

Title: PSI 2018/2019 - Classical Physics - Lecture 4

Date: Aug 17, 2018 09:00 AM

URL: <http://pirsa.org/18080012>

Abstract:

YESTERDAY CAN TRANSF GENERATED BY $F=F(q, Q, t)$

$$\Rightarrow \left[\begin{array}{l} p_j = \frac{\partial F}{\partial q_j}, \quad P_j = -\frac{\partial F}{\partial Q_j}, \quad H' = H + \frac{\partial F}{\partial t} \\ \frac{\partial H'}{\partial P_j} = \dot{Q}_j, \quad \frac{\partial H'}{\partial Q_j} = -\dot{P}_j \end{array} \right]$$

HAMILTON-JACOBI INGENIOUS IDEA

$$\boxed{H' = 0}$$

GENERATED BY $F=S$

• CAN EQS: $Q\dot{q} = \alpha\dot{q} = \text{CONST}$, $P_j = -\beta_j = \text{CONST}$.

• $P_j = -\frac{\partial S}{\partial Q_j} \Leftrightarrow \beta_j = \frac{\partial S(q^i, \alpha^i, t)}{\partial \alpha^i}$

... IMPLICIT SOLUTION

$$q^i = q^i(\alpha, \beta, t)$$

• $H' = 0 = \frac{\partial S}{\partial t} + H(q^i, \frac{\partial S}{\partial q^i}, t)$

HAMILTON-JACOBI EQ.

| PDE ... INTERESTED IN COMPLETE INTEGRAL

$$S = S(q^i, \alpha^i, t)$$

n -CONSTANTS.

$$0 = -E + \left(\frac{dS_0}{dx} \right)^2 \frac{1}{2m} + V$$

$$\Rightarrow S_0 = \int \sqrt{(E - V) 2m} dx$$

$$S = \frac{1}{3gm^2} (2m(E - V))^{3/2} - Et$$

↑
INTEGRATION CONSTANT

$$\frac{\partial S}{\partial E} = \beta = \frac{1}{mg} \sqrt{2m(E - V)} - t$$

INVERSION,

$$X = \frac{g}{2} (t + \beta)^2 - \frac{E}{mg}$$

• CONNECTION TO QM:

SCHRODINGER EQ: $i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x), \quad \hat{p} = -i\hbar \nabla$$

GEOMETRIC OPTICS (WKB) APPROXIMATION

$$\psi = \psi_0 e^{\frac{i}{\hbar} S(x,t)}$$

$$i\hbar \frac{\partial \psi}{\partial t} = - \frac{\partial S}{\partial t} \psi = \underbrace{\left(-\frac{\hbar^2}{2m} \nabla^2 + V \right)}_{\left(\frac{|\nabla S|^2}{2m} - \frac{i\hbar}{2m} \nabla^2 S + V \right)} \psi$$

• $\hbar \rightarrow 0$: $-\frac{\partial S}{\partial t} \psi = \left(\frac{|\nabla S|^2}{2m} + V \right) \psi$

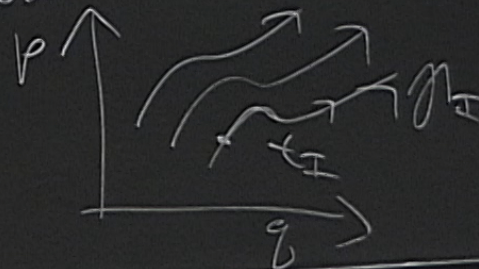
$$\left(\frac{|\nabla S|^2}{2m} + V + \frac{\partial S}{\partial t} = 0 \right) \quad \square$$

2) HIDDEN SYMMETRIES & INTEGRABLE SYSTEMS

- GIVEN A PHASE SPACE FUNCTION I
 $\Rightarrow$ HAMILTONIAN VECTOR

$$X_I = \left\{ \cdot, I \right\} = \frac{\partial I}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial I}{\partial q^i} \frac{\partial}{\partial p_i}$$

THIS GENERATES A FLOW IN PHASE SPACE
WITH TIME t_I .



