

Title: PSI 2018/2019 - Lie Groups and Lie Algebras - Lecture 6

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Abstract:

$$g = e^{i\xi T}$$

$$dg = e^{i\xi T} d\xi T$$

$$(\xi, T)(d\xi, T)$$

$$g^{-1}dg \in T^*(u) \otimes \mathcal{L}(g)$$

$$T_a^b \in T^*(u) \otimes T(u)$$

ONS. $\epsilon = \epsilon^* = \epsilon^T$

SU(2)

9)

$$2 = \psi_a, a=1,2$$

$$2^* = \psi_a^* = \psi^a$$

$$2 = 2^*$$

1)

$$\epsilon^{ab}, \epsilon_{ab}, \delta_b^a, \epsilon^{b'a'}$$

$$\chi^b = \epsilon^{ba} \psi_a \rightarrow \underbrace{(U^\dagger)_{b'}^b (U^\dagger)_{a'}^a}_{\delta_{a'}^c} U_{a'}^c \psi_c$$

$$= \epsilon^{b'a'} (U^\dagger)_{b'}^b \delta_{a'}^c \psi_c = (U^\dagger)_{b'}^b \epsilon^{b'c} \psi_c = (U^\dagger)_{b'}^b \chi^b$$

$SU(2)$ -tensor

$$\psi_a \equiv 2$$
$$\epsilon^{ab} \psi_b \equiv 2^*$$

$$\psi_{ab} = \psi_{[ab]} + \psi_{(ab)}$$

Check under $SU(2)$ $\psi_{[ab]} \neq \psi_{(ab)}$ don't mix.

$\psi_{abcd} \rightarrow$ no invariant irreducible.

$$\chi_a \psi^b \rightarrow$$

$$2 \otimes 2$$

$$\chi_a \psi^b = \underbrace{\delta^a_b \chi_a \psi^b}_1 \text{ transform as a singlet.}$$

$$1. \phi_{ab} = \chi_a \psi_b$$

$$\chi_a \psi_b = 2 \otimes 2$$

$$\chi_a \psi_b = \chi_{[a} \psi_{b]} + \chi_{(a} \psi_{b)}$$

$$2 \otimes 2 = 1 \oplus 3$$

$$\begin{aligned} \phi &\rightarrow \phi \\ \psi_a &\rightarrow U_a^b \psi_b \end{aligned}$$

Young tableau:

$$\psi_a = 2 = \begin{array}{|c|} \hline a \\ \hline \end{array}$$

$$\psi_{[ab]} = 1 = \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \quad \left. \vphantom{\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}} \right\} \text{antisymmetrized}$$

$$\psi_{(ab)} = 3 = \begin{array}{|c|c|} \hline a & b \\ \hline \end{array}$$

$$2 \otimes 2 = 1 \oplus 3$$

$$\square \times \square = \begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$$

For $SU(2)$



$$2 \otimes 2 \otimes 2 = ?$$



$$\begin{aligned}
 &= \boxed{a} \\
 &1 = \begin{bmatrix} a \\ b \end{bmatrix} \} \text{ antisymmetrized} \\
 &3 = \boxed{ab}
 \end{aligned}$$

For SU(2)

