

Title: PSI 2018/2019 - Classical Physics - Lecture 5

Date: Aug 20, 2018 09:00 AM

URL: <http://pirsa.org/18080013>

Abstract:

f) CONSTRAINTS

• LET NOT ALL q_i, p_i ARE INDEPENDENT

$\Rightarrow$ AT ANY INSTANT OF TIME WE HAVE

A CONSTRAINT

f) CONSTRAINTS

• LET NOT ALL q^i, p_i ARE INDEPENDENT

$\Rightarrow$ AT ANY INSTANT OF TIME WE HAVE

A CONSTRAINT

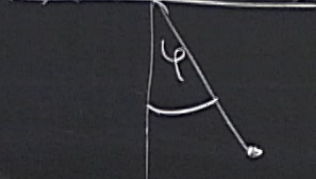
$$\Phi(q^i, p_i) = 0$$

CONSERVED QUANTITY. SYMMETRY OF
PHYSICAL SYSTEM.

PHYSICAL SYSTEM.

STANDARD EXAMPLE: PENDULUM

OCCAM'S DESCRIPTION



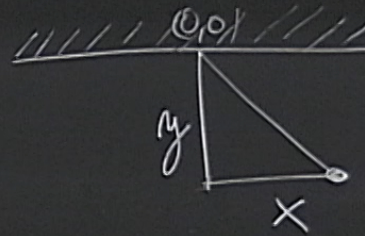
$n=1$ DOF

PHASE SPACE (φ, p_φ)

NO CONSTRAINT

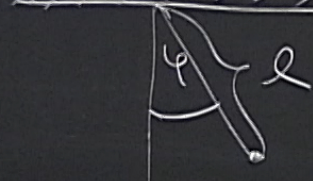
$(\Rightarrow)$

CONSTRAINT DESCRIPTION



STANDARD EXAMPLE: PENDULUM

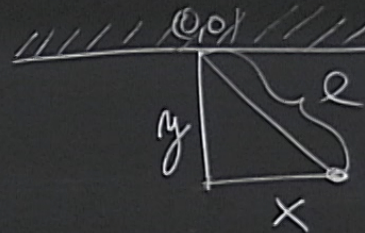
OCCAM'S DESCRIPTION



- $m=1$ DOF
- PHASE SPACE (ϕ, p_ϕ)
- NO CONSTRAINT

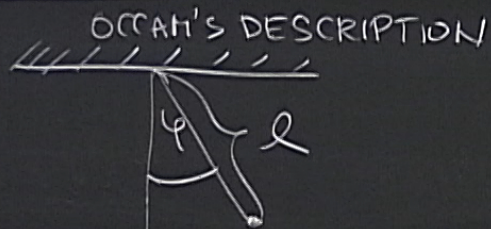
$(\Rightarrow)$

CONSTRAINT DESCRIPTION



- $m=2$ DOF
- PHASE SPACE (x, p_x, y, p_y)
- CONSTRAINT $x^2 + y^2 = l^2$

STANDARD EXAMPLE: PENDULUM

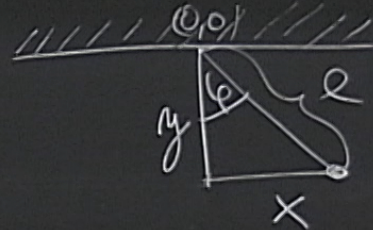


- $n=1$ DOF
- PHASE SPACE (ϕ, p_ϕ)
- NO CONSTRAINT

$(\Rightarrow)$

ELIMINATE
CONSTRAINT
 $\leftarrow$
 $x = l \sin \phi$
 $y = l \cos \phi$

CONSTRAINT DESCRIPTION



- $n=2$ DOF
- PHASE SPACE (x, p_x, y, p_y)
- CONSTRAINT $x^2 + y^2 = l^2$

• LET NOT ALL q_i, p_i ARE INDEPENDENT

⇒ AT ANY INSTANT OF TIME WE HAVE

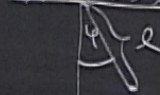
A CONSTRAINT

$$\Phi(q_i, p_i) = 0$$

CONSERVED QUANTITY. SYMMETRY OF PHYSICAL SYSTEM.

STANDARD EXAMPLE: PENDULUM

OTHER'S DESCRIPTION



• $m=1$ DOF

• PHASE SPACE (ϕ, p_ϕ)

NO CONSTRAINT

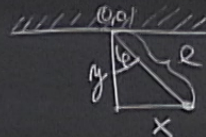
(=)

ELIMINATE
CONSTRAINT

$$x = l \sin \phi$$

$$y = l \cos \phi$$

CONSTRAINT DESCRIPTION



• $m=2$ DOF

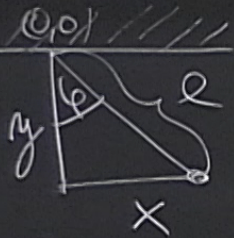
• PHASE SPACE (x, p_x, y, p_y)

• CONSTRAINT $x^2 + y^2 = l^2$

ADVANTAGE OF MORE COMPLICATED DESCRIPTION.

