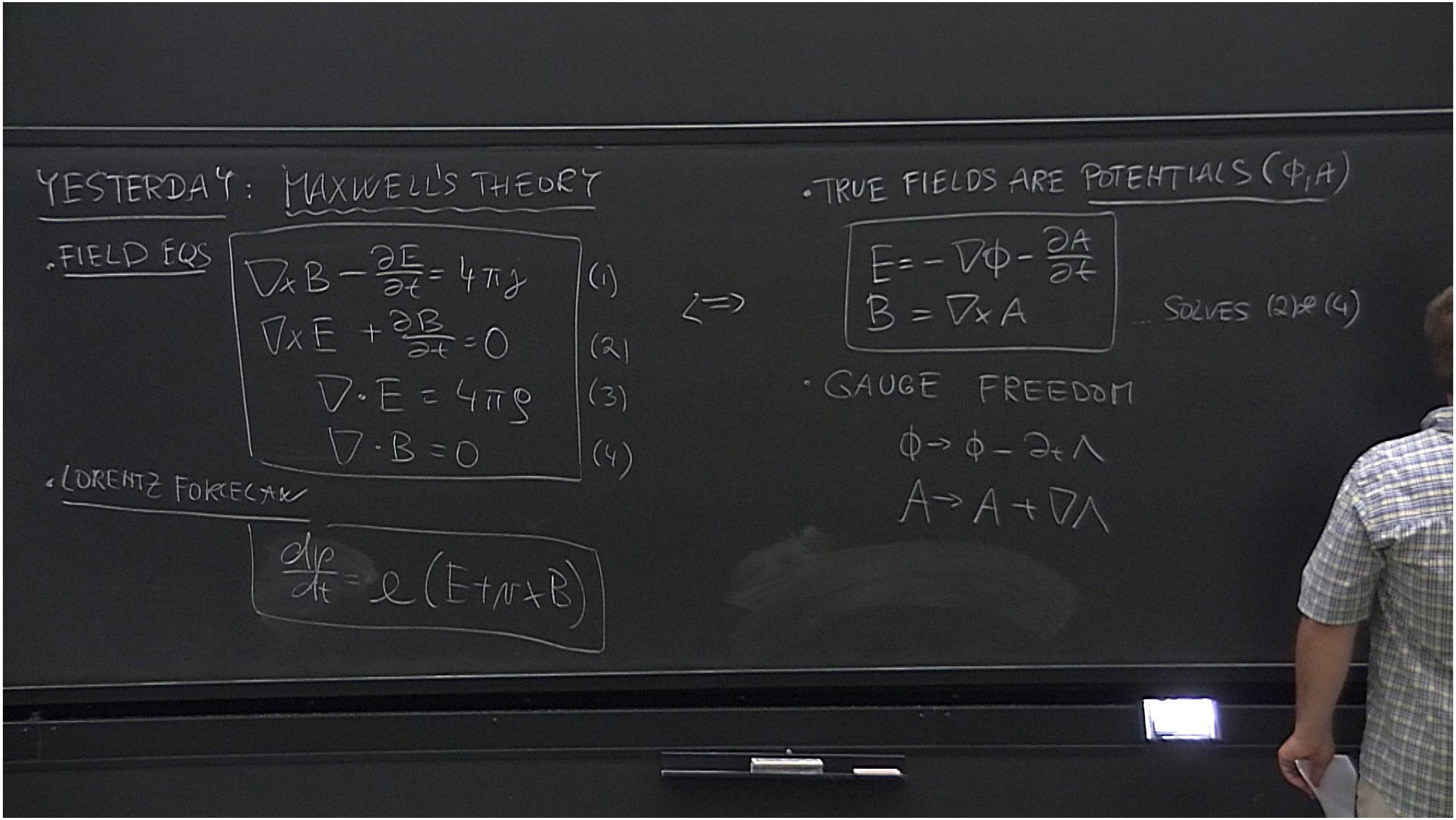
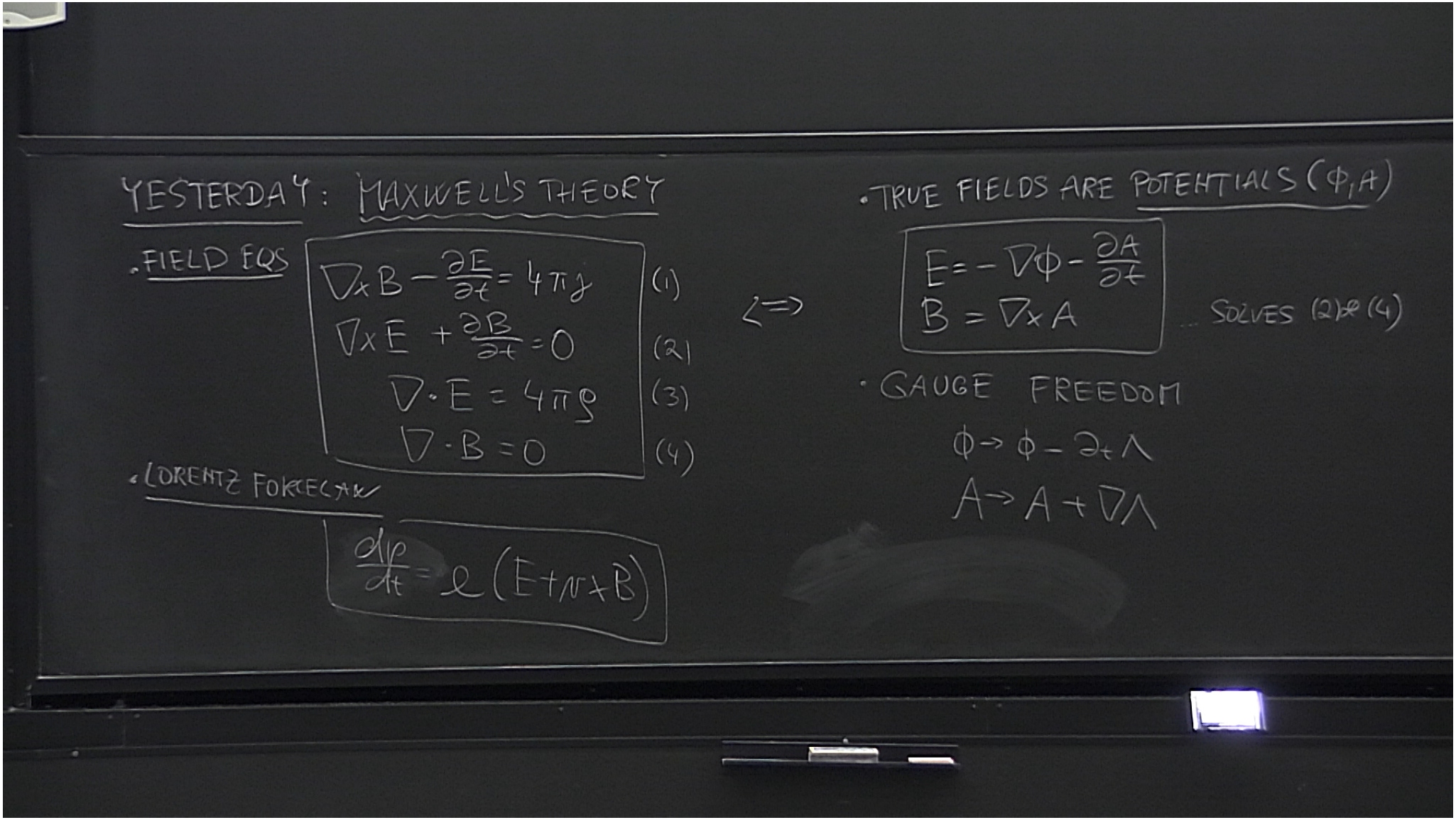
Title: PSI 2018/2019 - Classical Physics - Lecture 7

Date: Aug 22, 2018 09:00 AM

URL: http://pirsa.org/18080015

Abstract:


YESTERDAY: MAXWELL'S THEORY
• FIELD EQS
$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi j$ (1)
$\nabla \times E + \frac{\partial B}{\partial t} = 0$ (2)
$\nabla \cdot E = 4\pi \rho$ (3)
$\nabla \cdot B = 0$ (4)
• LORENTZ FORCE LAW
$\frac{dp}{dt} = e(E + v \times B)$
<=>
• TRUE FIELDS ARE POTENTIALS (ϕ, A)
$E = -\nabla\phi - \frac{\partial A}{\partial t}$
$B = \nabla \times A$
... SOLVES (2) & (4)
• GAUGE FREEDOM
$\phi \to \phi - \partial_t \Lambda$
$A \to A + \nabla\Lambda$


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FIELDS ARE POTENTIALS (ϕ, A)

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... SOLVES (2) & (4)

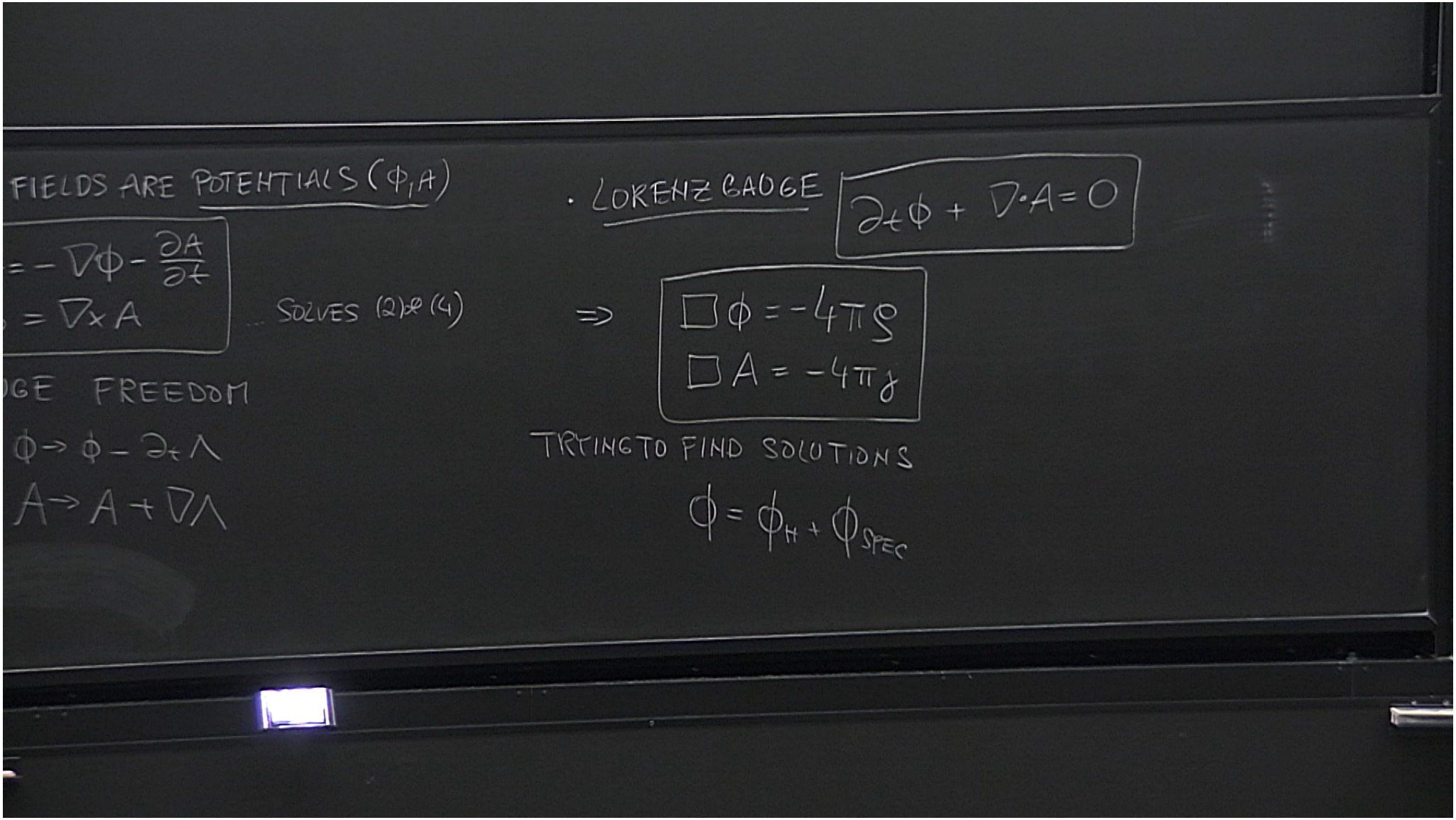
• LORENZ GAUGE $\partial_t \phi + \nabla \cdot A = 0$

$\Rightarrow \quad \Box \phi = -4$

GE FREEDOM

$\phi \to \phi - \partial_t \Lambda$

$A \to A + \nabla \Lambda$


FIELDS ARE POTENTIALS (φ, A)
$= -\nabla\phi - \frac{\partial A}{\partial t}$
$= \nabla \times A$
... SOLVES (2) & (4)
GE FREEDOM
$\phi \to \phi - \partial_t \Lambda$
$A \to A + \nabla\Lambda$
· LORENZ GAUGE
$\partial_t \phi + \nabla \cdot A = 0$
$\Rightarrow$
$\Box\phi = -4\pi\rho$
$\Box A = -4\pi j$
TRYING TO FIND SOLUTIONS
$\phi = \phi_H + \phi_{SPEC}$

ϕ SPEC. METHOD OF GREEN'S FUNCTIONS

$$\phi(x,t) = \int G(x,x',t,t')\, 4\pi\rho(x',t')\, dx'\, dt'$$

$$G(x,x',t,t') = G(\underbrace{x-x'}_{R}, \underbrace{t-t'}_{T})$$

$$\Box G(\vec{R},T) = -\delta(\vec{R})\,\delta(T)$$

$G(\vec{R},T)$ =


$G(\vec{R},T) = \frac{1}{(2\pi)^4} \int d\vec{k} d\omega \, G_{\vec{k},\omega} \, e^{i\vec{k}\cdot\vec{R} - i\omega T}$
$G_{k\omega} = \frac{1}{\vec{k}^2 - \omega^2}$