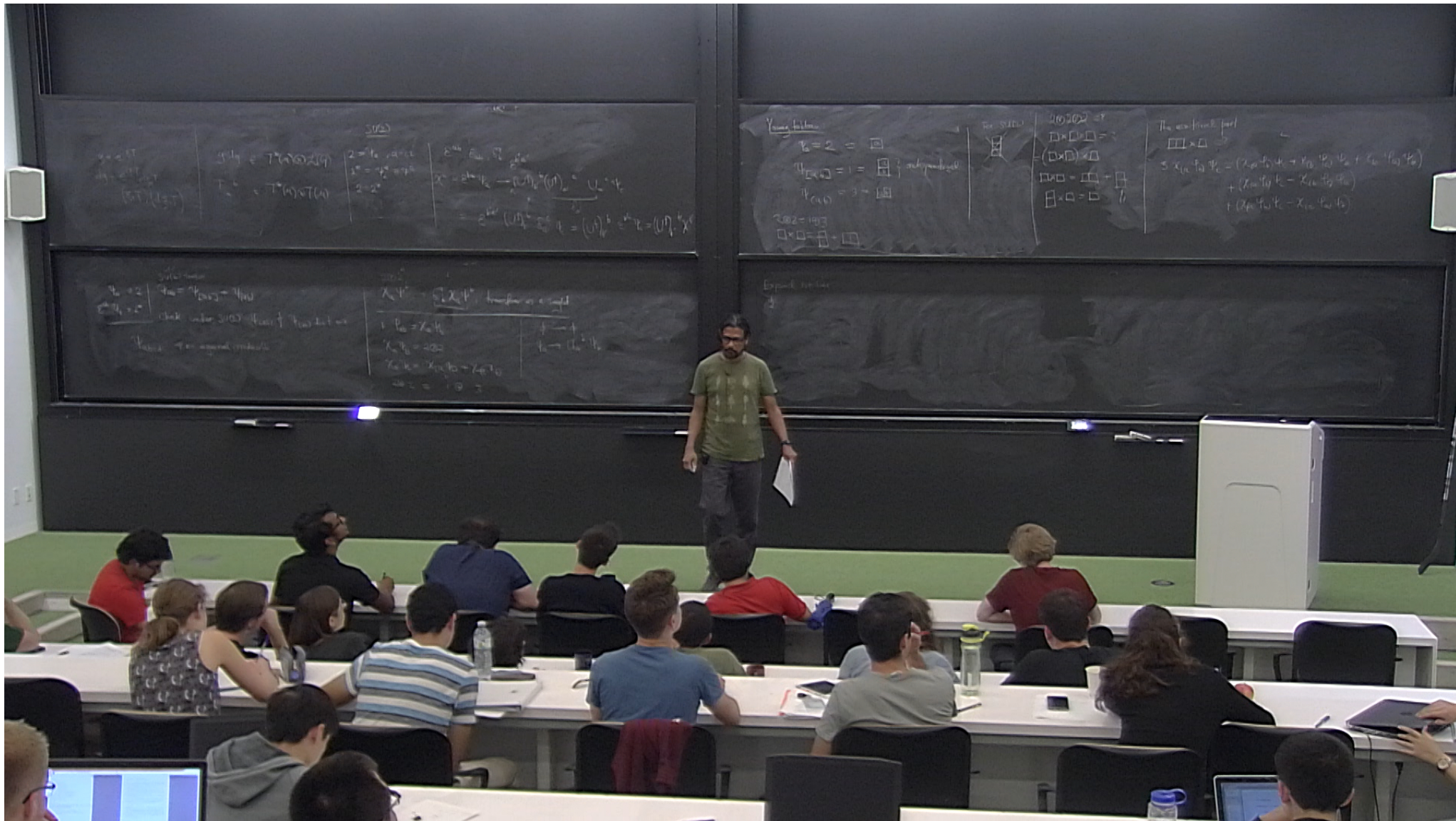


$$\begin{aligned}
 &2 \otimes 2 \otimes 2 = ? \\
 &\square \times \square \times \square = ? \\
 &= (\square \times \square) \times \square \\
 &\square \times \square = \square\square + \boxed{} \\
 &\boxed{} \times \square = \square \quad 1
 \end{aligned}$$

The non-trivial part

$$\square\square \times \square = ?$$

$$3 \chi_{(a \phi_b)} \psi_c = (\chi_{(a \phi_b)} \psi_c)$$



$$\square \times \square = \begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$$

Expand 1st line

$$\frac{1}{2} [\chi_a \psi_b \psi_c + \chi_b \psi_a \psi_c + \chi_b \psi_c \psi_a + \chi_c \psi_b \psi_a + \chi_c \psi_a \psi_b + \chi_a \psi_c \psi_b]$$

→ Symmetric in abc

$$\square \times \square = \begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$$

Expand 1st line

$$\frac{1}{2} [\chi_a \psi_b \psi_c + \chi_b \psi_a \psi_c + \chi_b \psi_c \psi_a + \chi_c \psi_b \psi_a + \chi_c \psi_a \psi_b + \chi_a \psi_c \psi_b]$$

→ Symmetric in abc

$$\psi(a, a_n) = n+1$$

$$\frac{1}{2} [\chi_a \psi_b \psi_c + \chi_b \psi_a \psi_c - \chi_b \psi_c \psi_a - \chi_c \psi_b \psi_a]$$

$$\square \times \square = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$$

