Title: PSI 2018/2019 - Classical Physics - Lecture 2

Date: Aug 15, 2018 09:00 AM

URL: http://pirsa.org/18080010

Abstract:

2) HAMILTONIAN MECHANICS

a) HAMILTON'S CANONICAL EQUATIONS (1834)

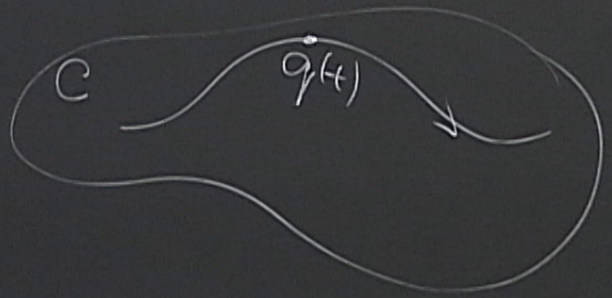
- CANONICAL MOMENTUM

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

CONFIGURATION SPACE ... C

"PHOTO OF THE STSTEM"

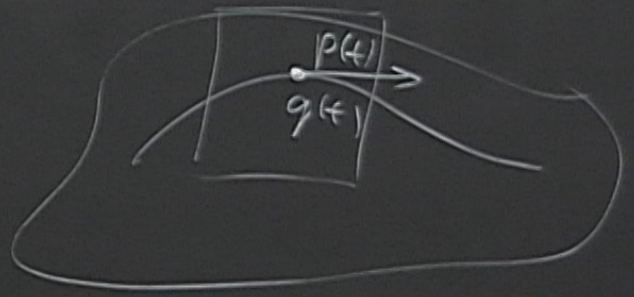
$\{q^i\}$ ... $\dim = n$



PHASE SPACE ... T^*C

DESCRIBES PHYSICAL STATES

$\{q^i, p_j\}$ ... $2n$





HGIAN

HAMILTONIAN = LEGENDRE TRANSFORM OF LAGRANGIAN

$$H(q^i, p_i, t) = p_i \dot{q}^i - L \Big|_{\dot{q} = \dot{q}(q, p, t)}$$

THEOREM: SYSTEM OF n (E-L) IS EQUIVALENT TO THE SYSTEM OF $2n$ HAMILTON'S EQS

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j}$$

PROOF: ACTION PRINCIPLE

$$0 = \delta S = \delta \int_1^2 (p\dot{q} - H)\,dt = \int_1^2 (\dot{q}\delta p + p\delta\dot{q} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p)\,dt$$

$$= [p\delta q]_1^2 + \int_1^2 \left(\left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right) dt$$

$\dot{q} - \frac{\partial H}{\partial p} = 0$ $\qquad$ $\dot{p} + \frac{\partial H}{\partial q} = 0$

- $\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

  $\dot{q} = \frac{\partial H}{\partial p}$ $\qquad$ $\dot{p} = -\frac{\partial H}{\partial q}$

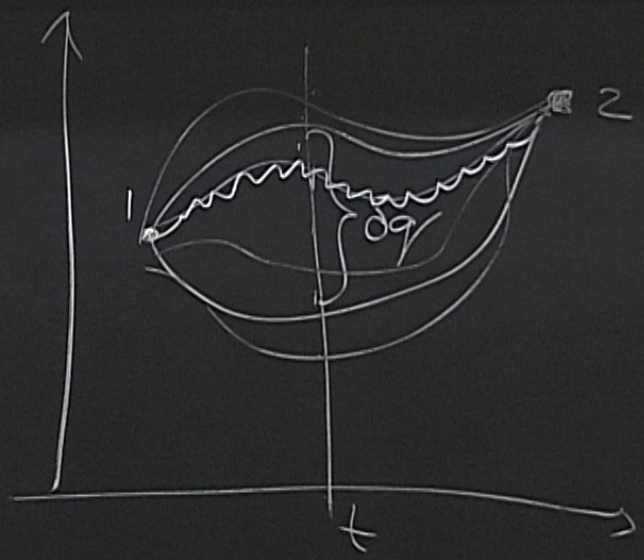
SO IF $H \neq H(t)$

$\Rightarrow \frac{dH}{dt} = 0$ (?)

