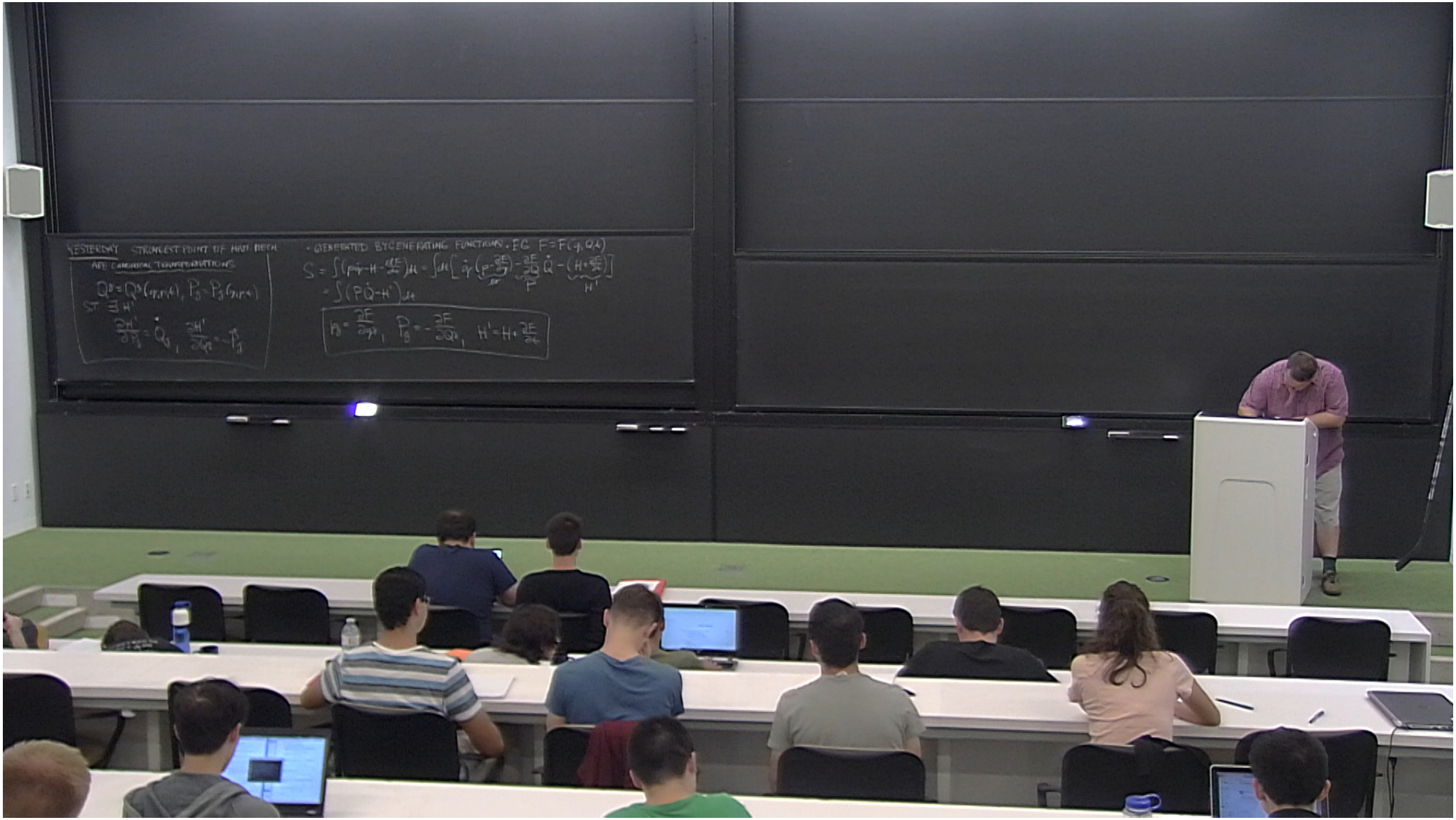
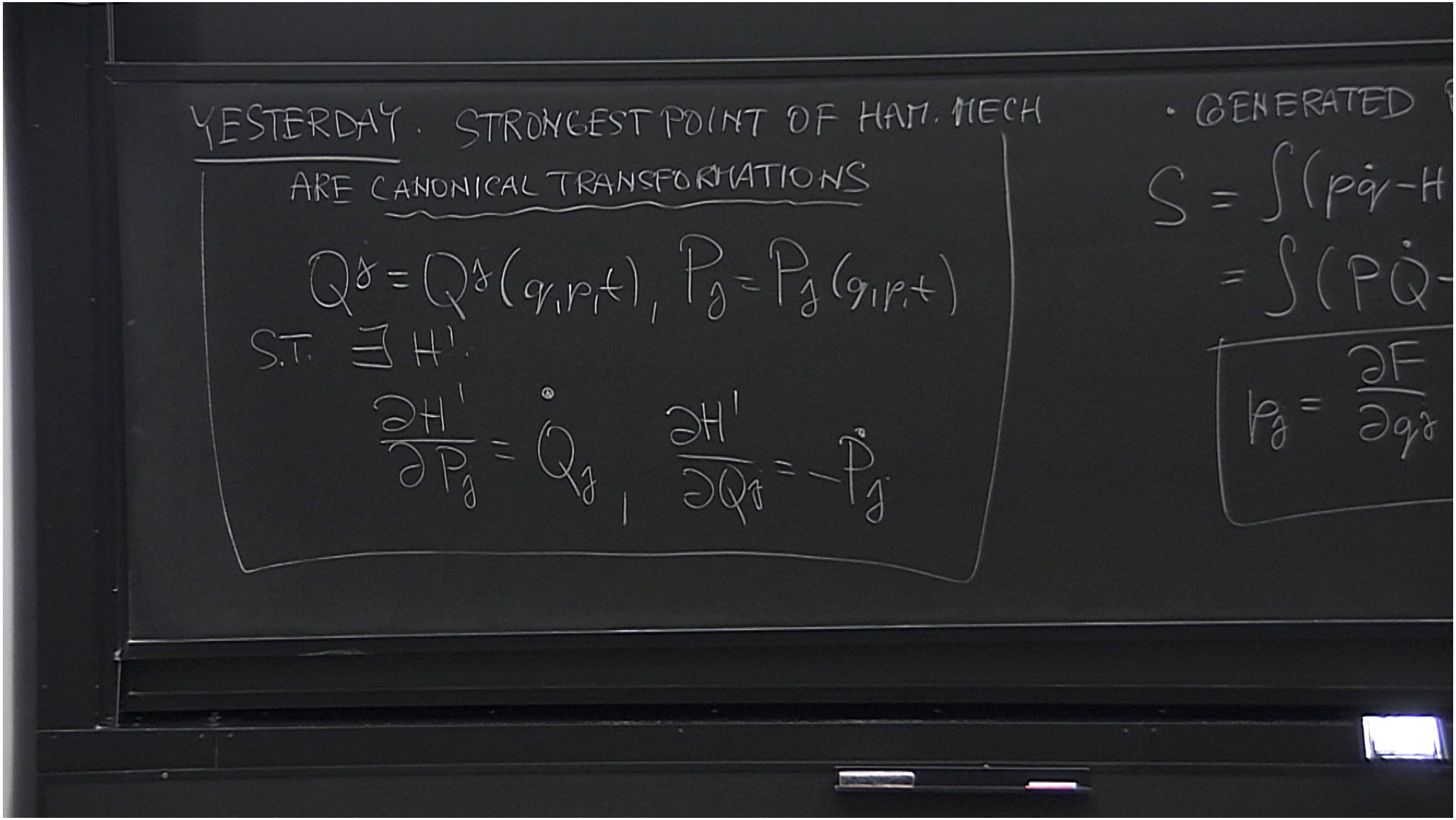
Title: PSI 2018/2019 - Classical Physics - Lecture 3

Date: Aug 16, 2018 09:00 AM

URL: http://pirsa.org/18080011

Abstract:


YESTERDAY STRONGEST POINT OF HAM. MECH
ARE CANONICAL TRANSFORMATIONS
GENERATED BY GENERATING FUNCTIONS. EG F=F(q,Q,t)


YESTERDAY. STRONGEST POINT OF HAM. MECH
ARE CANONICAL TRANSFORMATIONS
$Q^\gamma = Q^\gamma(q,p,t)$, $P_\gamma = P_\gamma(q,p,t)$
S.T. $\exists H'$.
$\frac{\partial H'}{\partial P_\gamma} = \dot{Q}_\gamma$, $\frac{\partial H'}{\partial Q_\gamma} = -\dot{P}_\gamma$
• GENERATED
$S = \int (p\dot{q} - H$
$= \int (P\dot{Q} -$
$p_\gamma = \frac{\partial F}{\partial q^\gamma}$

H

• GENERATED BY GENERATING FUNCTIONS. E.G $F = F(q, Q, t)$

$$S = \int \left( p\dot{q} - H - \frac{dF}{dt} \right) dt = \int dt \left[ \dot{q} \underbrace{\left( p - \frac{\partial F}{\partial q} \right)}_{0} - \underbrace{\frac{\partial F}{\partial Q}}_{P} \dot{Q} - \underbrace{\left( H + \frac{\partial F}{\partial t} \right)}_{H'} \right]$$

$$= \int \left( P\dot{Q} - H' \right) dt$$

$$p_j = \frac{\partial F}{\partial q_j}, \quad P_j = -\frac{\partial F}{\partial Q_j}, \quad H' = H + \frac{\partial F}{\partial t}$$

. MECH

• GENERATED BY GENERATING FUNCTIONS. E.G $F = F(q, Q, t)$

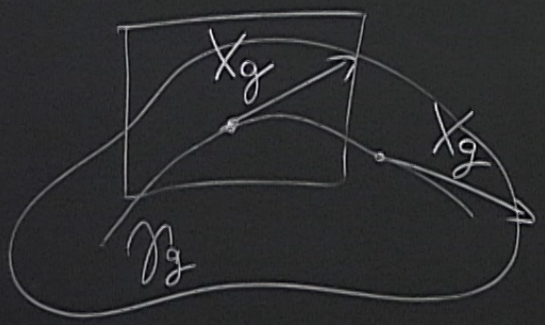
$$S = \int \left( p\dot{q} - H - \frac{dF}{dt} \right) dt = \int dt \left[ \dot{q} \underbrace{\left( p - \frac{\partial F}{\partial q} \right)}_{0} - \underbrace{\frac{\partial F}{\partial Q}}_{P} \dot{Q} - \underbrace{\left( H + \frac{\partial F}{\partial t} \right)}_{H'} \right]$$

$$\text{REQUIRE} = \int \left( P\dot{Q} - H' \right) dt$$

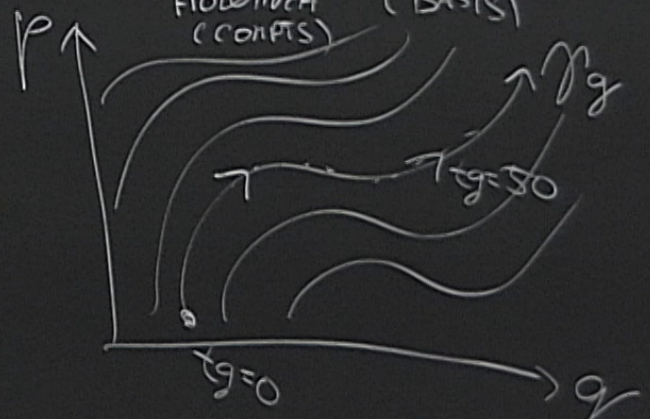
$$p_j = \frac{\partial F}{\partial q_j}, \quad P_j = -\frac{\partial F}{\partial Q_j}, \quad H' = H + \frac{\partial F}{\partial t}$$