Expand 1st line

$$\frac{1}{2} [\chi_a \psi_b \psi_c + \chi_b \psi_a \psi_c + \chi_b \psi_c \psi_a + \chi_c \psi_b \psi_a + \chi_c \psi_a \psi_b + \chi_a \psi_c \psi_b]$$

→ Symmetric in abc

$$\psi(a, a_n) = n+1$$

$$\left(\frac{1}{2} [\chi_a \psi_b \psi_c + \chi_b \psi_a \psi_c - \chi_b \psi_c \psi_a - \chi_c \psi_b \psi_a] \right) \epsilon^{ca} = \eta_b = 2$$

antisymmetric in c ↔ a and contract it with ϵ^{ca}

$$= \begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$$

$$\chi_c \psi_a \psi_b + \chi_b \psi_a \psi_c + \chi_c \psi_b \psi_a + \chi_c \psi_a \psi_b + \chi_a \psi_c \psi_b]$$

antisymmetric in abc

$$\chi_a) = n+1$$

$$[\chi_b \psi_c + \chi_b \psi_a \psi_c - \chi_b \psi_c \psi_a - \chi_c \psi_b \psi_a] \epsilon^{ca} = \eta_b = 2$$

antisymmetric in c and a and contract it with ϵ^{ca}

$$A_{cab} = -A_{acb}$$

121	211
122	212

$$\begin{array}{|c|} \hline \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|} \hline \square \\ \hline \end{array}$$

$$3 \otimes 2 =$$

$$+ \chi_a \psi_c \psi_b]$$

$$A_{cab} = -A_{acb}$$

$$\begin{array}{c|c} 121 & 211 \\ 122 & 212 \end{array}$$

$$= 2$$

$$\square \square \times \square = \square \square \square + \square$$

$$3 \otimes 2 = 4 \oplus 2$$

$$\text{For } SU(2) \quad \boxed{a_1 | a_2 | \dots | a_n} = n+1$$

$$\square \times \square \times \square = \square \square \square + \square + \square$$

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$$

antisymmetric in $c \neq a$ and column 1

SU(3) tensors:

$3 = \psi_a$ Fundamental

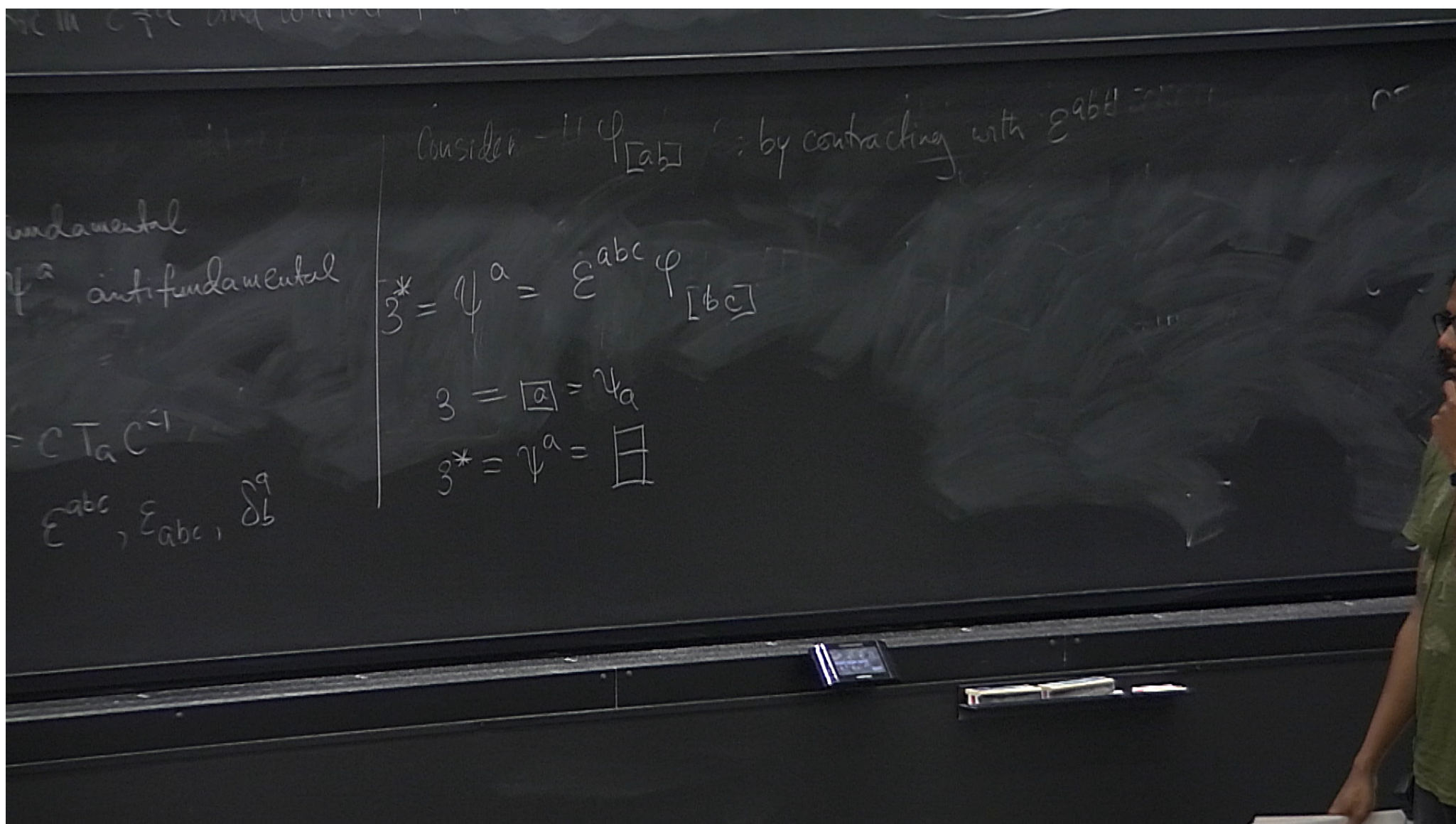
$3^* = \psi_a^* = \psi^a$ antifundamental