• EXTRA SYMMETRY

CONSTRAINT DESCRIPTION



• $m=2$ DOF

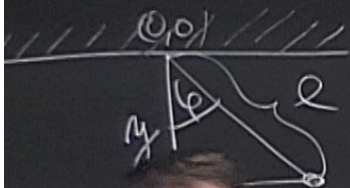
• PHASE SPACE (x, p_x, y, p_y)

• CONSTRAINT $x^2 + y^2 = l^2$

ADVANTAGE OF MORE COMPLICATED DESCRIPTION.

- EXTRA SYMMETRY
- CONSTRAINT FORCES (EG. STRING TENSION)

CONSTRAINT DESCRIPTION



ADVANTAGE OF MORE COMPLICATED DESCRIPTION.

- EXTRA SYMMETRY
- CONSTRAINT FORCES (EG. STRING TENSION)

$$(x, p_x, y, p_y)$$

$$x^2 + y^2 = r^2$$

• TO DEAL WITH SYMMETRIES .. METHOD OF LAGRANGE MULTIPLIERS

... ϕ_j ... $j=1, \dots, m$ SYMMETRIES

INSTEAD OF H CONSIDER TOTAL HAMILTONIAN

$$H_T = H + \sum_j \lambda_j(t) \phi_j$$

↑
LAGRANGE MULTIPLIERS
(ARBITRARY FUNCTIONS OF TIME)

• TO DEAL WITH SYMMETRIES .. METHOD OF LAGRANGE MULTIPLIERS

... ϕ_J ... $J=1, \dots, m$ SYMMETRIES

INSTEAD OF H CONSIDER TOTAL HAMILTONIAN

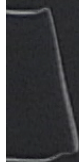
$$H_T = H + \sum_J \lambda_J(t) \phi_J$$

↑
LAGRANGE MULTIPLIERS
(ARBITRARY FUNCTIONS OF TIME)

ENLARGE MULTIPLIERS

ES

TONIAN



MULTIPLIERS

(FUNCTIONS OF TIME)

WE HAVE 2 POSSIBILITIES

POS 1: ENLARGE PHASE SPACE BY INCLUDING $\lambda(t)$
 $(N+m)$ DOF

CHANGE MULTIPLIERS

ES

LTOTAL

ERS
(TIME)

WE HAVE 2 POSSIBILITIES

POS 1: ENLARGE PHASE SPACE BY INCLUDING $\lambda_j(t)$

$(N+m)$ DOF $\rightarrow (n+m)(E-L)$ EQS

$$L_T = L - \lambda_j \phi_j$$

ENLARGE MULTIPLIERS

S

TONIAN

MULTIPLIERS
(FUNCTIONS OF TIME)

WE HAVE 2 POSSIBILITIES

POS 1: ENLARGE PHASE SPACE BY INCLUDING $\lambda(t)$

$(n+m)$ DOF $\rightarrow (n+m)(E-L)$ EQS.

$$L_T = L - \lambda \phi$$

$$\delta S = \int (\delta L - \lambda \delta \phi - \phi \delta \lambda) dt$$

ANCE MULTIPLIERS

S

TONIAN

MULTIPLIERS
(FUNCTIONS OF TIME)

WE HAVE 2 POSSIBILITIES

POS 1: ENLARGE PHASE SPACE BY INCLUDING $\lambda_j(t)$

$(n+m)$ DOF $\rightarrow (n+m)$ (E-L) EQS.

$$L_T = L - \lambda_j \phi_j, \quad \phi_j = \phi_j(q)$$

$$\delta S = \int (\delta L - \lambda_j \delta \phi_j - \phi_j \delta \lambda_j) dt$$

ANCE MULTIPLIERS

S

TONIAN

MULTIPLIERS
(FUNCTIONS OF TIME)

