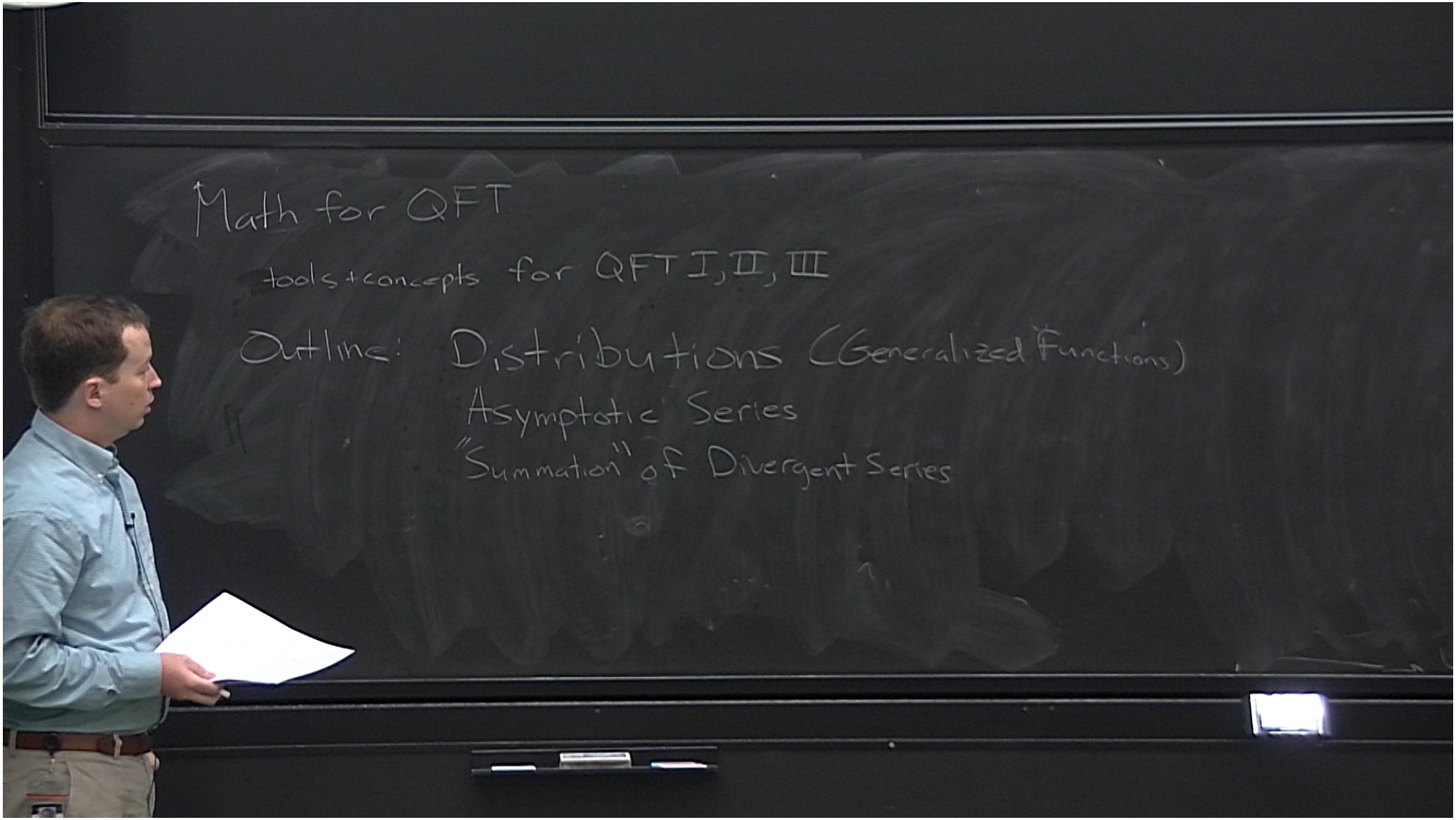
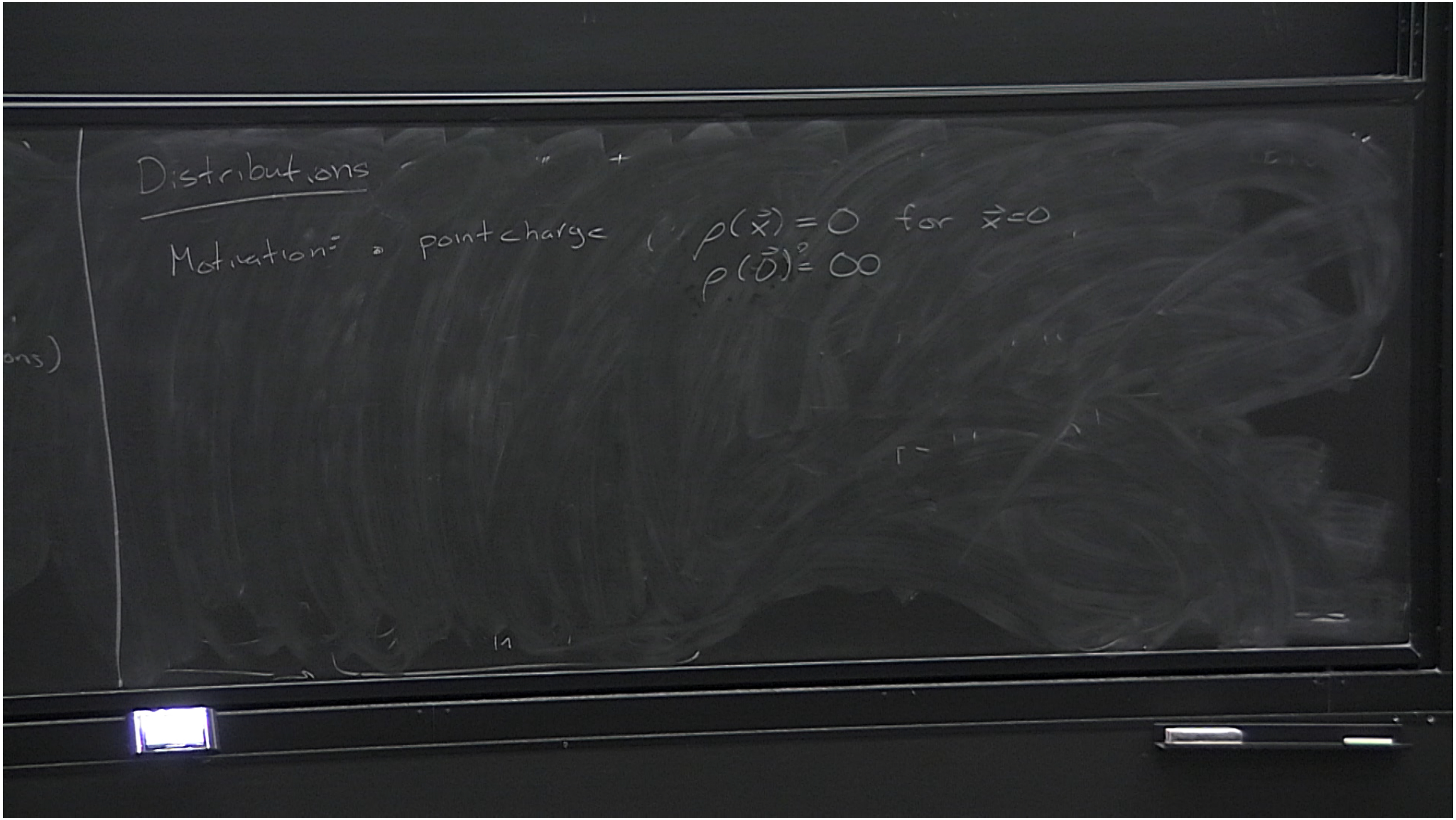
Title: PSI 2018/2019 - Functions, "Functions", etc. - Lecture 1