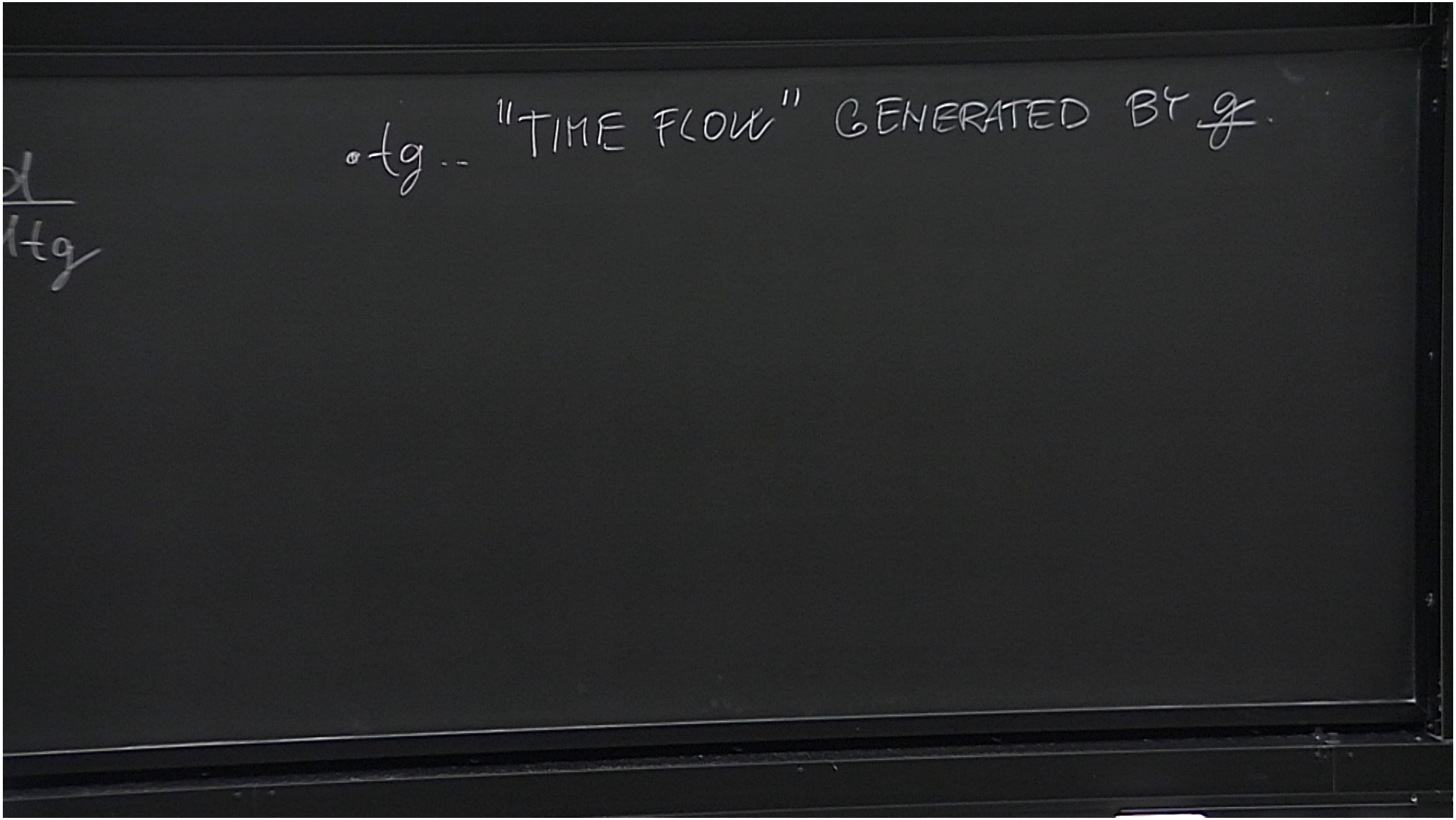
• GIVEN ANY PHASE SPACE FUNCTION $g$.

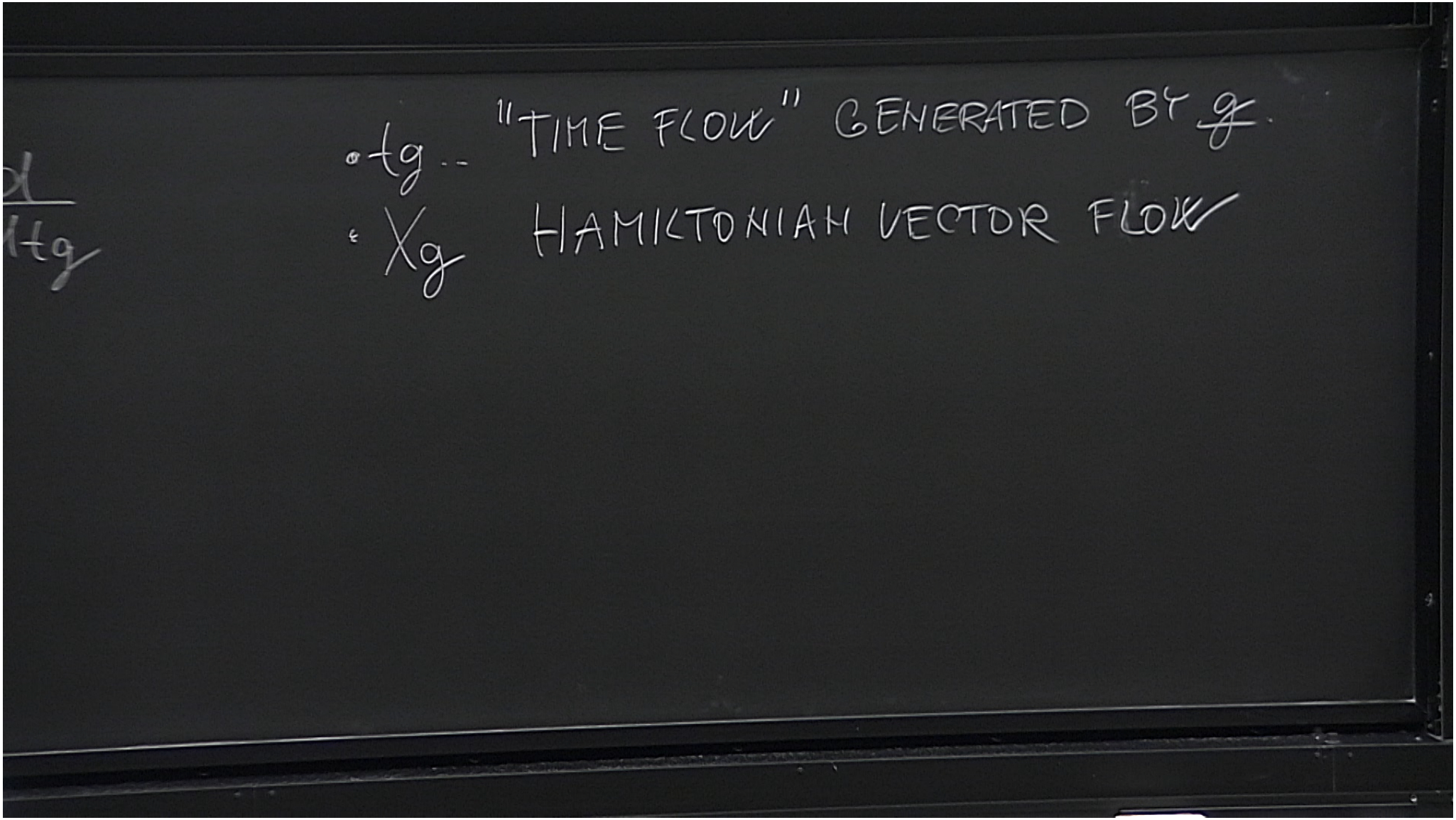
$$g \to \hat{g} = \{\cdot, g\} = \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i} \overset{\text{HOW MUCH (COMPTS)}}{\overset{\text{WHICH DIRECTION (BASIS)}}{=}} X_g = \frac{d}{dt_g}$$




$\Leftrightarrow$




$\cdot t_g$ -- "TIME FLOW" GENERATED BY $g$.


"TIME FLOW" GENERATED BY g.
HAMILTONIAN VECTOR FLOW

$\frac{d}{dt_g}$

- $t_g$ -- "TIME FLOW" GENERATED BY $g$.
- $X_g$ HAMILTONIAN VECTOR FLOW OF $g$.

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- TRUE HAMILTONIAN $H$ IS NOT SPECIAL AT ALL (EXCEPT $t = t_H$)