WE HAVE 2 POSSIBILITIES

POS 1: ENLARGE PHASE SPACE BY INCLUDING $\lambda_j(t)$

$(n+m)$ DOF $\rightarrow (n+m)$ (E-L) EQS

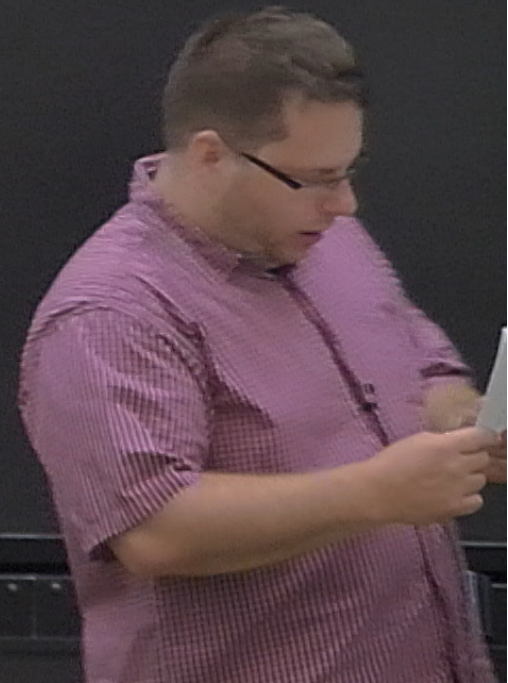
$$L_T = L - \lambda_j \phi_j, \quad \phi_j = \phi_j(q)$$

$$\delta S = \int (\delta L - \lambda_j \delta \phi_j - \phi_j \delta \lambda_j) dt$$

$$= \int \left(\left(\frac{\partial L_T}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L_T}{\partial \dot{q}^i} \right) \right) \delta q^i - \phi_j \delta \lambda_j \right) dt$$

$$(E-L): \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = 0, \quad \Phi_J = 0 \right]$$

• CONSTRAINTS ARE RECOVERED AS $(E-L)$.



(E-L): $\boxed{\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = 0, \quad \Phi_J = 0} \quad (\Leftrightarrow)$

- CONSTRAINTS ARE RECOVERED AS (E-L)
- CAN SOLVE FOR q^i, λ_J
- CALCULATE CONSTRAINT FORCES

$$\left(\frac{\partial L_T}{\partial \dot{q}^i} \right) = 0, \quad \Phi_J = 0 \quad \Leftrightarrow \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i} - \lambda_J \frac{\partial \Phi}{\partial q^i}$$

RECOVERED AS $(E-L)$.

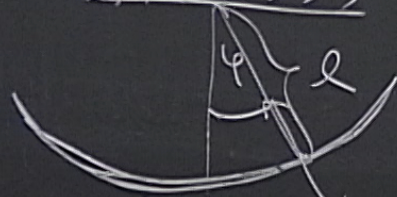
q^i, λ_J

CONSTRAIN FORCES

$$\Leftrightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i} - \underbrace{\lambda \frac{\partial \phi}{\partial q^i}}_{\substack{\text{CONSTRAINT FORCE} \\ \text{(PERPENDICULAR TO CONSTRAINT SURFACE)}}}$$

STANDARD EXAMPLE: PENDULUM

ODDAN'S DESCRIPTION



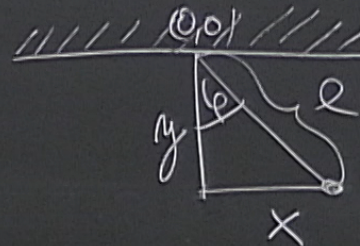
- $n=1$ DOF
- PHASE SPACE (φ, p_φ)
- NO CONSTRAINT

$(=)$

ELIMINATE
CONSTRAINT

$$\begin{aligned} x &= l \cos \varphi \\ y &= l \sin \varphi \end{aligned}$$

CONSTRAINT DESCRIPTION



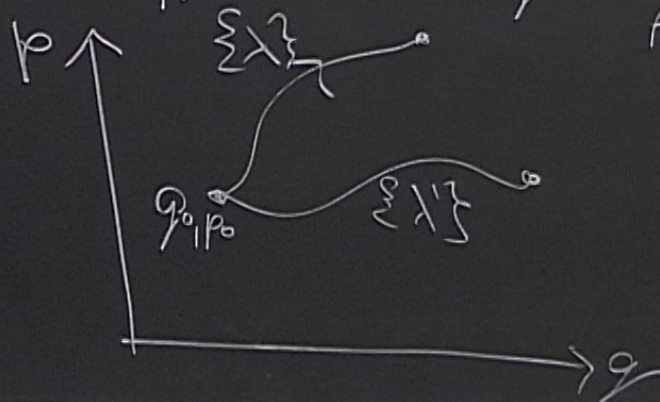
- $n=2$ DOF
- PHASE SPACE (x, p_x, y, p_y)
- CONSTRAINT $x^2 + y^2 = l^2$

ADVANTAGE

POSSIBILITY 2:

• LET'S KEEP $\lambda_j = \lambda_j(H)$ ARBITRARY.