$\dot{q})\,dt$

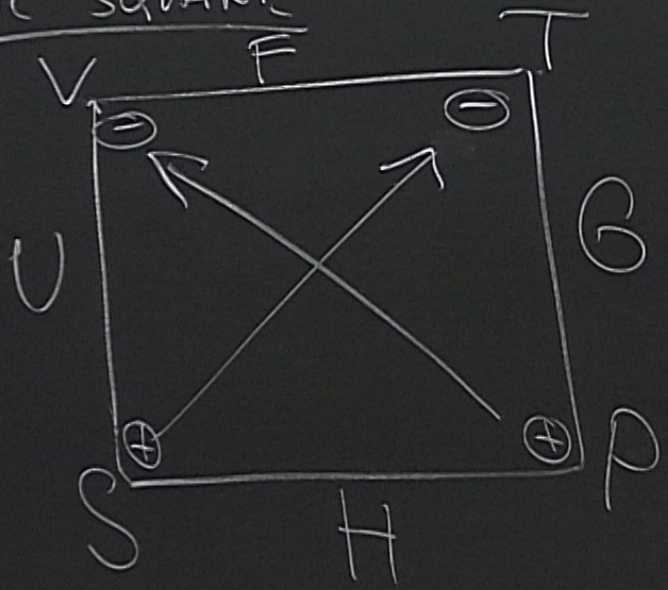
IF $H \neq H(t)$

$=0$?



INTERMEZZO: LEGENDRE TRANSFORM IN TDS.