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- IT'S OBVIOUS

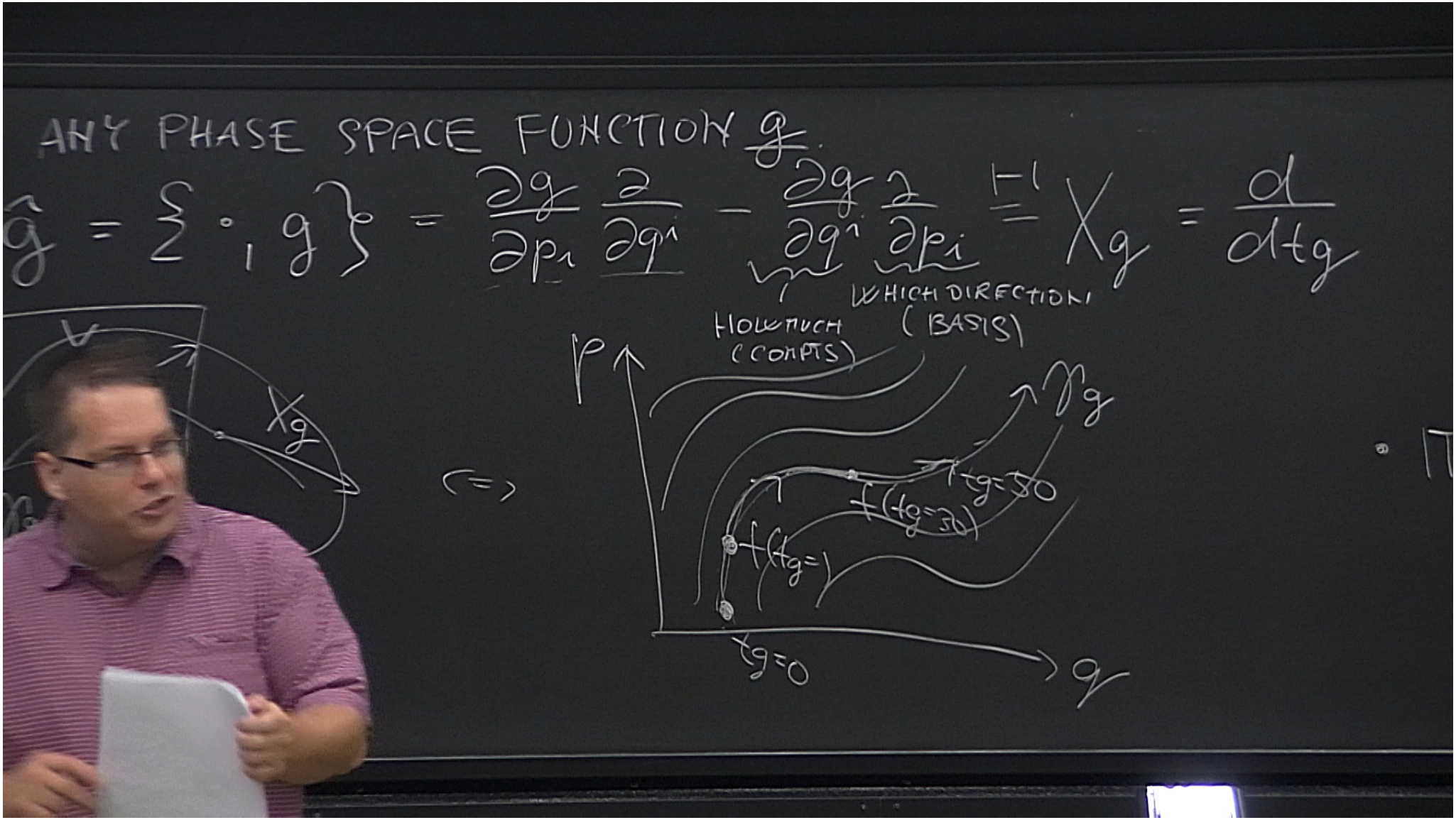
$$\{f, g\} = \frac{df}{dt_g}$$

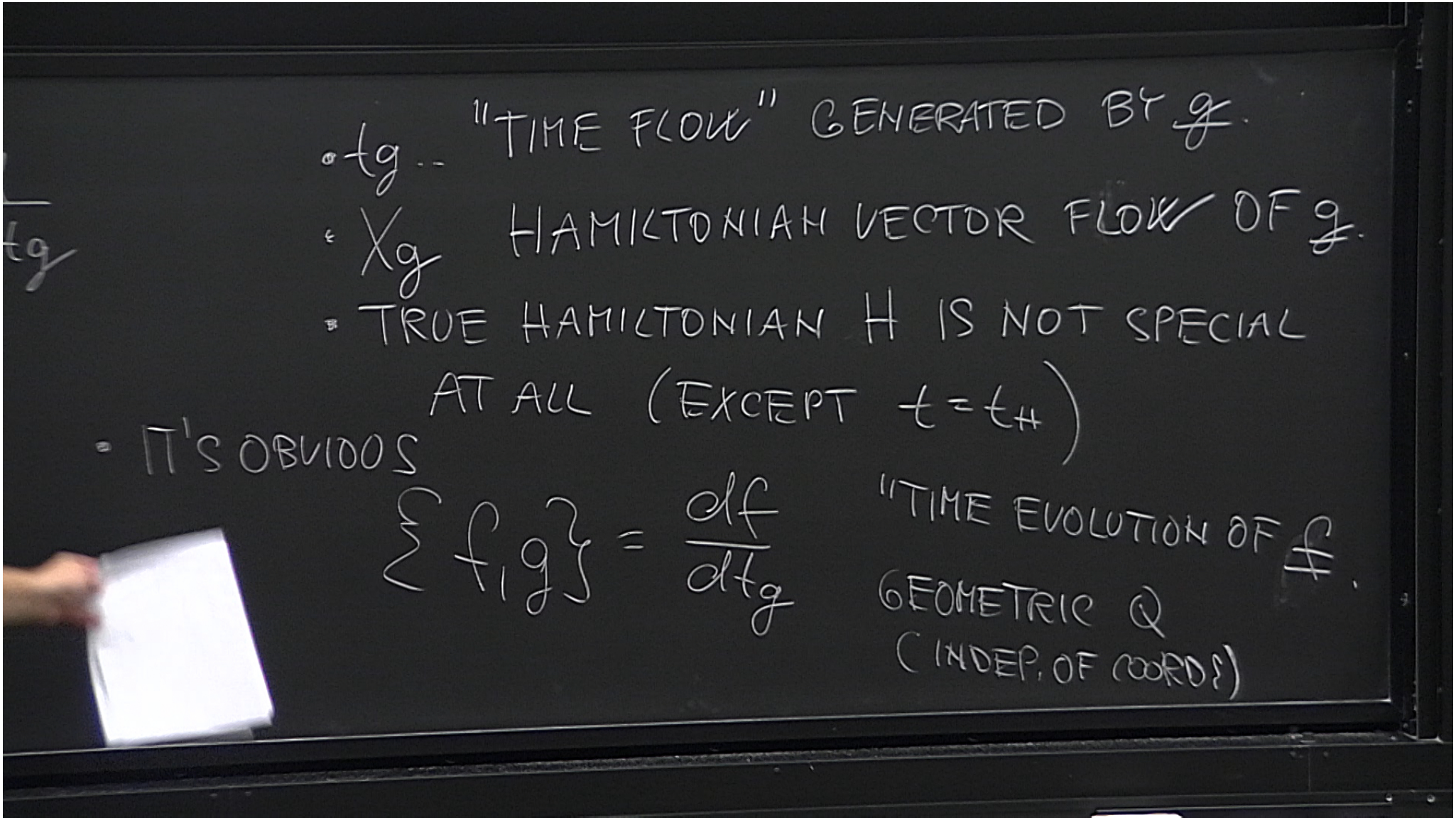
$\frac{d}{dt_g}$

- $t_g$ .. "TIME FLOW" GENERATED BY $g$.
- $X_g$ HAMILTONIAN VECTOR FLOW OF $g$.
- TRUE HAMILTONIAN $H$ IS NOT SPECIAL AT ALL (EXCEPT $t = t_H$)
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$$\{f, g\} = \frac{df}{dt_g}$$

"TIME EVOLUTION OF $f$".


ANY PHASE SPACE FUNCTION g.
$\hat{g} = \{\cdot, g\} = \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i} \equiv X_g = \frac{d}{dt_g}$
HOW MUCH (COMPTS)
WHICH DIRECTION (BASIS)
$X_g$
$\Leftrightarrow$
p
q
$\gamma_g$
$t_g=30$
$t_g=0$


$t_g$ -- "TIME FLOW" GENERATED BY $g$.
$X_g$ HAMILTONIAN VECTOR FLOW OF $g$.
TRUE HAMILTONIAN $H$ IS NOT SPECIAL AT ALL (EXCEPT $t = t_H$)
IT'S OBVIOUS
$\{f, g\} = \frac{df}{dt_g}$
"TIME EVOLUTION OF $f$"
GEOMETRIC Q
(INDEP. OF COORDS!)

$$\frac{df}{dt} = \{f, H\} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p}$$

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\frac{df}{dt} = \{f, H\} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p}$$

$\dot{q} = \frac{\partial H}{\partial p}$ $\quad \dot{p} = -\frac{\partial H}{\partial q}$

- CANONICAL TRANSF. PRESERVE THE FORM OF POISSON BRACKET. CAN TR. $\downarrow$

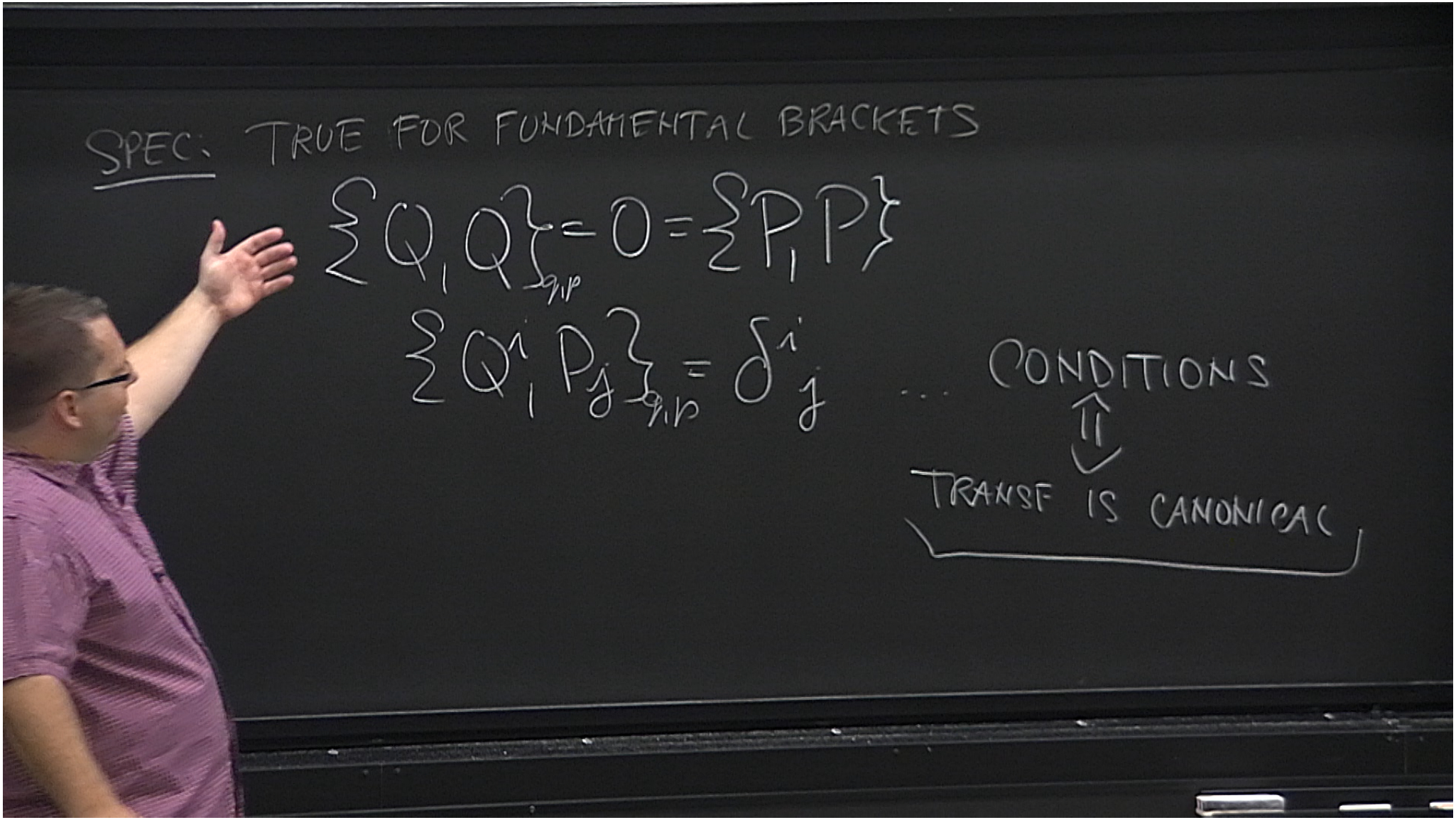
$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}$$

SPEC: TRUE FOR FUNDAMENTAL BRACKETS

$$\{Q_i, Q_j\} = 0 = \{P_i, P_j\}$$

$$\{Q^i, P_j\} = \delta^i_j$$

$$\delta - \frac{\partial f}{\partial p}\frac{\partial g}{\partial Q}$$

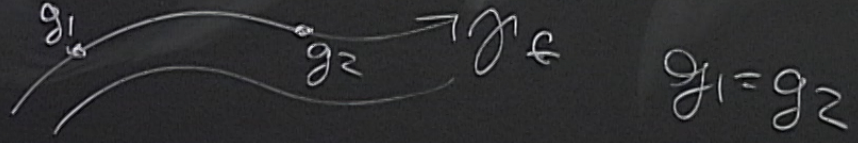

SPEC: TRUE FOR FUNDAMENTAL BRACKETS
$\{Q_i, Q_j\}_{q,p} = 0 = \{P_i, P_j\}$
$\{Q^i, P_j\}_{q,p} = \delta^i_j$
... CONDITIONS
⇕
TRANSF IS CANONICAL

