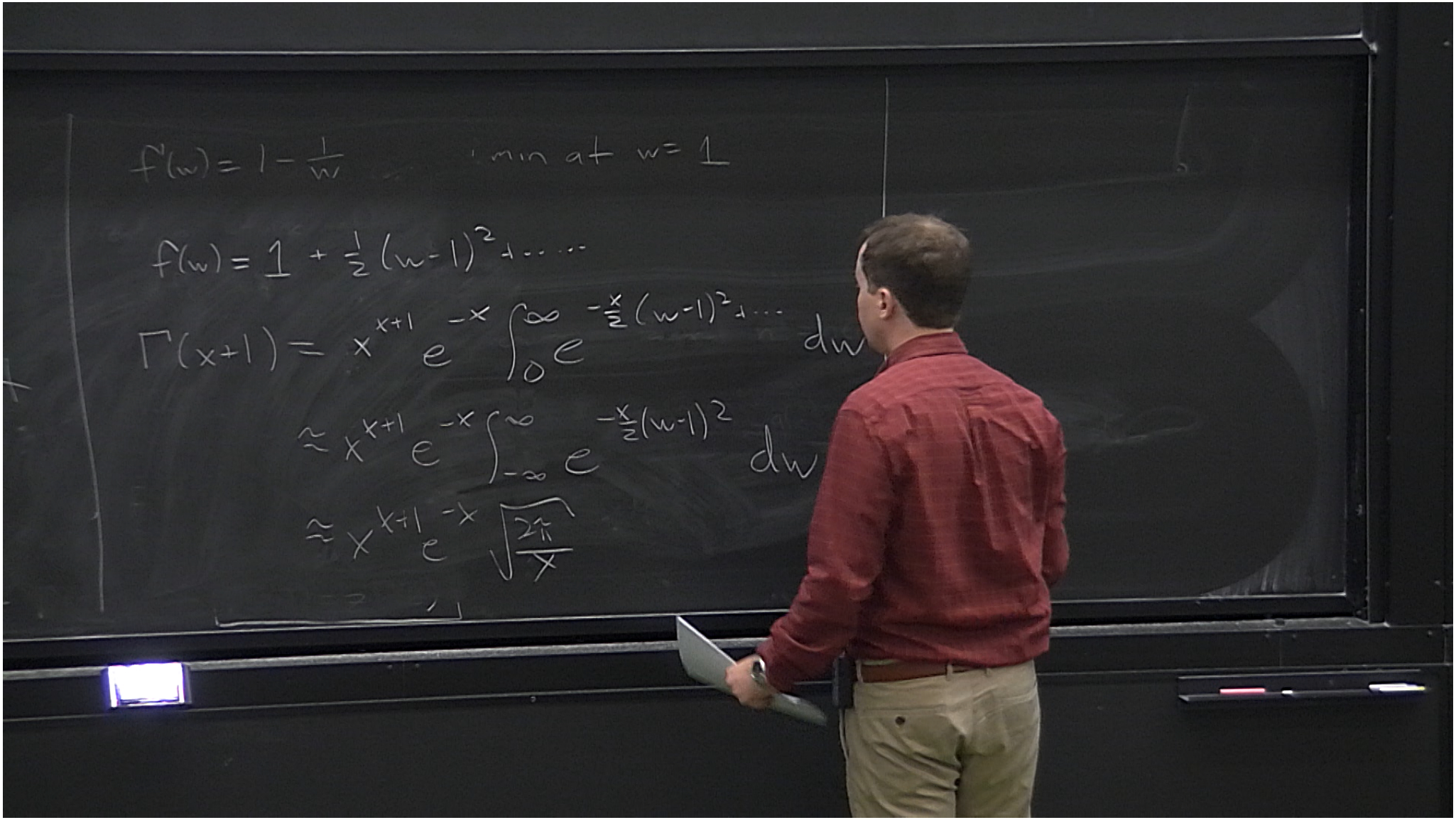
$t = wx$

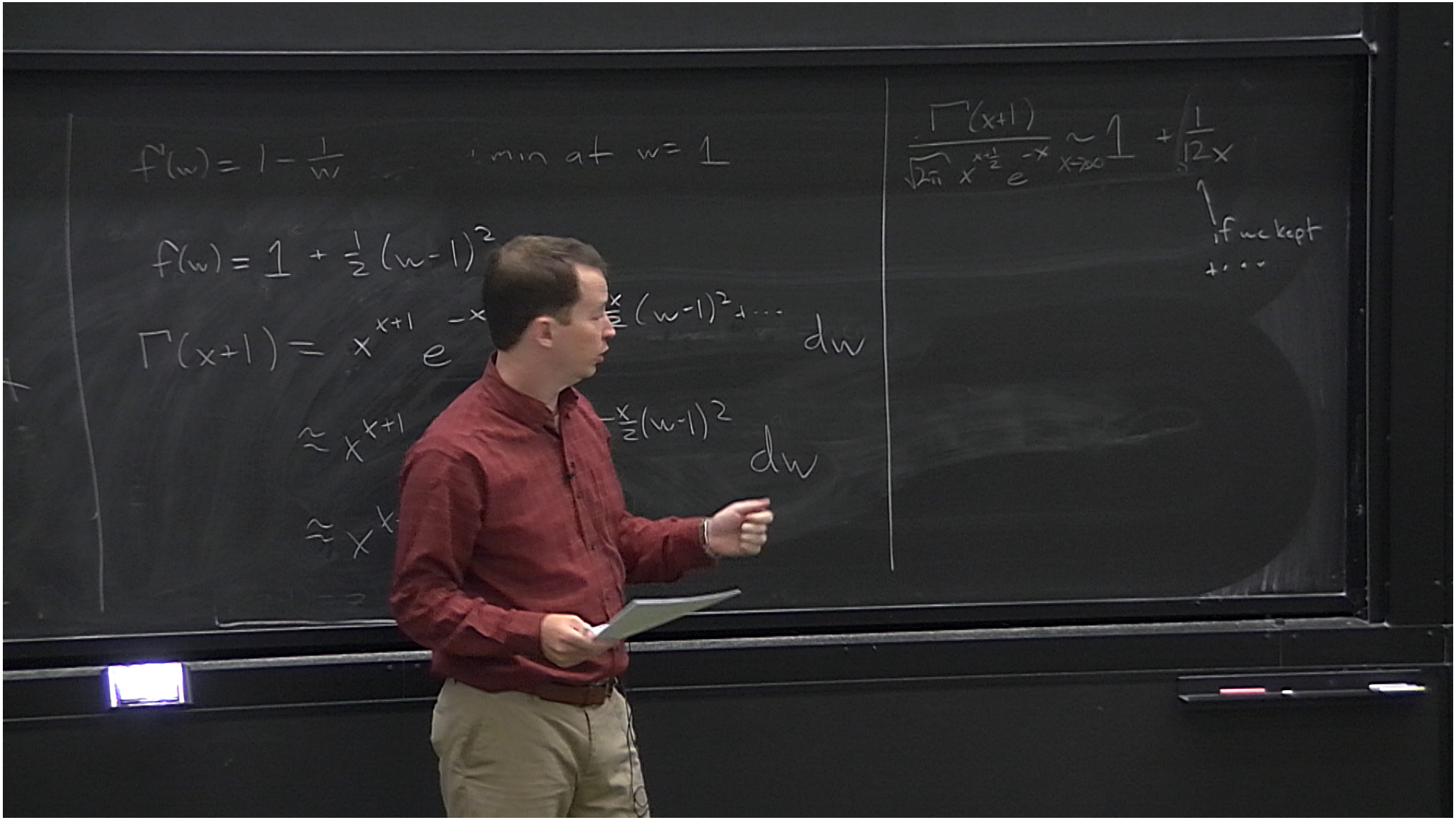
$$\Gamma(x+1) = x^{x+1} \int_0^\infty e^{-x(w - \ln w)} \, dw$$