MAGIC SQUARE

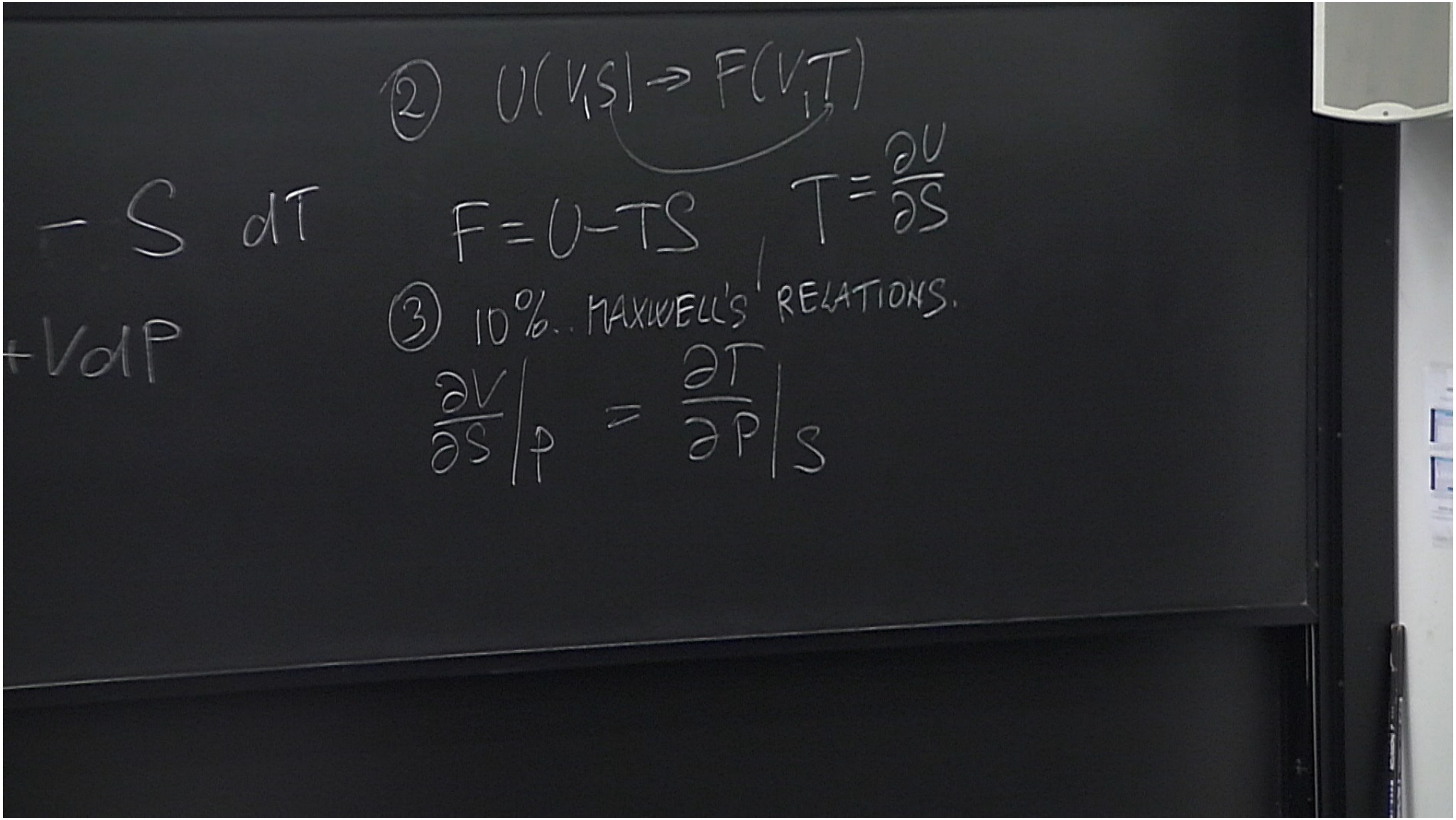


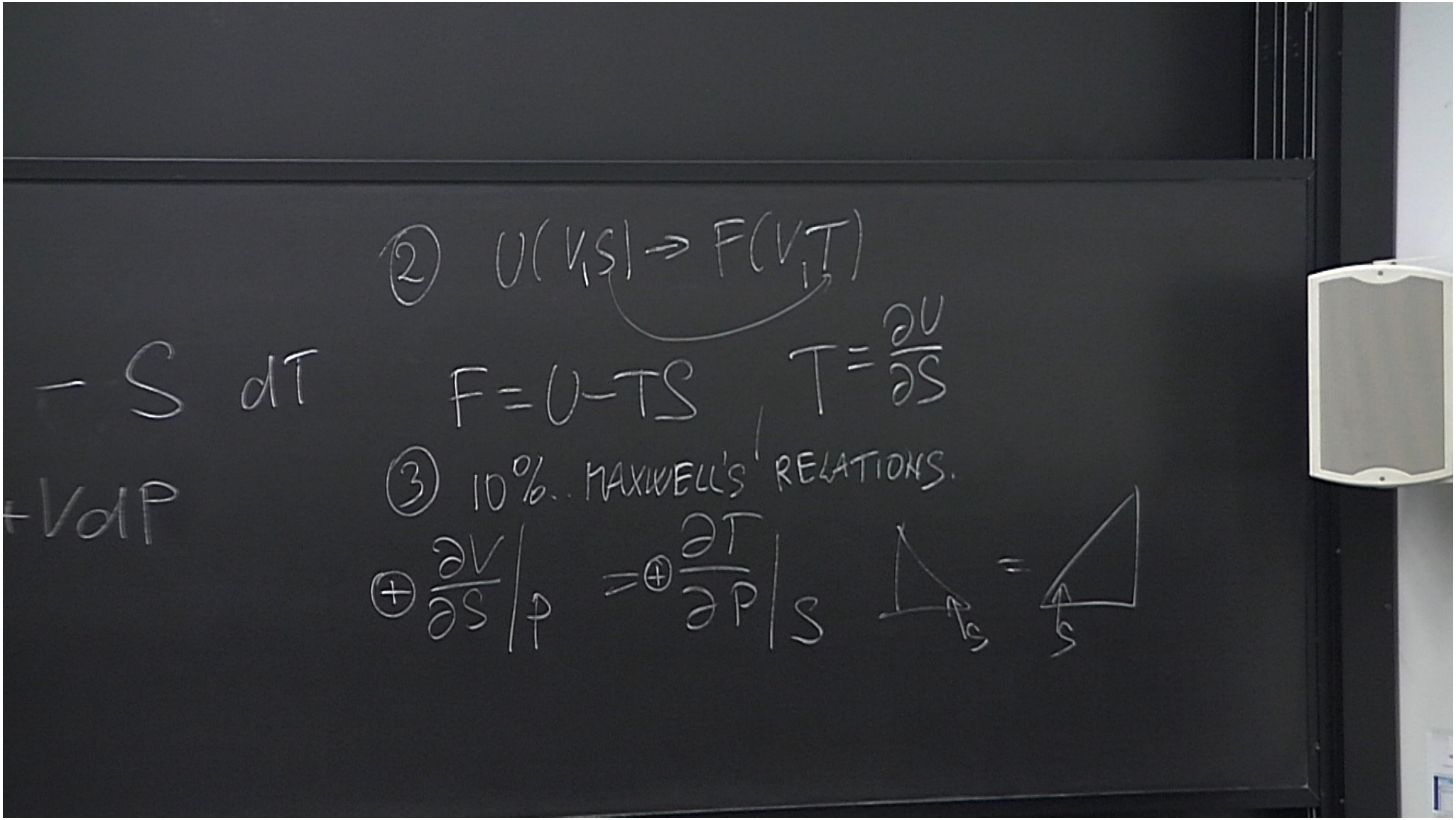
① $F = F(V,T)$

$dF = -P\,dV - S\,dT$

$dG = -S\,dT + V\,dP$

95%

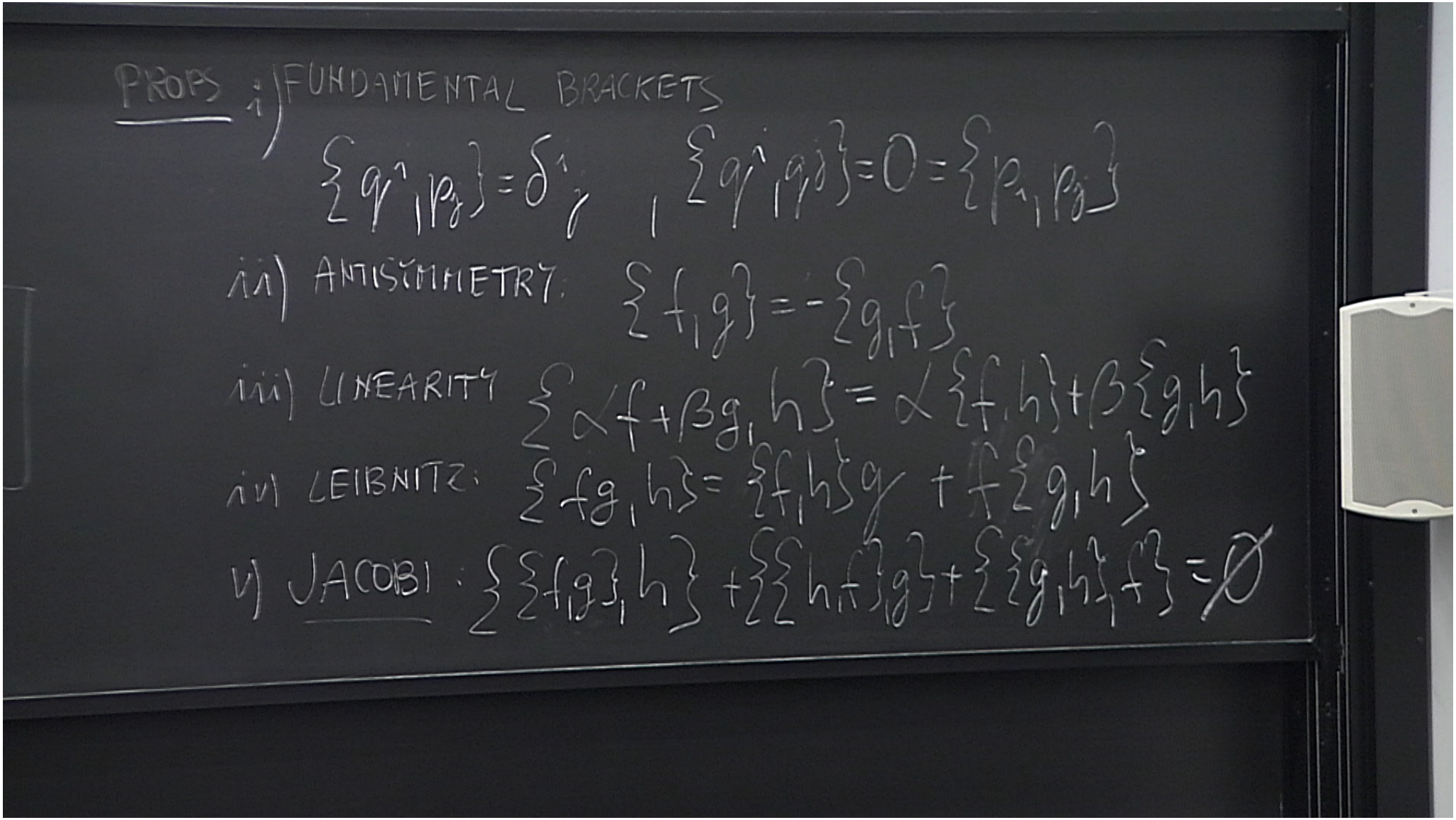

② U(V,S) → F(V,T)
$-S\,dT$
$F = U - TS$, $T = \frac{\partial U}{\partial S}$
$+V\,dP$
③ 10%.. MAXWELL'S RELATIONS.
$\left.\frac{\partial V}{\partial S}\right|_P = \left.\frac{\partial T}{\partial P}\right|_S$