- FLOW GENERATED BY $f$ IS A SYMMETRY $g$

$$\{g, f\} = \frac{dg}{dt_f} = 0$$

- FLOW GENERATED BY $f$ IS A SYMMETRY $g$

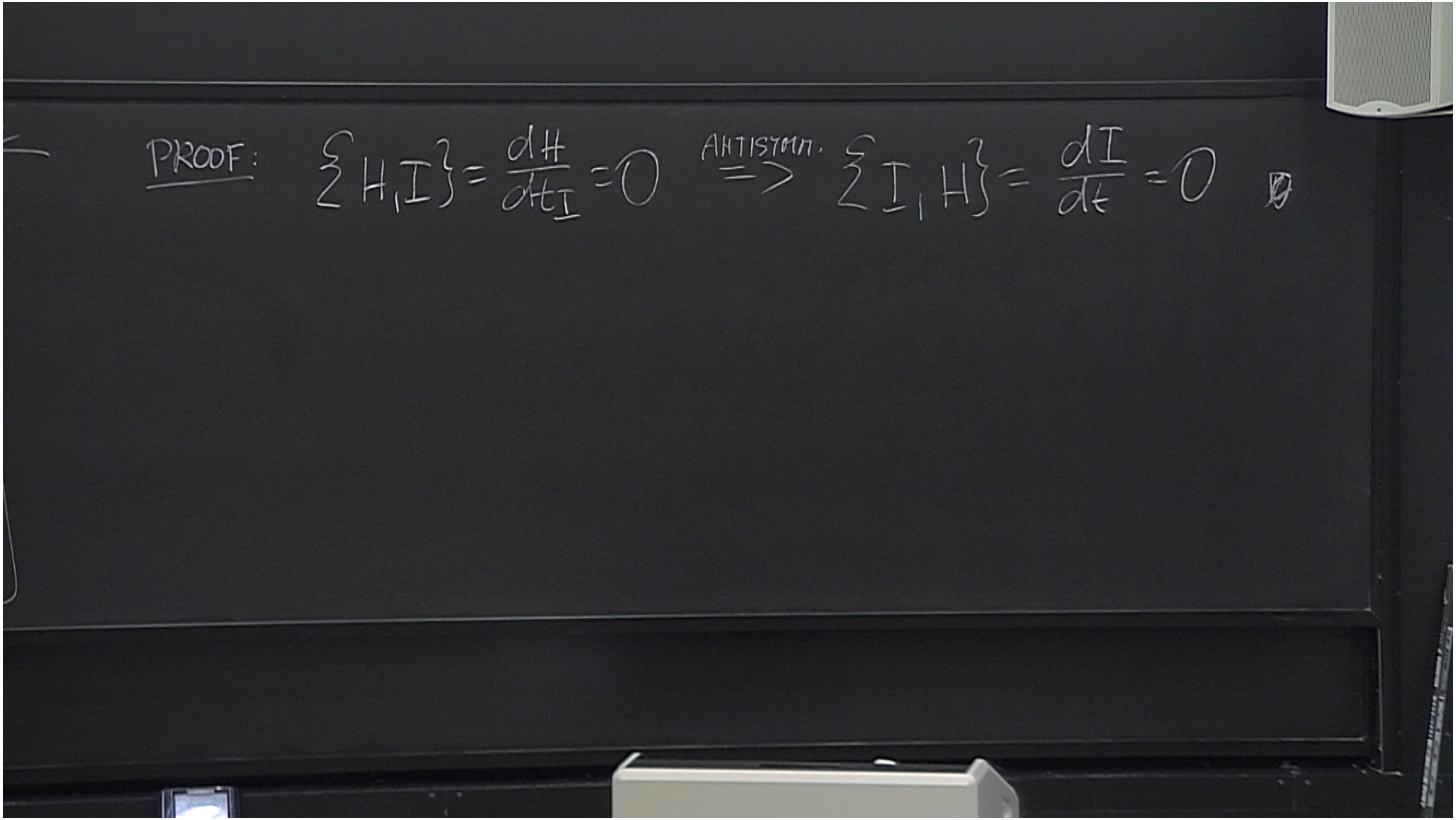
$$\{g, f\} = \frac{dg}{dt_f} = 0$$

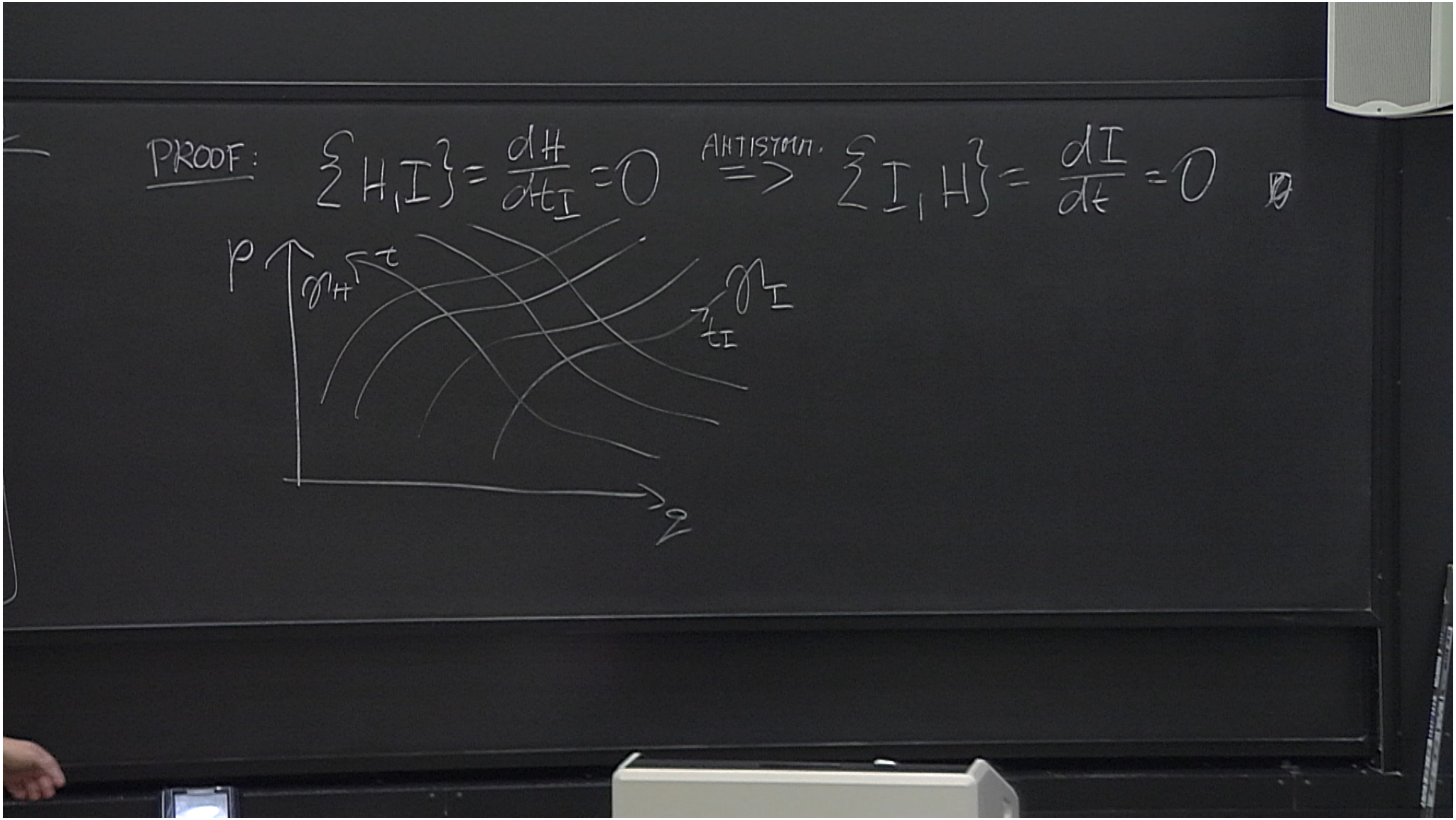



$g_1 = g_2$

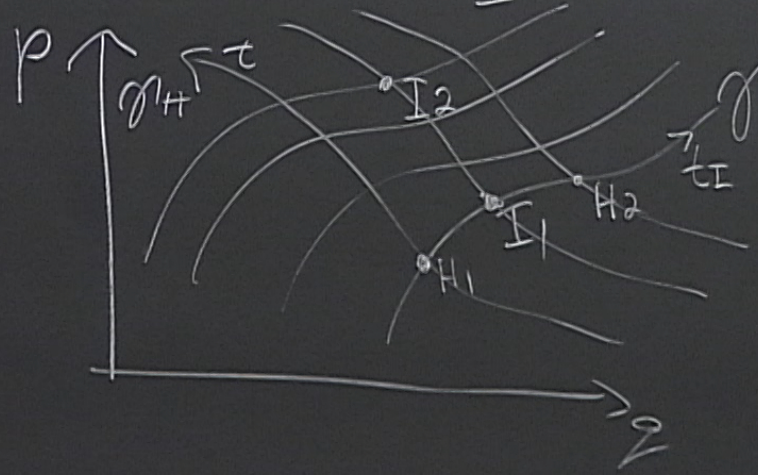
NOETHER'S (HAMILTONIAN VERSION)

LET THERE BE A SYMMETRY OF HAM H. GENERATED BY $I$ => $I$ IS CONSERVED Q


PROOF: $\{H, I\} = \frac{dH}{dt_I} = 0$ ANTISYMM. $\Rightarrow \{I, H\} = \frac{dI}{dt} = 0$ ∎

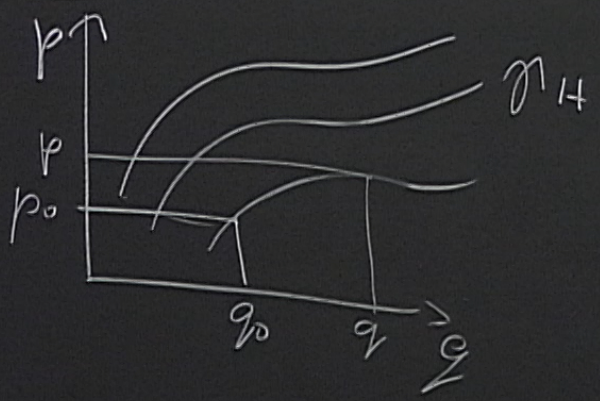

PROOF: $\{H, I\} = \frac{dH}{dt_I} = 0$ ANTISYMM. $\Rightarrow$ $\{I, H\} = \frac{dI}{dt} = 0$
p
$\pi_H$
t
$\pi_I$
$t_I$
q

PROOF: $\{H, I\} = \frac{dH}{dt_I} = 0$ ANTISYMM. $\Rightarrow$ $\{I, H\} = \frac{dI}{dt} = 0$



