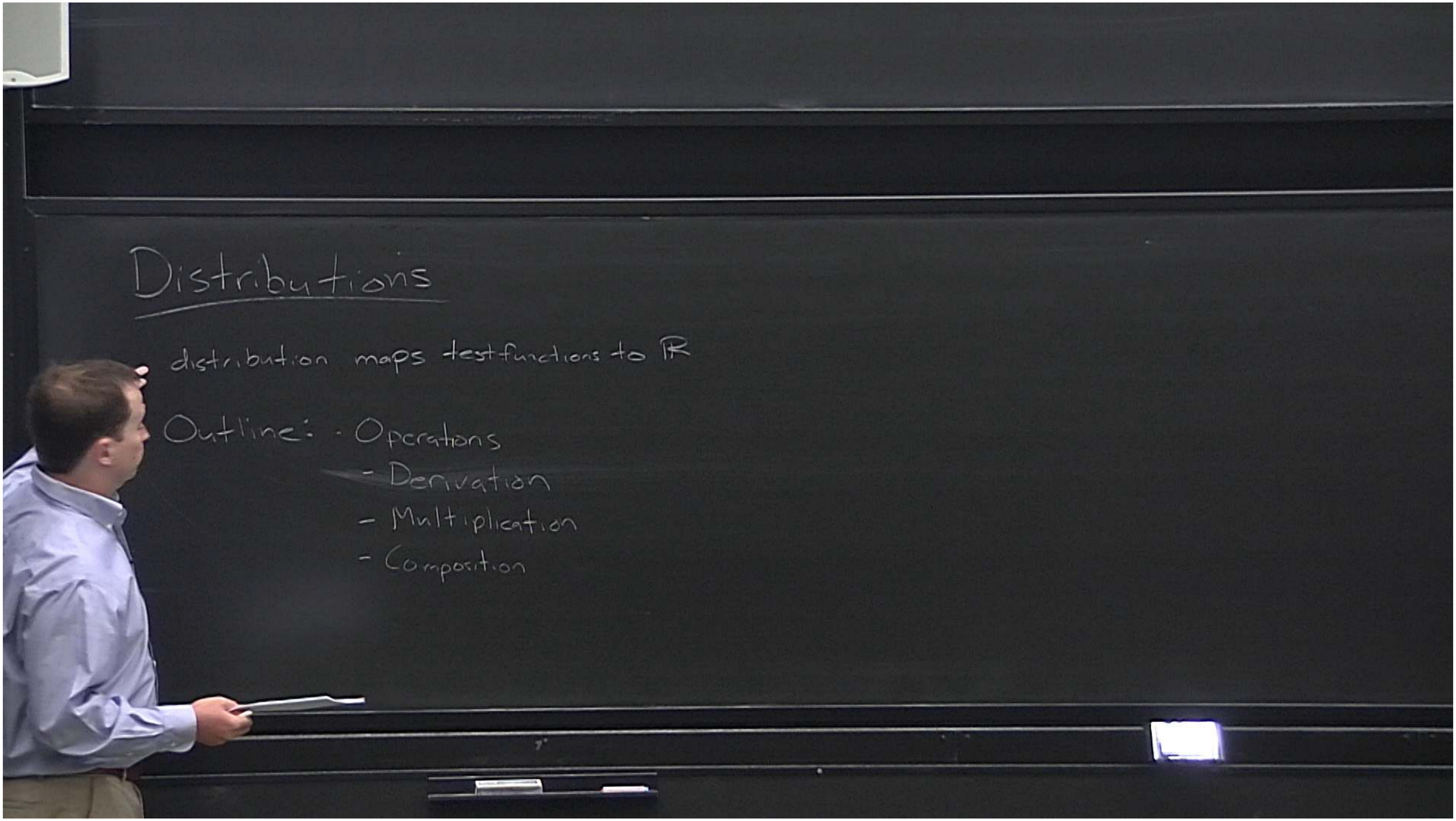
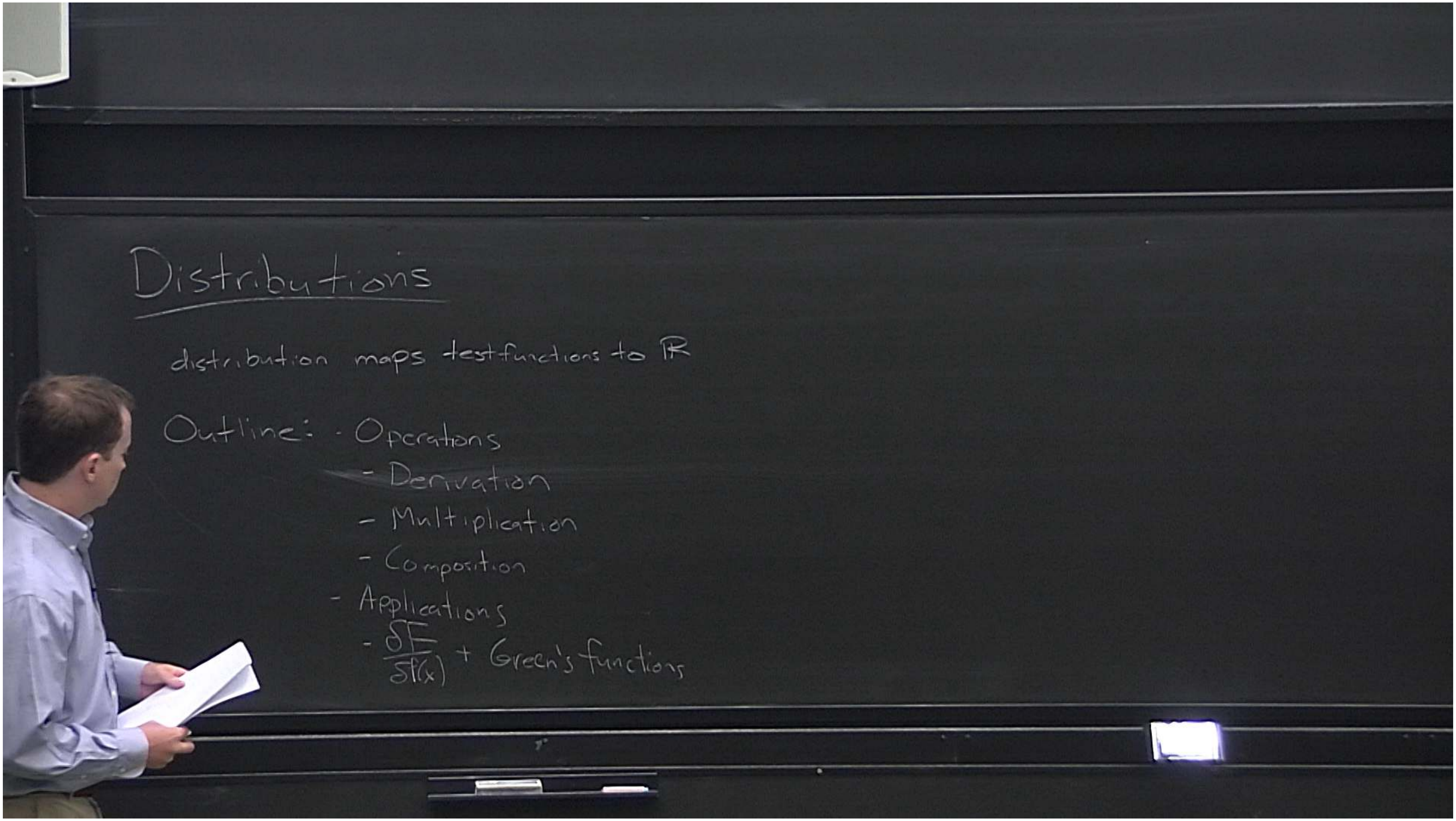
Title: PSI 2018/2019 - Math for QFT - Lecture 2