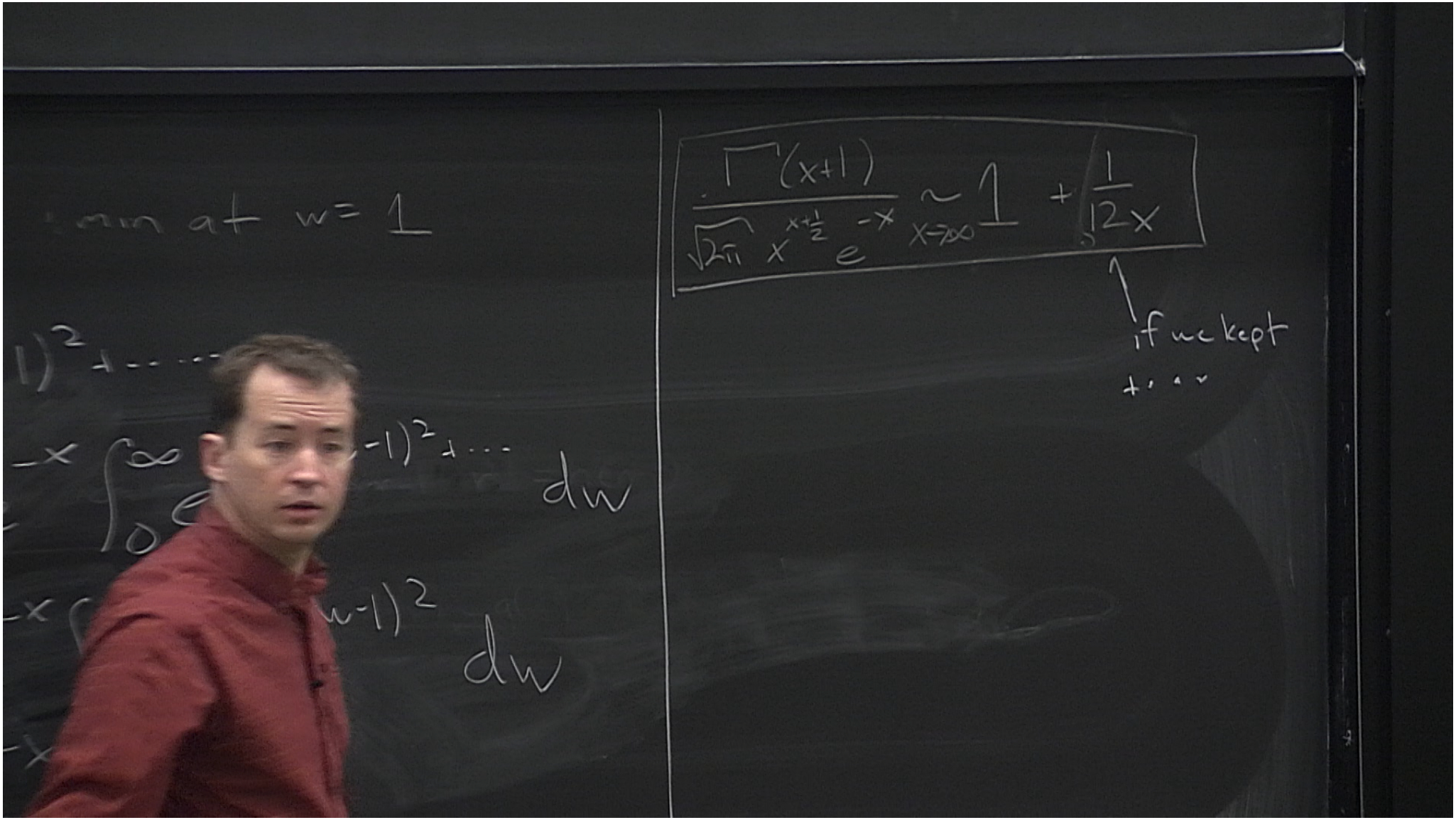
$$= x^{x+1} \int_0^\infty e^{-x f(w)} \, dw$$




$f'(w) = 1 - \frac{1}{w}$
min at $w = 1$
$f(w) = 1 + \frac{1}{2}(w-1)^2 + \cdots$
$\Gamma(x+1) = x^{x+1} e^{-x} \int_0^\infty e^{-\frac{x}{2}(w-1)^2 + \cdots} dw$
$\approx x^{x+1} e^{-x} \int_{-\infty}^{\infty} e^{-\frac{x}{2}(w-1)^2} dw$


$f'(w) = 1 - \frac{1}{w}$ ... min at $w = 1$
$f(w) = 1 + \frac{1}{2}(w-1)^2 + \cdots$
$\Gamma(x+1) = x^{x+1} e^{-x} \int_0^\infty e^{-\frac{x}{2}(w-1)^2 + \cdots} dw$
$\approx x^{x+1} e^{-x} \int_{-\infty}^\infty e^{-\frac{x}{2}(w-1)^2} dw$
$\approx x^{x+1} e^{-x} \sqrt{\frac{2\pi}{x}}$