② U(V,S) → F(V,T)
$-S\,dT$
$F = U - TS$
$T = \frac{\partial U}{\partial S}$
③ 10%.. MAXWELL'S RELATIONS.
$+V\,dP$
$\oplus \left.\frac{\partial V}{\partial S}\right|_P = \oplus \left.\frac{\partial T}{\partial P}\right|_S$
P_2
S

b) POISSON BRACKETS

DEF: LET f, g BE 2 PHASE SPACE FUNCTION $\Rightarrow$ THEIR CANONICAL POISSON BRACKET IS A NEW FUNCTION

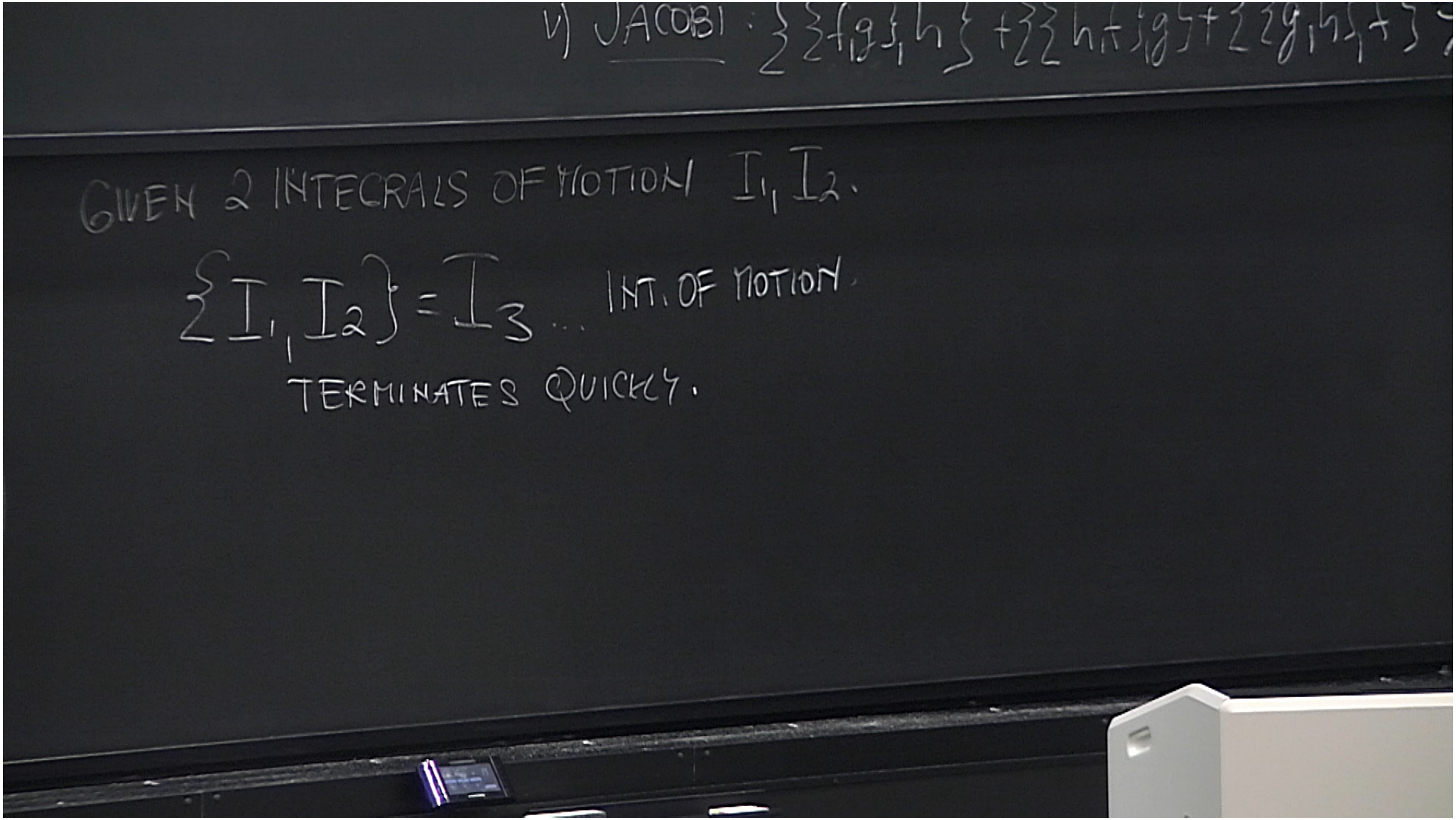
$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}$$


PROPS i) FUNDAMENTAL BRACKETS
$\{q^i, p_j\} = \delta^i_j$, $\{q^i, q^j\} = 0 = \{p_i, p_j\}$
ii) ANTISYMMETRY: $\{f, g\} = -\{g, f\}$
iii) LINEARITY $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$
iv) LEIBNITZ: $\{fg, h\} = \{f, h\}g + f\{g, h\}$
v) JACOBI: $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

SPECIFICALLY. INTEGRAL OF MOTION I.

$$\frac{dI}{dt} = 0 \rightarrow \{I, H\} = 0$$


v) JACOBI
GIVEN 2 INTEGRALS OF MOTION I_1, I_2.
$\{I_1, I_2\} = I_3$ ... INT. OF MOTION.
TERMINATES QUICKLY.