$\Rightarrow$ $n(E-L) \dots q^i$... CONTAIN m
ARBITRARY FUNCTIONS λ_j

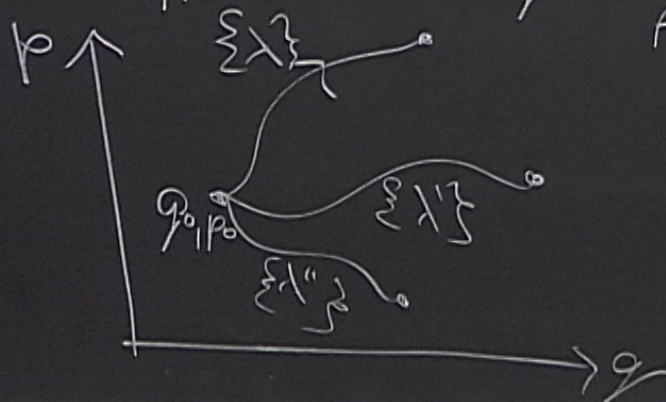


POSSIBILITY 2:

• LET'S KEEP $\lambda_j = \lambda_j(H)$ ARBITRARY.

$\Rightarrow$ $m(E-L) \dots q^i$... CONTAIN m
ARBITRARY FUNCTIONS λ_j

NON-UNIQUE?

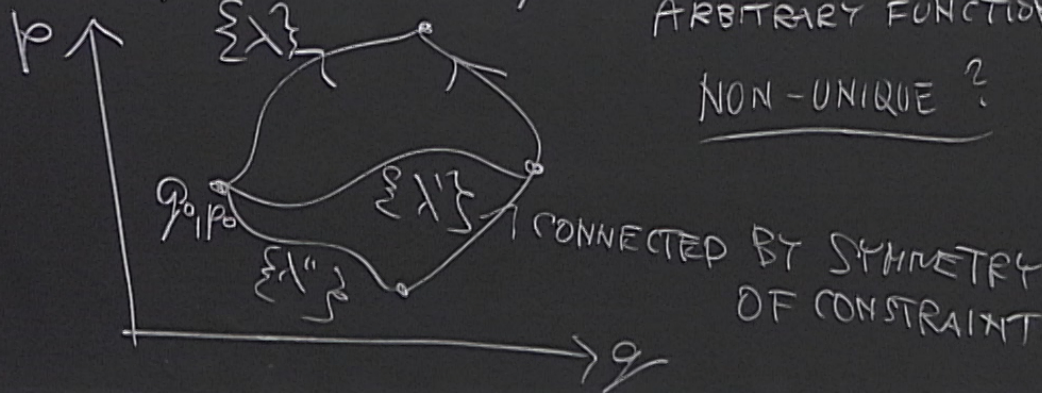


POSSIBILITY 2:

• LET'S KEEP $\lambda_j = \lambda_j(t)$ ARBITRARY.

$\Rightarrow$ $m(E-L) \dots q^i$... CONTAIN m
ARBITRARY FUNCTIONS λ_j

NON-UNIQUE?



"EQUIVALENCE CLASS OF SOLUTIONS"

PREDICTIONS ARE STILL POSSIBLE,
(RELATIONAL QUESTIONS)

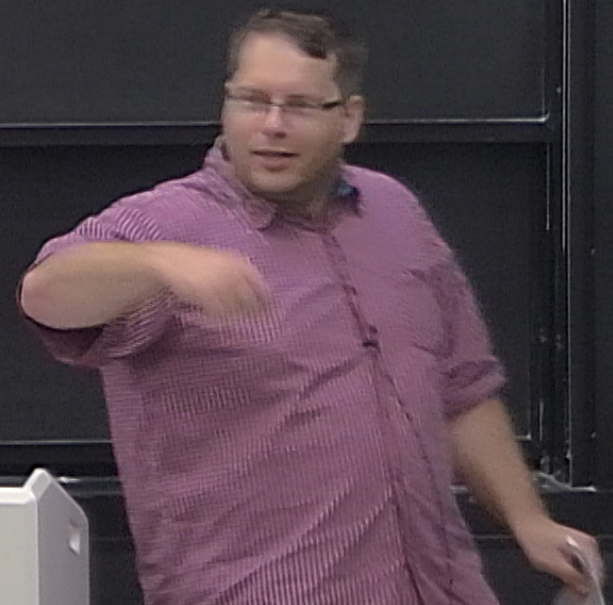
EX. TOY MODEL:

MOTION OF FREE PARTICLE USING
"LOOSY CLOCK"

CTIONS λ

?

TRY
AINT.