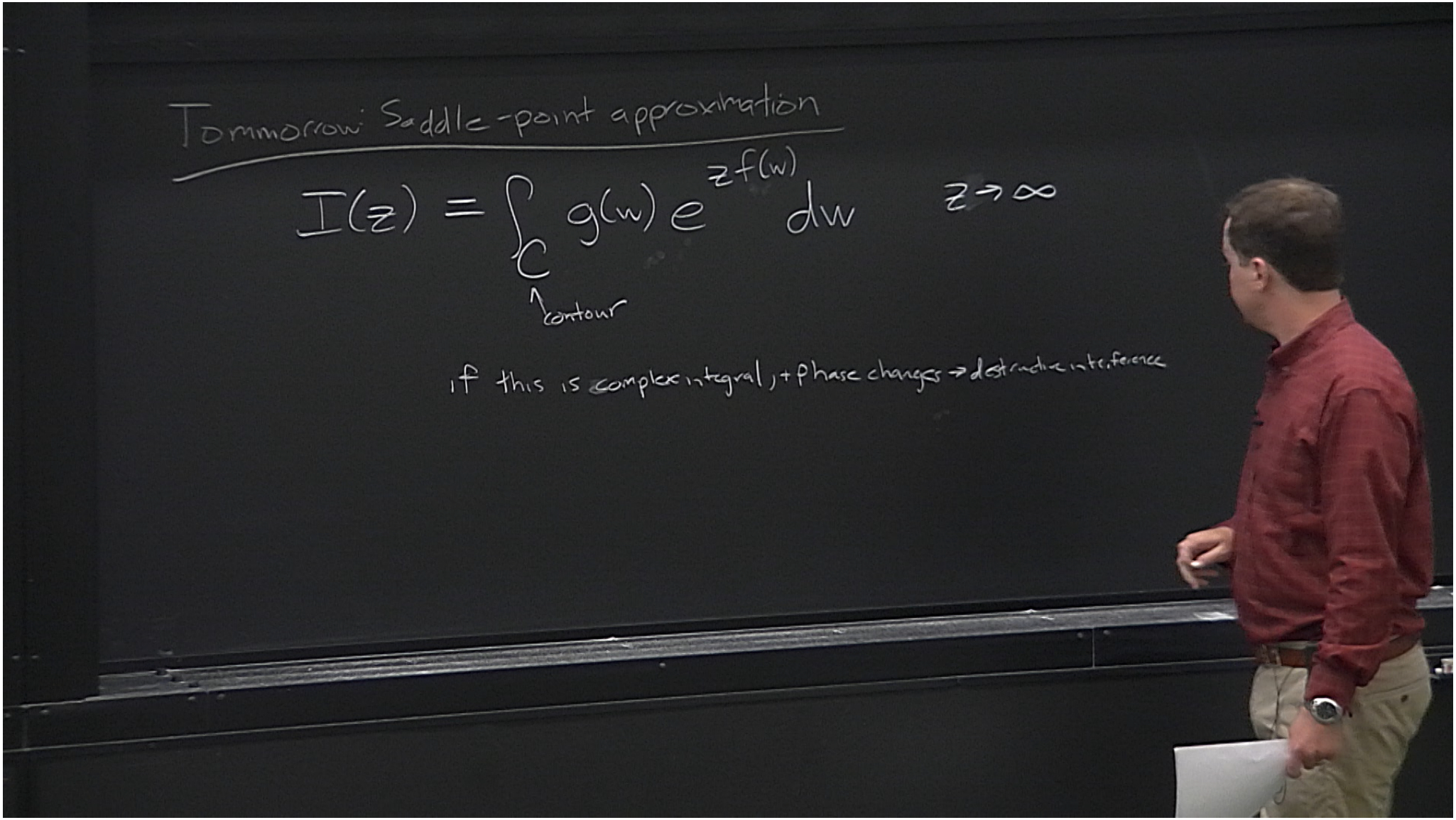

$f'(w) = 1 - \frac{1}{w}$
min at $w = 1$
$f(w) = 1 + \frac{1}{2}(w-1)^2$
$\Gamma(x+1) = x^{x+1} e^{-x}$
$\frac{x}{2}(w-1)^2 + \cdots$
$dw$
$\approx x^{x+1}$
$-\frac{x}{2}(w-1)^2$
$dw$
$\approx x^{x}$
$\frac{\Gamma(x+1)}{\sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x}} \underset{x\to\infty}{\sim} 1 + \frac{1}{12x}$
if we kept
+ ...


min at w = 1
dw
dw
$\frac{\Gamma(x+1)}{\sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x}} \underset{x\to\infty}{\sim} 1 + \frac{1}{12x}$
if we kept
+ ...

Tommorrow: Saddle-point approximation

$$I(z) = \int_C g(w)\, e^{z f(w)}\, dw \qquad z \to \infty$$

$C$: contour


Tommorrow: Saddle-point approximation
$I(z) = \int_C g(w) e^{z f(w)} dw \quad z \to \infty$
contour
if this is complex integral, + phase changes → destructive interference