$q_i(p,t)$ PROOF: ACTION PRINCIPLE

S.THAT ONE CAN FIND NEW HAMILTONIAN $H'(Q_j, P_j, t)$
PRESERVING THE FORM OF HAMILTON'S EQS.

$$\dot{P}_j = -\frac{\partial H'}{\partial Q_j}, \quad \dot{Q}_j = \frac{\partial H'}{\partial P_j}$$

$\vec{F} = m\vec{a}$
↑ REAL FORCE

- NOT EVERY TRANSF IS CANONICAL (NOT A PROOF)

$H = \frac{p^2}{2m}$ $\quad Q = \sqrt{q}, \; P = p \quad$ $H' = \frac{P^2}{2m}$ GUESS:

$$\frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} = 2Q\dot{Q} \neq \dot{Q}$$

↑ CAN EQ.

$\Rightarrow \frac{dH}{dt} = 0$ (?)



- CANONICAL TRANSFORMATIONS ARE GENERATED BY GENERATING FUNCTIONS

$$F = F(\text{OLD}, \text{NEW})$$

$$\text{OLD} \in (q, p)$$

$$\text{NEW} \in (Q, P)$$

... 4 POSSIBILITIES.

LET'S PICK $F(q, Q, t)$ FOR SUFFICIENTLY COMPLICATED H

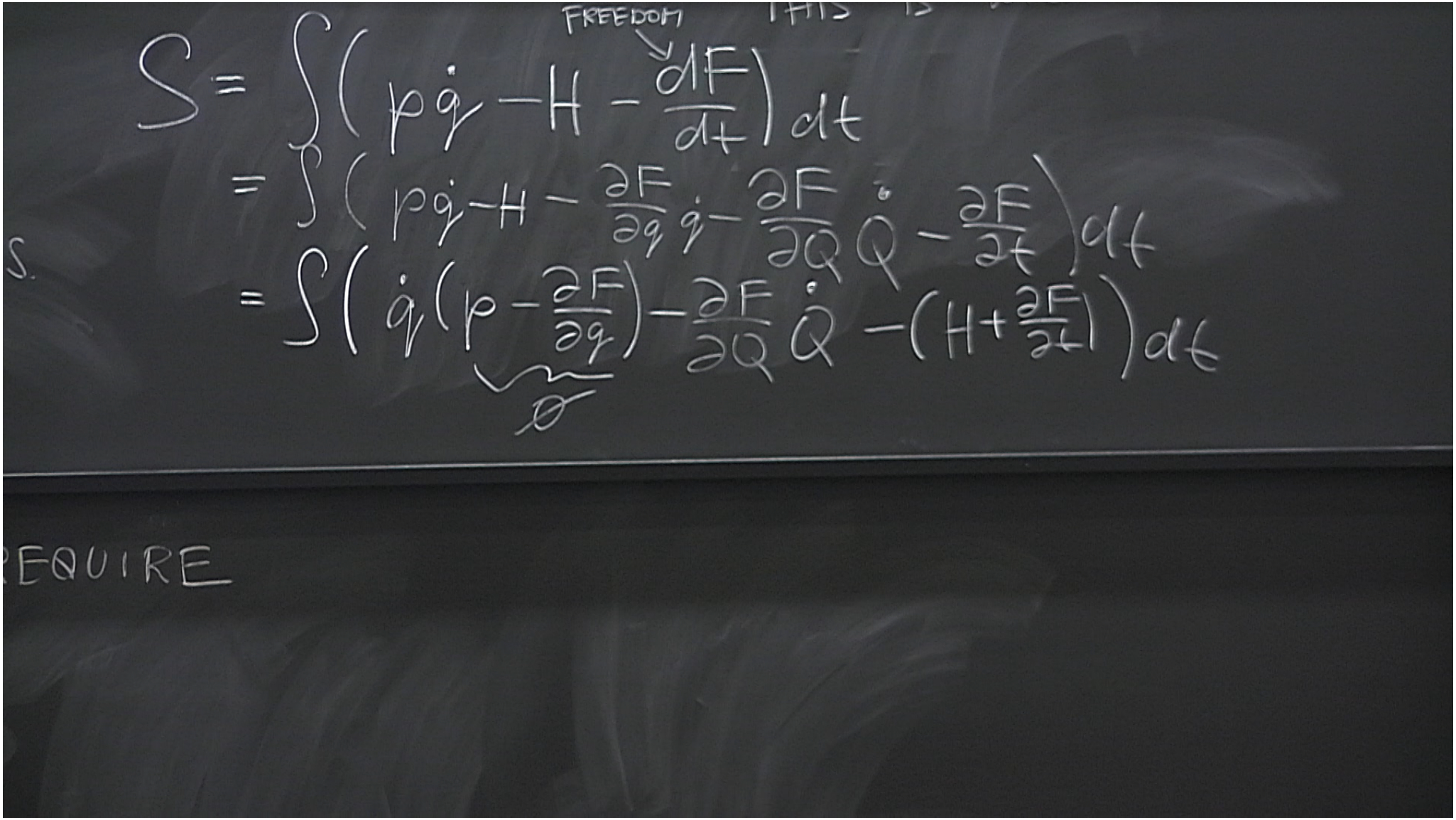
FREEDOM

THIS IS W.L.O.G.

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt$$

$$= \int \left(p\dot{q} - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \right) dt$$

$$= \int \left(\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right) \right) dt$$