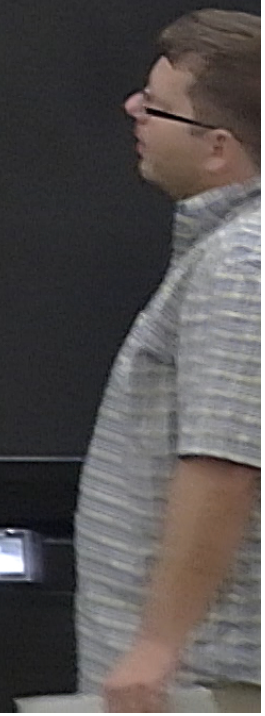
ϕ_{SPEC} · METHOD OF GREEN'S FUNCTIONS

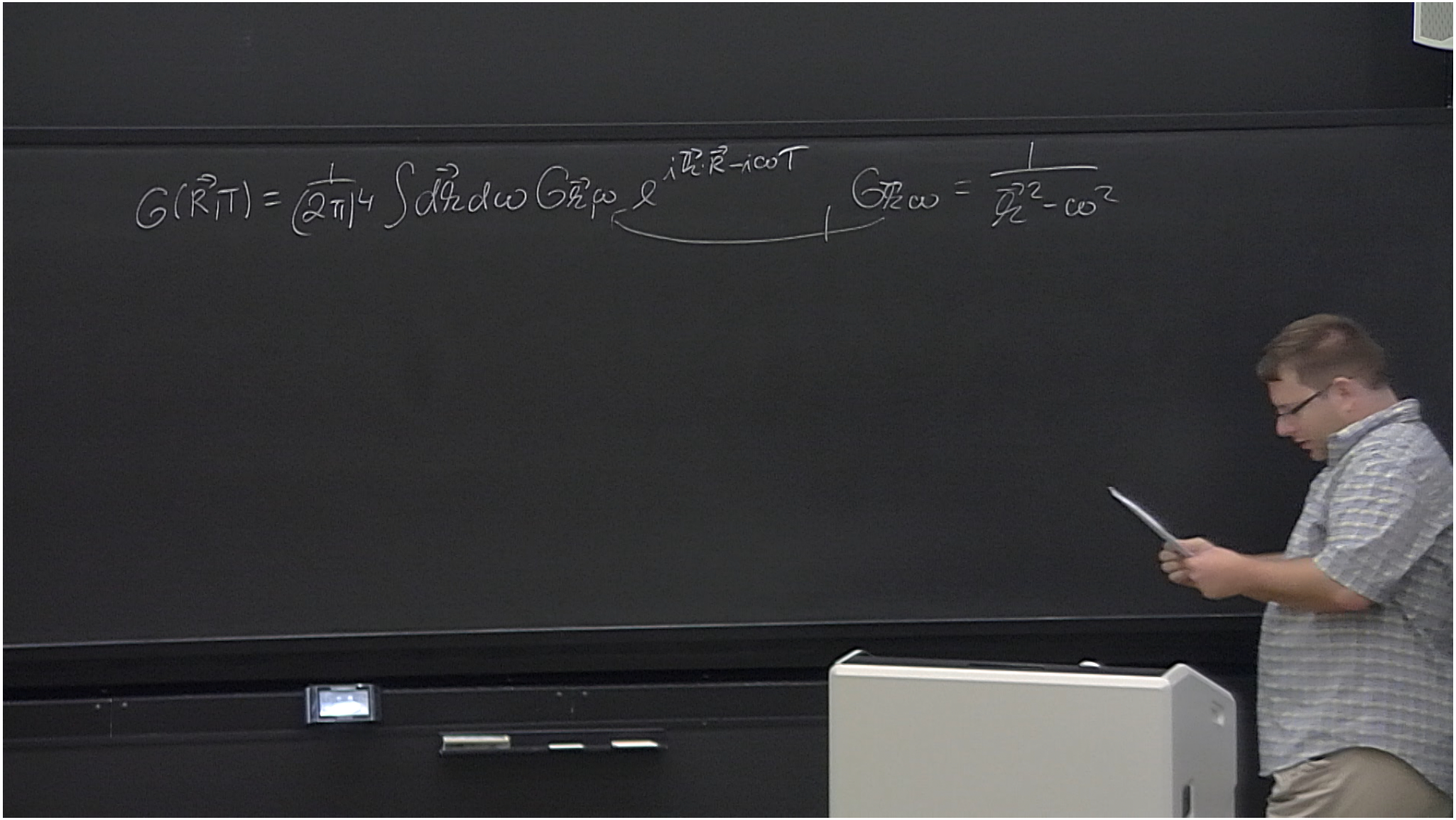
$$\phi_{SPEC}(r,t) = \int G(r,r',t,t')\, 4\pi\varrho(r',t')\, dr'\, dt'$$

$$G(r,r',t,t') = G\left(\underbrace{r-r'}_{R}, \underbrace{t-t'}_{T}\right)$$

$$\Box G(\vec{R},T) = -\delta(\vec{R})\,\delta(T)$$

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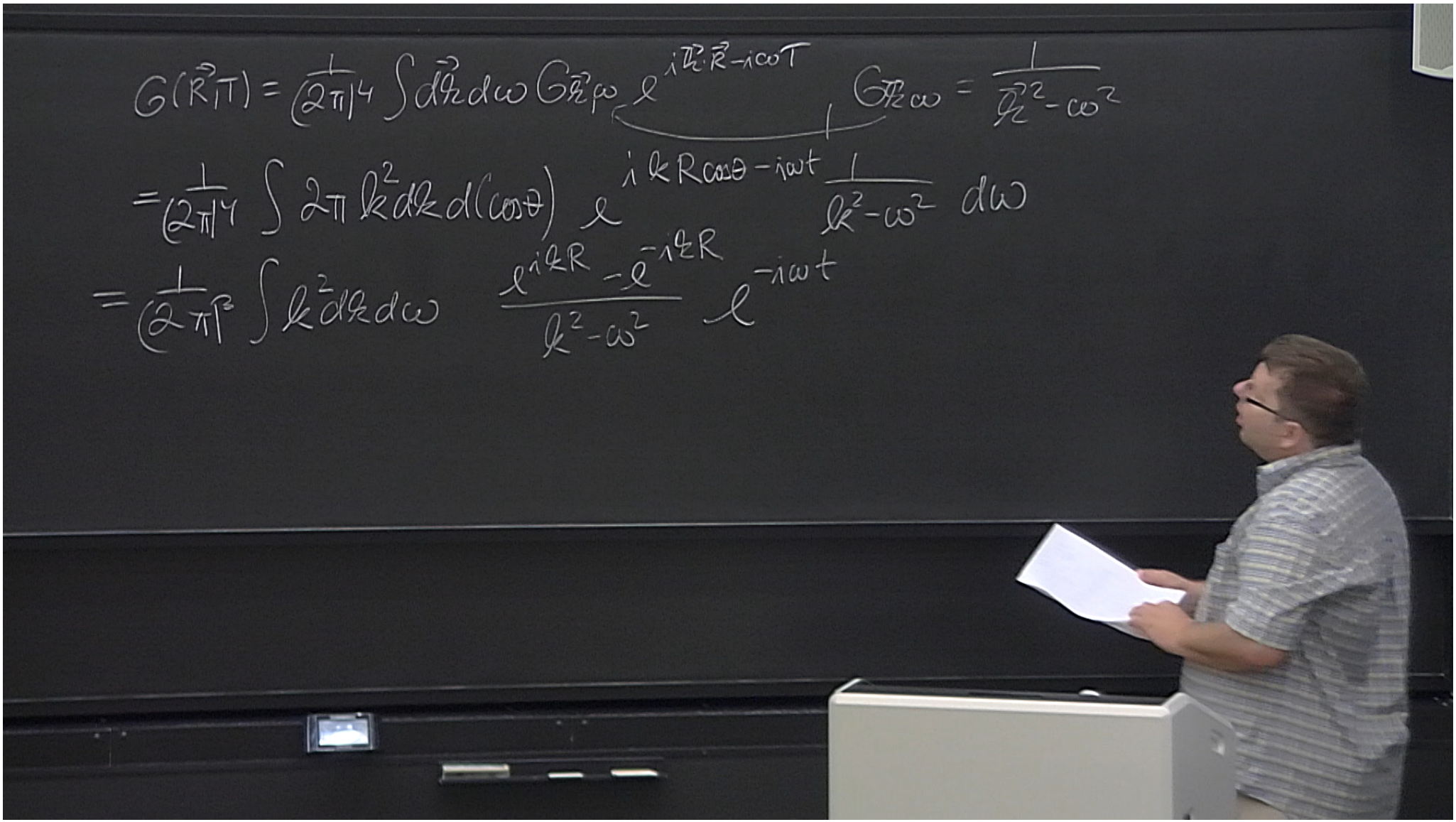
$$G(\vec{R},T) = \frac{1}{(2\pi)^4} \int d\vec{k}\, d\omega\, G_{\vec{k},\omega}\, e^{i\vec{k}\cdot\vec{R} - i\omega T}, \qquad G_{\vec{k},\omega} = \frac{1}{\vec{k}^2 - \omega^2}$$

$$= \frac{1}{(2\pi)^4} \int 2\pi k^2 dk\, d(\cos\theta)\, e^{ikR\cos\theta - i\omega t} \frac{1}{k^2 - \omega^2}\, d\omega$$

$$G(\vec{R},T) = \frac{1}{(2\pi)^4} \int d\vec{k}\, d\omega\, G_{\vec{k},\omega}\, e^{i\vec{k}\cdot\vec{R} - i\omega T}, \qquad G_{\vec{k},\omega} = \frac{1}{\vec{k}^2 - \omega^2}$$

$$= \frac{1}{(2\pi)^4} \int 2\pi k^2 dk\, d(\cos\theta)\, e^{ikR\cos\theta - i\omega t} \frac{1}{k^2 - \omega^2}\, d\omega$$

$$= \frac{1}{(2\pi)^3} \int k^2 dk\, d\omega\, \frac{e^{ikR} - e^{-ikR}}{k^2 - \omega^2}\, e^{-i\omega t}$$



$$G(\vec{R},T) = \frac{1}{(2\pi)^4}\int d\vec{k}\,d\omega\, G_{\vec{k}\omega}\, e^{i\vec{k}\cdot\vec{R}-i\omega T} \qquad G_{\vec{k}\omega} = \frac{1}{\vec{k}^2-\omega^2}$$

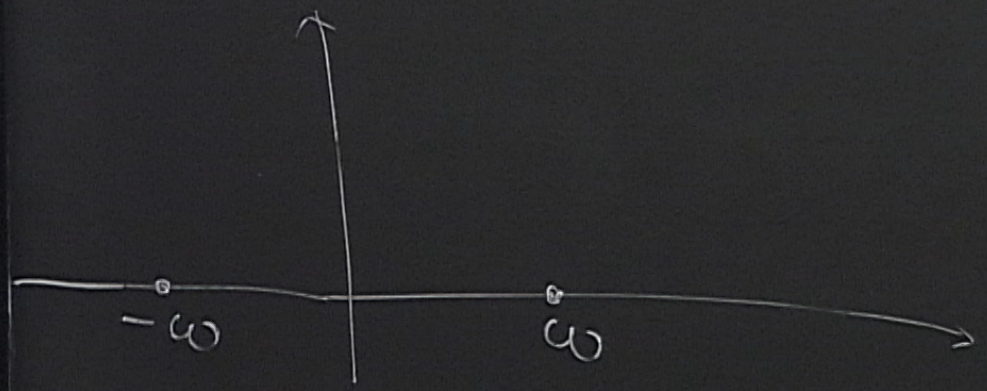
$$= \frac{1}{(2\pi)^4}\int 2\pi k^2 dk\, d(\cos\theta)\, e^{ikR\cos\theta - i\omega T}\frac{1}{k^2-\omega^2}\, d\omega$$

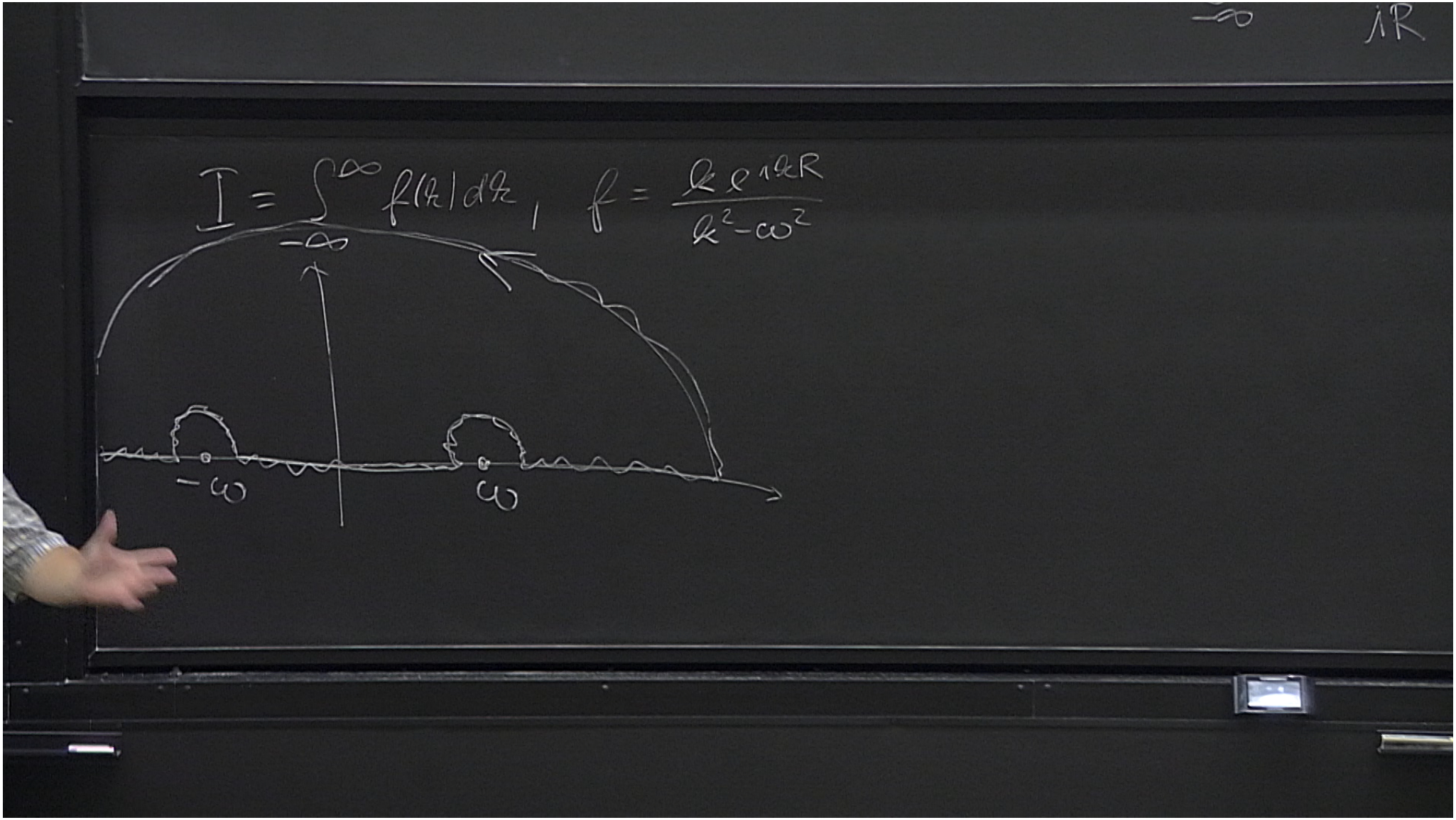
$$= \frac{1}{(2\pi)^3}\int k^2 dk\, d\omega \frac{e^{ikR}-e^{-ikR}}{(iR\vec{k})(k^2-\omega^2)}\, e^{-i\omega T}$$