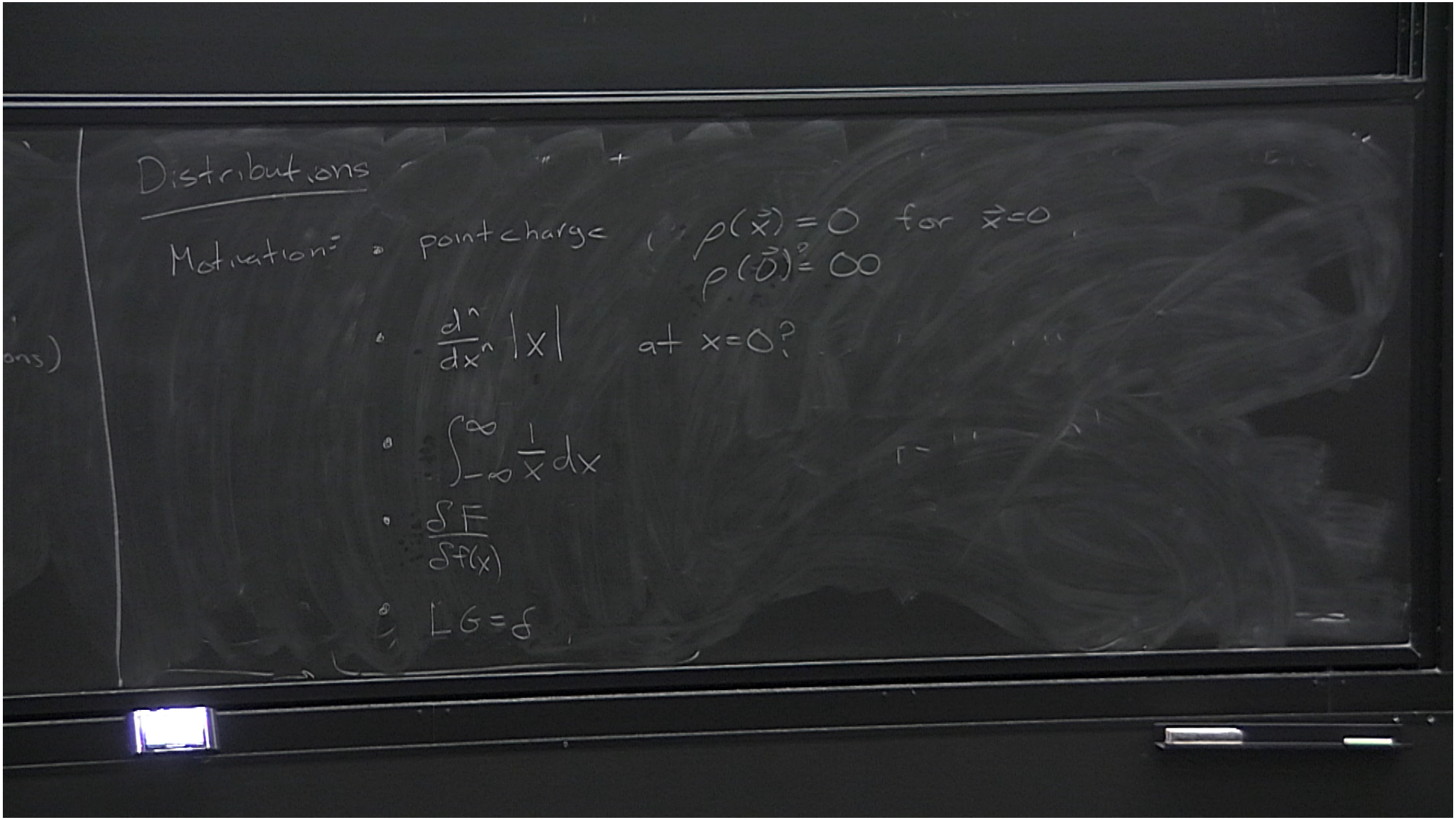
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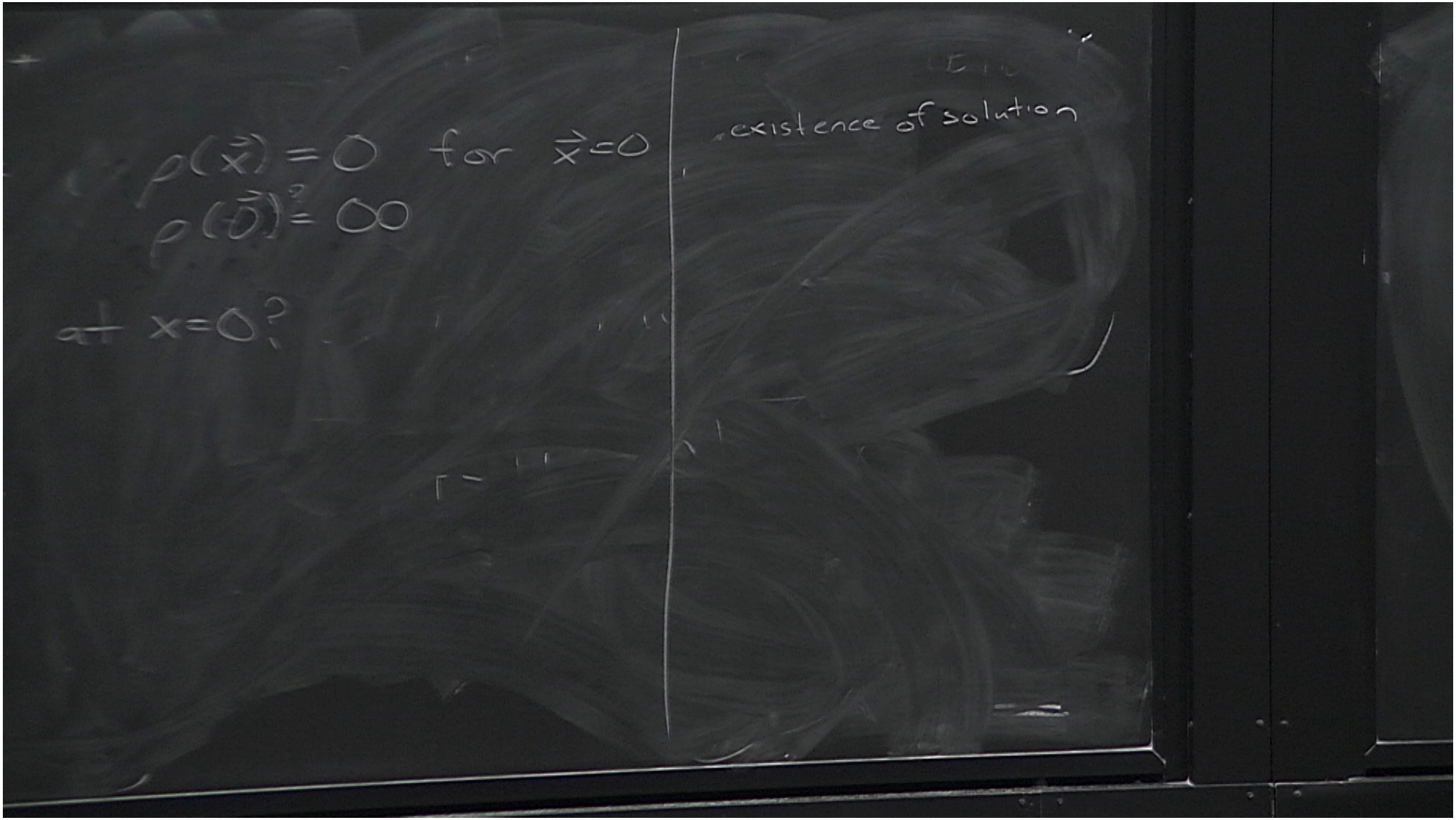
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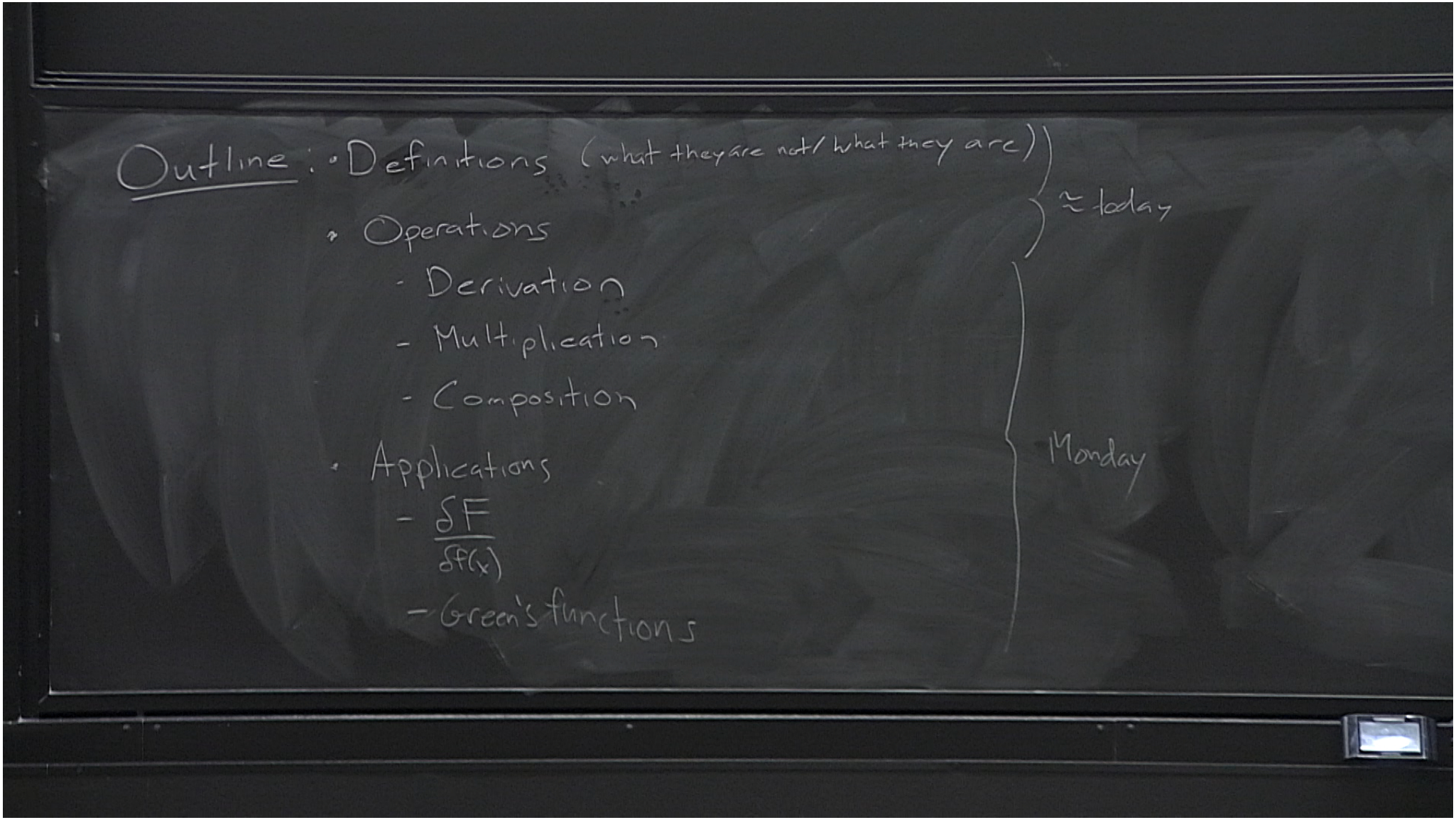
Abstract:

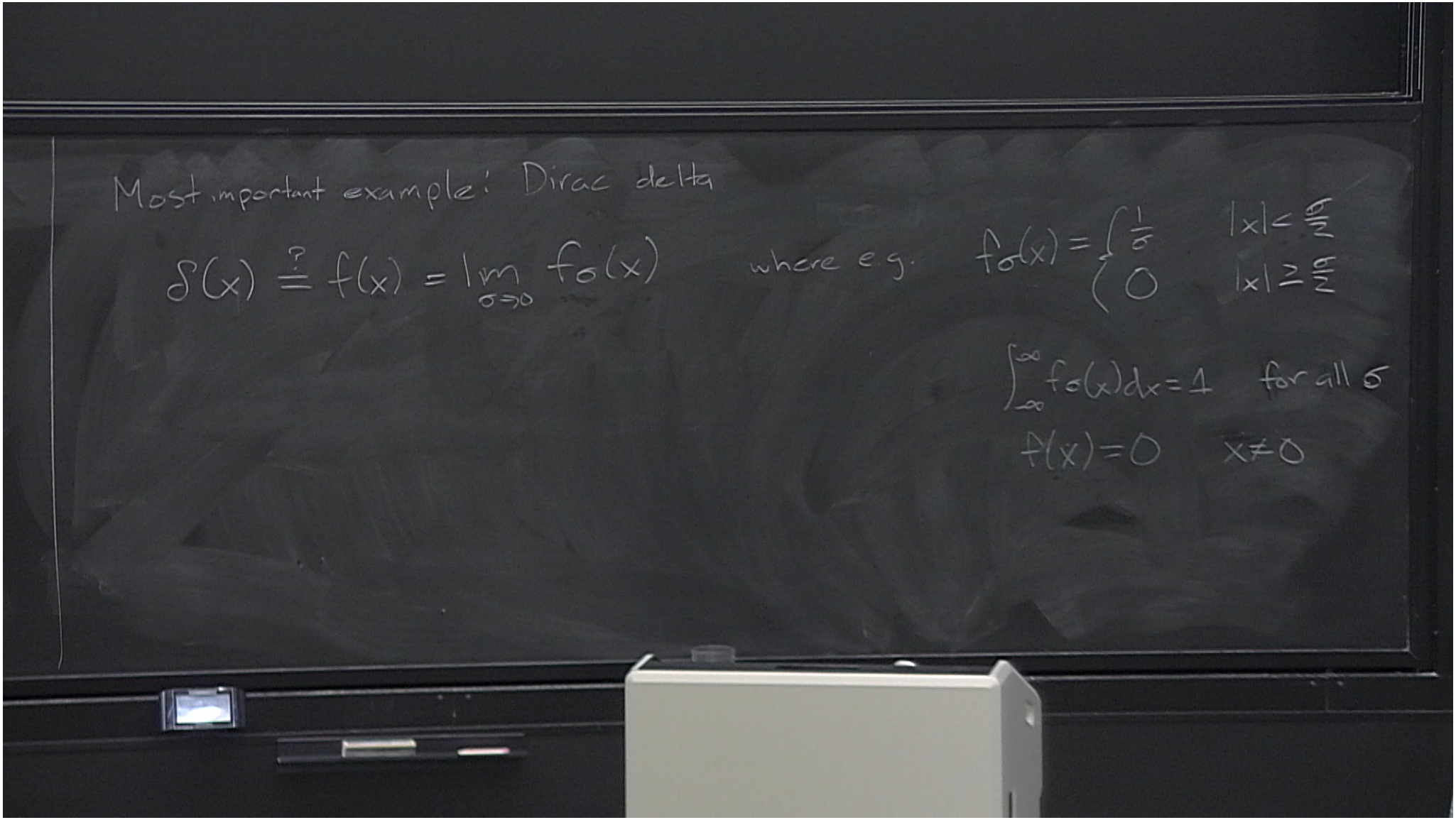

Math for QFT
tools + concepts for QFT I, II, III
Outline: Distributions (Generalized Functions)
Asymptotic Series
"Summation" of Divergent Series