"EQUIVALENCE CLASS OF SOLUTIONS"

PREDICTIONS ARE STILL POSSIBLE,
(RELATIONAL QUESTIONS)

EX: TOY MODEL:

MOTION OF FREE PARTICLE USING
"LOOSE CLOCK"

$$\boxed{H=0} \quad \Phi \quad \text{FIND}$$

$H_T = H + \lambda \Phi$... DEFINES
EVOLUTION.
CAN WE TAKE PREDICTIONS?
CAN WE AGREE ON THEM?

OF CONSTRAINT.

TYPICAL ORIGIN OF SOME CONSTRAINTS

LEGENDRE TRANSFORM FROM $L \rightarrow H$,

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \quad \dots \quad \text{MUST BE INVERTABLE}$$
$$\dot{q}^i = \dot{q}^i(p_i, q, t)$$

TYPICAL ORIGIN OF SOME CONSTRAINTS

LEGENDRE TRANSFORM FROM $L \rightarrow H$,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

MUST BE INVERTABLE

$$\dot{q}^i = \dot{q}_i(p_i, q, t)$$

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

IF DEGENERATE

$\Rightarrow$ NOT INVERTABLE.

ZERO EIGENVALUES $\rightarrow$ # OF CONSTRAINTS.

$\dots \phi_J \dots J=1, \dots, m$ SYMMETRIES

III) CLASSICAL FIELD THEORY

DISCRETE SYSTEM

a) VIBRATIONS OF STRING

• TO DEAL WITH SYMMETRIES .. METHOD OF LAGRANGE MULTIPLIERS
 $\dots \phi_J \dots J=1, \dots, m$ SYMMETRIES