Date: Aug 20, 2018 10:30 AM

URL: http://pirsa.org/18080018

Abstract:


Distributions
distribution maps testfunctions to $\mathbb{R}$
Outline: · Operations
- Derivation
- Multiplication
- Composition


Distributions
distribution maps testfunctions to $\mathbb{R}$
Outline: - Operations
- Derivation
- Multiplication
- Composition
- Applications
- $\frac{\delta F}{\delta f(x)}$ + Green's functions

Derivation

$$u'(x) \stackrel{?}{=} \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$$

Derivation

$$\cancel{u'(x) \overset{?}{=} \lim_{\epsilon \to 0} \frac{u(x+\epsilon)-u(x)}{\epsilon}}$$

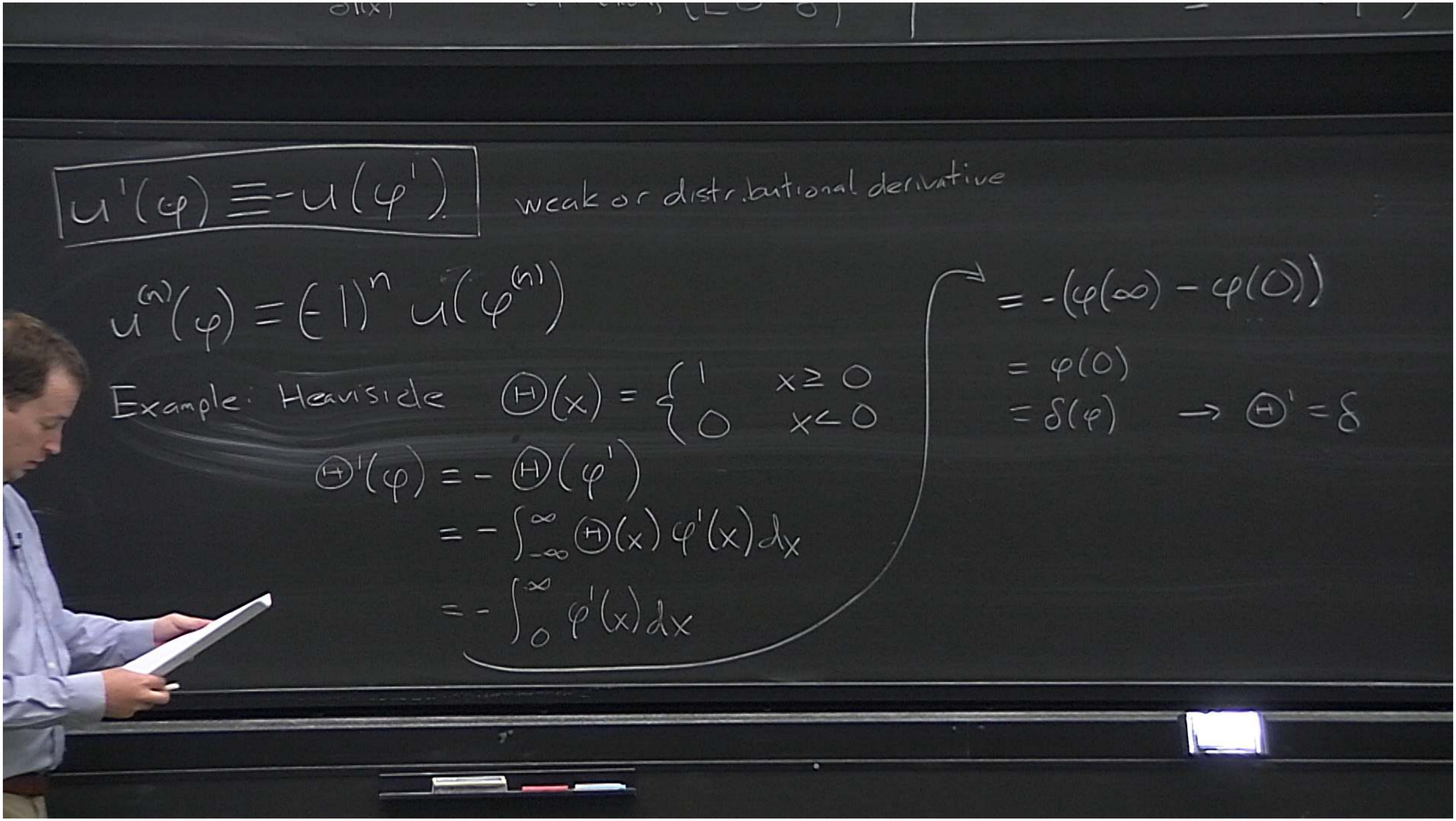
dead end

$$u'(\varphi) = \int_{-\infty}^{\infty} u'(x)\,\varphi(x)\,dx$$

if u is a function this line makes sense

$$= \overbrace{\cancel{u(x)\varphi(x)\Big|_{-\infty}^{\infty}}}^{\to 0 \text{ compact support}} - \int_{-\infty}^{\infty} u(x)\,\varphi'(x)\,dx$$