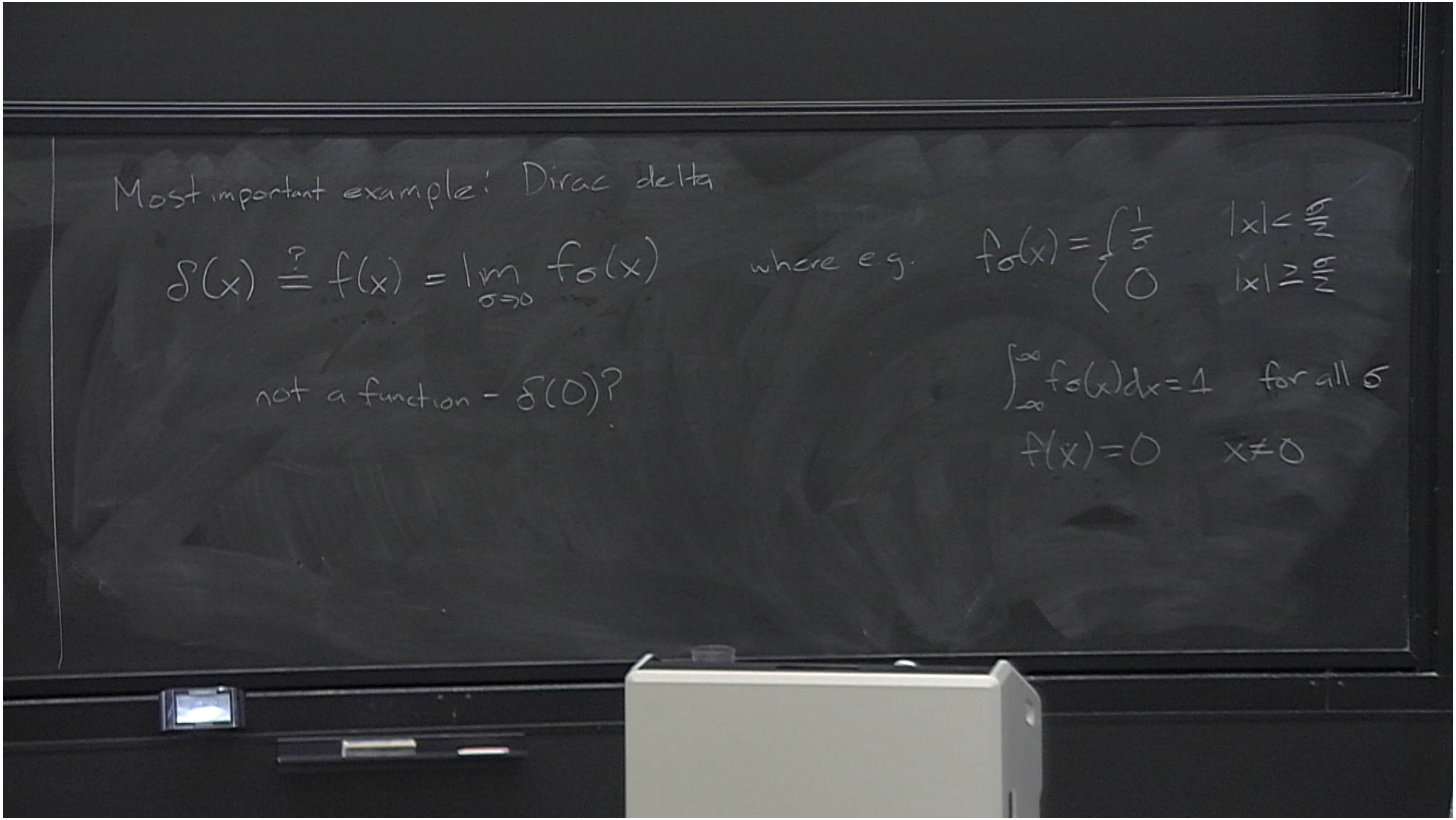

Distributions
Motivation: • point charge ($\rho(\vec{x}) = 0$ for $\vec{x} \neq 0$
$\rho(\vec{0}) = \infty$

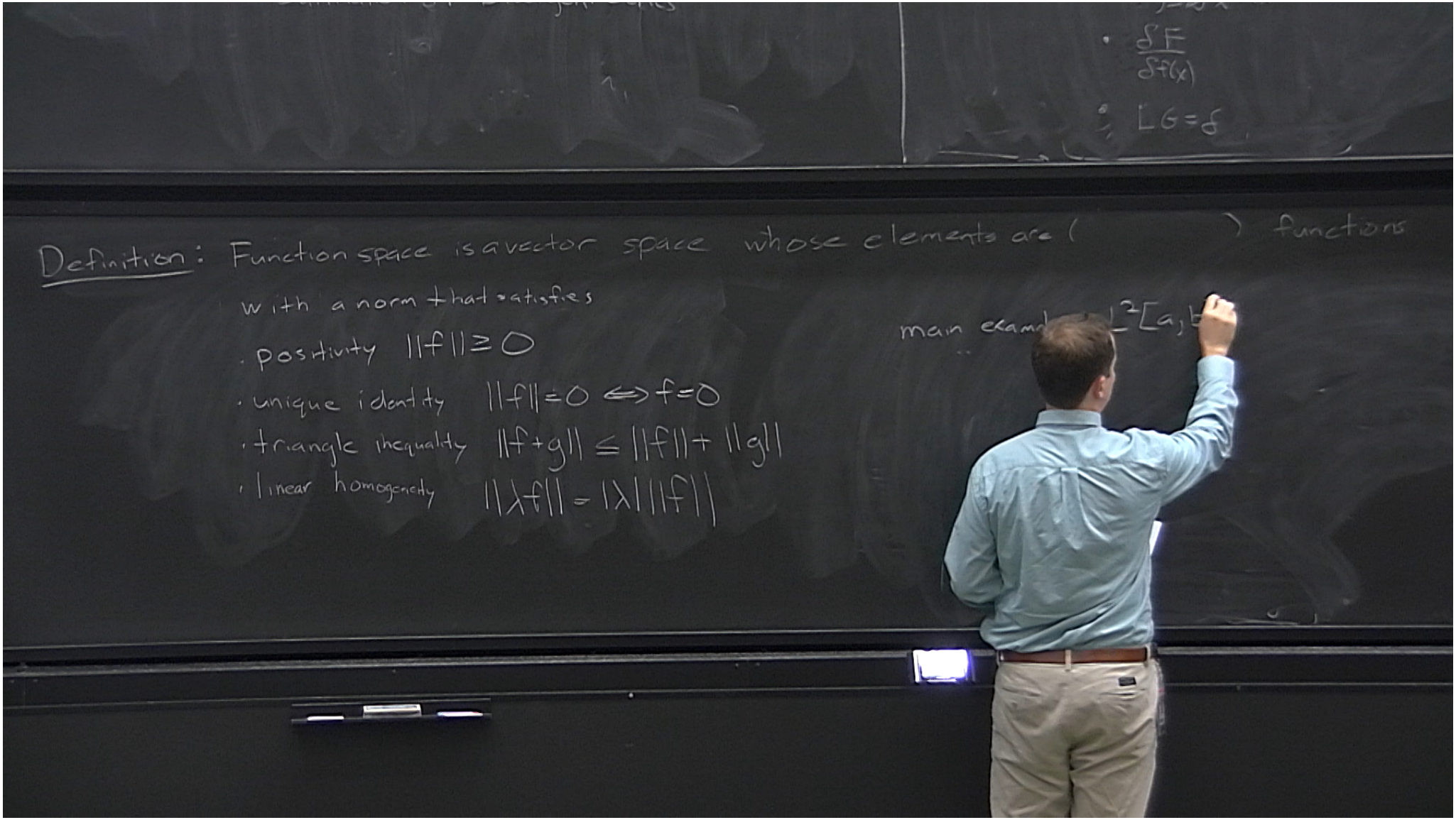

Distributions
Motivation:
• point charge ($\rho(\vec{x}) = 0$ for $\vec{x} \neq 0$, $\rho(\vec{0}) \stackrel{?}{=} \infty$
• $\frac{d^n}{dx^n}|x|$ at $x=0$?
• $\int_{-\infty}^{\infty} \frac{1}{x} dx$
• $\frac{\delta F}{\delta f(x)}$
• $LG = \delta$