$$\tilde{T}_a = -T_a^*$$

No C st $\tilde{T}_a = C T_a C^{-1}$

Invariant tensors. $\epsilon^{abc}, \epsilon_{abc}, \delta_b^a$

Consider $\psi \psi^a \epsilon_{ab}$



fundamental
 ψ^a antifundamental

$$= C T_a C^{-1}$$

$$\epsilon^{abc}, \epsilon_{abc}, \delta^a_b$$

Consider $\psi_{[ab]}$: by contracting with ϵ^{abd}

$$\bar{3}^* = \psi^a = \epsilon^{abc} \psi_{[bc]}$$

$$3 = \boxed{a} = \psi_a$$

$$\bar{3}^* = \psi^a = \boxed{}$$

$$\epsilon^{abc} \psi_{[bc]} \rightarrow$$

III CFC and ...

fundamental

ψ^a antifundamental

$$= C T_a C^{-1}$$

$$\epsilon^{abc}, \epsilon_{abc}, \delta^a_b$$

Consider $\psi_{[ab]}$ by contracting with ϵ^{ab}

$$\bar{3}^* = \psi^a = \epsilon^{abc} \psi_{[bc]}$$

$$3 = \psi_a = \psi_a$$

$$\bar{3}^* = \psi^a = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

$$1 = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

$$\epsilon^{abc} \psi_{[bc]} \rightarrow$$

Examples:

1. $3 \otimes 3 = ?$

$$\psi_a \chi_b = \underbrace{\psi_{[a} \chi_{b]}}_3 + \underbrace{\psi_{(a} \chi_{b)}}_6$$

$$\square \times \square = \begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$$