$$= -u(\varphi')$$


$\boxed{u'(\varphi) \equiv -u(\varphi')}$ weak or distributional derivative
$u^{(n)}(\varphi) = (-1)^n u(\varphi^{(n)})$
Example: Heaviside $\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$
$\Theta'(\varphi) = -\Theta(\varphi')$
$= -\int_{-\infty}^{\infty} \Theta(x)\varphi'(x)dx$
$= -\int_{0}^{\infty} \varphi'(x)dx$
$= -(\varphi(\infty) - \varphi(0))$
$= \varphi(0)$
$= \delta(\varphi) \rightarrow \Theta' = \delta$

$$\boxed{u'(\varphi) \equiv -u(\varphi')}$$ weak or distributional derivative

$$u^{(n)}(\varphi) = (-1)^n u(\varphi^{(n)})$$

Example: Heaviside $\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$\begin{aligned} \Theta'(\varphi) &= -\Theta(\varphi') \\ &= -\int_{-\infty}^{\infty} \Theta(x)\,\varphi'(x)\,dx \\ &= -\int_{0}^{\infty} \varphi'(x)\,dx \end{aligned}$$

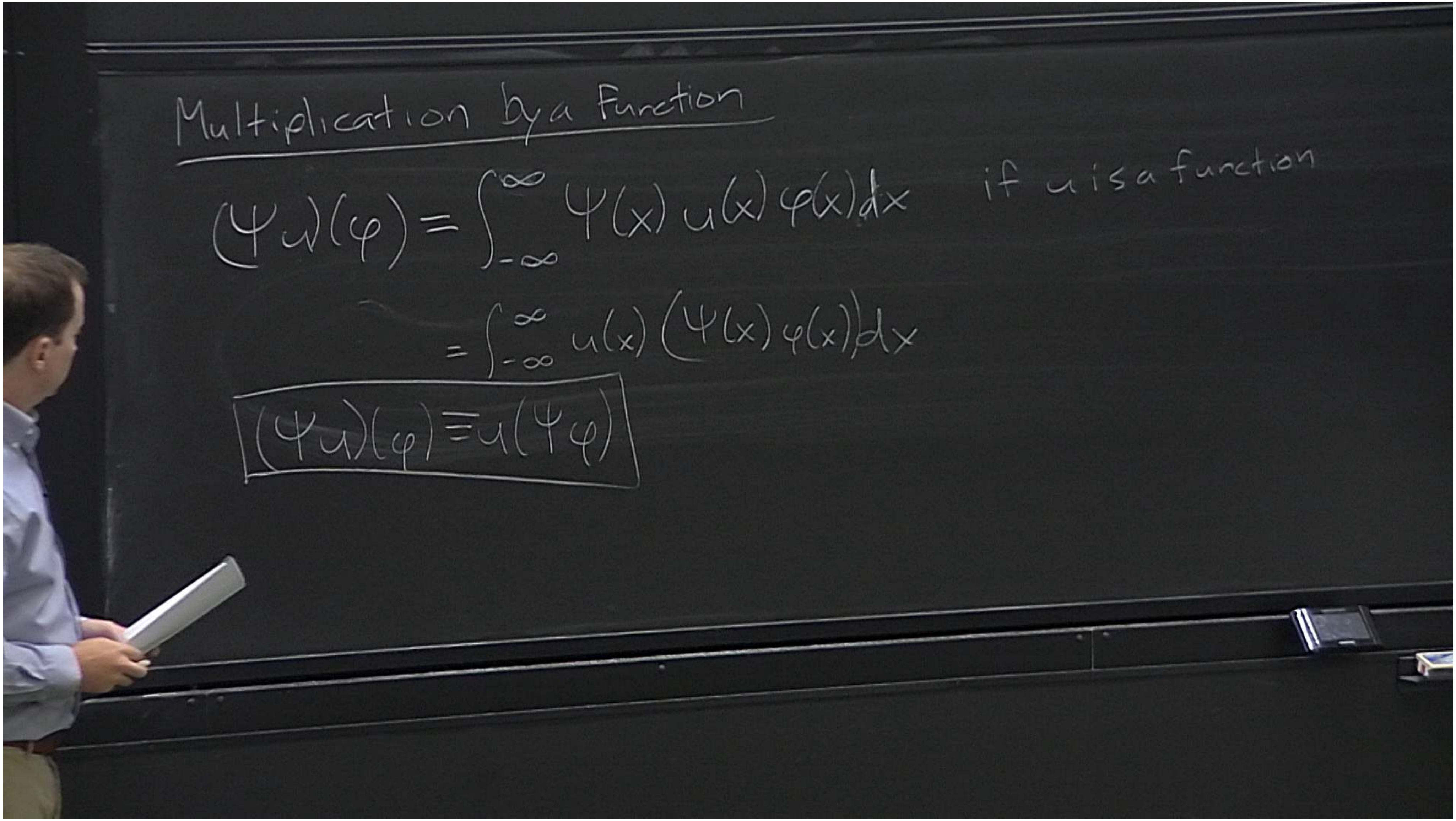
$$\begin{aligned} &= -(\varphi(\infty) - \varphi(0)) \\ &= \varphi(0) \\ &= \delta(\varphi) \quad \rightarrow \quad \Theta' = \delta \end{aligned}$$

$$\rightarrow \quad \Theta' = \delta$$

$$(\ln|x|)(\varphi) \equiv \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\varphi(x)\ln|x|\,dx$$

$$(\ln|x|)'(\varphi) = -\lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\varphi'(x)\ln|x|\,dx$$

$$= \lim_{\epsilon\to 0}\Big[\underbrace{\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{1}{x}\varphi(x)dx}_{(PV\frac{1}{x})(\varphi)} + \underbrace{\varphi(\epsilon)\ln|\epsilon| - \varphi(-\epsilon)\ln|\epsilon|}_{\approx 2\varphi'(0)\epsilon\ln|\epsilon| \to 0}\Big]$$