$\rho(\vec{x}) = 0$ for $\vec{x} = 0$
$\rho(\vec{0}) \stackrel{?}{=} \infty$
at $x = 0$?
existence of solution


Outline:
• Definitions (what they are not/ what they are)
• Operations
- Derivation
- Multiplication
- Composition
• Applications
- $\frac{\delta F}{\delta f(x)}$
- Green's functions
≃ today
Monday


Most important example: Dirac delta
$\delta(x) \stackrel{?}{=} f(x) = \lim_{\sigma \to 0} f_\sigma(x)$
where e.g. $f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$
$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1$ for all σ
$f(x) = 0 \quad x \neq 0$


Most important example: Dirac delta
$\delta(x) \stackrel{?}{=} f(x) = \lim_{\sigma \to 0} f_\sigma(x)$
where e.g. $f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$
not a function - $\delta(0)$?
$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1$ for all σ
$f(x) = 0 \quad x \neq 0$

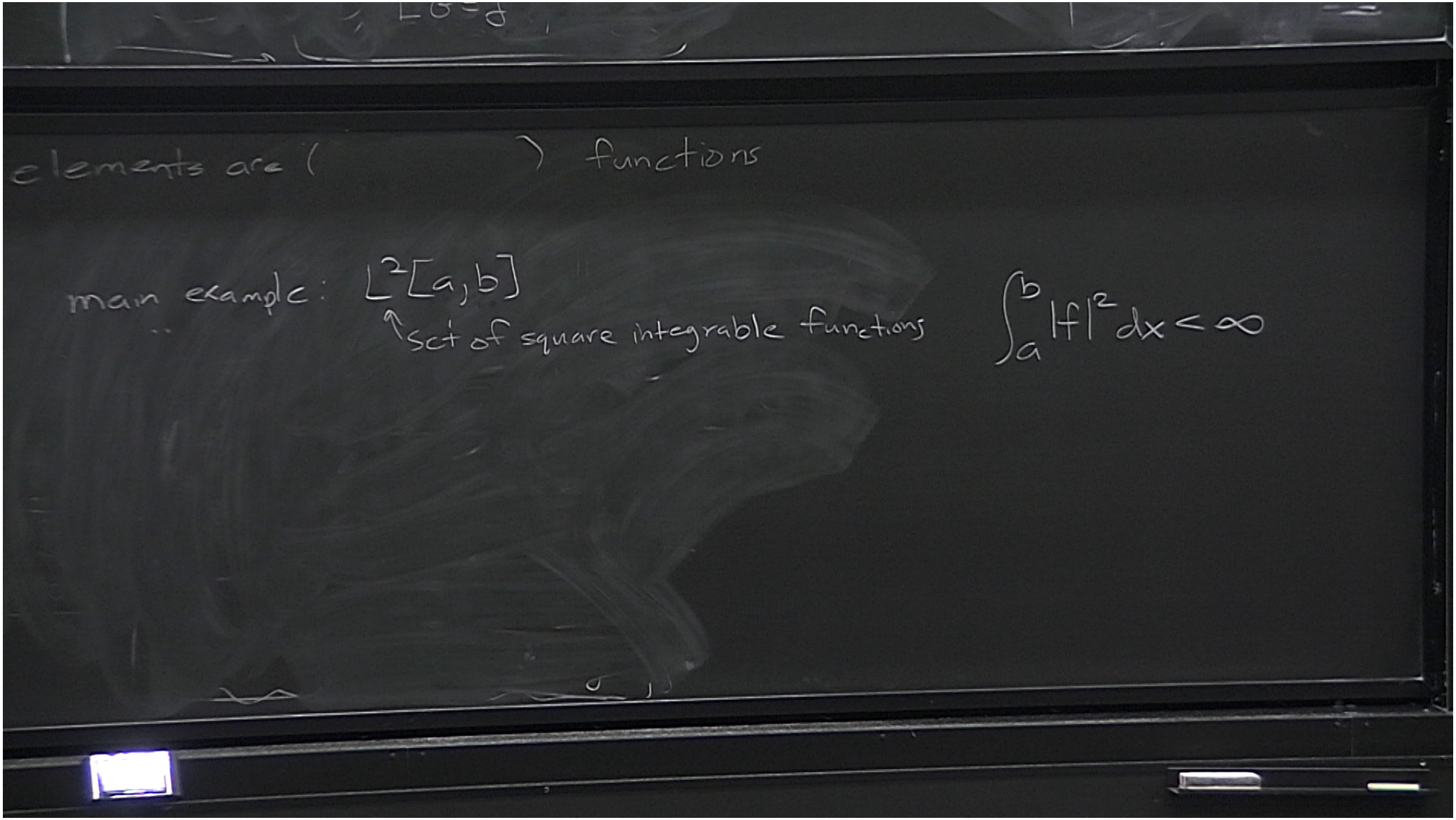

$\frac{\delta F}{\delta f(x)}$
$LG=\delta$
Definition: Function space is a vector space whose elements are ( ) functions
with a norm that satisfies
· positivity $\|f\| \geq 0$
· unique identity $\|f\|=0 \Leftrightarrow f=0$
· triangle inequality $\|f+g\| \leq \|f\|+\|g\|$
· linear homogeneity $\|\lambda f\| = |\lambda| \|f\|$
main exam
$^2[a,$

on: Function space is a vector space whose elements are (

with a norm that satisfies

- positivity $\|f\| \geq 0$
- unique identity $\|f\| = 0 \iff f = 0$
- triangle inequality $\|f+g\| \leq \|f\| + \|g\|$
- linear homogeneity $\|\lambda f\| = |\lambda| \|f\|$

main example:


elements are ( ) functions
main example: $L^2[a,b]$
↑ set of square integrable functions
$\int_a^b |f|^2 dx < \infty$

elements are equivalence classes functions

main example: $L^2[a,b]$ ↑ set of square integrable functions $\int_a^b |f|^2 dx < \infty$