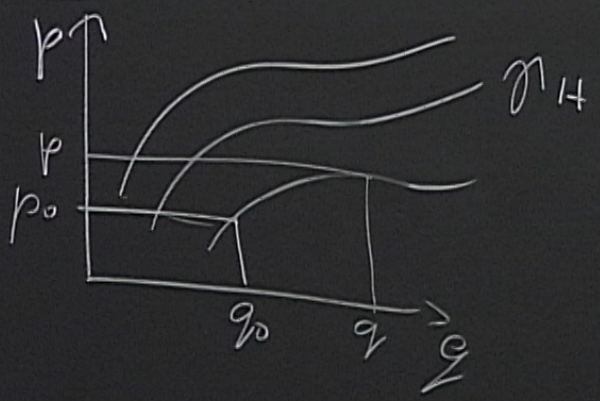

$H_1 = H_2 \Rightarrow I_1 = I_2$

TIME EVOLUTION = CANONICAL TRANSFORMATION




$(q_0, p_0) \to (q, p)$

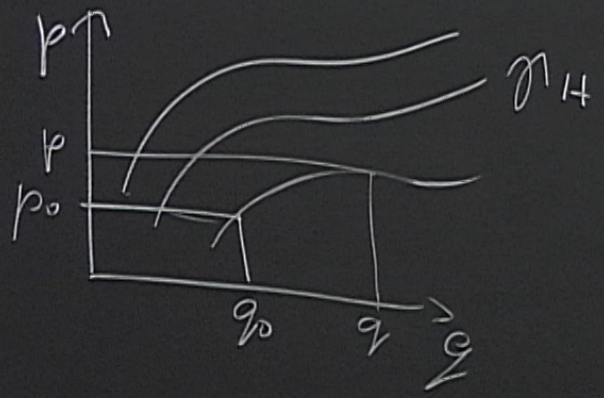
# TIME EVOLUTION = CANONICAL TRANSFORMATION




$(q_0, p_0) \to (q, p)$

GENERATED BY $F = -S$

# TIME EVOLUTION = CANONICAL TRANSFORMATION



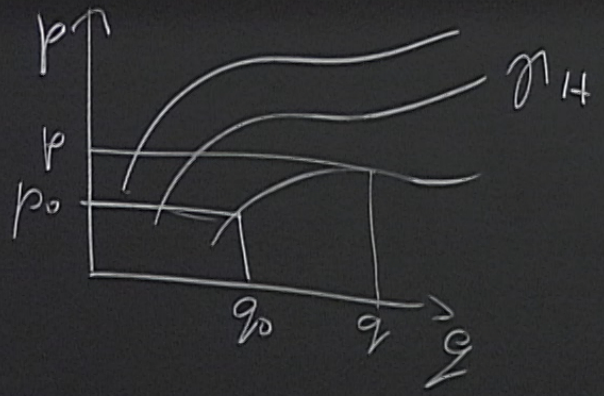

$(q_0, p_0) \to (q, p)$

GENERATED BY $\boxed{F = -S}$

S... HAMILTON'S FUNCTION

GENERATED BY $I$ => $I$ IS CONSERVED Q.

TIME EVOLUTION = CANONICAL TRANSFORMATION




$(q_0, p_0) \to (q, p)$

GENERATED BY $F = -S$

S... HAMILTON'S FUNCTION

= "ACTION ON SHELL - FUNCTION OF COORDINATES & TIME"

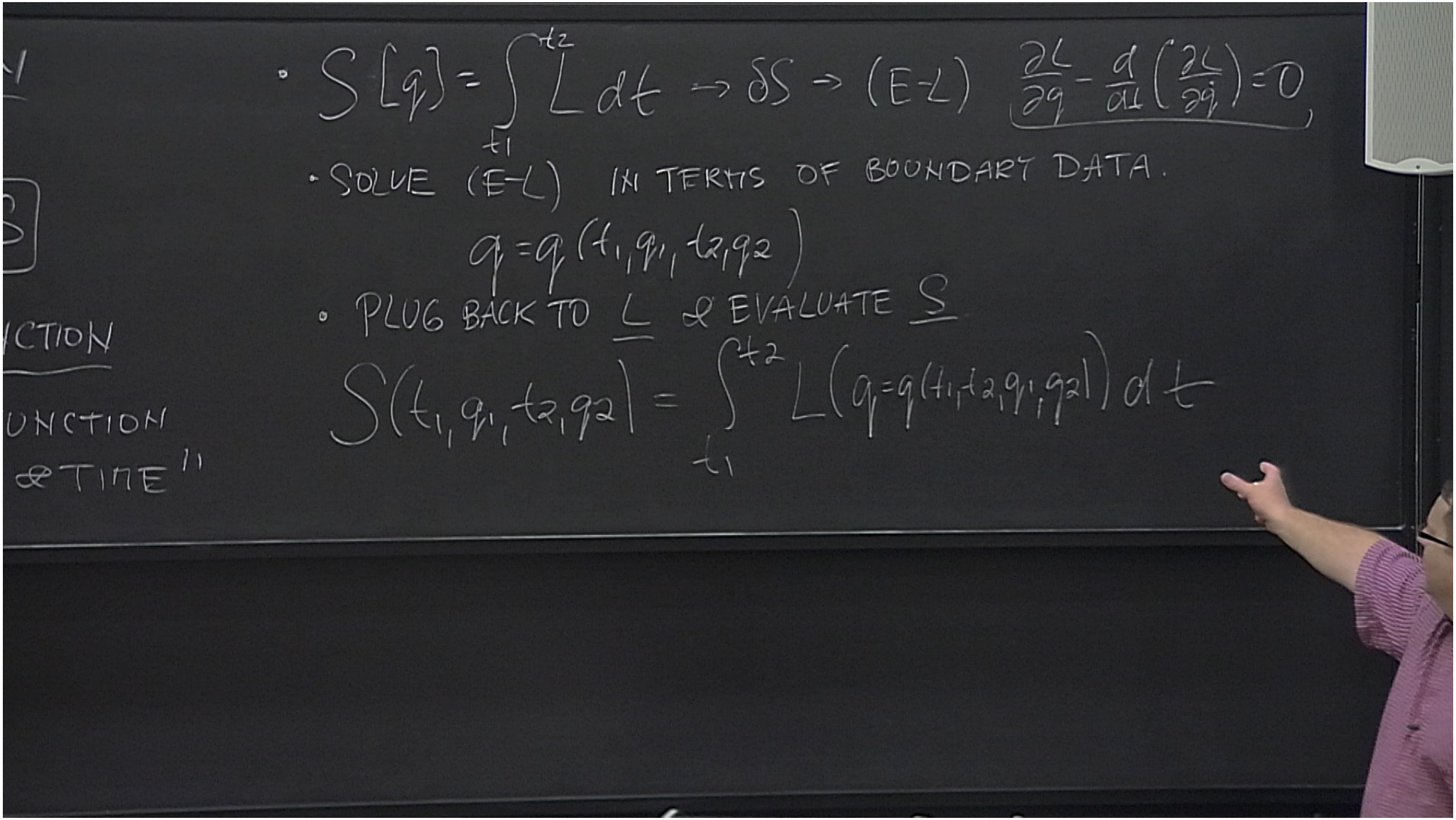
$\cdot S[q] = \int_{t1}^{t2} L\,dt \rightarrow \delta S \rightarrow (E\text{-}L)\ \underline{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0,}$

• SOLVE (E-L) IN TERMS OF BOUNDARY DATA.

CTION

UNCTION

& TIME"


$S[q] = \int_{t_1}^{t_2} L\,dt \rightarrow \delta S \rightarrow (E\text{-}L)$ $\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$
• SOLVE (E-L) IN TERMS OF BOUNDARY DATA.
$q = q(t_1, q_1, t_2, q_2)$
• PLUG BACK TO L & EVALUATE S
$S(t_1, q_1, t_2, q_2) = \int_{t_1}^{t_2} L(q = q(t_1, t_2, q_1, q_2))\,dt$

CTION

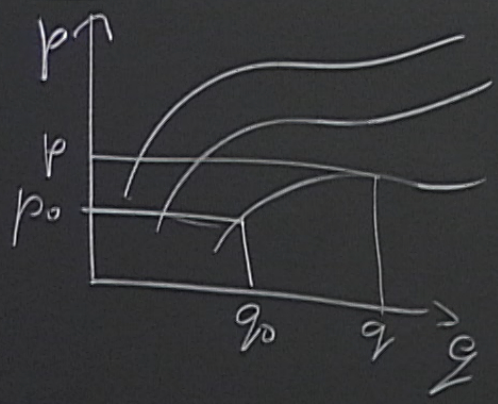
UNCTION
& TIME"

• PLUG BACK TO $L$ & EVALUATE $S$

$$S(t_1, q_1, t_2, q_2) = \int_{t_1}^{t_2} L(q = q(t_1, t_2, q_1, q_2, t))\, dt$$

GET THERE BE A SYMMETRY OF HAM H.
GENERATED BY I. => I IS CONSERVED Q.

TIME EVOLUTION = CANONICAL TRANSFORMATION




$\gamma_H$

$(q_0, p_0) \to (q, p)$

GENERATED BY $F = -S$

S.. HAMILTON'S FUNCTION

= "ACTION ON SHELL – FUNCTION OF COORDINATES & TIME"

$S = \int (p\dot{q} - H)\, dt$

(A)

- $S[q] =$
- SOLVE (E
- PLUG BA

$S(t_1, q_1$

$S = \int (p\dot{q} - H)\, dt$

= "ACTION ON SHELL – FUNCTION OF COORDINATES & TIME"

$S(t_1, q_1, t_2$

L&L $dS = p\,dq - H\,dt\Big|_{t_2} - (p\,dq - H\,dt)\Big|_{t_1}$

$$\int (dS + p_1\,dq_1 - H_1\,dt_1) = \int p\,dq - H\,dt$$

$q_1, t_1$

$S = \int (p\dot{q} - H)\,dt$

= "ACTION ON SHELL – FUNCTION OF COORDINATES & TIME"

$S(t_1, q_1, t_2$

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$$\int (dS + p_1\,dq_1 - H_1\,dt_1) = \int p\,dq - H\,dt$$