Multiplication by a Function
$(\psi u)(\varphi) = \int_{-\infty}^{\infty} \psi(x)\, u(x)\, \varphi(x)\, dx$ if u is a function
$= \int_{-\infty}^{\infty} u(x) \big(\psi(x)\, \varphi(x)\big) dx$
$\boxed{(\psi u)(\varphi) \equiv u(\psi\varphi)}$

Multiplication by a Function

$$(\psi u)(\varphi) = \int_{-\infty}^{\infty} \psi(x)\, u(x)\, \varphi(x)\, dx$$

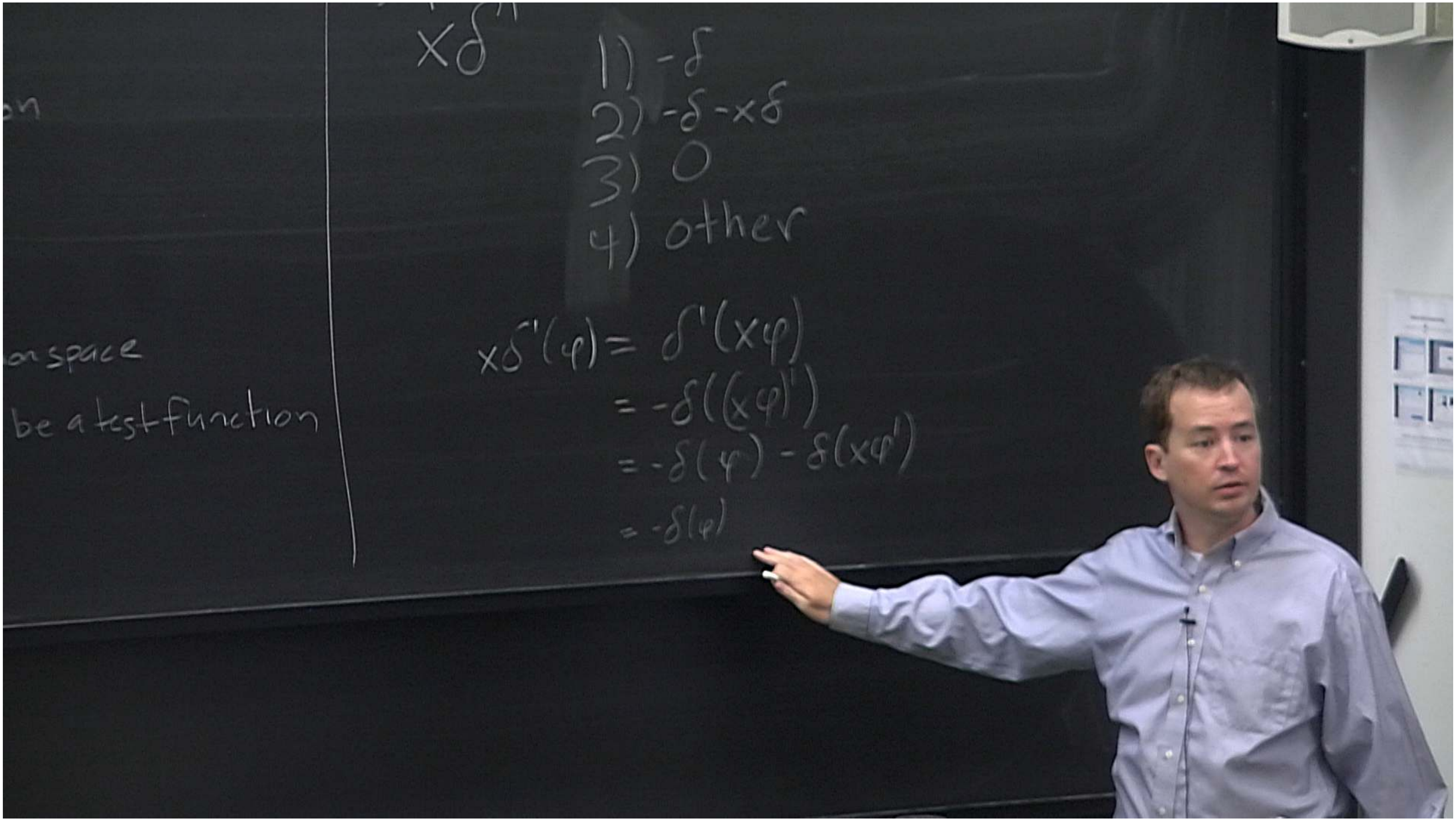
if u is a function

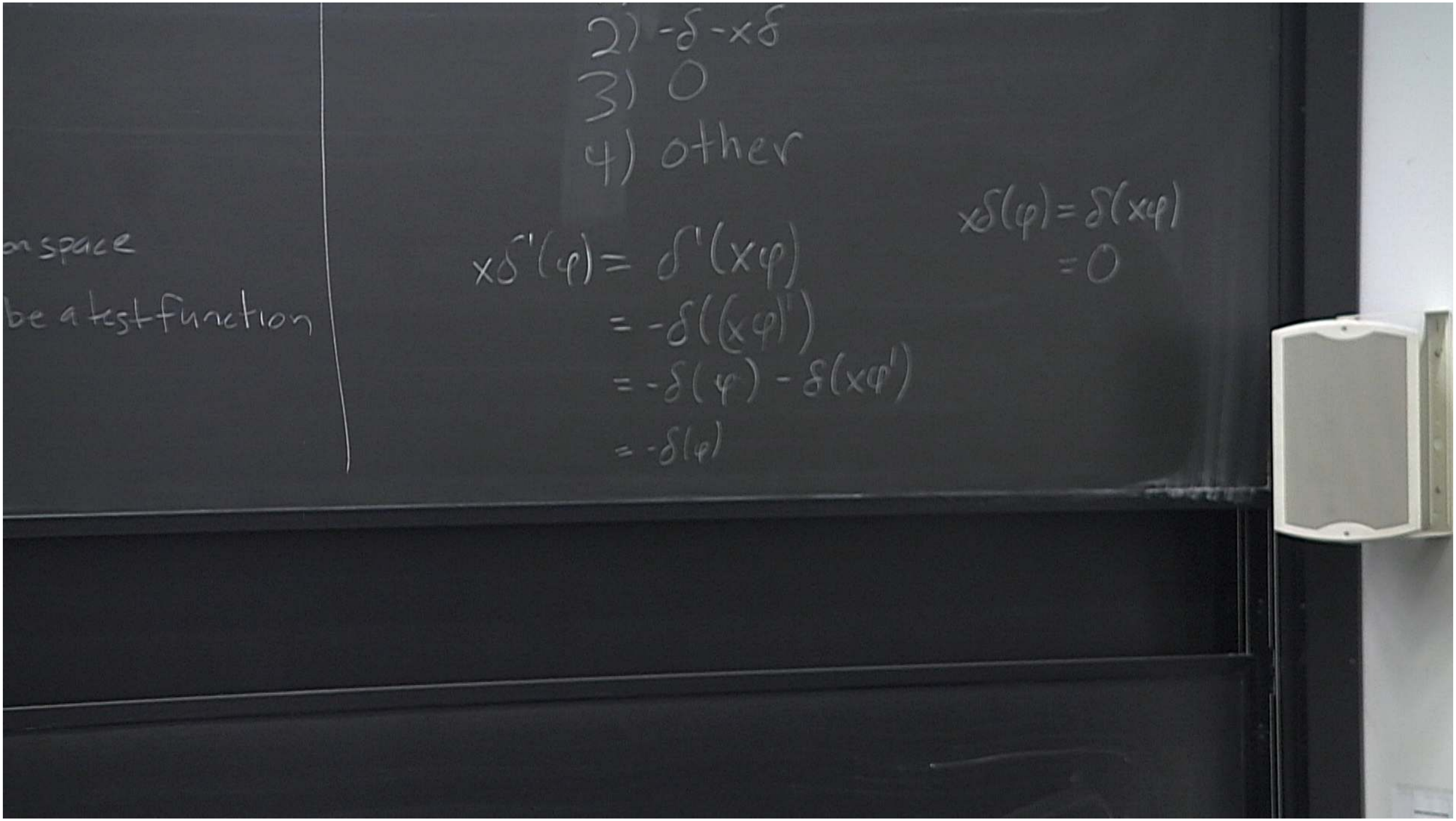
$$= \int_{-\infty}^{\infty} u(x) \left(\psi(x)\, \varphi(x)\right) dx$$

$$\boxed{(\psi u)(\varphi) \equiv u(\psi \varphi)}$$

ψ should be C^∞ for our test function space

but in general we need $\psi\varphi$ to be a test fu


xδ'
1) -δ
2) -δ-xδ
3) 0
4) other
on space
be a test function
xδ'(φ)= δ'(xφ)
= -δ((xφ)')
= -δ(φ) - δ(xφ')
= -δ(φ)


2) $-\delta - x\delta$
3) 0
4) other
on space
be a test function
$x\delta(\varphi) = \delta(x\varphi)$
$= 0$
$x\delta'(\varphi) = \delta'(x\varphi)$
$= -\delta((x\varphi)')$
$= -\delta(\varphi) - \delta(x\varphi')$
$= -\delta(\varphi)$

Composition

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\,\varphi(x)\,dx$$

, if u is func.

$$y = f(x) \qquad g(y) = x$$

$$= \int_{-\infty}^{\infty} u(y)\,\underbrace{\varphi(g(y))\,|g'(y)|}_{\text{test function}}\,dy$$