$f_1(x) = 0 \quad \|f_1\| = 0$

$f_2(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases} \quad \|f_2\| = 0$

$$\|f_\sigma\|^2 = \int_{-\infty}^{\infty} |f_\sigma(x)|^2 \, dx$$

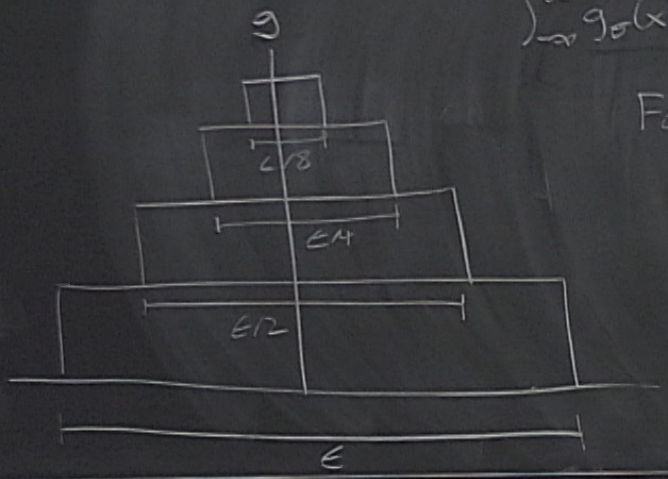
$$= \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{\sigma^2} \, dx$$

$$= \frac{1}{\sigma}$$

$$\|f\| \to \infty$$

Choose some g_σ so that $\lim_{\sigma \to 0} g_\sigma(x) = 0$ for $x \neq 0$

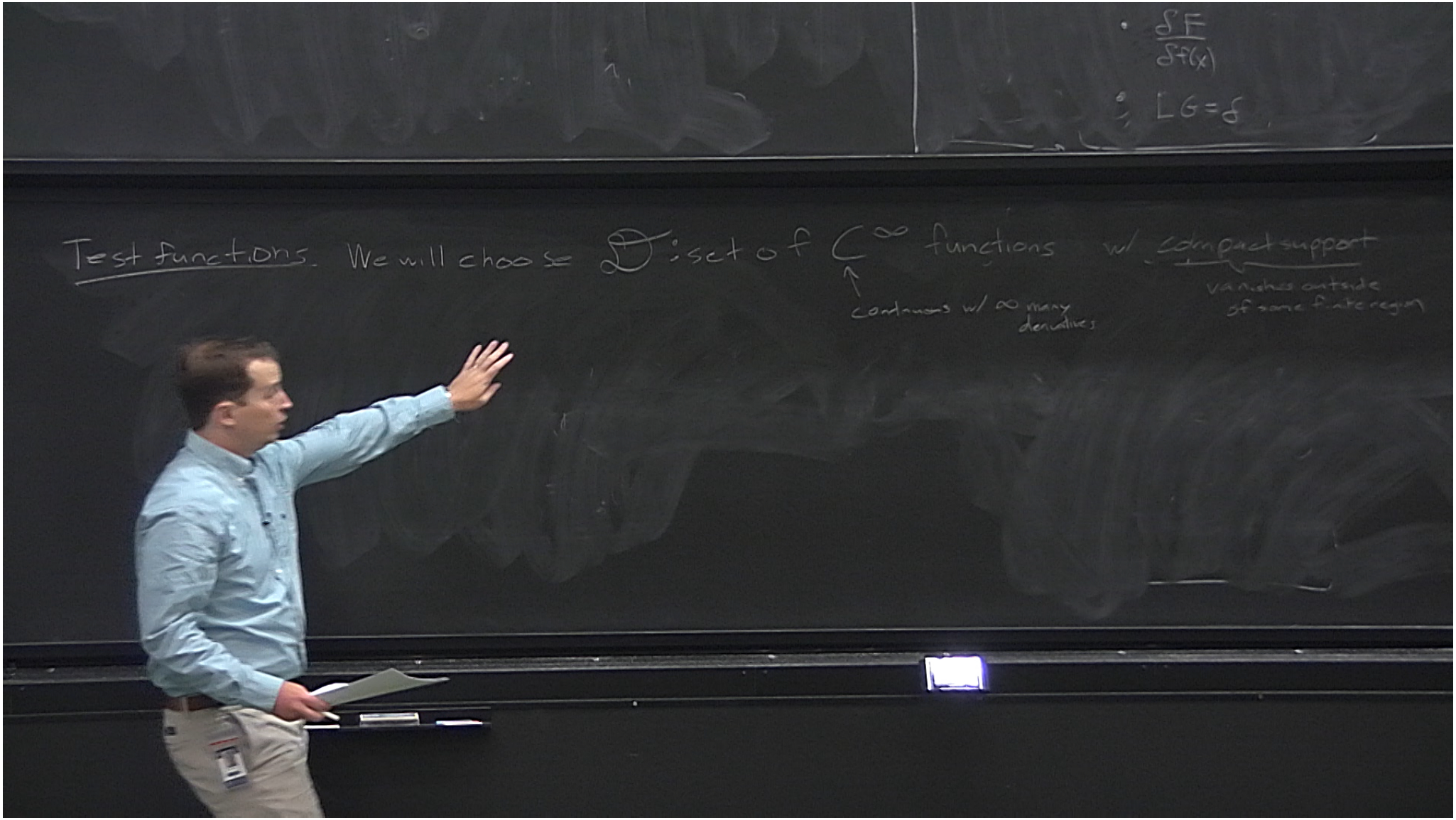
$\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$ for all $\sigma > 0$

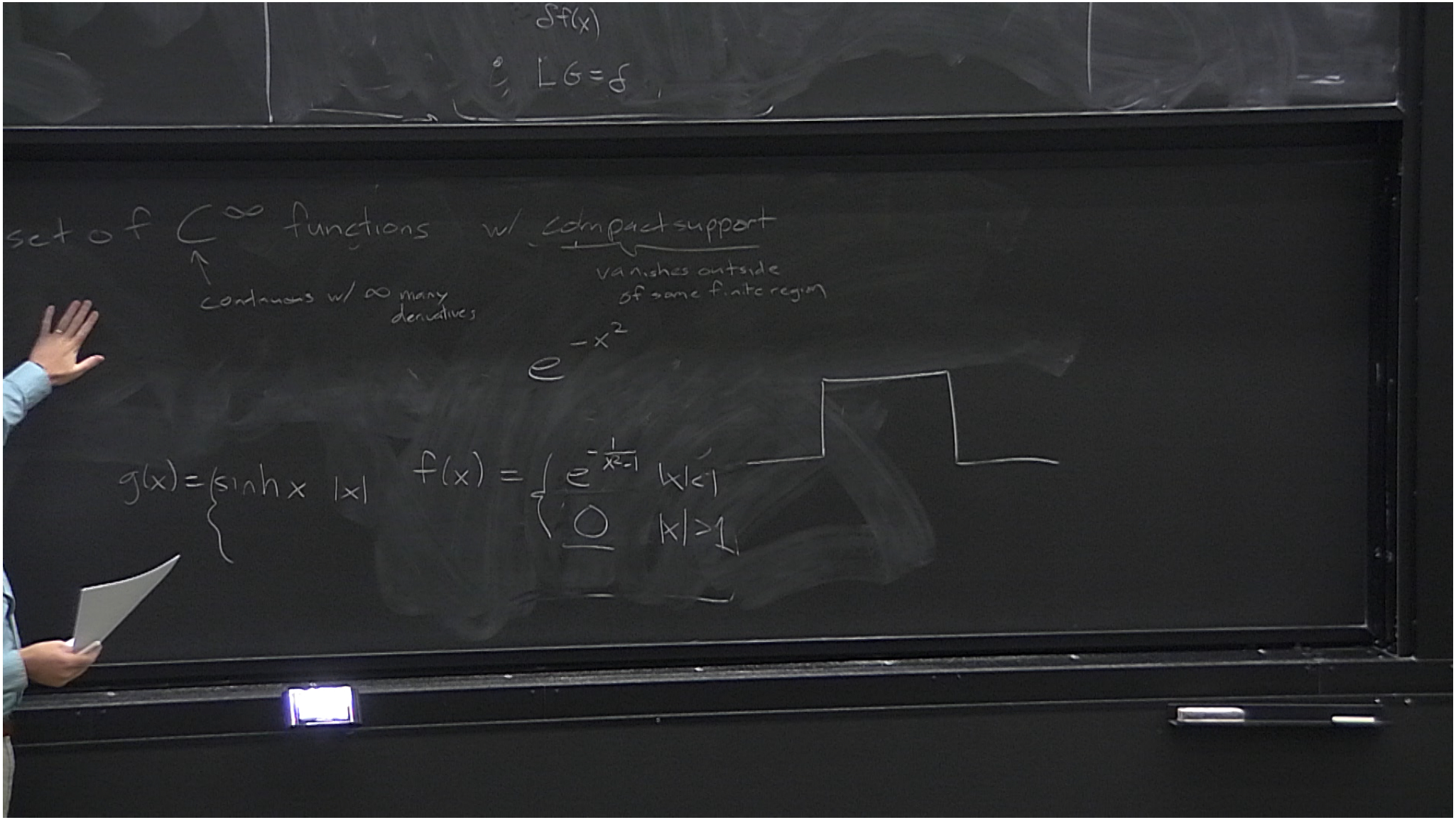


For each ϵ and each x we can find σ_0 so that $g_\sigma(x)$ lies inside the boxes for all $\sigma < \sigma_0$