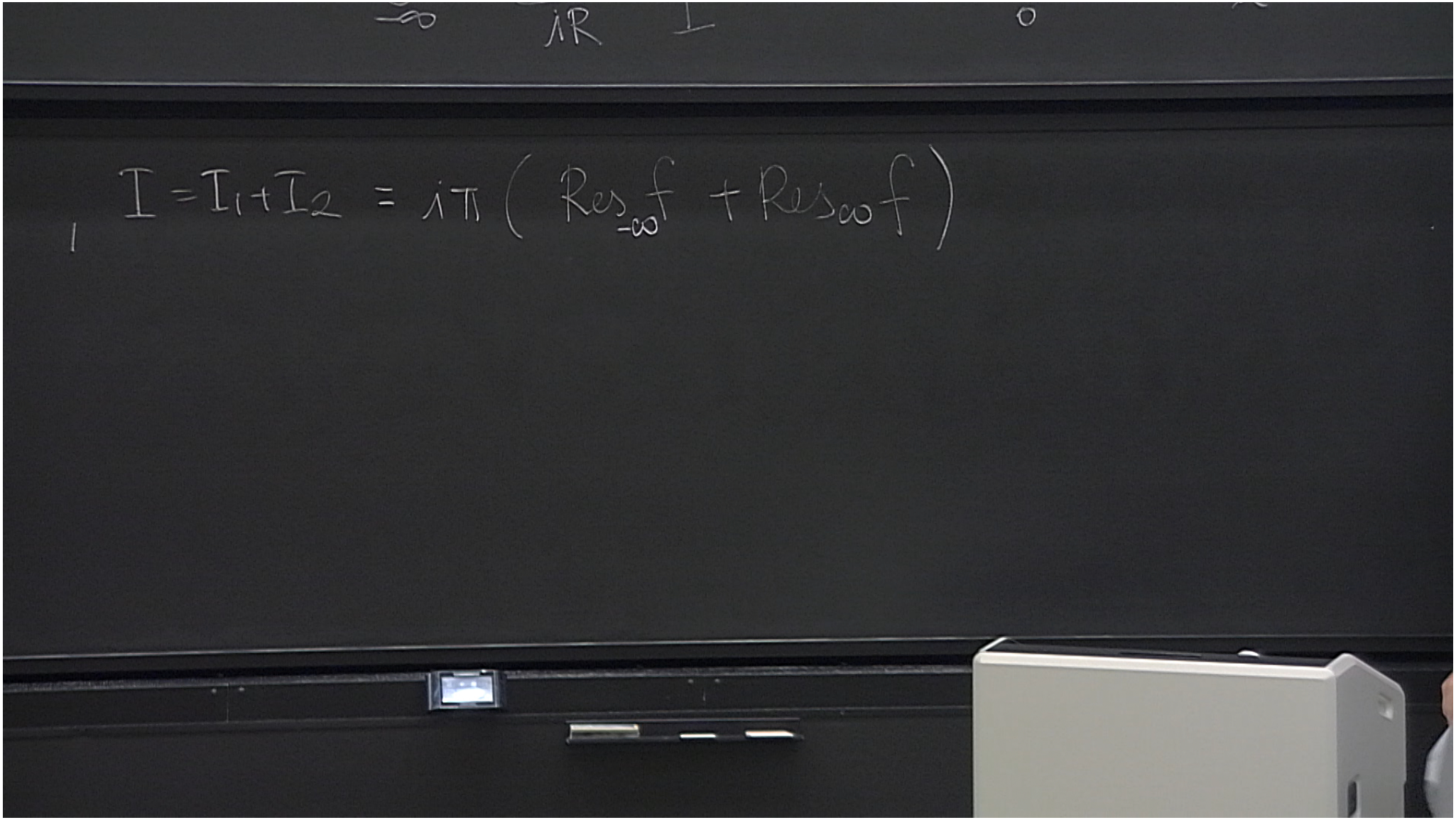
$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega T}}{iR}\, I \qquad I = \int dk\, k \frac{e^{ikR}-e^{-ikR}}{k^2-\omega^2}$$

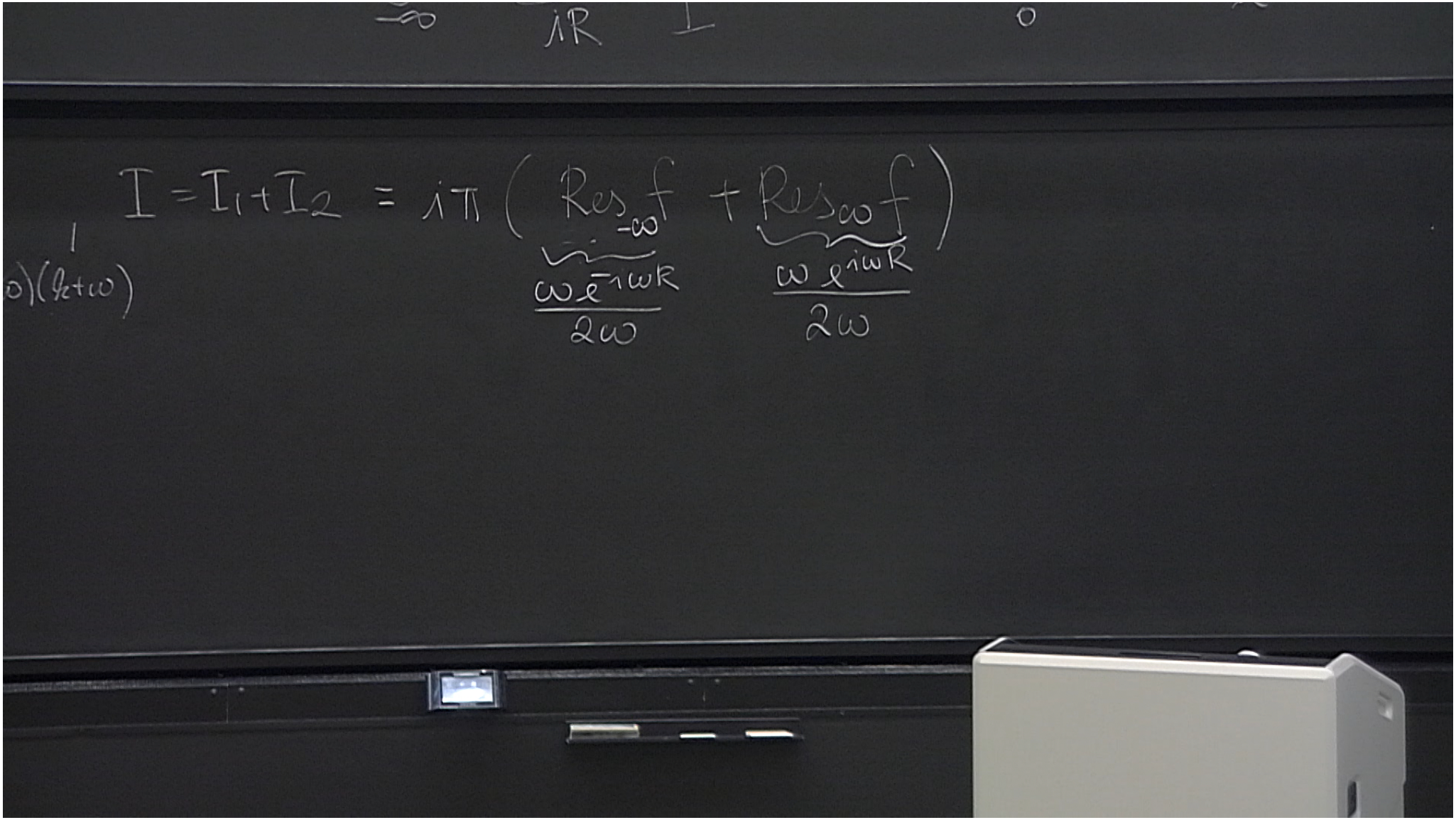
$-\infty$ iR

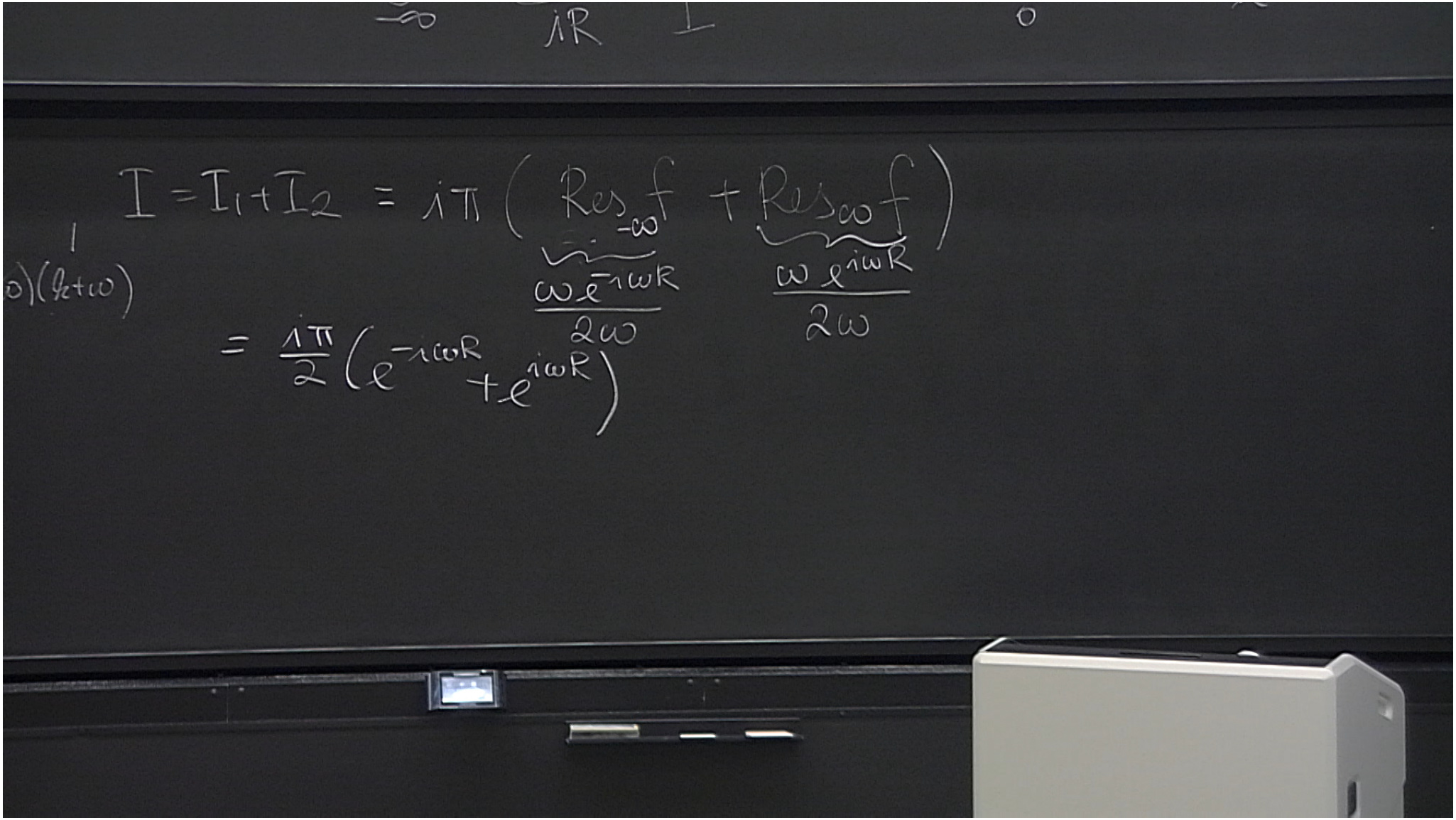
$$I = \int_{-\infty}^{\infty} f(k)\,dk, \quad f = \frac{k\, e^{ikR}}{k^2 - \omega^2}$$




$-\infty$
iR
$I = \int_{-\infty}^{\infty} f(k)\,dk$, $f = \frac{k e^{ikR}}{k^2 - \omega^2}$
$-\omega$
ω


$-\infty$
iR
I
0
$I = I_1 + I_2 = i\pi \left(\mathrm{Res}_{-\infty} f + \mathrm{Res}_{\infty} f \right)$


∞
iR
I
0
$I = I_1 + I_2 = i\pi \left(\underbrace{\text{Res}_{-\omega} f}_{\frac{\omega e^{-i\omega R}}{2\omega}} + \underbrace{\text{Res}_{\omega} f}_{\frac{\omega e^{i\omega R}}{2\omega}} \right)$
$\omega)(k+\omega)$