test function – if f is C^∞

$y = f(x)$ has a unique solution

Composition

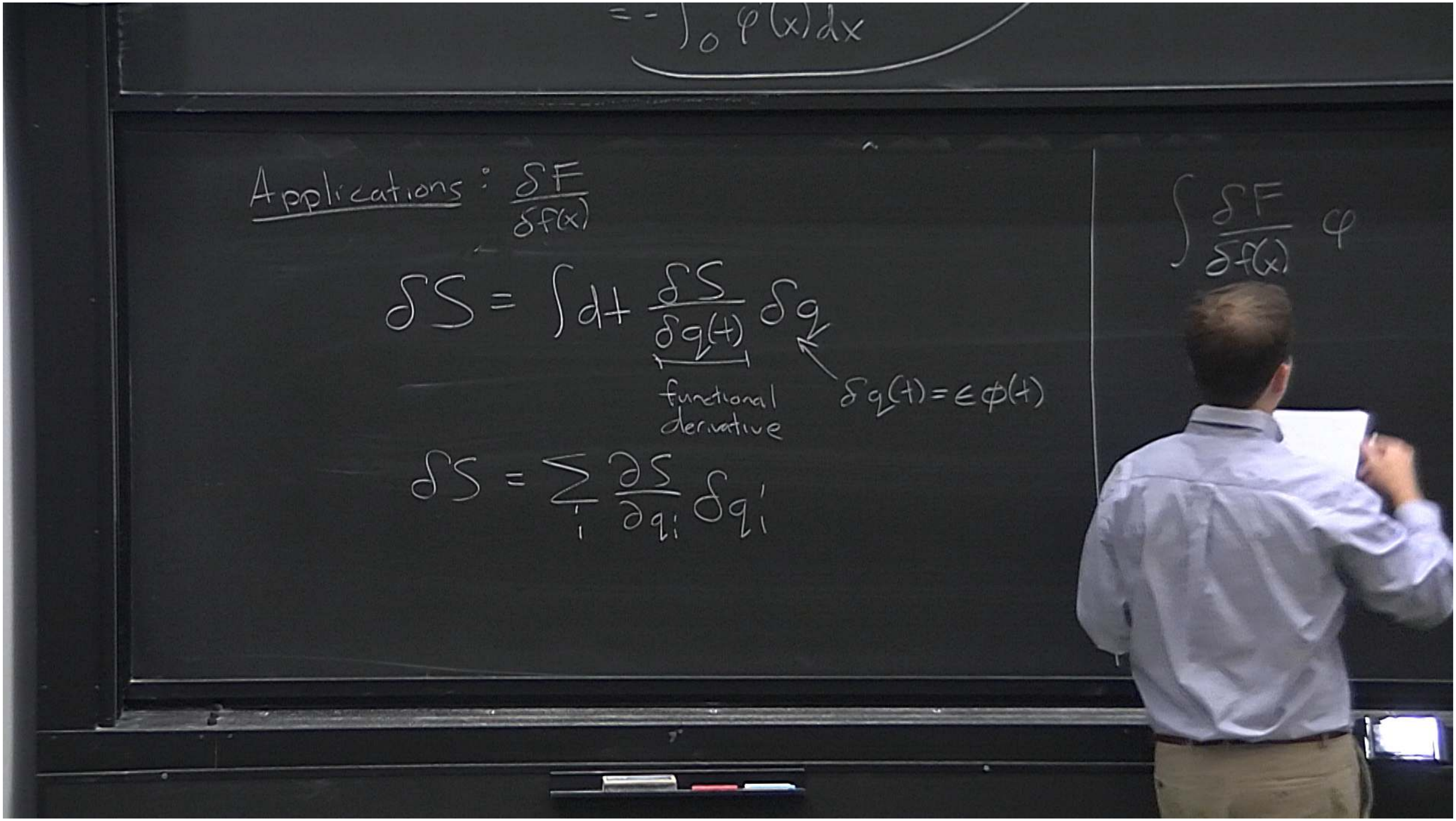
$$(u\circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\,\varphi(x)\,dx$$ if u is func.

$y=f(x)$ $\quad g(y)=x$

$$= \int_{-\infty}^{\infty} u(y)\,\underbrace{\varphi(g(y))\,|g'(y)|}_{\text{test function}}\,dy$$

- if f is C^∞
- $y=f(x)$ has a unique solution

$$\boxed{(u\circ f)(\varphi) \equiv u\big((\varphi\circ g)\,|g'|\big)}$$


= - ∫0 φ(x) dx
Applications : δF/δf(x)
δS = ∫dt δS/δq(t) δq
functional derivative
δq(t) = ε φ(t)
δS = Σi ∂S/∂qi δqi
∫ δF/δf(x) φ

$\left(PV\frac{1}{x}\right)(\varphi)$ $\approx 2\varphi'(0)$