$$3 \otimes 3 = 3^* \oplus 6$$

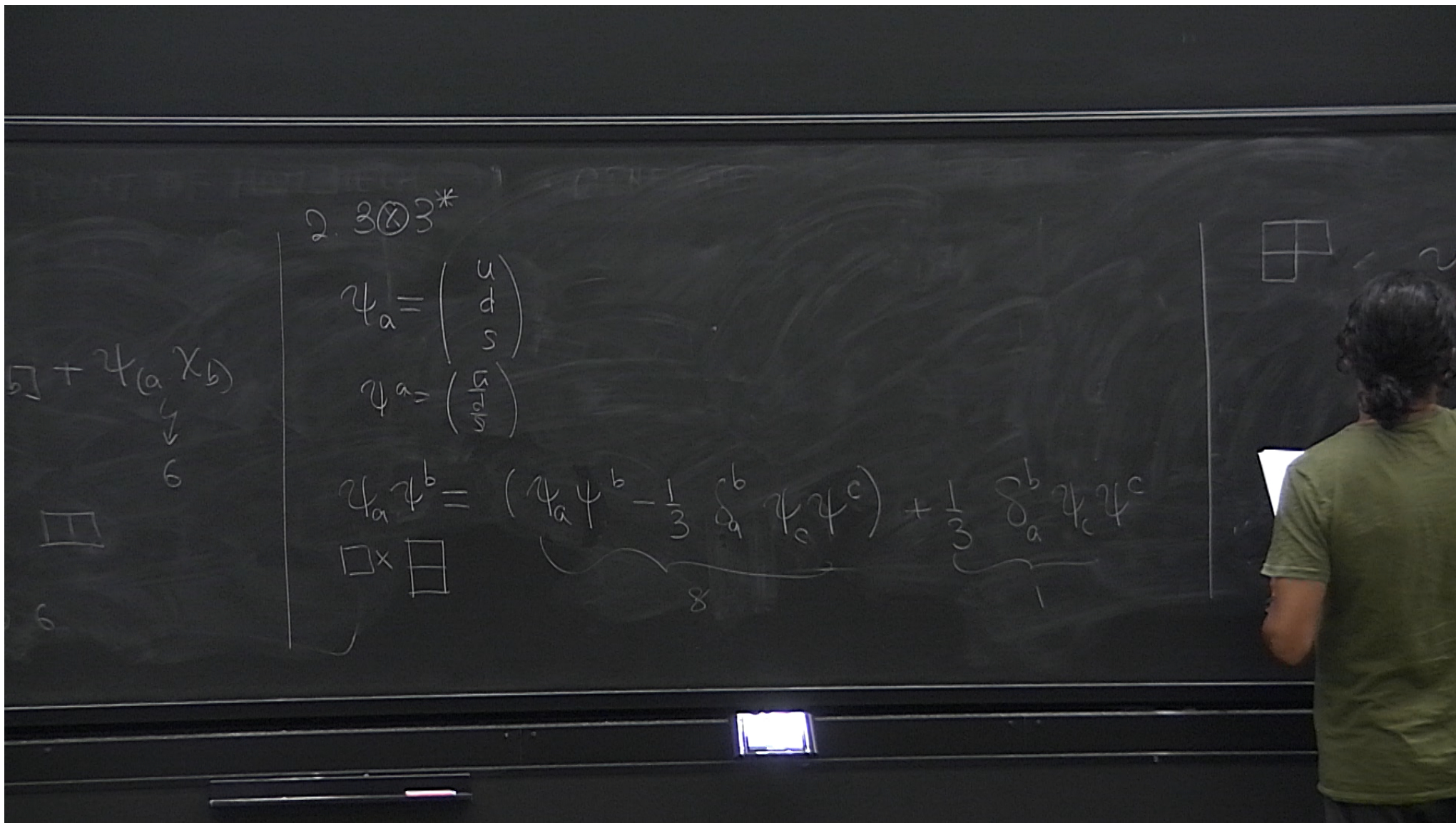
2. $3 \otimes 3^*$

$$\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$\psi^a = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \end{pmatrix}$$

$$\psi_a \psi^b = \left(\psi_a \psi^b - \frac{1}{3} \delta_a^b \psi_c \psi^c \right) + \underbrace{\frac{1}{3} \delta_a^b \psi_c \psi^c}_8$$

$$\square \times \begin{array}{|c|} \hline \square \\ \hline \end{array}$$



$$P_a^b = \psi_a \psi^b - \frac{1}{3} \delta_a^b \psi_c \psi^c$$

$$= \begin{pmatrix} \frac{\pi_0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi_0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta_8}{\sqrt{6}} \end{pmatrix}$$

$$3 \times 3 \times 3 = ?$$

$$\square \times \square \times \square = \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline \end{array} + \begin{array}{|c|} \hline \\ \hline \end{array}$$