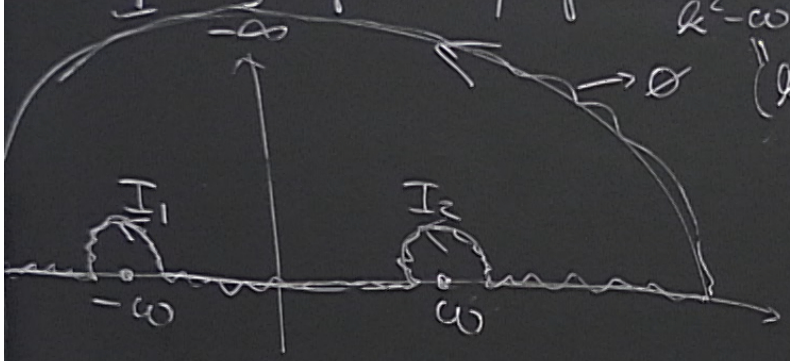

∞
iR
I
0
$I = I_1 + I_2 = i\pi \left(\underbrace{\text{Res}_{-\omega} f}_{\frac{\omega e^{-i\omega R}}{2\omega}} + \underbrace{\text{Res}_{\omega} f}_{\frac{\omega e^{i\omega R}}{2\omega}} \right)$
$= \frac{i\pi}{2} \left(e^{-i\omega R} + e^{i\omega R} \right)$


$I = \int_{-\infty}^{\infty} f(k)\,dk$, $f = \frac{k\, e^{ikR}}{k^2-\omega^2}$, $I = I_1 + I_2 = i\pi\left(\underbrace{\mathrm{Res}_{-\omega} f}_{\frac{\omega e^{-i\omega R}}{2\omega}} + \underbrace{\mathrm{Res}_{\omega} f}_{\frac{\omega e^{i\omega R}}{2\omega}}\right)$
$= \frac{i\pi}{2}\left(e^{-i\omega R} + e^{i\omega R}\right)$
$G = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{d\omega}{iR}\,\frac{i\pi}{2}\left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)}\right)$
I_1
I_2
$-\omega$
ω

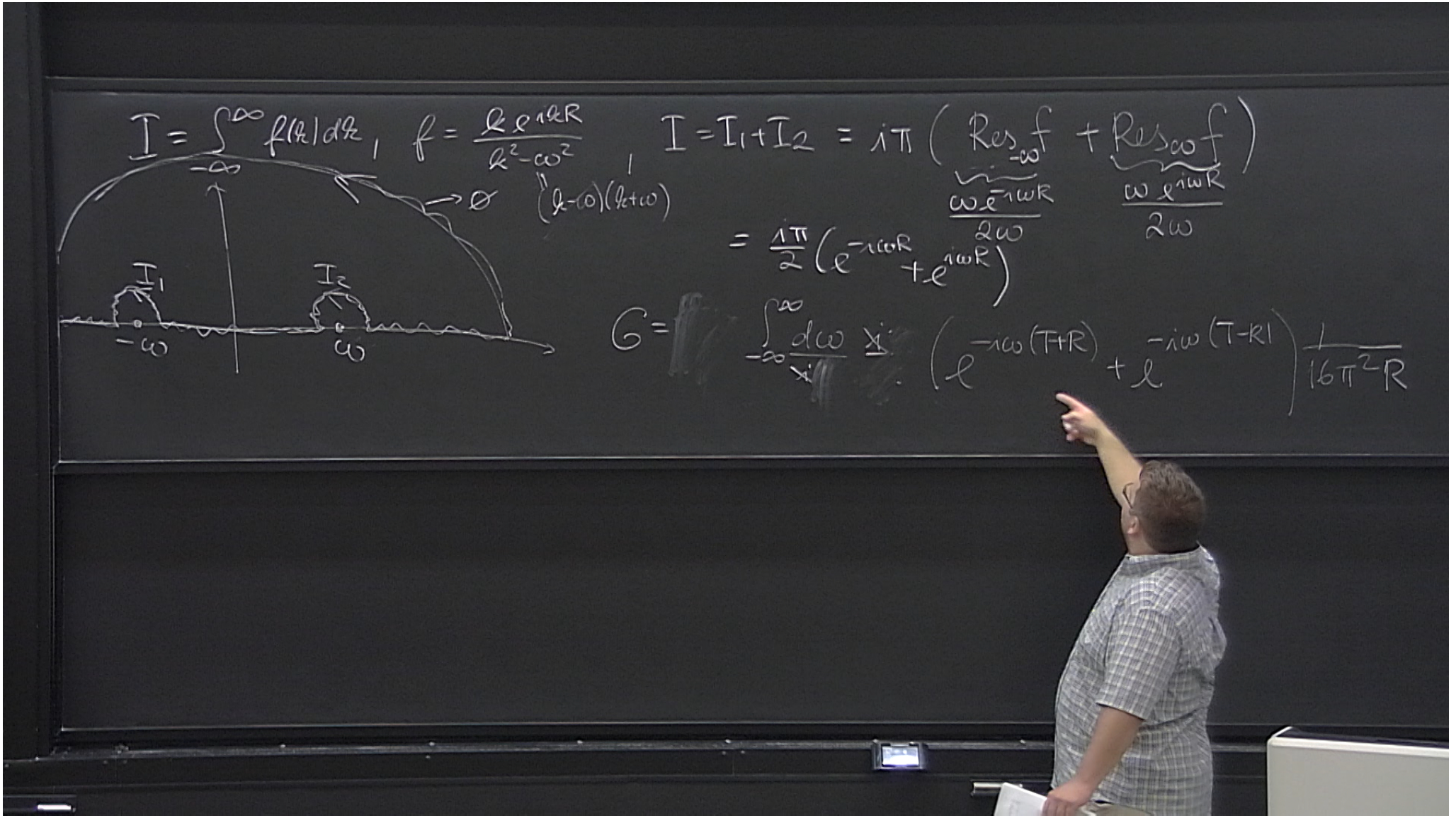
$$I = \int_{-\infty}^{\infty} f(k)\,dk, \quad f = \frac{k\,e^{ikR}}{k^2-\omega^2} = \frac{k\,e^{ikR}}{(k-\omega)(k+\omega)}$$

$$I = I_1 + I_2 = i\pi\left(\underbrace{\mathrm{Res}_{-\omega} f}_{\frac{\omega e^{-i\omega R}}{2\omega}} + \underbrace{\mathrm{Res}_{\omega} f}_{\frac{\omega e^{i\omega R}}{2\omega}}\right)$$

$$= \frac{i\pi}{2}\left(e^{-i\omega R} + e^{i\omega R}\right)$$



$$G = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{d\omega}{iR}\,\frac{i\pi}{2}\left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)}\right)$$


$I = \int_{-\infty}^{\infty} f(k)\,dk$, $f = \frac{k\, e^{ikR}}{k^2-\omega^2}$, $I = I_1 + I_2 = i\pi\left(\underset{-\omega}{\mathrm{Res}}\, f + \underset{\omega}{\mathrm{Res}}\, f\right)$
$\to 0$
$(k-\omega)(k+\omega)$
$\frac{\omega e^{-i\omega R}}{2\omega}$ $\frac{\omega e^{i\omega R}}{2\omega}$
$= \frac{i\pi}{2}\left(e^{-i\omega R} + e^{i\omega R}\right)$
I_1 I_2
$-\omega$ ω
$G = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)}\right) \frac{1}{16\pi^2 R}$

$$G = \frac{1}{8\pi R} \underbrace{\frac{1}{2\pi} \int d\omega \left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)} \right)}$$

$$\delta(T+R) \qquad \delta(T-R)$$

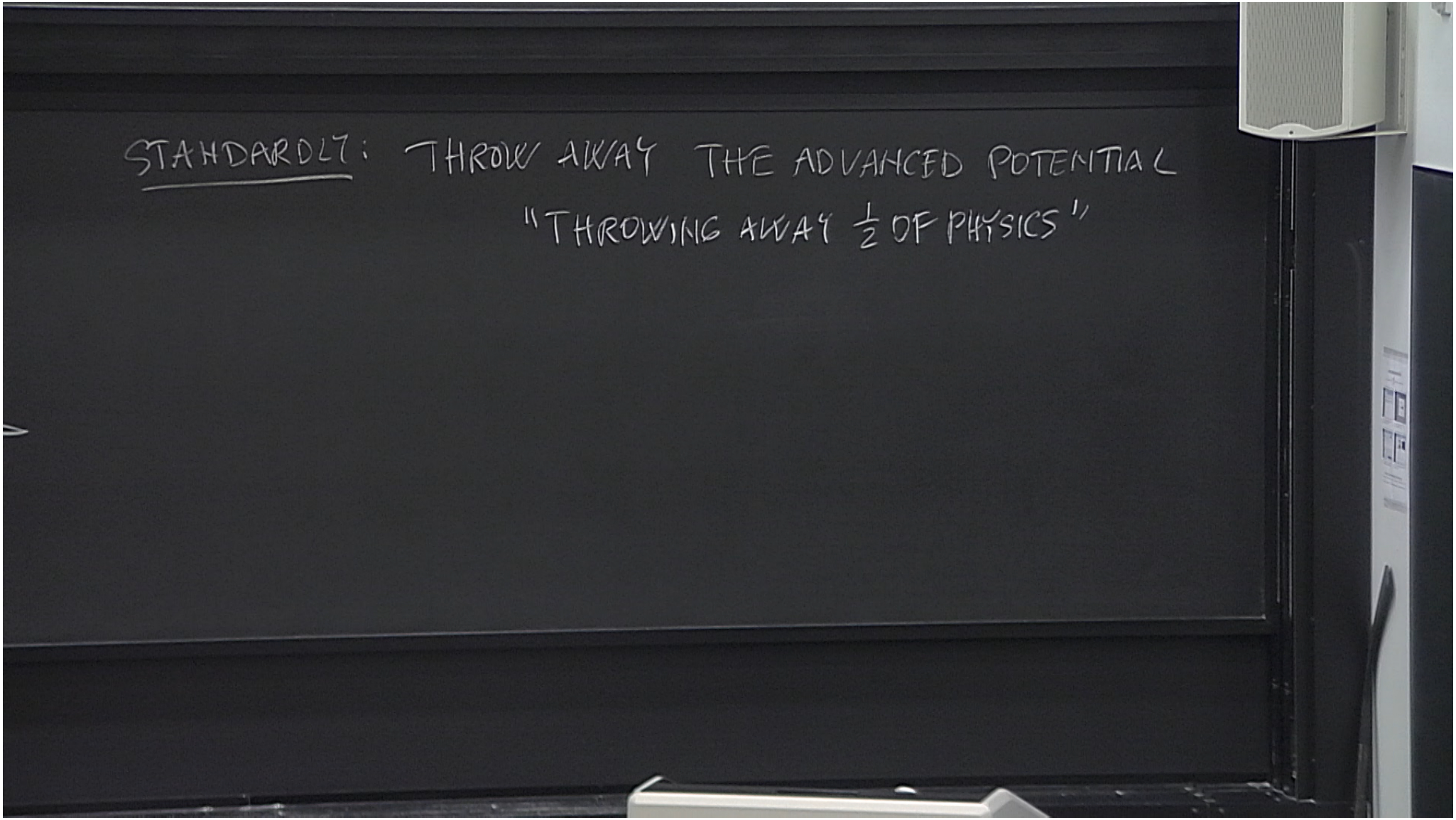
$$= \frac{1}{8\pi R} \left(\delta(T+R) + \delta(T-R) \right)$$

$$G = \frac{1}{8\pi R}\,\frac{1}{2\pi}\int d\omega \left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)} \right)$$

$$e^{-i\omega(T+R)} \rightarrow \delta(T+R) \qquad e^{-i\omega(T-R)} \rightarrow \delta(T-R)$$

$$= \frac{1}{8\pi R}\left(\delta(T+R) + \delta(T-R) \right) = \frac{1}{4\pi}\,\delta(R^2 - T^2)$$

$\delta(T+R)$: ADVANCED $\qquad$ $\delta(T-R)$: RETARDED


STANDARD 7: THROW AWAY THE ADVANCED POTENTIAL
"THROWING AWAY ½ OF PHYSICS"

$$G = \frac{1}{8\pi R}\,\frac{1}{2\pi}\int d\omega \left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)} \right)$$

STANDARD

$$e^{-i\omega(T+R)} \rightarrow \delta(T+R) \qquad e^{-i\omega(T-R)} \rightarrow \delta(T-R)$$

$$= \frac{1}{8\pi R}\left(\delta(T+R) + \delta(T-R) \right) = \frac{1}{4\pi}\,\delta(R^2 - T^2)$$

$\delta(T+R)$ → ADVANCED → PROBLEMATIC ?