FREEDOM
$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt$
$= \int \left(p\dot{q} - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \right) dt$
$= \int \left(\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right) \right) dt$
σ
EQUIRE

FREEDOM

THIS IS

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt$$

$$= \int \left(p\dot{q} - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \right) dt$$

$$= \int \left(\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right) \right) dt$$

$\underbrace{\qquad}_{0}$ $\underbrace{\qquad}_{P}$

ITIES.

REQUIRE

FREEDOM

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt$$

$$= \int \left(p\dot{q} - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \right) dt$$

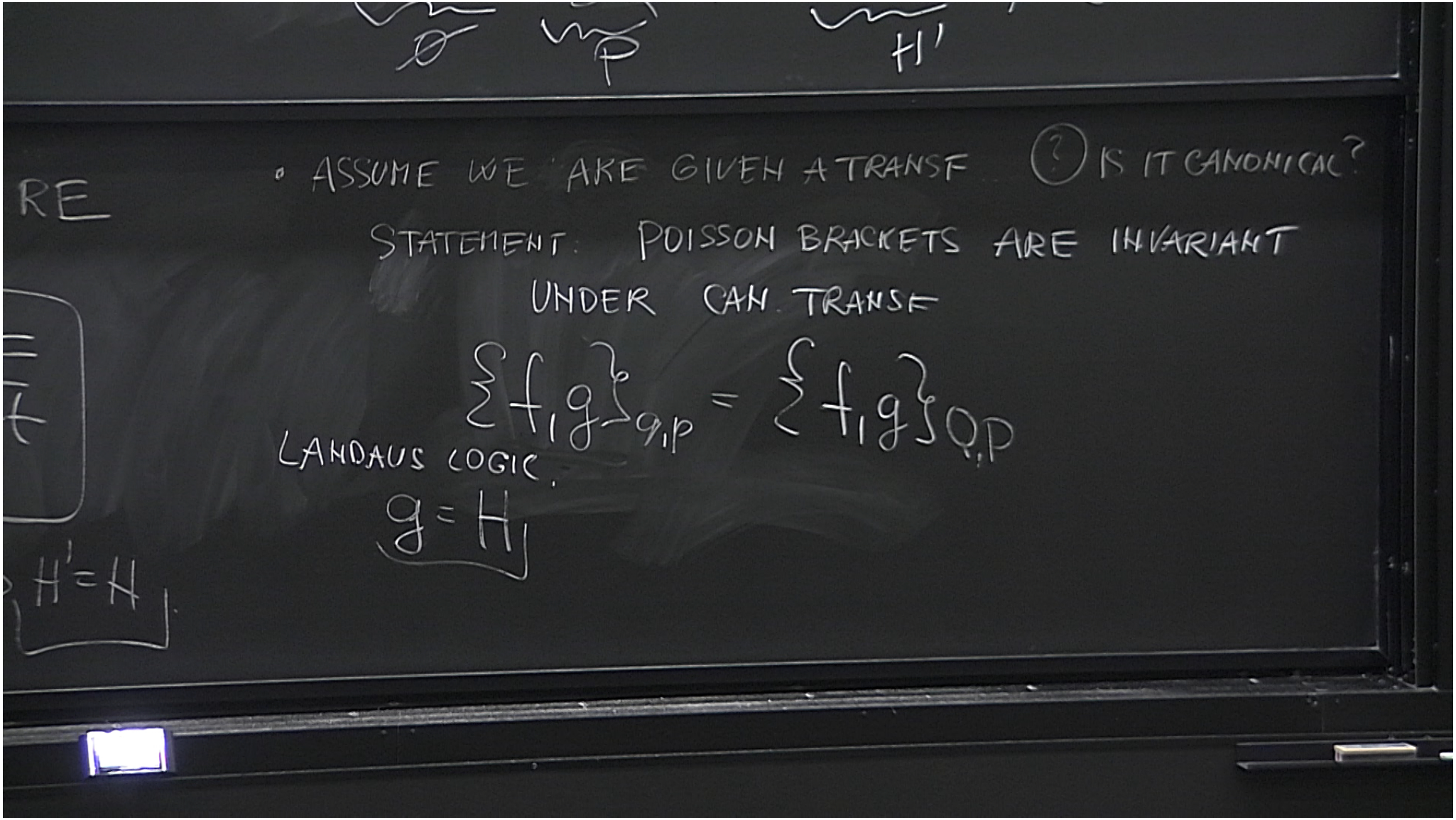
$$= \int \left(\dot{q}\underbrace{\left(p - \frac{\partial F}{\partial q}\right)}_{\emptyset} - \underbrace{\frac{\partial F}{\partial Q}}_{P}\dot{Q} - \underbrace{\left(H + \frac{\partial F}{\partial t}\right)}_{H'} \right) dt$$