$$\int \frac{\delta F}{\delta f(x)}\,\varphi(x)\,dx = \lim_{\epsilon\to 0} \frac{F[f+\epsilon\varphi]-F[f]}{\epsilon}$$

Example: $F[f] = f^2(x_0)$

$\phi(t)$

$$\int \boxed{\frac{\delta F}{\delta f(x)}}\,\phi(x)\,dx = \lim_{\epsilon\to 0} \frac{\cancel{f^2(x_0)} + 2\epsilon f(x_0)\,\phi(x_0) + \epsilon^2\phi^2(x_0) - \cancel{f^2(x_0)}}{\epsilon}$$

$$= 2f(x_0)\,\phi(x_0)$$

$$= \int \boxed{2f(x_0)\,\delta(x-x_0)}\,\phi(x)\,dx$$

$(PV\frac{1}{x})(\varphi)$ $\approx 2\varphi'(0)$

$\phi(t)$

$$\int \frac{\delta F}{\delta f(x)} \varphi(x)\, dx = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\phi] - F[f]}{\epsilon}$$

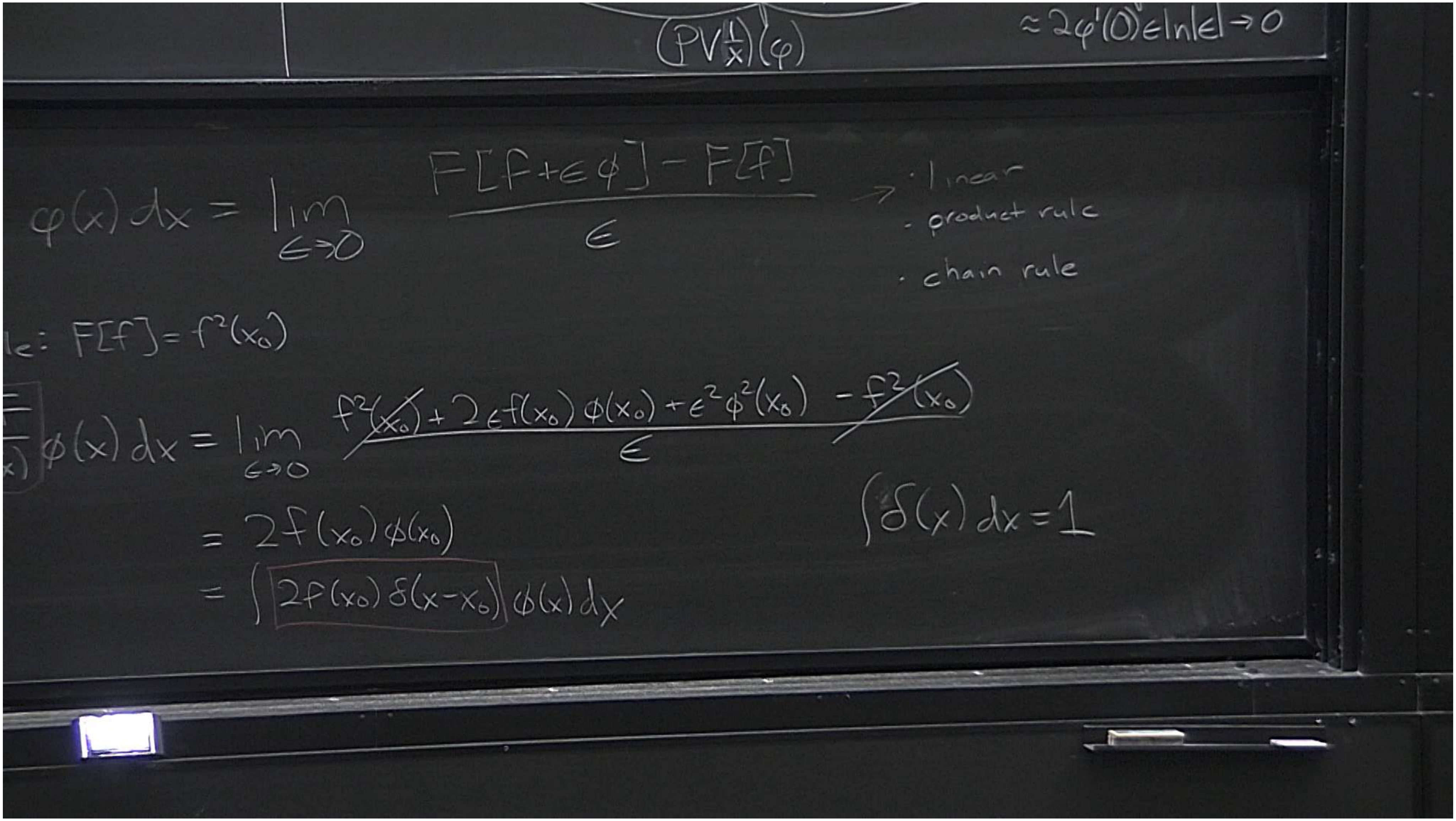
$\to$
- linear
- product rule
- chain rule