$\int_{-\infty}^{\infty} g(x)\,dx < A = \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \cdots = 2\epsilon$

$\int_{-\infty}^{\infty} g(x)\,dx = 0 \neq 1 = \lim_{\sigma \to 0} \int_{-\infty}^{\infty} g_\sigma(x)\,dx$ order of lim + int matters!


$\frac{\delta F}{\delta f(x)}$
$LG = \delta$
Test functions. We will choose $\mathcal{D}$: set of C^∞ functions w/ compact support
continuous w/ ∞ many derivatives
vanishes outside of some finite region


δf(x)
LG=δ
set of C^∞ functions w/ compact support
continuous w/ ∞ many derivatives
vanishes outside of some finite region
e^{-x^2}
$g(x)=\begin{cases}\sinh x & |x|\\ \end{cases}$
$f(x)=\begin{cases} e^{-\frac{1}{x^2-1}} & |x|<1 \\ 0 & |x|>1 \end{cases}$

$\mathcal{D}^*$ = dual vector space of linear functionals of $\mathcal{D}$ to $\mathbb{R}$

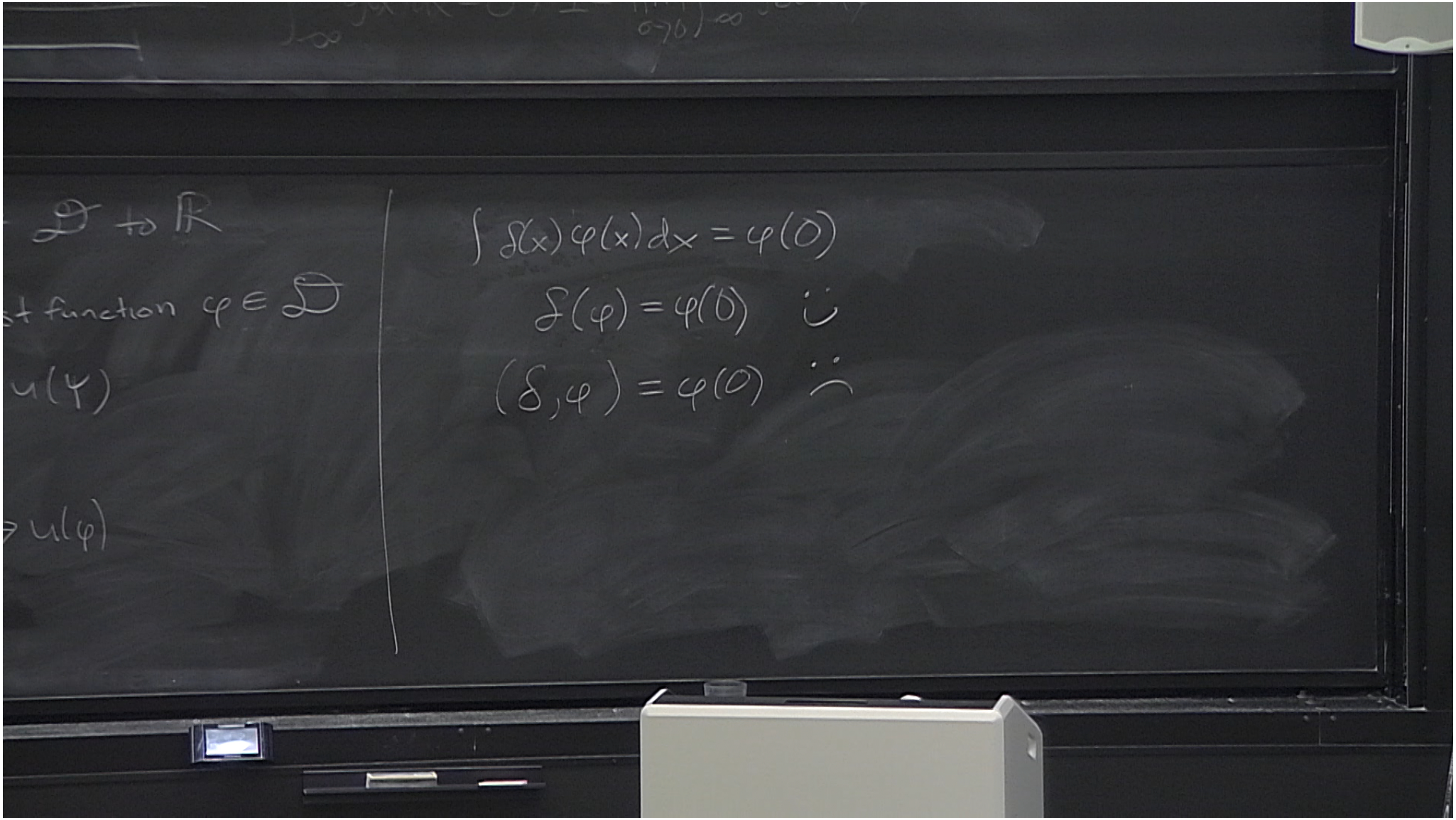
$u \in \mathcal{D}^*$ if $u(\varphi) \in \mathbb{R}$ for any test function $\varphi \in \mathcal{D}$