$q_1, t_1 \to q, t$

$S = \int (p\dot{q} - H)\, dt$

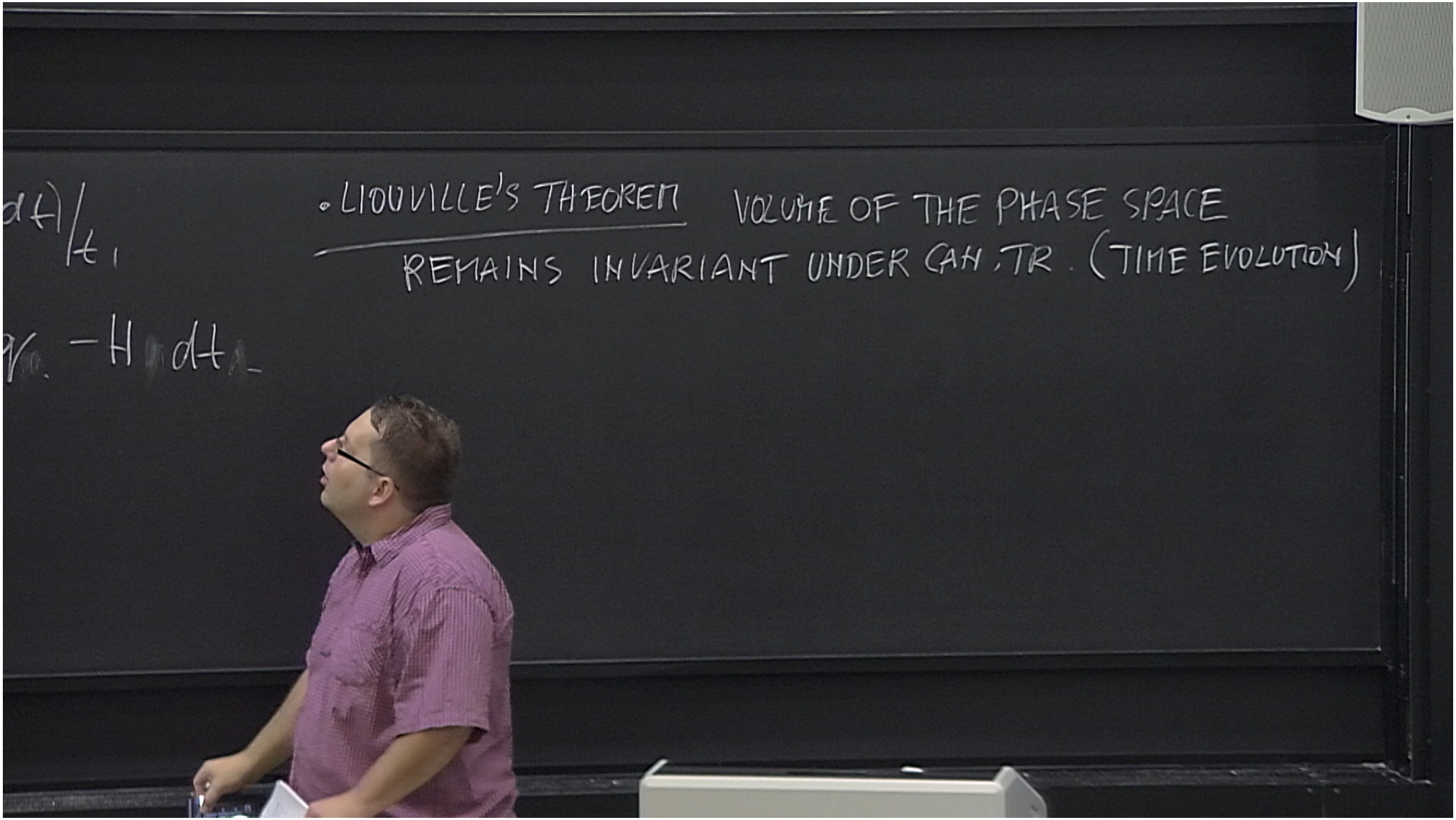
= "ACTION ON SHELL – FUNCTION OF COORDINATES & TIME"

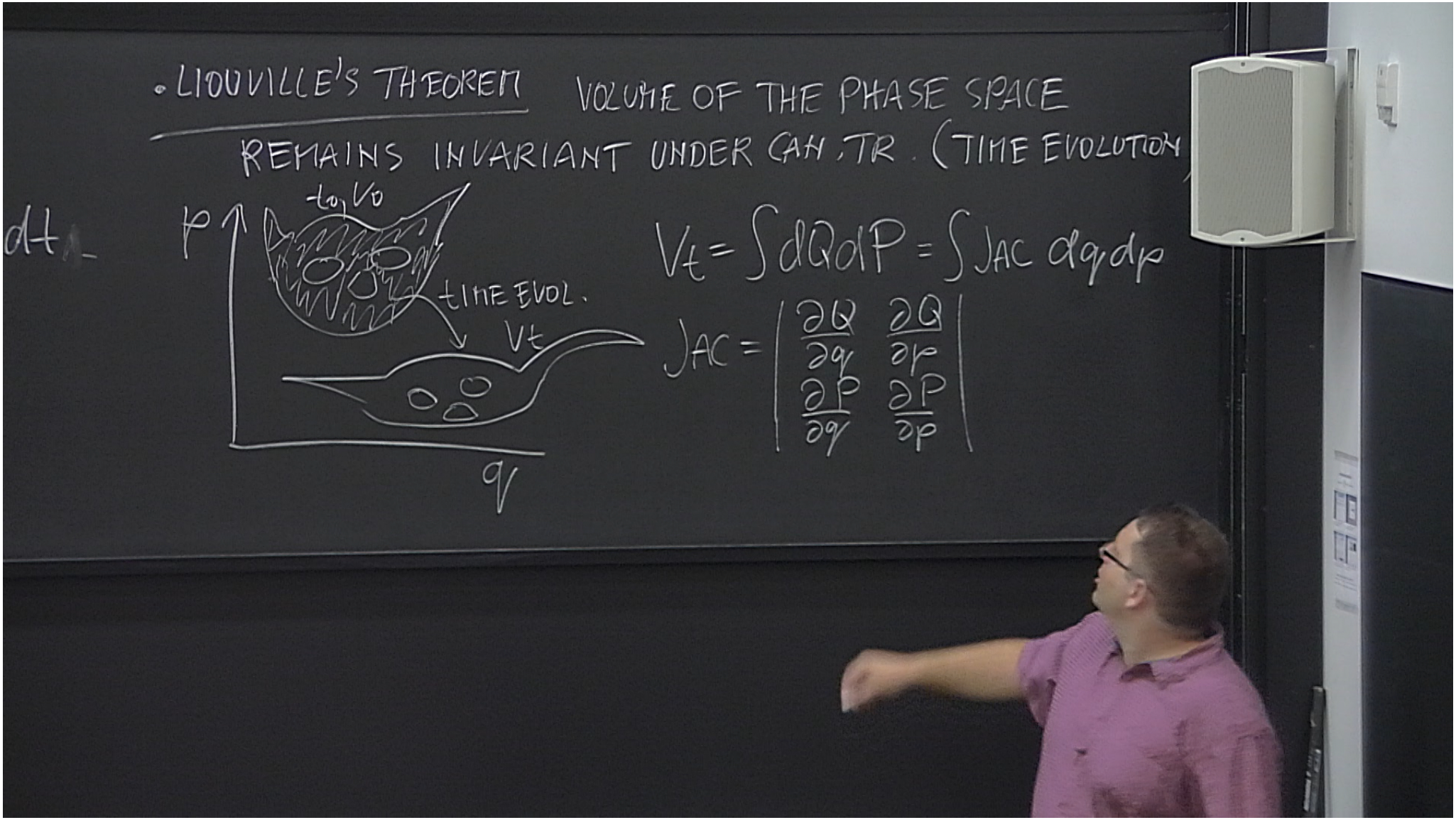
$S(t_1, q_1, t_2$

L & L $\underline{dS = p\,dq - H\,dt\,|_{t_2} - (p\,dq - H\,dt)\,|_{t_1}}$

$\int (dS + p_1\,dq_1 - H_1\,dt_1) = \int p\,dq - H\,dt$

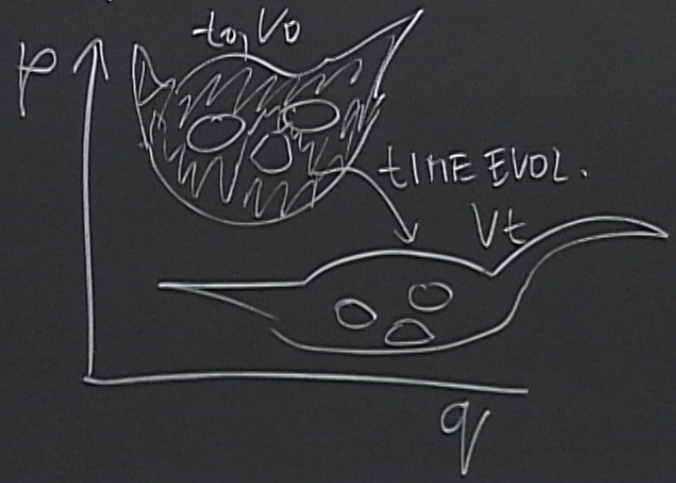
$q_1, t_1 \to q, t$ $\quad \underline{F = -S}$


• LIOUVILLE'S THEOREM VOLUME OF THE PHASE SPACE
REMAINS INVARIANT UNDER CAN. TR. (TIME EVOLUTION)


• LIOUVILLE'S THEOREM VOLUME OF THE PHASE SPACE
REMAINS INVARIANT UNDER CAN. TR. (TIME EVOLUTION
dt
p
$t_0$, $V_0$
time evol.
$V_t$
q
$V_t = \int dQ\,dP = \int JAC\, dq\,dp$
$JAC = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix}$

• LIOUVILLE'S THEOREM VOLUME OF THE PHASE SPACE

REMAINS INVARIANT UNDER CAN. TR. (TIME EVOLUTION)