$$3 \otimes 3 \otimes 3$$

$$10 \oplus$$

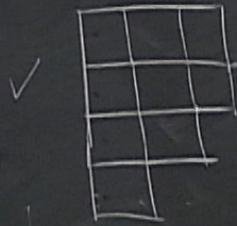
$$8 \oplus$$

$$8 \oplus$$

$$1$$

Young's Tableaux

1. In a legal Young's tableau the length of any row cannot be larger than



2. The number of boxes in a column for $SU(n)$ cannot be more than n due to

antisymmetric in $c \rightarrow a$ and column $1 \rightarrow 2$

3. Suppose we have a Young's tableau with n boxes. We have a rank n tensor to the tableau.

a_1	a_2	a_3	a_4
a_5	a_6	a_7	
a_8			

$$\psi_{a_1 \dots a_8}$$

$$\psi(a_1 a_2 a_3 a_4) (a_5 a_6 a_7) a_8$$

3a) We first symmetrize the indices which lie on each row

3b) Antisymmetrize the columns

4. How to compute the dimension of a tableau

$SU(N)$

N	$N+1$	$N+2$
$N-1$	N	
$N-2$	$N-1$	
$N-3$		

6	4	1
4	2	
3	1	
1		

$SU(3)$:

3	4
2	

3	1
1	

$$= \frac{3 \times 4 \times 2}{3 \times 1 \times 1} = 8$$

5. How to expand the product of 2 tableaux as a sum of tableaux

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline a & a \\ \hline b & \\ \hline b & \\ \hline \end{array}$$

$T_1 \quad T_2$

1. Label the rows of T_2 by a for the first row, b for the second row, c for the third...
2. Add one box from T_2 to T_1 at a time starting from the top row making sure the following

Each tableau
#1 $\wedge$ must be a legal Young tableau at each stage

2. More than one box with the same label must not appear in the same column

3. For each box assign a number $n_a = \#$ of a 's above and right to it

Similarly assign a " $n_b = \#$ of b 's "

$$n_a \geq n_b \geq n_c \geq \dots$$

r in the same column
ght to it

C. If two tableaux are the same then they are
counted the same if their labels are the same.

Otherwise they refer to distinct irreps.

D. Cancel columns with n boxes for $SU(n)$ because
they are trivial.

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \square$$

Example: $SU(3)$ $\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline a & a \\ \hline b & \\ \hline \end{array}$

$$= \left(\begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline & \\ \hline \end{array} + \begin{array}{|c|} \hline \\ \hline a \\ \hline \end{array} \right) \times \begin{array}{|c|c|} \hline a \\ \hline b \\ \hline \end{array}$$

$$= \left(\begin{array}{|c|c|c|} \hline & a & a \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline & \\ \hline a & \\ \hline \end{array} \right) \times \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$+ \left(\begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} \right) \times \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline a & & \\ \hline \end{array}$$

Example: $SU(3)$ $\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline a & \\ \hline b & \\ \hline \end{array}$

$$= \left(\begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline & \\ \hline \end{array} + \begin{array}{|c|} \hline \\ \hline a \\ \hline \end{array} \right) \times \begin{array}{|c|c|} \hline a & \\ \hline b & \\ \hline \end{array}$$

$$= \left(\begin{array}{|c|c|c|} \hline & a & a \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & a \\ \hline & \\ \hline a & \\ \hline \end{array} \right) \times \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$+ \left(\begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} + \begin{array}{|c|} \hline \\ \hline a \\ \hline \end{array} \right) \times \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$+ \left(\begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} + \begin{array}{|c|} \hline \\ \hline a \\ \hline \end{array} \right) \times \begin{array}{|c|} \hline b \\ \hline \end{array}$$

$$\begin{aligned}
 &= \left(\begin{array}{|c|c|c|} \hline & a & a \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline a & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & a \\ \hline a & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & a & \\ \hline & & \\ \hline \end{array} \begin{array}{|c|} \hline b \\ \hline \end{array} + \text{th} \right) \\
 &= \begin{array}{|c|c|c|} \hline & a & a \\ \hline b & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & a & a \\ \hline b & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & a & \\ \hline a & b & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & a & \\ \hline b & a & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & a & \\ \hline a & b & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & a & \\ \hline a & b & \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|} \hline & a & a \\ \hline b & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline a & a & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & a & \\ \hline a & b & \\ \hline \end{array} + \begin{array}{|c|c|} \hline a \\ \hline a \\ \hline \end{array} + \begin{array}{|c|c|} \hline a \\ \hline b \\ \hline \end{array} + 1 \\
 &= 27 \oplus 10 \oplus 10^* \quad +8+8+1
 \end{aligned}$$