REQUIRE

$$= \int \left(P\dot{Q} - H'(P,Q,t) \right) dt \quad \ldots \text{REQUIRE}$$

$$p = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q}, \quad H' = H + \frac{\partial F}{\partial t}$$

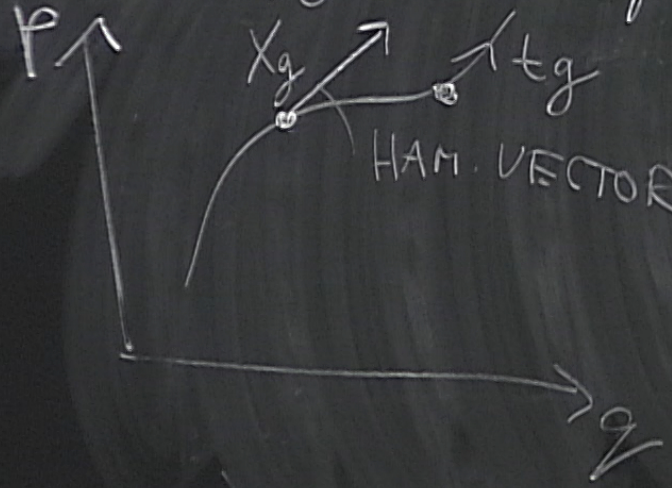
$F \neq F(t) \Rightarrow H' = H$


H'
RE
• ASSUME WE ARE GIVEN A TRANSF ? IS IT CANONICAL?
STATEMENT: POISSON BRACKETS ARE INVARIANT
UNDER CAN. TRANSF
$\{f,g\}_{q,p} = \{f,g\}_{Q,P}$
LANDAUS LOGIC:
$g = H$
$H' = H$

SPECIFICALLY INTEGRAL OF MOTION !

.. GENERATES "TIME FLOW t_g"

$$\{f, g\} = \frac{df}{dt_g} = X_g f$$



... COORDINATE INDEPENDENT.

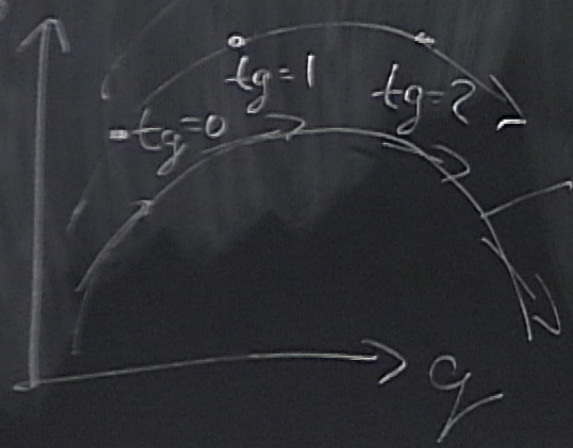
HAM. VECTOR FLOW:

$$X_g = \{\cdot, g\} = \frac{\partial \cdot}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial \cdot}{\partial p}$$

$$= \frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}$$

. DIRECTIONAL DERIVATIVE

= VECTOR



INTEGRAL CURVE.