WE HAVE 2 POSSIBILITIES

POS 1: ENLARGE PHASE SPACE

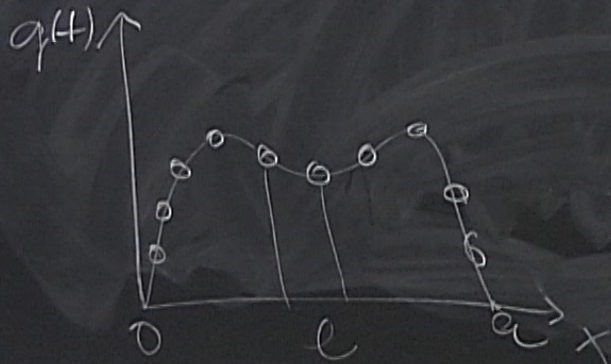
III) CLASSICAL FIELD THEORY

DISCRETE SYSTEM

CONTINUOUS

a) VIBRATIONS OF STRING

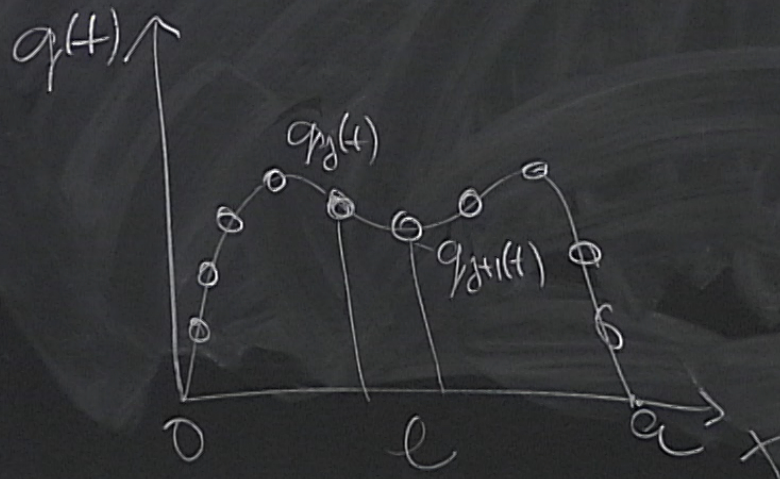
N BEADS, SEPARATED BY $\underline{L}$



... ϕ_j ... $j=1, \dots, m$ SYMMETRIES

III) CLASSICAL FIELD THEORY

a) VIBRATIONS OF STRING



DISCRETE SYSTEM

N BEADS, SEPARATED BY $\underline{\ell}$

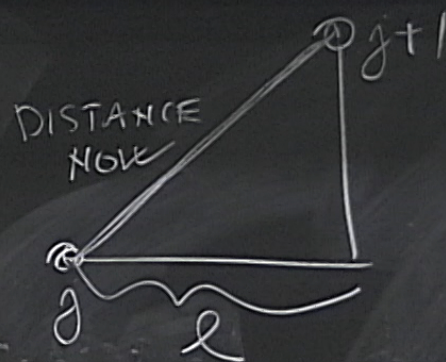
LABEL BEADS BY POSITION

$q_j(t)$ DISPLACEMENT OF j -TH BEAD

$$T_K = \frac{1}{2} \sum_{j=1}^N m \dot{q}_j^2$$

$$V = \sum_{j=1}^N T(\Delta l)_j$$

↑
TENSION

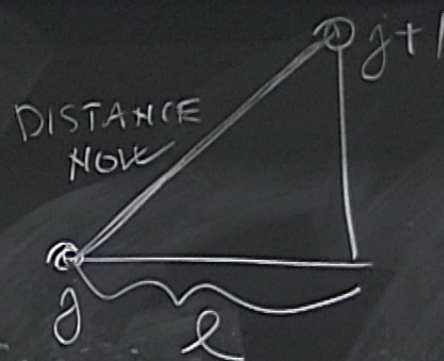


$$\begin{aligned} \Delta l &= \sqrt{l^2 + (q_{j+1} - q_j)^2} - l \\ &= \frac{1}{2} l \left(\frac{q_{j+1} - q_j}{l} \right)^2 \end{aligned}$$

$$T_K = \frac{1}{2} \sum_{j=1}^N m \dot{q}_j^2$$

$$V = \sum_{j=1}^N T (\Delta l)_j$$

↑
TENSION



$$(\Delta l)_j = \sqrt{l^2 + (q_{j+1} - q_j)^2} - l$$

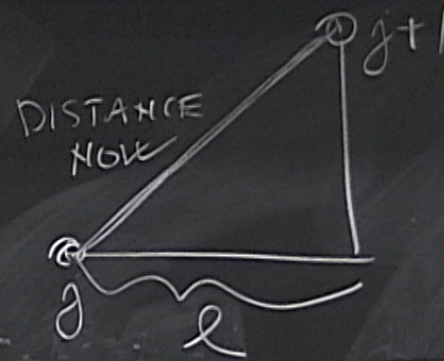
$$= \frac{1}{2} l \left(\frac{q_{j+1} - q_j}{l} \right)^2$$

$$T_K = \frac{1}{2} \sum_{j=1}^N m \dot{q}_j^2$$

$$V = \sum_{j=1}^N T (\Delta l)_j$$

↑
TENSION

$$= \frac{1}{2} \sum_j \ell T \left(\frac{q_{j+1} - q_j}{\ell} \right)^2$$



$$(\Delta l)_j = \sqrt{\ell^2 + (q_{j+1} - q_j)^2} - \ell$$

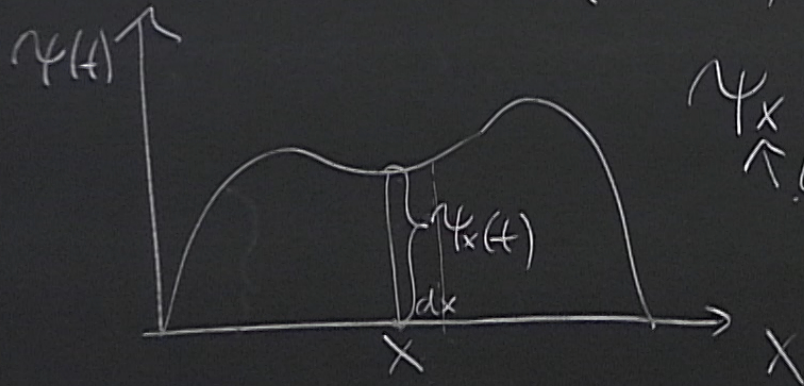
$$= \frac{1}{2} \ell \left(\frac{q_{j+1} - q_j}{\ell} \right)^2$$

PLIERS

WE HAVE 2 POSSIBILITIES.

POS 1: ENLARGE PHASE SPACE BY INCLUDING $\lambda(t)$

CONTINUOUS ($N \rightarrow \infty, R \rightarrow \infty$)



$\psi_x(t)$

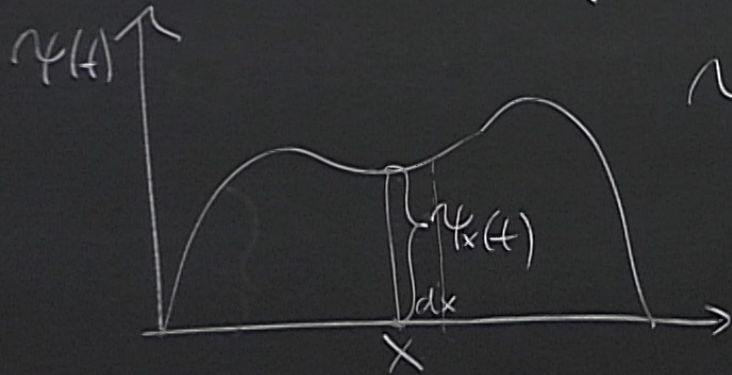
↑ LABEL WHERE I AM (Δ)