$\delta(T-R)$ → RETARDED → USED TO

$$= \frac{1}{8\pi R}\left(\delta(T+R) + \delta(T-R)\right) = \frac{1}{4\pi}\delta(R^2-T^2)$$

C) FROM MAXWELL TO RELATIVITY

SYMMETRIES OF MAXWELL'S THEORY

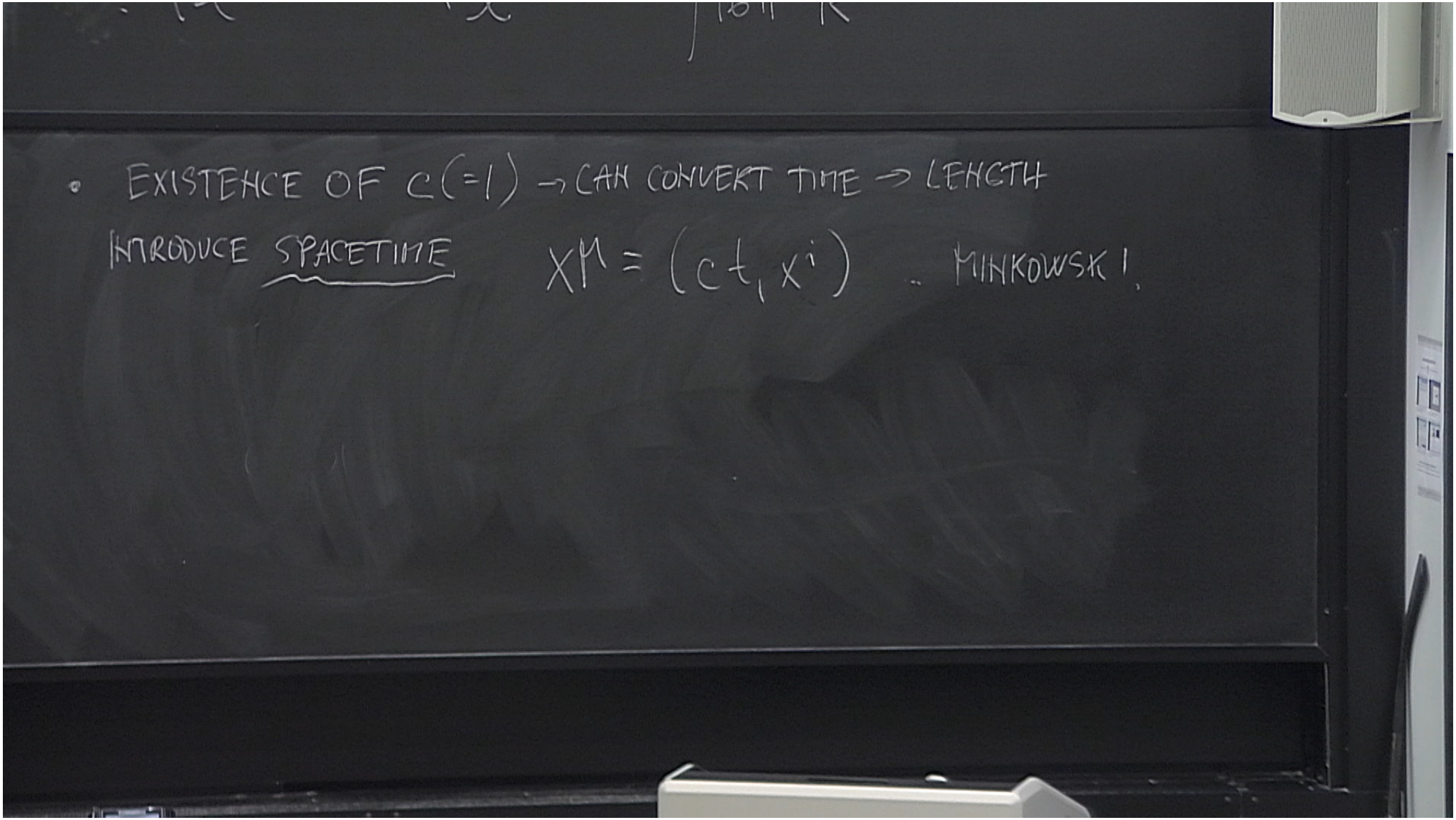
GAUGE T

INVARIANT UNDER POINCARE (LORENTZ + TRANSLATIONS)

c) FROM MAXWELL TO RELATIVITY

SYMMETRIES OF MAXWELL'S THEORY

- GAUGE ↑
- INVARIANT UNDER POINCARE (LORENTZ + TRANSLATIONS) → SR
- CONFORMAL S.
- $\rho = 0 = j$ EM DUALITY $E \to B$, $B \to -E$


• EXISTENCE OF c(=1) → CAN CONVERT TIME → LENGTH
INTRODUCE SPACETIME $X^\mu = (ct, x^i)$ .. MINKOWSKI

- EXISTENCE OF $c(=1)$ → CAN CONVERT TIME ⇒ LENGTH

INTRODUCE SPACETIME $X^\mu = (ct, x^i)$ _ MINKOWSKI

$\mu = 0,1,2,3$; $i = 1,2,3$

μ ↓ SPACETIME i ↓ SPATIAL

(GREEKS MORE CLEVER THAN ROMANS)

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\right)$$

$$\left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)}\right)\frac{1}{16\pi^2 R}$$

EM DUALITY $E \to B, B \to -E$

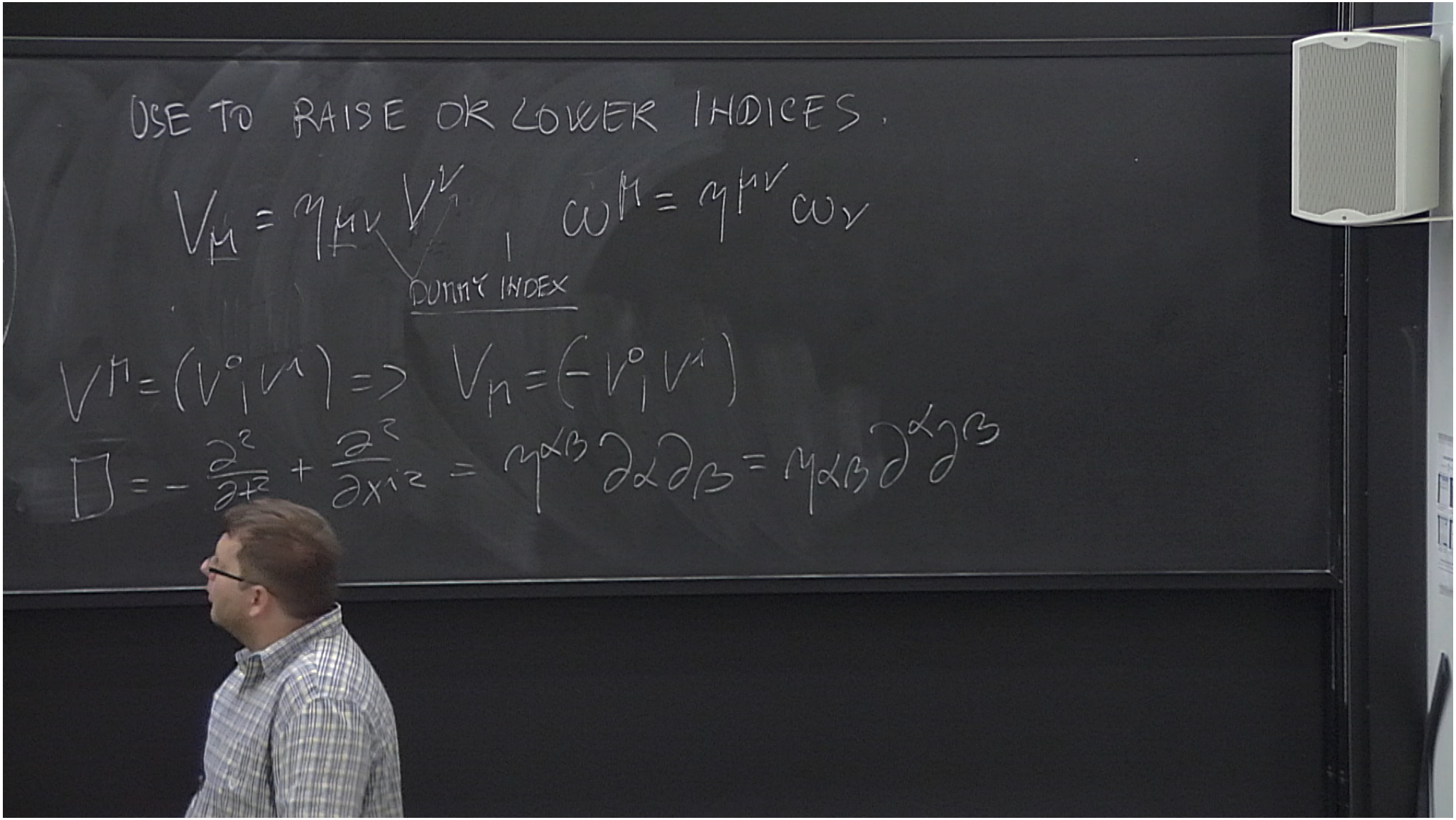
$\partial_\mu = \partial$

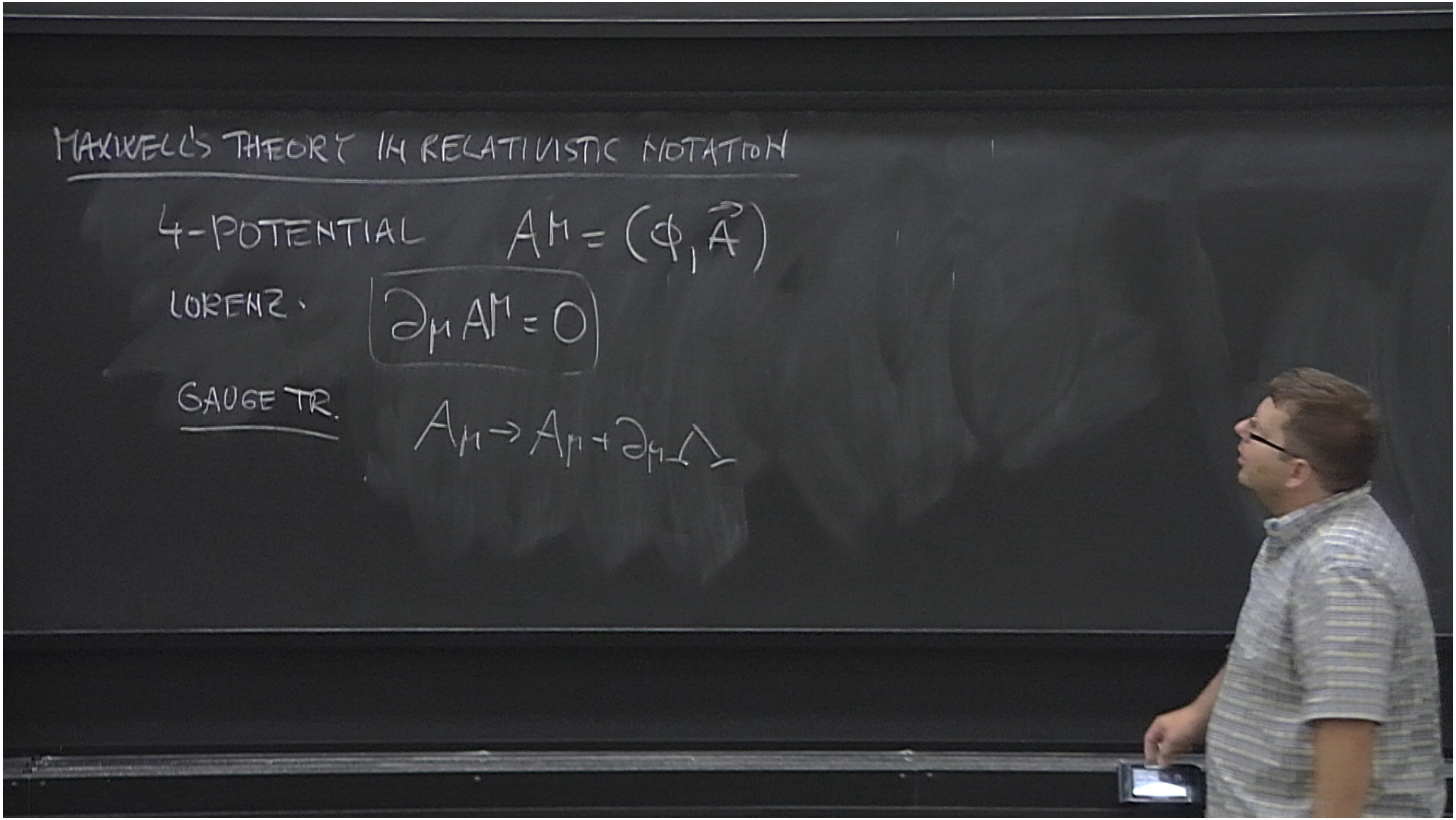
$= \frac{1}{8\pi R}\left(\delta(T+R) + \delta(T-R)\right) = \frac{1}{\ldots}\delta(R^2-T^2)$

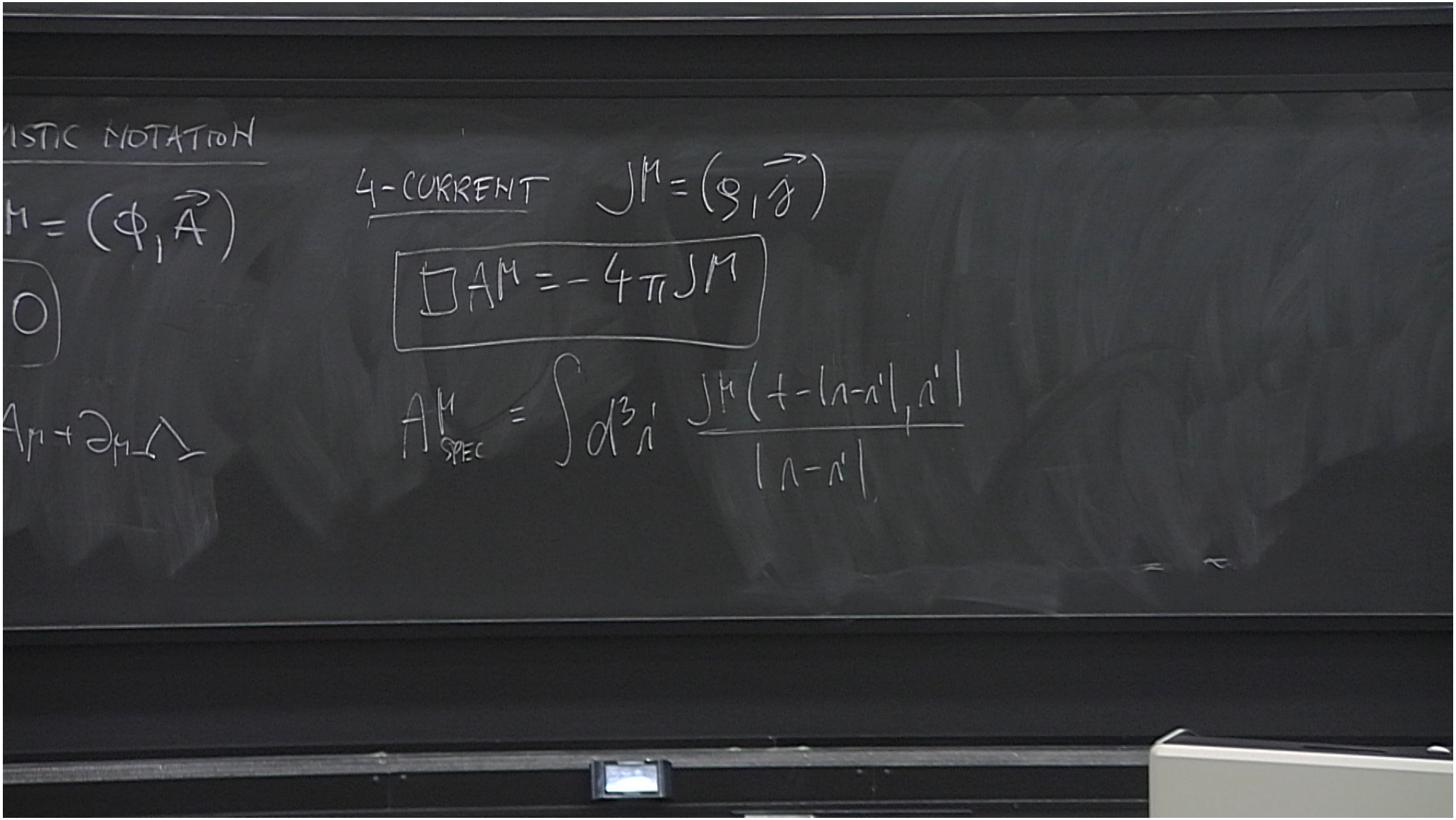
MATRICES:

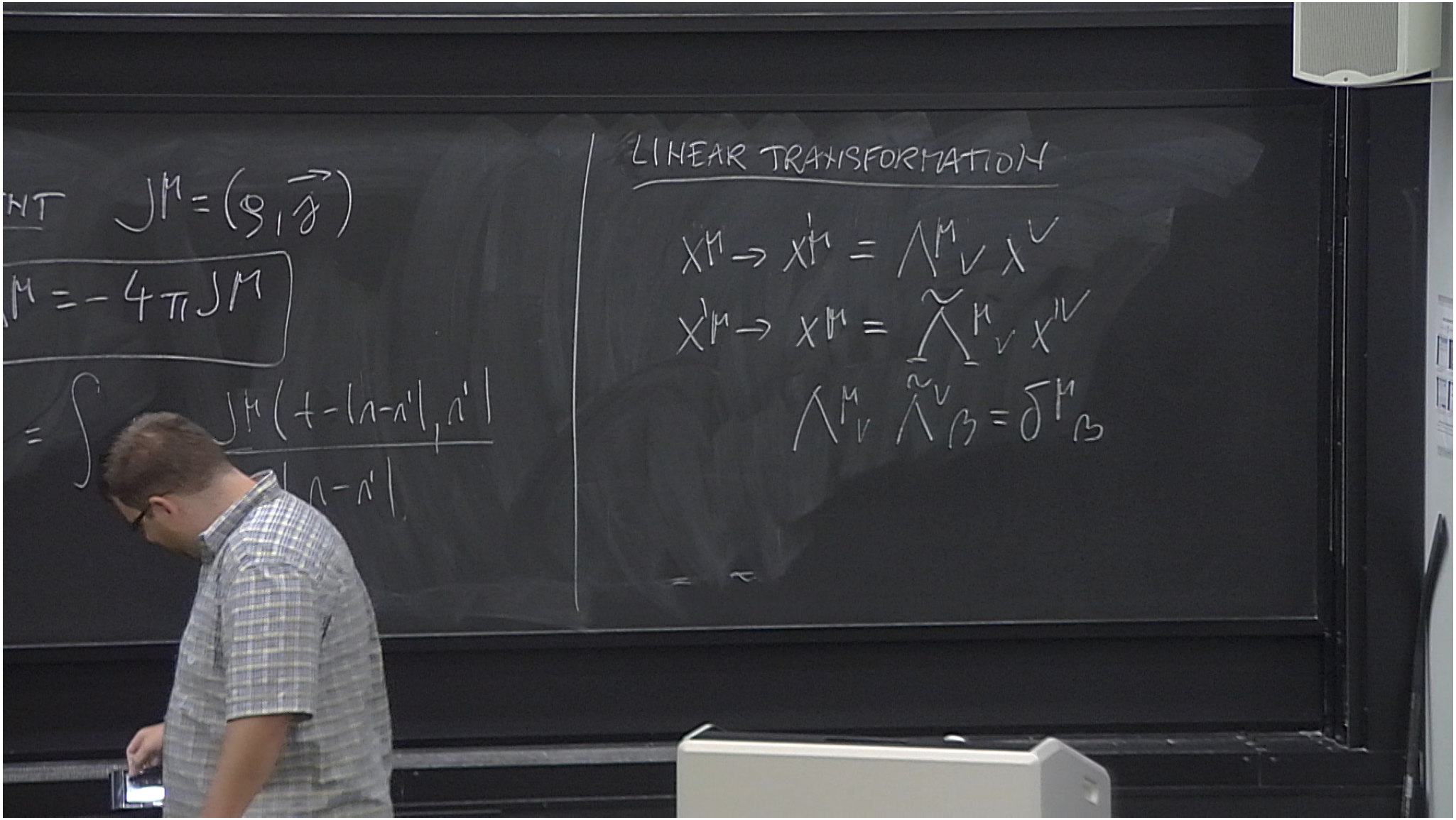
$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \eta^{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

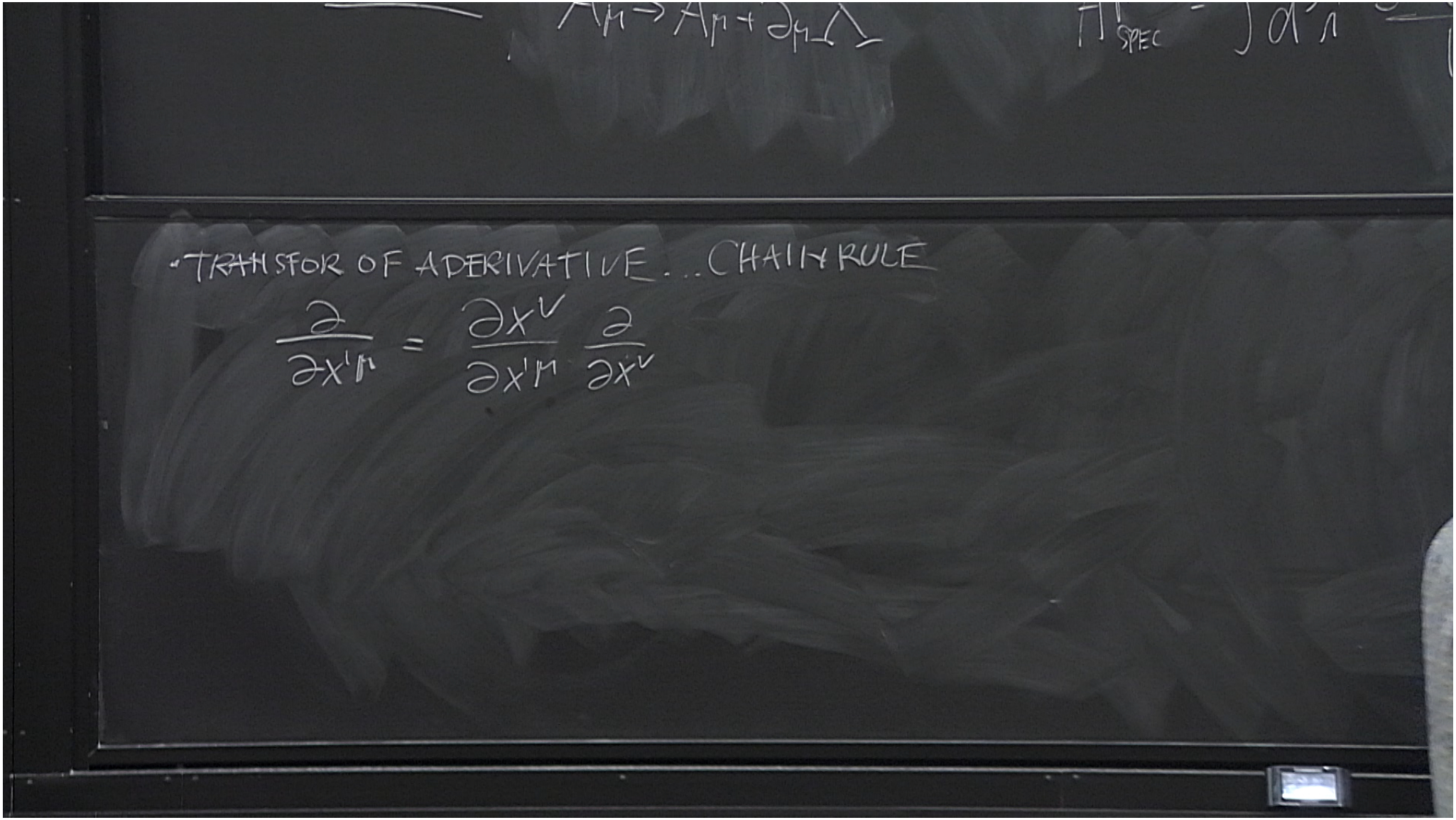
$\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^\gamma_\alpha$

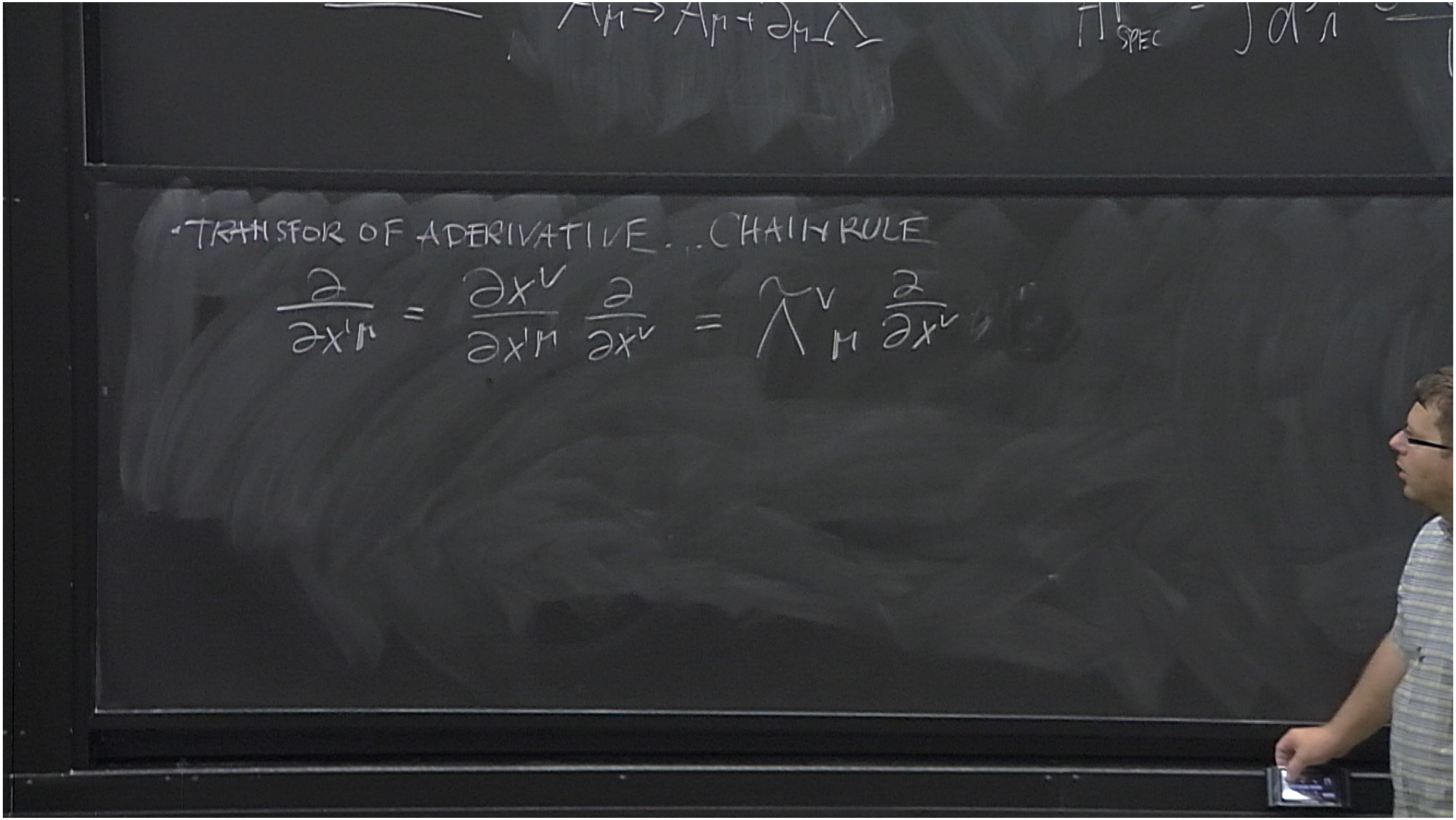

USE TO RAISE OR LOWER INDICES.
$V_\mu = \eta_{\mu\nu} V^\nu$, $\omega^\mu = \eta^{\mu\nu} \omega_\nu$
DUMMY INDEX
$V^\mu = (V^0, V^1) \Rightarrow V_\mu = (-V^0, V^1)$
$\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^{12}} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \eta_{\alpha\beta}\partial^\alpha\partial^\beta$


MAXWELL'S THEORY IN RELATIVISTIC NOTATION
4-POTENTIAL $A^\mu = (\phi, \vec{A})$
LORENZ: $\partial_\mu A^\mu = 0$
GAUGE TR. $A_\mu \to A_\mu + \partial_\mu \Lambda$


ISTIC NOTATION
M = (Φ, A)
0
A_M + ∂_M Λ
4-CURRENT J^M = (ρ, j)
□A^M = −4π J^M
A^M_SPEC = ∫ d³x' J^M(t − |x − x'|, x') / |x − x'|


LINEAR TRANSFORMATION


$A_\mu \to A_\mu + \partial_\mu \Lambda$
$\Pi^{ij}_{SPEC} = \int d^3 n$
• TRANSFOR OF A DERIVATIVE ... CHAIN RULE
$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}$


$A_\mu \to A_\mu + \partial_\mu \Lambda$
Π^{SPEC}
$\int d^n$
TRANSFOR OF A DERIVATIVE... CHAIN RULE
$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \Lambda^\nu_\mu \frac{\partial}{\partial x^\nu}$

• TRANSFOR OF A DERIVATIVE ... CHAIN RULE

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \tilde{\Lambda}^\nu{}_\mu \frac{\partial}{\partial x^\nu}$$

• SAME FORM

$$\partial_\mu A^\mu = 0 = \partial'_\mu A'^\mu$$

$$\boxed{A^{\mu'} = \Lambda^\mu{}_\nu A^\nu}$$

A CO-VECTOR (A 1-FORM)

$\omega_\mu = \tilde{\Lambda}^{\nu}{}_{\mu}\,\omega_\nu$ ... AS DERIVATIVE

A VECTOR $V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu}$ ... AS POTENTIAL.