- LET THIS BE A SYMMETRY OF H .

$$\{H, I\} = \frac{dH}{dt_I} = 0$$

SUCH SYMMETRY IS AT THE LEVEL
OF PHASE SPACE:

- PHASE SPACE IS EQUIPPED WITH
CANONICAL PROJECTION

$$\pi: (q, p) \rightarrow q$$

BE ASYMMETRY OF H .

$$\{H, I\} = \frac{dH}{dt_I} = 0$$

ASYMMETRY IS AT THE LEVEL

BASE SPACE:

SPACE IS EQUIPPED WITH

LOCAL PROJECTION

$\pi(p) \rightarrow q$

$$\pi^*: X_I = \pi^*(X_I) = \underbrace{\frac{\partial I}{\partial p_i}}_{\text{"q-DIR"}} \underbrace{\frac{\partial I}{\partial q_i}}_{\text{"p-DIR"}}$$

MAP "KILLS" P-DIRECTION

$$\pi^*(X_I) = X_I = \underbrace{\frac{\partial I}{\partial p_i}}_{X_I^{\wedge}} \frac{\partial I}{\partial q_i}$$

SMALL

X_I^i

IS IT A VECTOR IN
CONFIGURATION
SPACE

$X_I^i = X_I^i(q) \dots$ CANNOT
DEPEND
ON p .

X_I^i

$X_I^i(q) \dots$

VECTOR FIELD ON CONF. SPACE.
... ISOMETRY ...

$X_I^i(q, p) \dots$

NOT A VECTOR IN CONF. SPACE.

HIDDEN (DYNAMICAL) SYMMETRY

BTW:

ENERGY.

$$\frac{p^2}{2m} = I$$

$$X_I = \{ \cdot, I \}$$

$q \rightarrow$

SMALL
↓

X_I

IS IT A VECTOR IN
CONFIGURATION
SPACE

$X_I^{\dot{}} = X_I^{\dot{}}(q) \dots$ CANNOT
DEPEND
ON p .

$X_I^{\dot{}}$

$X_I^{\dot{}}(q) \dots$

VECTOR FIELD ON CONF. SPACE.
... ISOMETRY ...

$X_I^{\dot{}}(q, p) \dots$

NOT A VECTOR IN CONF SPACE.