PLIERS

WE HAVE 2 POSSIBILITIES

POS 1: ENLARGE PHASE SPACE BY INCLUDING $\lambda(t)$

CONTINUOUS ($N \rightarrow \infty, Q \rightarrow \infty$)



$$\psi_x(t) = \psi(x, t) \dots \text{FIELD}$$

↑ LABEL WHERE I AM (Δ)

(FIELD = PHYSICAL ENTITY
THAT DEPENDS ON SPACE & TIME)

$(M_1 + m)$ DOF $\rightarrow (M + m)(E - () E)$

$$T_K = \frac{1}{2} \sum_{\nu=1}^N \ell \mu \dot{\psi}^2$$

$$m = \mu \ell$$

$$(M_1 + M_2) \text{ DOF} \rightarrow (M_1 + M_2) (E - \frac{1}{2} E_{\text{CM}})$$

$$T_K = \frac{1}{2} \sum_{j=1}^N \ell \mu \dot{\psi}_j^2$$

$$m = \mu \ell$$

$$\lim_{\substack{N \rightarrow \infty \\ \ell \rightarrow 0}} \rightarrow \frac{1}{2} \int dx \mu \dot{\psi}(x,t)^2$$

$$V = \lim_{\substack{N \rightarrow \infty \\ \ell \rightarrow 0}} \frac{1}{2} \sum \ell T \left(\frac{\psi_x - \psi_{x+\ell}}{\ell} \right)^2$$

$(M+m)$ DOF $\rightarrow (M+m)(E-L)$ FOS

$$T_K = \frac{1}{2} \sum_{j=1}^N \ell \mu \dot{\psi}_j^2$$

$$m = \mu \ell$$

$$\lim_{\substack{N \rightarrow \infty \\ \ell \rightarrow 0}} \rightarrow \frac{1}{2} \int dx \mu \dot{\psi}(x,t)^2$$

$$V = \lim_{\substack{N \rightarrow \infty \\ \ell \rightarrow 0}} \frac{1}{2} \sum \ell T \left(\frac{\partial \psi}{\partial x} \right)^2 = \frac{1}{2} \int dx T (\psi_x)^2$$

$$T_K = \frac{1}{2} \left(\sum_{j=1}^N \ell \right) \mu \dot{\psi}^2$$

$$m = \mu \ell$$

$$\lim_{\substack{N \rightarrow \infty \\ \ell \rightarrow 0}} \rightarrow \frac{1}{2} \int dx \mu \dot{\psi}^2(x,t)$$

$$V = \lim_{\substack{N \rightarrow \infty \\ \ell \rightarrow 0}} \frac{1}{2} \left(\sum \ell \right) T \left(\frac{\partial \psi}{\partial x} \right)^2 = \frac{1}{2} \int dx T (\psi_{,x})^2$$

$$L = T_K - V = \frac{1}{2} \int dx (\mu \dot{\psi}^2 - T (\psi_{,x})^2)$$

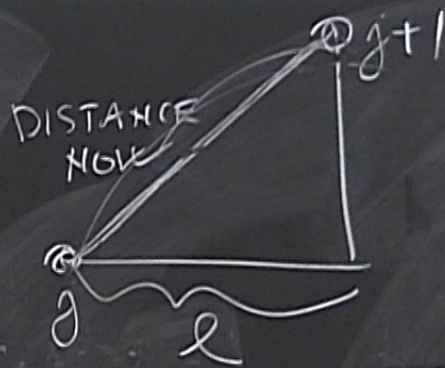
$$T_K = \frac{1}{2} \sum_{j=1}^N m \dot{q}_j^2$$

$$V = \sum_{j=1}^N T (\Delta l)_j$$

↑
TENSION

$$= \frac{1}{2} \sum_j \ell T \left(\frac{q_{j+1} - q_j}{\ell} \right)^2$$

$$L = T_K - V$$



$$(\Delta l)_j = \sqrt{\ell^2 + (q_{j+1} - q_j)^2} - \ell$$

$$= \frac{1}{2} \ell \left(\frac{q_{j+1} - q_j}{\ell} \right)^2$$

$$T_K$$

$$\lim_{\substack{N \rightarrow \infty \\ \ell \rightarrow 0}}$$

$$V$$

$$L$$

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{\frac{1}{2} (m \dot{\psi}^2 - T (\psi_x)^2)}_{\mathcal{L}(\psi, \dot{\psi}, \psi_x, t, x)}$$

LAGRANGIAN DENSITY

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{\frac{1}{2} (m \dot{\psi}^2 - T (\psi_x)^2)}_{\mathcal{L}(\psi, \dot{\psi}, \psi_x, t, x)} = \int dt dx \mathcal{L}$$