SCALAR $S(x^\mu) = S(x'^\mu)$

TENSOR ... E.G. $T'^{\mu\nu}{}_{\alpha} = \Lambda^{\mu}{}_{\hat{\mu}}\,\Lambda^{\nu}{}_{\hat{\nu}}\,\tilde{\Lambda}^{\hat{\alpha}}{}_{\alpha}\,T^{\hat{\mu}\hat{\nu}}{}_{\hat{\alpha}}$

1)

$V^\mu = (V^0, \vec{V}) \Rightarrow V_\mu = (-V^0, \vec{V})$

$\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^{12}} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \eta_{\alpha\beta}\partial^\alpha\partial$

TENSOR $F_{\alpha\beta}$

WANT $\Box$: PRESERVES FORM.

$$\Box = \eta_{\alpha\beta}\partial^\alpha\partial^\beta = \eta'_{\alpha\beta}\partial'^\alpha\partial'^\beta = \eta_{\alpha\beta}\partial'^\alpha\partial'^\beta$$

$$= \eta_{\alpha\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta\partial^\gamma\partial^\delta$$

$$\eta_{\gamma\delta}$$

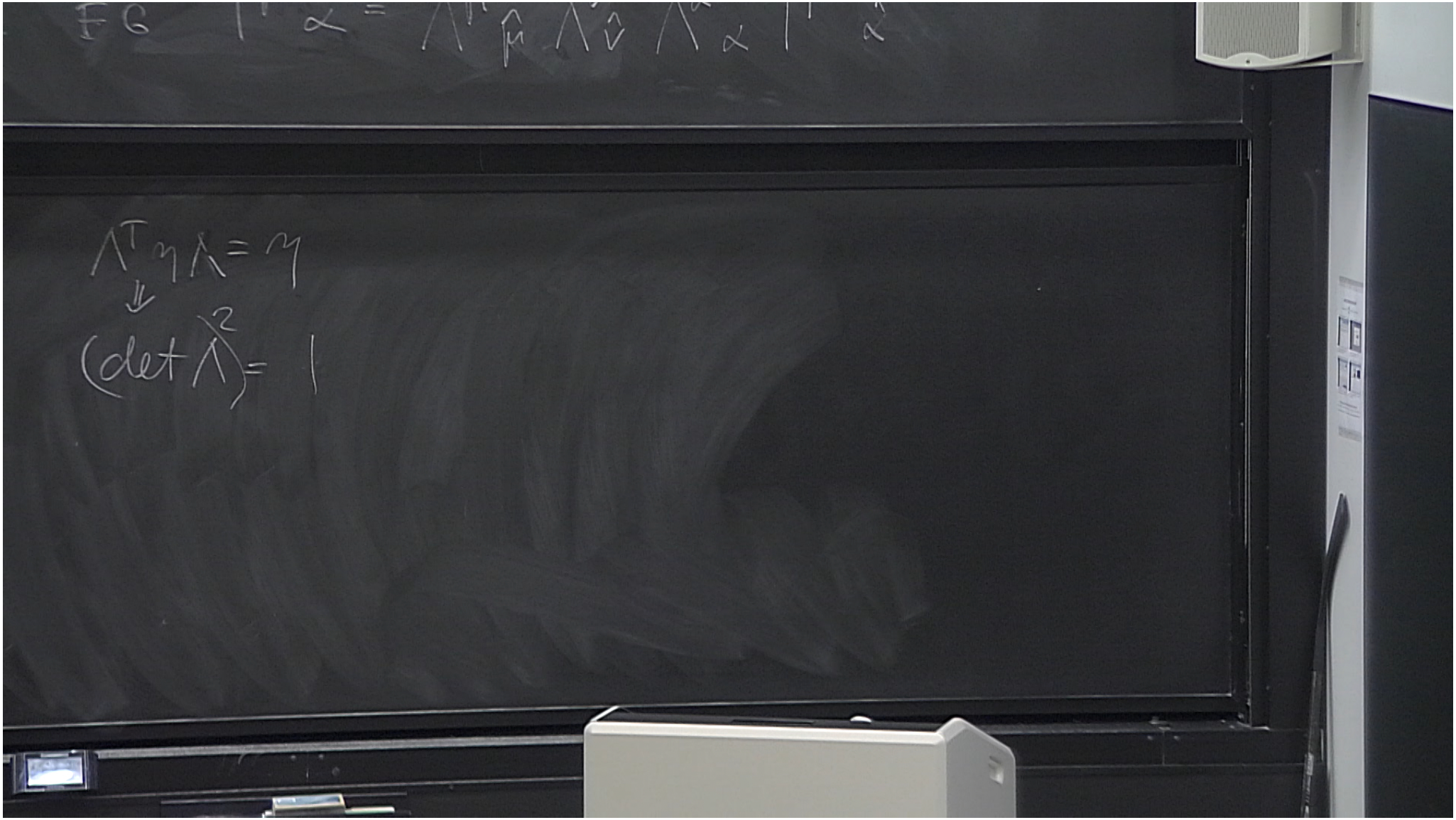
TENSOR F G

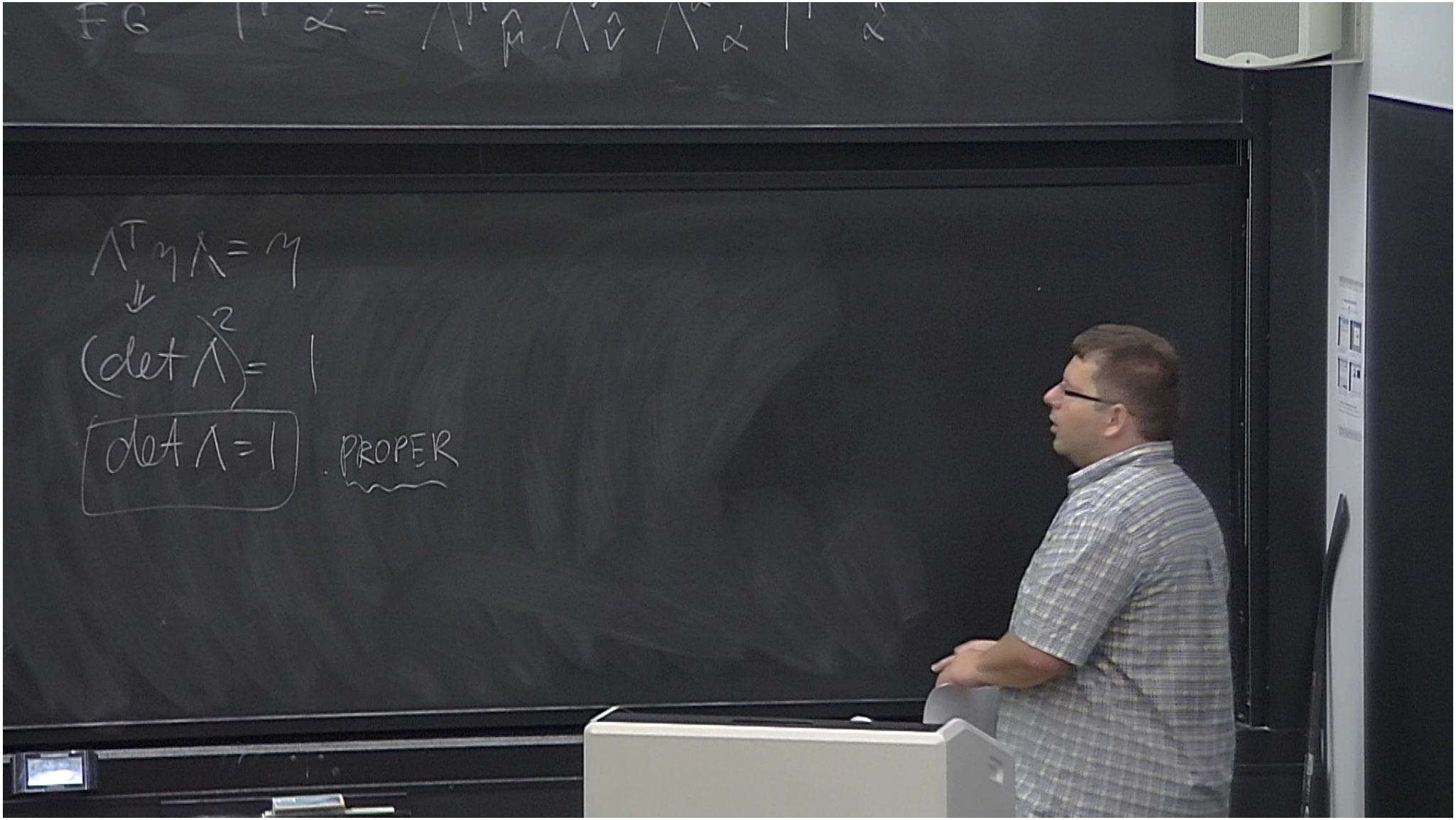
WANT $\Box$: PRESERVES FORM.

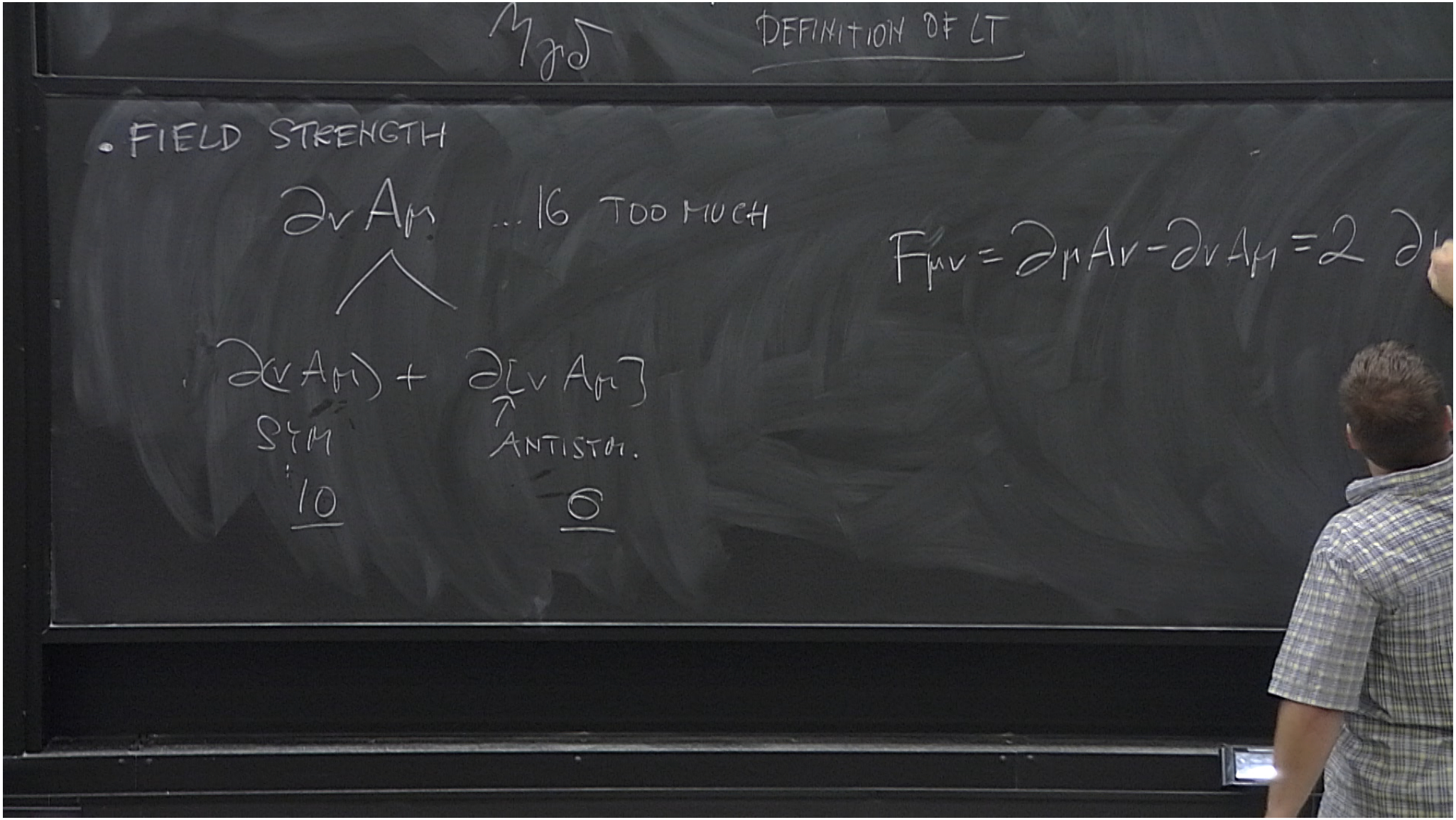
$$\Box = \eta_{\alpha\beta}\partial^\alpha\partial^\beta = \eta'_{\alpha\beta}\partial'^\alpha\partial'^\beta = \eta_{\alpha\beta}\partial'^\alpha\partial'^\beta$$