Example: $F[f] = f^2(x_0)$

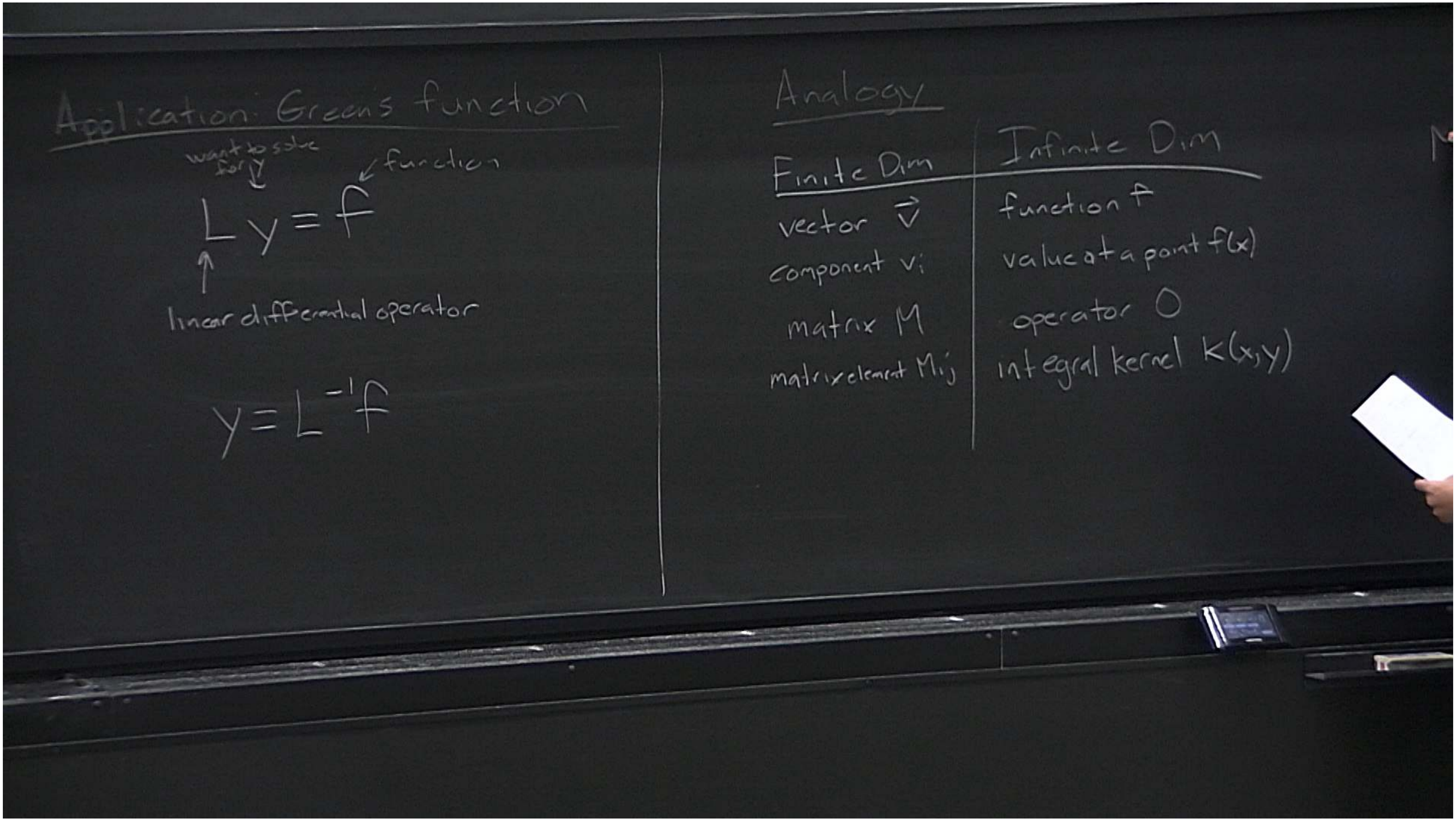
$$\int \boxed{\frac{\delta F}{\delta f(x)}} \phi(x)\, dx = \lim_{\epsilon \to 0} \frac{\cancel{f^2(x_0)} + 2\epsilon f(x_0)\phi(x_0) + \epsilon^2\phi^2(x_0) - \cancel{f^2(x_0)}}{\epsilon}$$