LAGRANGIAN DENSITY

ACTION PRINCIPLE: $\delta S = 0 = \int dt \int dx \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta \dot{\psi} + \frac{\partial \mathcal{L}}{\partial \psi_x} \delta \psi_x \right]$

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{\frac{1}{2} (m \dot{\psi}^2 - T (\psi_x)^2)}_{\mathcal{L}(\psi, \dot{\psi}, \psi_x, t, x)} = \int dt dx \mathcal{L}$$

LAGRANGIAN DENSITY

ACTION PRINCIPLE: $\delta S = 0 = \int dt \int dx \left[\underbrace{\frac{\partial \mathcal{L}}{\partial \psi}}_{\theta} \delta \psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta \dot{\psi} + \frac{\partial \mathcal{L}}{\partial \psi_x} \delta \psi_x \right]$

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{\frac{1}{2} (m \dot{\psi}^2 - T (\psi_x)^2)}_{\mathcal{L}(\psi, \dot{\psi}, \psi_x, t, x)} = \int dt dx \mathcal{L}$$

LAGRANGIAN DENSITY

ACTION PRINCIPLE: $\delta S = 0 = \int dt \int dx \left[\underbrace{\frac{\partial \mathcal{L}}{\partial \psi}}_{\theta} \delta \psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\psi}}}_{p^t = m \dot{\psi}} \delta \dot{\psi} + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_x}}_{p^x = -T \psi_x} \delta \psi_x \right]$

TYPICAL ORIGIN OF SOME CONSTRAINTS

ZERO EIGENVALUES $\rightarrow$ # OF CONS

LEGENDRE TRANSFORM FROM $L \rightarrow H$,

$$= - \int_{t_1}^{t_2} dt \int_0^a dx \left(\dot{p}^t + p_{ix}^x \right) \delta \psi + \int_0^a dx \left[p^t \delta \psi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[p^x \delta \psi \right]_0^a dt$$

TYPICAL ORIGIN OF SOME CONSTRAINTS

ZERO EIGENVALUES $\rightarrow$ # OF CONS

$$= - \int_{t_1}^{t_2} dt \int_0^a dx \left(\dot{p}^t + p_{ix}^x \right) \delta \psi + \int_0^a dx \left[p^t \delta \psi \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[p^x \delta \psi \right]_0^a dt = 0$$

Hence can be

$$= - \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{(\dot{p}^t + p_{ix}^x)}_{=0} \delta\psi + \int_0^a dx [p^t \delta\psi]_{t_1}^{t_2} + \int_{t_1}^{t_2} [p^x \delta\psi]_0^a dt = 0$$

Hence can be

$$= - \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{(\dot{p}^t + p_{ix}^x)}_{=0} \delta\psi + \int_0^a dx [p^t \delta\psi]_{t_1}^{t_2} + \int_{t_1}^{t_2} [p^x \delta\psi]_0^a dt = 0$$

$\psi_{,xx} - \frac{1}{n^2} \ddot{\psi} = 0, \quad n = \sqrt{T/\mu}$

Hence cancel

$$= - \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{(\dot{p}^t + p_{ix}^x)}_{=0} \delta\gamma + \int_0^a dx \underbrace{[p^t \delta\gamma]}_{\text{FIXED END CONDITION}} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} [p^x \delta\gamma]_0^a dt = 0$$

$\gamma_{,xx} - \frac{1}{n^2} \ddot{\gamma} = 0$

$v = \sqrt{T/\mu}$

"INITIAL DATA"

 $\delta\gamma(t_1) = \delta\gamma(t_2) = 0$

$\frac{t=t_1}{t=t_2}$


OF CONSTRAINT

III - (1) (1) FIND

$$+ \int_0^a [p^t \delta \psi]_{t_1}^{t_2} + \int_{t_1}^{t_2} [p^x \delta \psi]_0^a dt = 0$$

FIXED END
CONDITION/INTIAL

FIXED END IN SPACE

BOUNDARY DATA

"INITIAL DATA"

$$\delta \psi(t_1) = \delta \psi(t_2) = 0$$

$$\begin{array}{c} t=t_1 \\ \text{---} \\ t=t_2 \end{array}$$

2 POSSIBILITIES TO MAKE
[p^x \delta \psi]_0^a VANISH

i) DIRICHLET $\delta \psi|_0 = 0 = \delta \psi|_a$

$$\dot{\psi}|_0 = 0 = \dot{\psi}|_a$$

ii) NEUMANN

$$p^x|_0 = 0 = p^x|_a$$

GENERATE
INVERTABLE,