$$V_t = \int dQ\,dP = \int JAC\; dq\,dp$$

$$JAC = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \{Q,P\} = 1$$

$$= \int dq\,dp = V_0$$

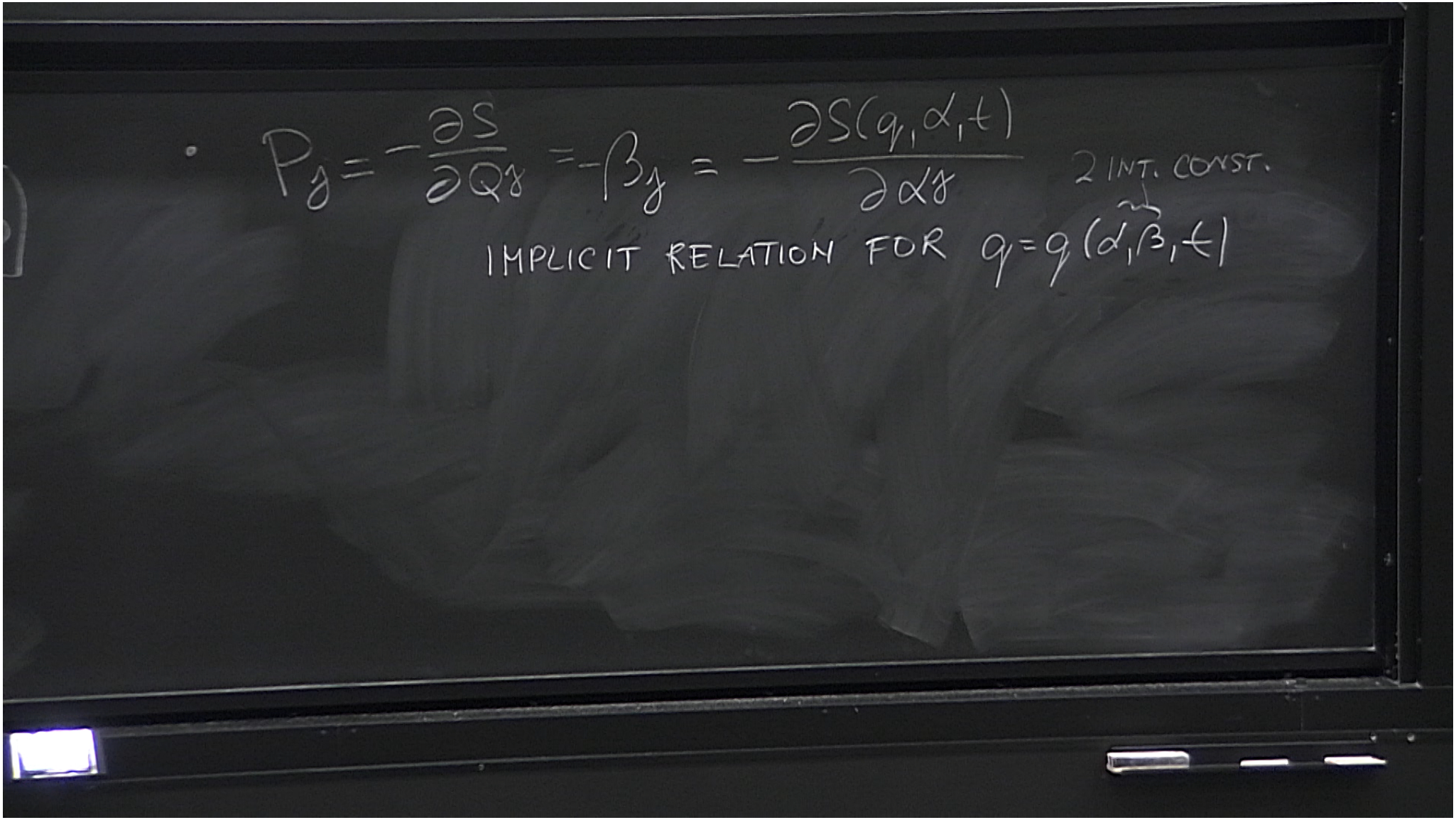
d) HAMILTON-JACOBI THEORY

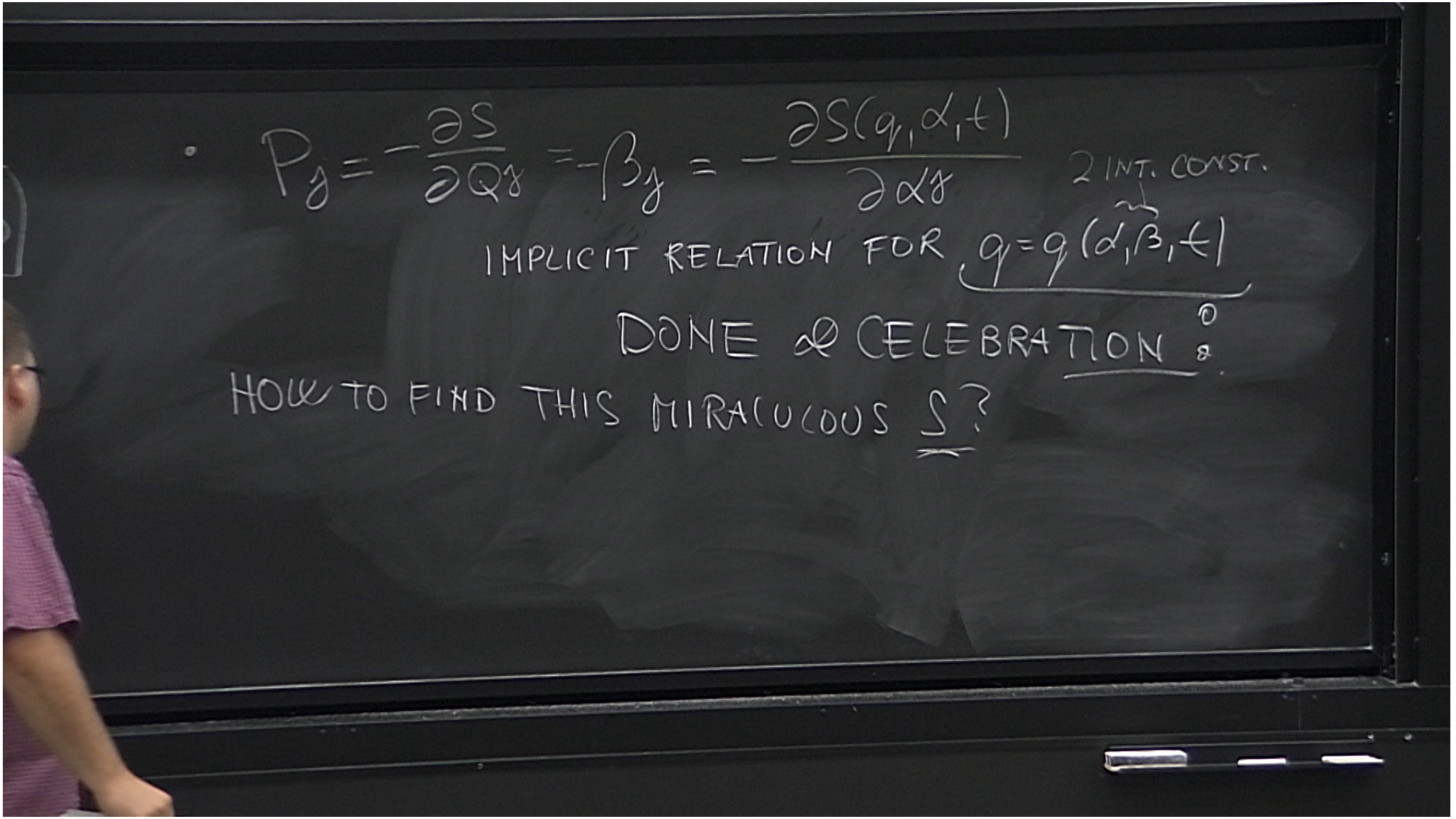
INGENIOUS IDEA. SUCH CAN TRANSF SO THAT $H' = 0$

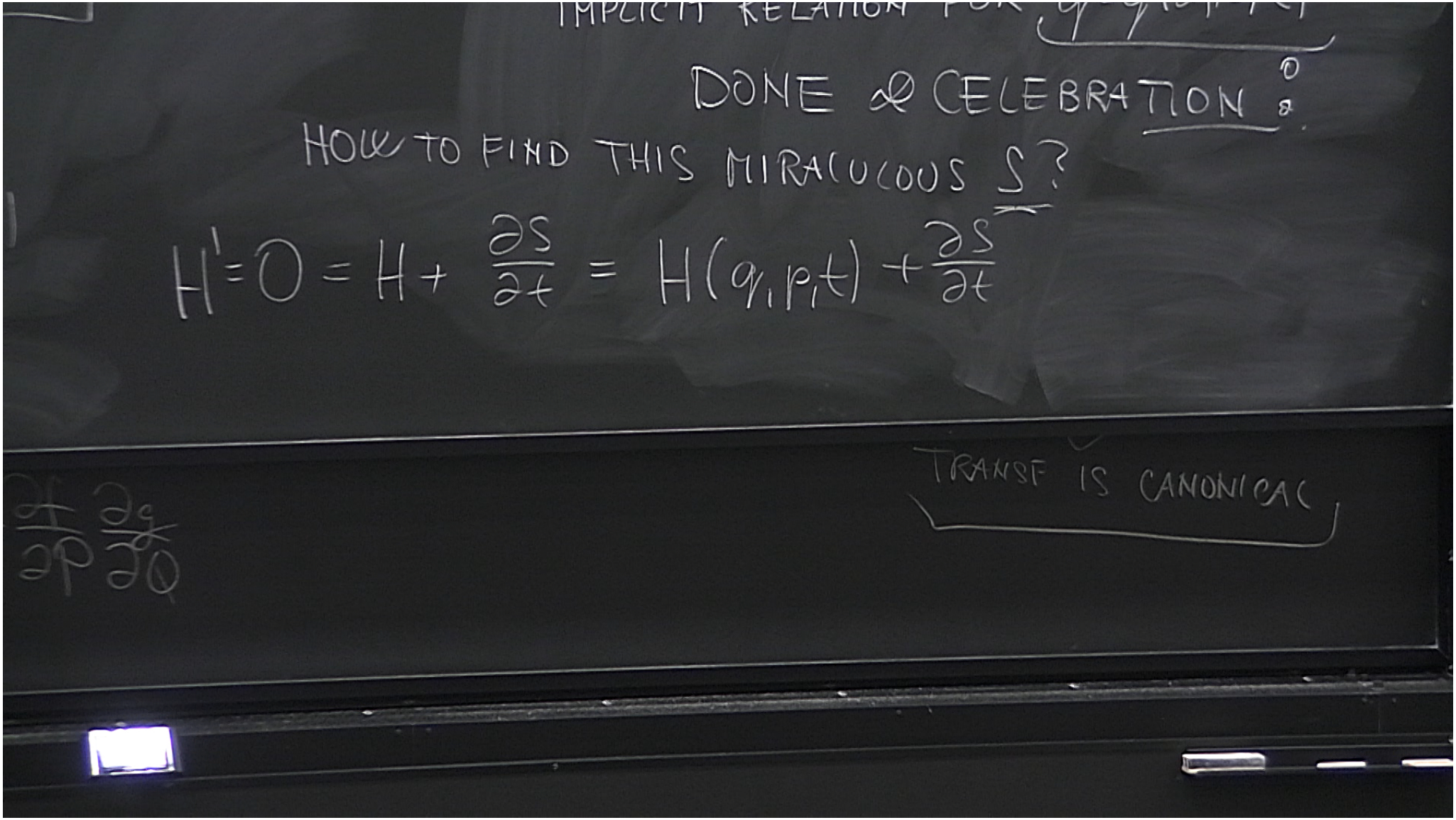
$\Rightarrow$ HAMILTON'S EQS ARE TRIVIAL

$$Q^\alpha = \alpha^\alpha = \text{CONST}$$

$$P_\alpha = -\beta_\alpha = \text{CONST}$$


$P_\gamma = -\frac{\partial S}{\partial Q_\gamma} = -\beta_\gamma = -\frac{\partial S(q,\alpha,t)}{\partial \alpha_\gamma}$
2 INT. CONST.
IMPLICIT RELATION FOR $q = q(\alpha,\beta,t)$


$P_\gamma = -\frac{\partial S}{\partial Q_\gamma} = -\beta_\gamma = -\frac{\partial S(q,\alpha,t)}{\partial \alpha_\gamma}$
2 INT. CONST.
IMPLICIT RELATION FOR $q = q(\alpha,\beta,t)$
DONE & CELEBRATION !
HOW TO FIND THIS MIRACULOUS $S$ ?


IMPLICIT RELATION FOR
DONE & CELEBRATION !
HOW TO FIND THIS MIRACULOUS S ?
$H' = 0 = H + \frac{\partial S}{\partial t} = H(q, p, t) + \frac{\partial S}{\partial t}$
TRANSF IS CANONICAL

$\frac{\partial Q_\gamma}{\partial t}$, $H = H + \frac{\partial}{\partial t}$

$= 0$

$P_\gamma = -\frac{\partial S}{\partial Q_\gamma} = -\beta_\gamma = -\frac{\partial S(q,\alpha,t)}{\partial \alpha_\gamma}$ 2 INT. CONST.

IMPLICIT RELATION FOR $q = q(\alpha, \beta, t)$

DONE & CELEBRATION !!

HOW TO FIND THIS MIRACULOUS $S$?