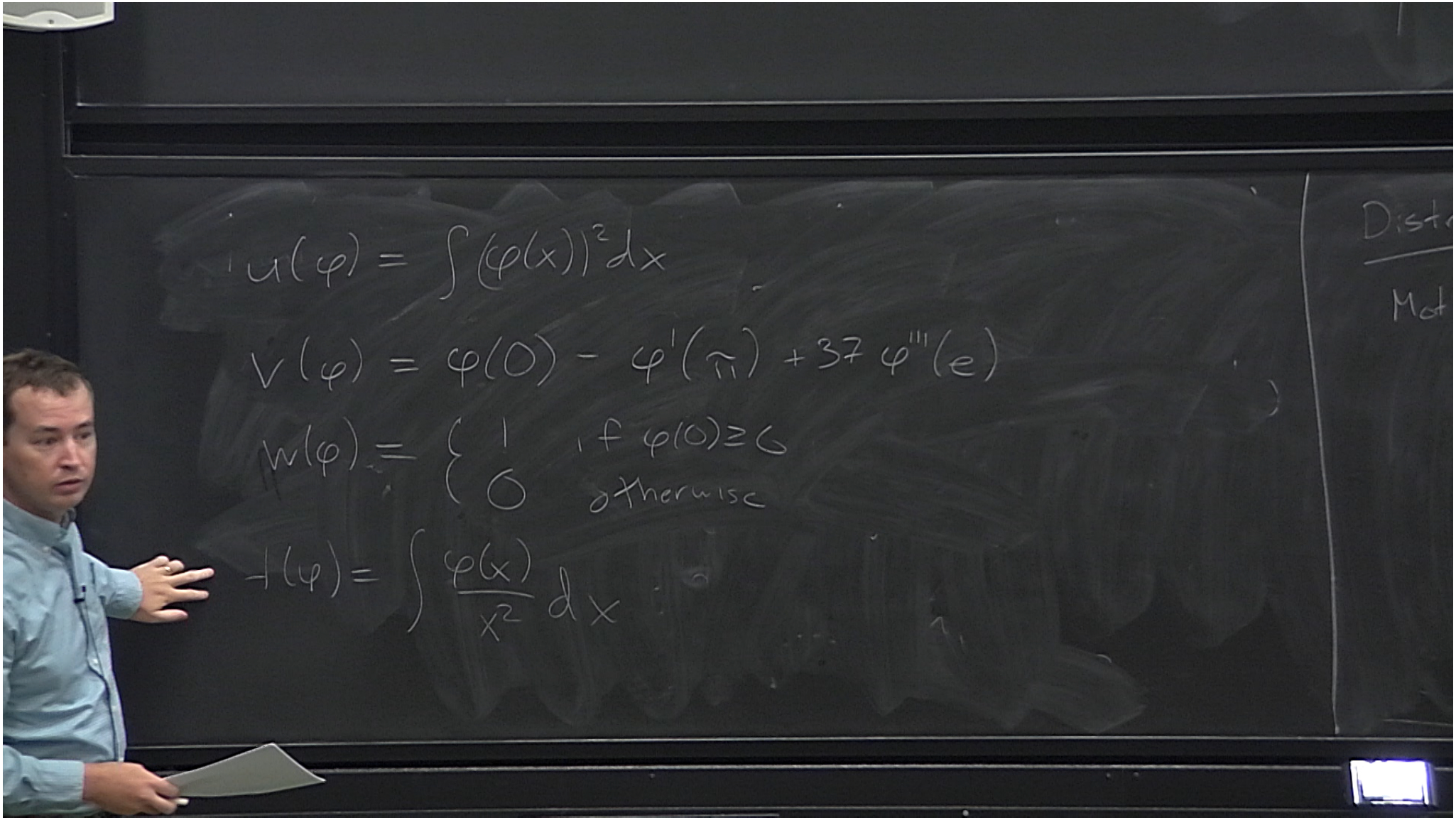
linear: $u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi)$

$\mathcal{D}'$ = distributions - continuous dual space

↑ If $\varphi_n \xrightarrow{\text{uniform}} \varphi$ then $u(\varphi_n) \to u(\varphi)$

$u \in \mathcal{D}'$ if u is
- finite
- linear
- continuous


$\mathcal{D}$ to $\mathbb{R}$
st function $\varphi \in \mathcal{D}$
$u(\varphi)$
$u(\varphi)$
$\int \delta(x)\varphi(x)dx = \varphi(0)$
$\delta(\varphi) = \varphi(0)$
$(\delta, \varphi) = \varphi(0)$


$u(\varphi) = \int (\varphi(x))^2 dx$
$V(\varphi) = \varphi(0) - \varphi'(\pi) + 37\,\varphi'''(e)$
$w(\varphi) = \begin{cases} 1 & \text{if } \varphi(0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$
$J(\varphi) = \int \frac{\varphi(x)}{x^2} dx$
Dist
Mat

$u(\varphi) = \int (\varphi(x))^2 dx$ not linear $u(\varphi+4) \neq u(\varphi)+u(4)$

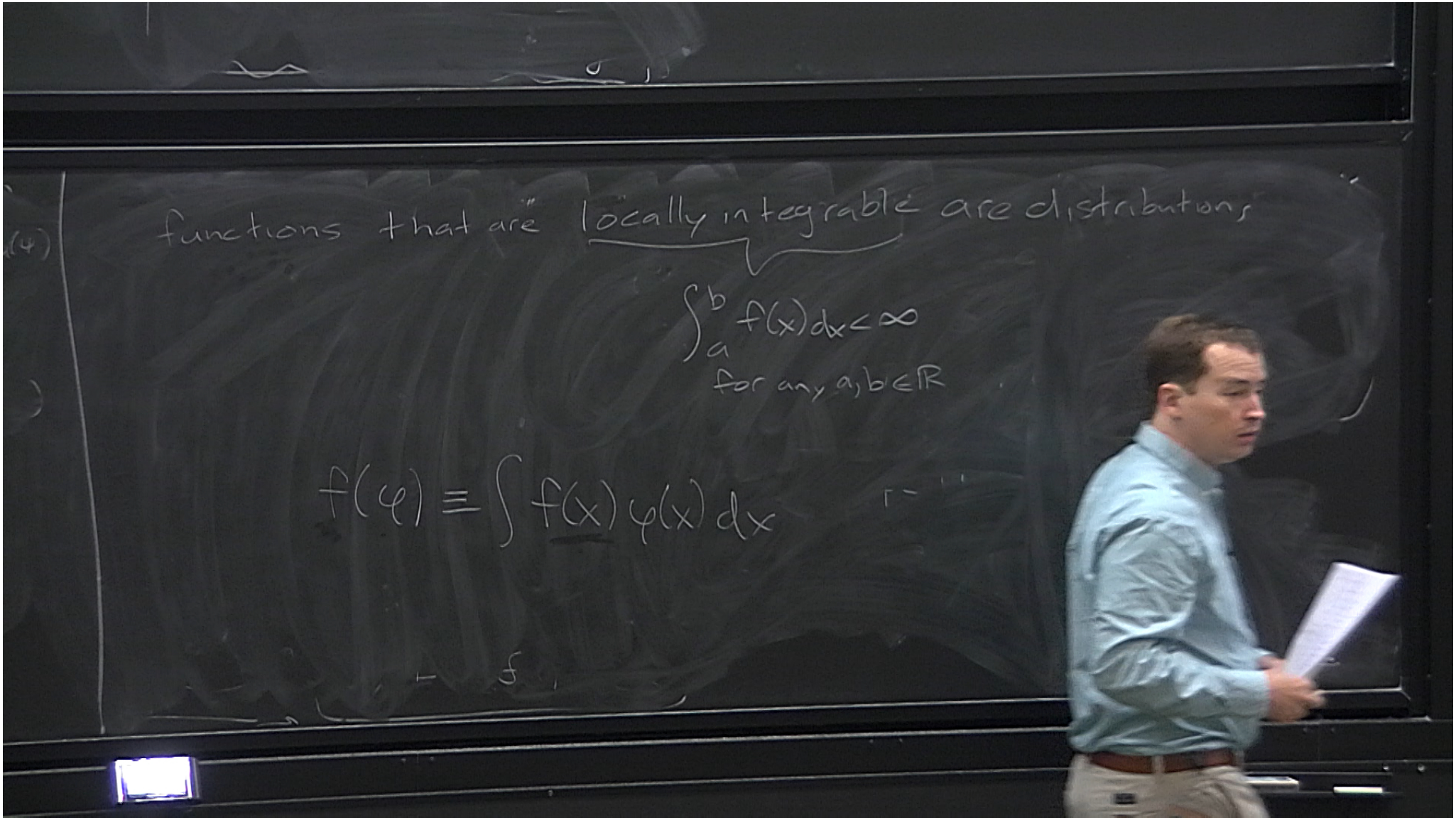
$V(\varphi) = \varphi(0) - \varphi'(\pi) + 37\,\varphi'''(e)$

$w(\varphi) = \begin{cases} 1 & \text{if } \varphi(0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$ not continuous, not linear

$t(\varphi) = \int \frac{\varphi(x)}{x^2} dx$ not finite

Dist

Mat


functions that are locally integrable are distributions
$\int_a^b f(x)\,dx < \infty$
for any $a, b \in \mathbb{R}$
$f(\varphi) \equiv \int f(x)\,\varphi(x)\,dx$