HIDDEN (DYNAMICAL) SYMMETRY

BTW: ENERGY. $\frac{p^2}{2m} = I$

$$X_I = \{ \cdot, I \} = \frac{p}{m} \frac{\partial}{\partial q} + 0 = X_I$$

ANOTHER EXAMPLE: CONSIDER THE KEPLER PROBLEM

$$\vec{F} = -\frac{k}{r^2} \hat{r}$$

ISOMETRIES: $\vec{L}$

DYNAMICAL SYMMETRY, E

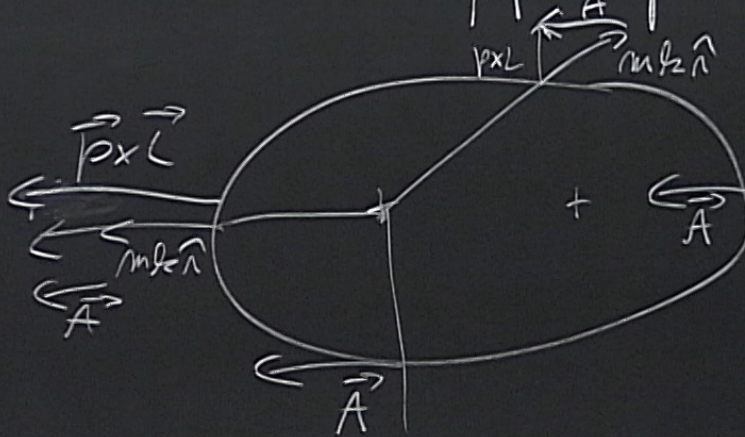
WE ALSO HAVE

$$\vec{A} = \vec{p} \times$$

KEPLER PROBLEM

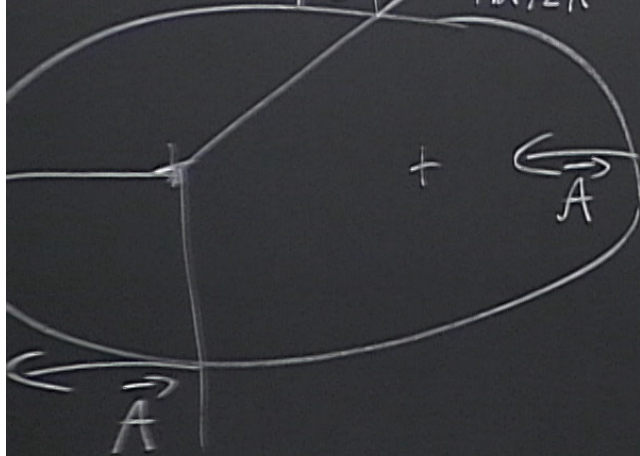
WE ALSO HAVE LAPLACE-RUNGE-LENZ VECTOR

$$\vec{A} = \vec{p} \times \vec{L} - m k \hat{r}$$



WE ALSO HAVE LAPLACE-RUNGE-LENZ VECTOR

$$\vec{A} = \vec{p} \times \vec{L} - m k \hat{n}$$



$$\vec{A} \cdot \vec{L} = 0, \quad A^2 = m^2 k^2 + 2mEL^2$$

$$\begin{array}{ccccccc} 3 & + & 1 & + & 3 & - & 2 & = & 5 & \text{INDEP. INTEGRALS} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \text{OF MOTION} \\ \vec{L} & & E & & \vec{A} & & \text{CONSTRAINTS} & & & \end{array}$$

ANOTHER EXAMPLE: CONSIDER THE KEPLER PROBLEM

WE ALSO HAVE

$$\vec{F} = -\frac{k}{r^2} \hat{n}$$

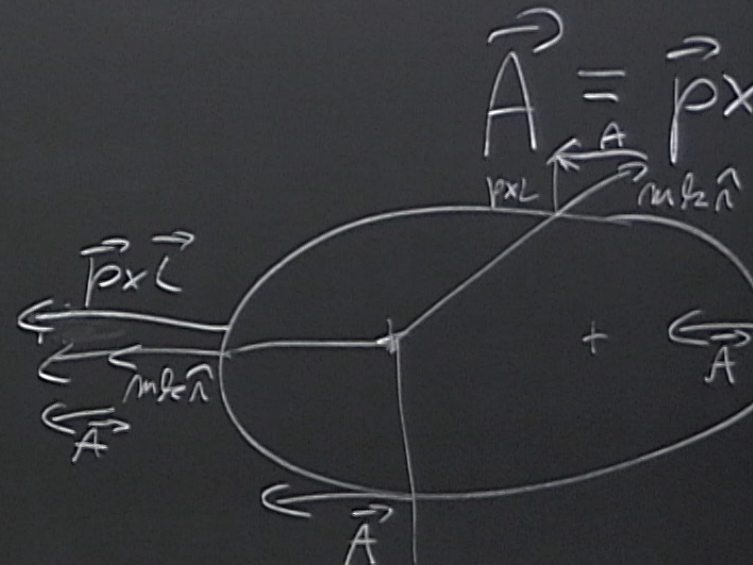
ISOMETRIES: $\vec{L}$

DYNAMICAL SYMMETRY: E

NEED ONLY 3... INTEGRABLE

IF MORE ... SUPERINTEGRABLE

5... MAXIMUM - MAXIMALLY SUPERINTEGRABLE.