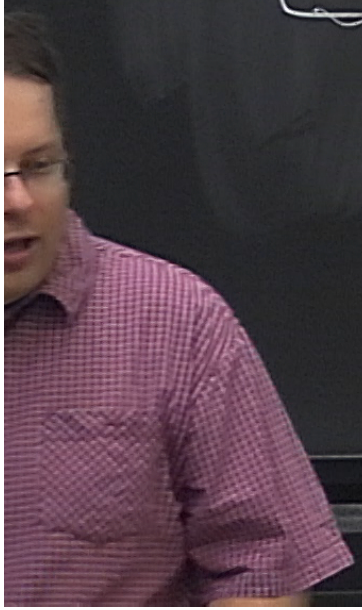
$$H' = 0 = H + \frac{\partial S}{\partial t} = H(q, p, t) + \frac{\partial S}{\partial t}$$

$p = \frac{\partial S}{\partial q}$

S. SOLUTION TO

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0$$

HAMILTON-JACOBI

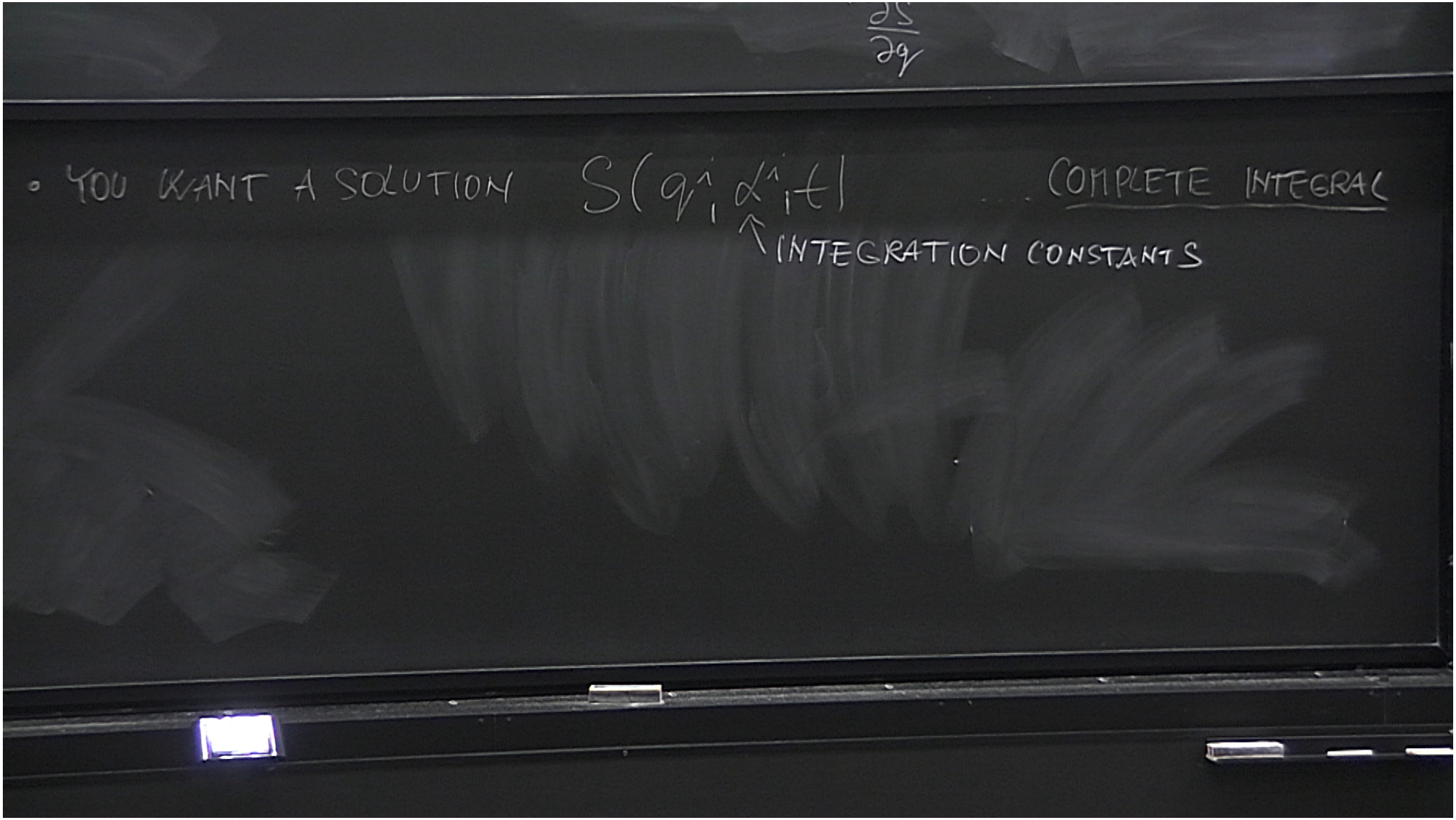


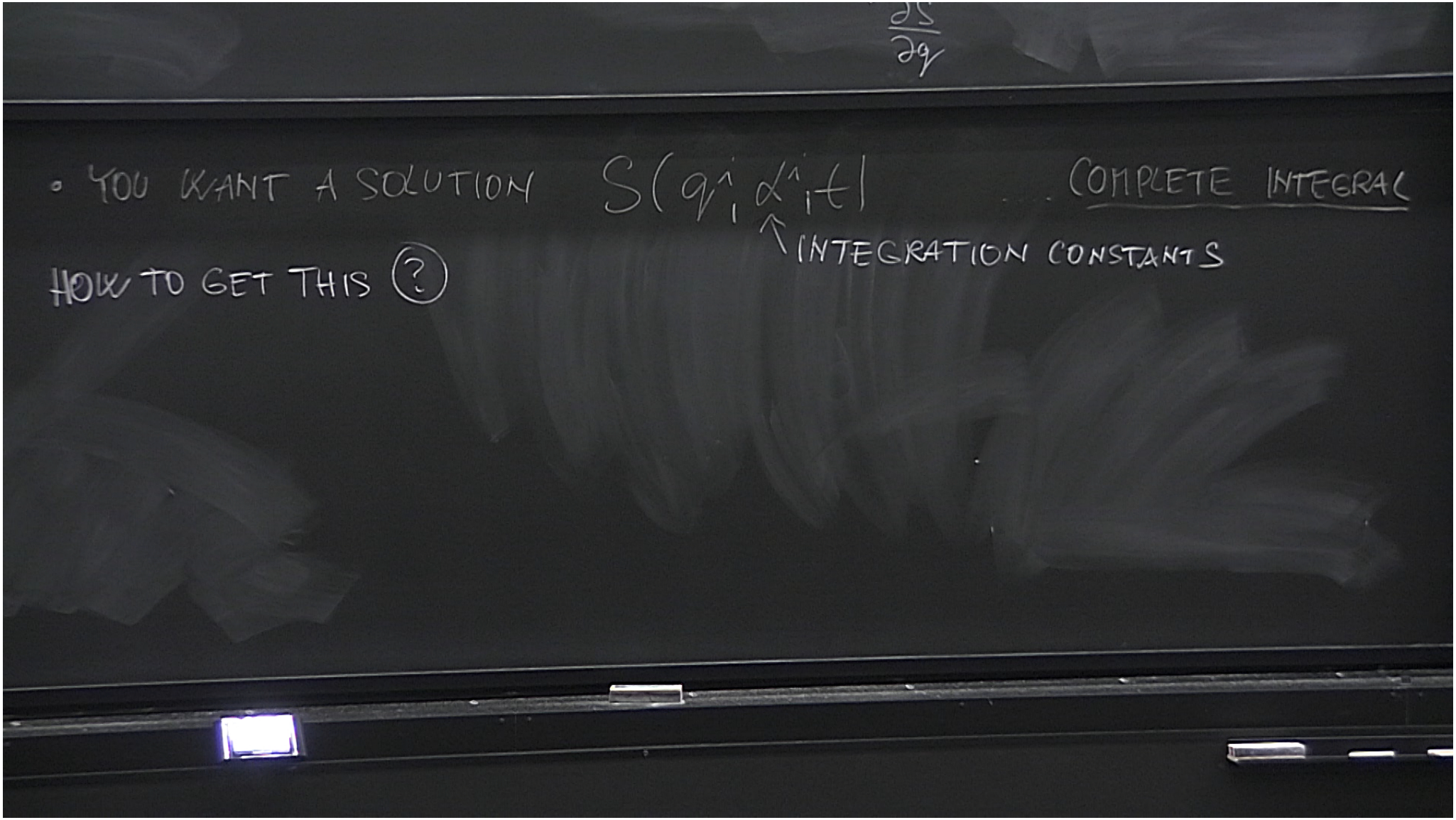
S. SOLUTION TO

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0$$

HAMILTON-JACOBI

1 PDE $S(q_i, t)$


$\frac{\partial S}{\partial q}$
• YOU WANT A SOLUTION $S(q_i, \alpha_i, t)$ ... COMPLETE INTEGRAL
↑ INTEGRATION CONSTANTS


$\frac{\partial S}{\partial q}$
• YOU WANT A SOLUTION $S(q_i, \alpha_i, t)$ ... COMPLETE INTEGRAL
↑ INTEGRATION CONSTANTS
HOW TO GET THIS (?)

$\frac{\partial S}{\partial q}$

- YOU WANT A SOLUTION $S(q^i, \alpha^i, t)$ ... COMPLETE INTEGRAL

↑ INTEGRATION CONSTANTS

HOW TO GET THIS (?)

YOU CAN TRY SEPARATION OF VARIABLES.

$H \neq H(t)$ $\quad S(q^i, t) = S_0(q^i) - Et$

↑ INT. CONST

$H \neq H(q_c)$ $\quad S(q^i, t) = \alpha_c q_c + S_0(q^i)$

$q^i \neq q_c$

$\frac{\partial S}{\partial q}$

- YOU WANT A SOLUTION $S(q^i, \alpha^i, t)$ ... COMPLETE INTEGRAL

↑ INTEGRATION CONSTANTS

HOW TO GET THIS (?)

YOU CAN TRY SEPARATION OF VARIABLES.

$H \neq H(t)$ $\quad S(q^i, t) = S_0(q^i) - Et$

↑ INT. CONST

$H \neq H(q_c)$ $\quad S(q^i, t) = \alpha_c q_c + S_0(q^i)$

↑ $\quad q^i \neq q_c$

$= \frac{\partial S}{\partial q}$

- YOU WANT A SOLUTION $S(q^i, \alpha^i, t)$ ... COMPLETE INT

↑ INTEGRATION CONSTANTS

HOW TO GET THIS (?)

YOU CAN TRY SEPARATION OF VARIABLES.

$H \neq H(t)$ $\quad S(q^i, t) = S_0(q^i) - Et$

↑ INT. CONST

$H \neq H(q_c)$ $\quad S(q^i, t) = \alpha_c q_c + S_0(q^i)$

↑ $\quad q^i \neq q_c$

IN GENERAL TRY $S(q^1, q^2, \ldots) = \sum_i S_i(q^i) = S_1(q^1) + S_2(q^2) \ldots$

$\frac{\partial S}{\partial q}$

- YOU WANT A SOLUTION $S(q^i, \alpha^i, t)$ ... COMPLETE INTEGRAL

↑ INTEGRATION CONSTANTS

HOW TO GET THIS (?)

YOU CAN TRY SEPARATION OF VARIABLES.

$H \neq H(t)$ $\quad S(q^i, t) = S_0(q^i) - Et$

↑ INT. CONST

$H \neq H(q_c)$ $\quad S(q^i, t) = \alpha_c q_c + S_0(q^i)$

↑ $\quad q^i \neq q_c$

IN GENERAL TRY $S(q^1, q^2, \ldots) = \sum_i S_i(q^i) = S_1(q^1) + S_2(q^2) \ldots$

BE CAREFUL NOT TO TRY FOREVER !!

$\frac{\partial S}{\partial q}$

- YOU WANT A SOLUTION $S(q_i, \alpha_i, t)$ COMPLETE INTEGRAL

↑ INTEGRATION CONSTANTS

HOW TO GET THIS (?)

YOU CAN TRY SEPARATION OF VARIABLES.

$H \neq H(t)$ $\quad S(q_i, t) = S_0(q_i) - Et$

↑ INT CONST

$H \neq H(q_c)$ $\quad S(q_i, t) = p\, q_c + S_0(q_i)$ $\quad q_i \neq q_c$

IN GENERAL TRY $S(q_1, q_2, \ldots) = \sum S_i(q_i) = S_1(q_1) + S_2(q_2) \ldots$

BE CAREFUL NOT TO TRY FOREVER !!