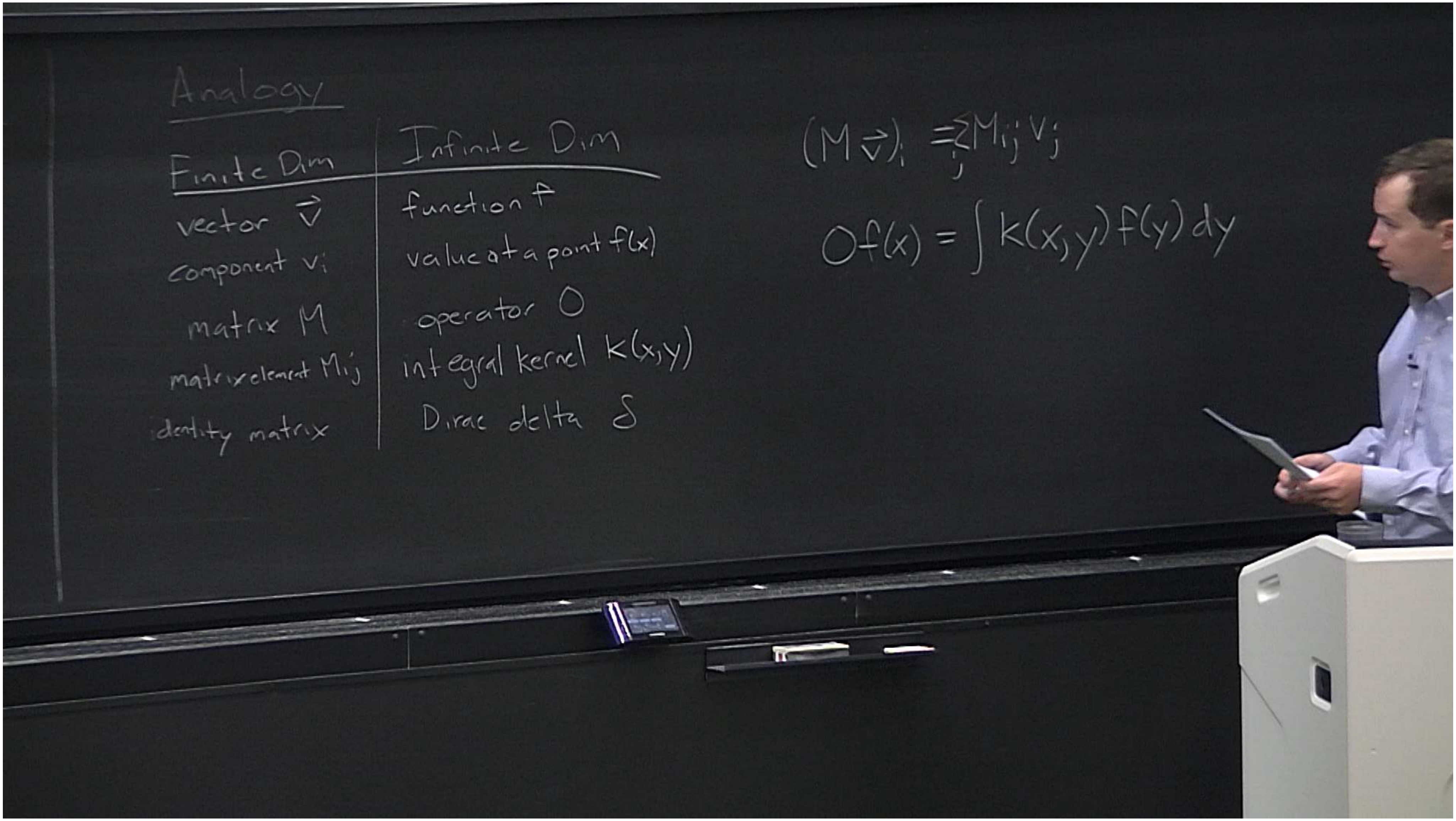
$$= 2f(x_0)\phi(x_0)$$

$$= \int \boxed{2f(x_0)\delta(x-x_0)}\, \phi(x)\, dx$$



$\approx 2\varphi'(0)\epsilon\ln|\epsilon| \to 0$
$(PV\frac{1}{x})(\varphi)$
$\varphi(x)\,dx = \lim_{\epsilon\to 0} \frac{F[f+\epsilon\phi]-F[f]}{\epsilon}$
· linear
· product rule
· chain rule
le: $F[f]=f^2(x_0)$
$\phi(x)\,dx = \lim_{\epsilon\to 0} \frac{\cancel{f^2(x_0)}+2\epsilon f(x_0)\phi(x_0)+\epsilon^2\phi^2(x_0) - \cancel{f^2(x_0)}}{\epsilon}$
$= 2f(x_0)\phi(x_0)$
$= \int \boxed{2f(x_0)\delta(x-x_0)}\,\phi(x)\,dx$
$\int \delta(x)\,dx = 1$


Application: Green's function
want to solve for y
function
Ly = f
linear differential operator
y = L^{-1} f
Analogy
Finite Dim
Infinite Dim
vector v
function f
component v_i
value at a point f(x)
matrix M
operator O
matrix element M_ij
integral kernel K(x,y)


Analogy
Finite Dim
Infinite Dim
vector $\vec{v}$
function f
component v_i
value at a point f(x)
matrix M
operator O
matrix element M_{ij}
integral kernel k(x,y)
identity matrix
Dirac delta δ
$(M\vec{v})_i = \sum_j M_{ij} v_j$
$Of(x) = \int k(x,y) f(y)\, dy$

...cation: Green's function

want to solve for y; function f

$$Ly = f$$

L: linear differential operator

$$y = L^{-1} f$$

$$L L^{-1} = \delta$$

$$L_x G(x;\xi) = \delta(x-\xi)$$

Analogy

Finite Dim	Infinite Dim
vector $\vec{v}$	function f
component v_i	value at a point $f(x)$
matrix M	operator O
matrix element M_{ij}	integral kernel $K(x,y)$
identity matrix	Dirac delta δ

sometimes subtleties

$$(M\vec{v})_i = \sum_j$$

$$Of(x) =$$