$H \neq 0$ ∂S
DEF: DYNAMICAL SYSTEM WITH n DOF IS
COMPLETELY (LIOUVILLE) INTEGRABLE
 IT POSSESSES n GLOBAL INDEPENDENT
 CONSERVED QUANTITIES $F_i(q, p) = f_i$
 $\{H, F_i\} = 0$, THAT ARE IN INVOLUTION
 $\{F_i, F_j\} = 0 \quad \forall i, j$

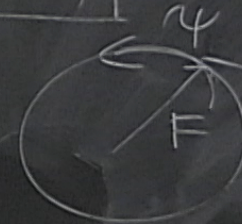
$$\Psi = \Psi_0 e^{\frac{i}{\hbar} S(x, t)}$$

LIIOVILLE'S THEOREM THE SOLUTION OF EQM OF A COMPLETELY
 INTEGRABLE SYSTEM CAN BE OBTAINED BY
QUADRATURE (FINITE # OF ALGEBRAIC OPS.
 & INTEGRATIONS)

• $(q, p) \xrightarrow{\text{CAN TRANSF}} (\underline{F_i}, \psi_i)$ $H = H(F_i)$

$$\dot{\psi}_i = \{H, \psi_i\} = \frac{\partial H}{\partial F_i} = \Omega_i = \text{CONST}$$

$$\psi_i = \psi_i(0) + \Omega_i t$$



$$\left(\frac{DS}{dm} + V + \frac{\partial S}{\partial t} = 0 \right)$$

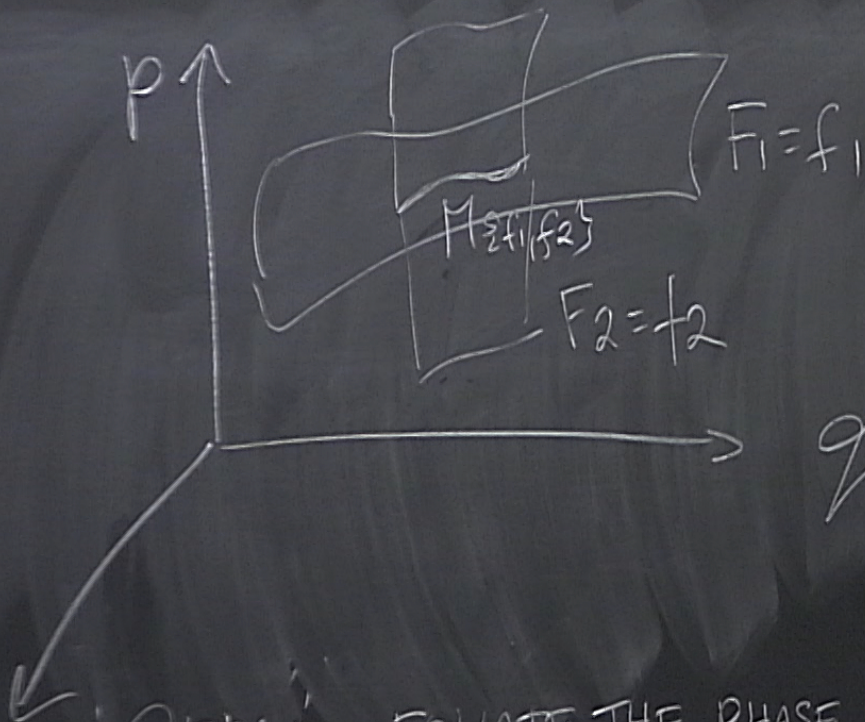
DYNAMICAL SYMMETRY, F

$\int \dots$
 $\leftarrow$ $\int \dots$

WHAT "GLOBAL & INDEPENDENT" MEANS?

EACH INTEGRAL DEFINES HYPERSURFACE
IN PHASE SPACE

E.G. $F_1(q, p) = 5$



MOTION HAS TO STAY
IN $M_{\{f_1, f_2\}}$

• REQUIRE $M_{\{f_1, \dots, f_n\}} = M$

• ONLY IF HYPERSURFACES ARE
NEVER TANGENT

... INDEPENDENCE!

GLOBAL = FOLIATE THE PHASE SPACE

ER INTEGRABLE.

IN PHASE SPACE

E.G. $F(q,p) = 5$

∝ MOTION IS CONFINED TO THIS
HYPER SURFACE.

KAM

CLASSICAL CHAOS

GLOBAL L = F

IF MORE ... SUPERINTEGRABLE
5 ... MAXIMUM - MAXIMALLY SUPERINTEGRABLE.