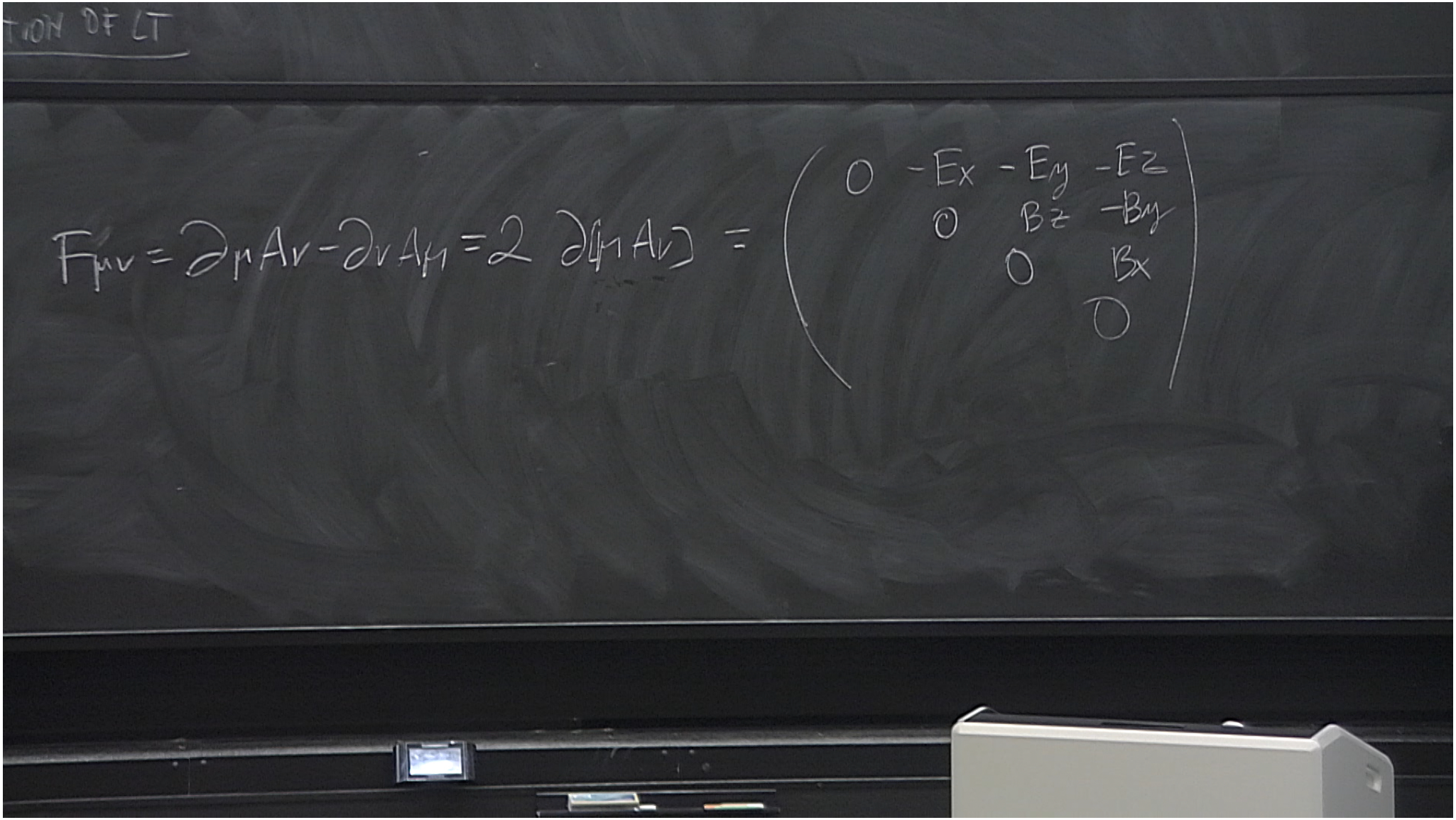
$$= \eta_{\alpha\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta\partial^\gamma\partial^\delta$$

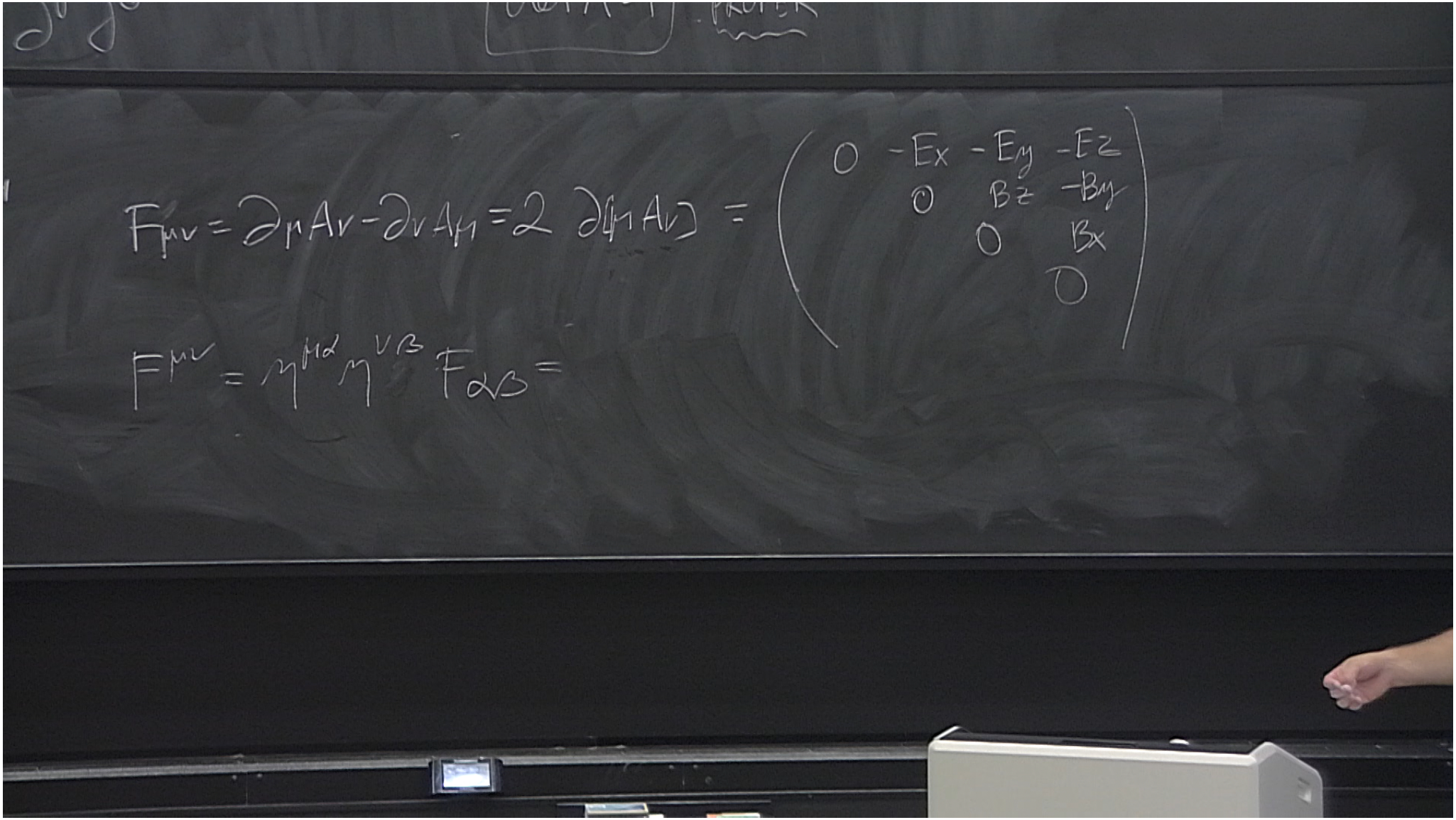
$\eta_{\gamma\delta}$ DEFINITION OF LT


$\Lambda^T \eta \Lambda = \eta$
$(\det \Lambda)^2 = 1$


$\Lambda^T \eta \Lambda = \eta$
$(\det \Lambda)^2 = 1$
$\det \Lambda = 1$
PROPER


$M_{\mu\nu}$
DEFINITION OF LT
• FIELD STRENGTH
$\partial_\nu A_\mu$ ... 16 TOO MUCH
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\,\partial$
$\partial_{(\nu} A_{\mu)} + \partial_{[\nu} A_{\mu]}$
SYM
ANTISYM.
10
6


TION OF LT
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\,\partial_{[\mu} A_{\nu]} =$
$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & B_z & -B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$

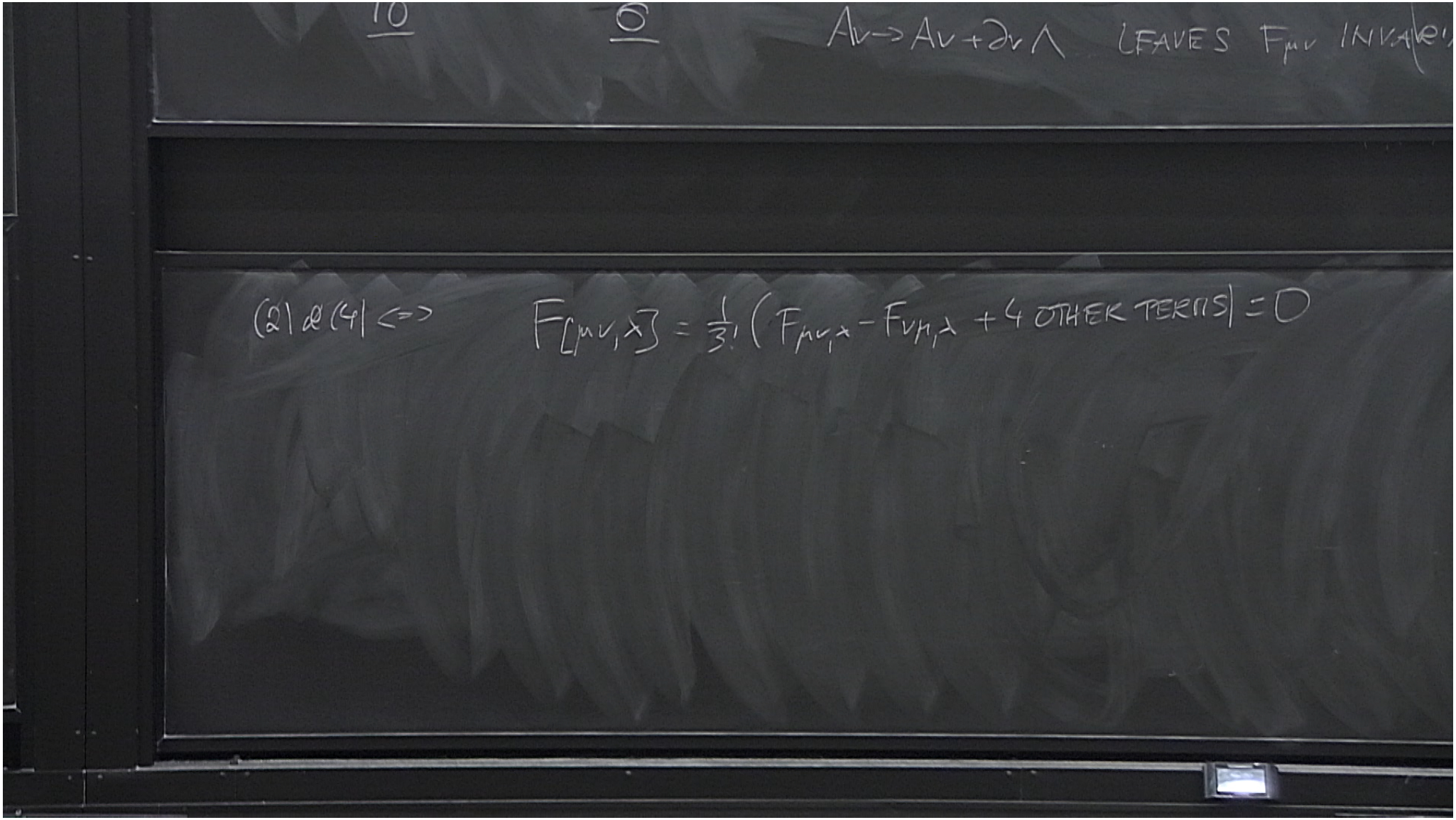

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\,\partial_{[\mu} A_{\nu]} =$
$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & B_z & -B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$
$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta} =$

MUCH

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\,\partial_{[\mu} A_{\nu]} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & B_z & -B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ & 0 & B_z & -B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

$A_\nu \to A_\nu + \partial_\nu \Lambda$ LEAVES $F_{\mu\nu}$ INVARIANT


10
6
$A_\nu \to A_\nu + \partial_\nu \Lambda$ LEAVES $F_{\mu\nu}$ INVARI
(2) & (4) $\Leftrightarrow$ $F_{[\mu\nu,\lambda]} = \frac{1}{3!}(F_{\mu\nu,\lambda} - F_{\nu\mu,\lambda}$ + 4 OTHER TERMS) = 0

10 6 $A_\nu \to A_\nu + \partial_\nu \Lambda$ LEAVES $F_{\mu\nu}$ INVARIA

(2) & (4) $\Leftrightarrow$ $F_{[\mu\nu,\lambda]} = \frac{1}{3!}(F_{\mu\nu,\lambda} - F_{\nu\mu,\lambda} + 4 \text{ OTHER TERMS}) = 0$

$F_{[\mu\nu,\lambda\sigma]} = F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0$

10 6 $A_\nu \to A_\nu + \partial_\nu \Lambda$ LEAVES $F_{\mu\nu}$ INVARIANT

(2) & (4) $\Leftrightarrow$ $F_{[\mu\nu,\lambda]} = \frac{1}{3!}(F_{\mu\nu,\lambda} - F_{\nu\mu,\lambda} + 4 \text{ OTHER TERMS}) = 0$

$$\boxed{0 = F_{[\mu\nu,\lambda]}} = F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0$$

(1) & (3)

$$\boxed{\partial_\mu F^{\mu\nu} = -4\pi J^\nu}$$

LORENTZ.

$$\boxed{\frac{dp^\mu}{d\tau} = e F^{\mu\nu} u_\nu}$$