

Title: PSI 2018/2019 - Lie Groups and Lie Algebras - Lecture 5

Date: Aug 15, 2018 10:30 AM

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Abstract:

$$\# \quad X \rightarrow X' = UXU^\dagger$$

$$U \in U(2)$$

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$$\# X \rightarrow X' = UXU^\dagger$$

$$U \in U(2)$$

Abelian invariant subalgebra: Subalgebra that commutes with every element

Semi-simple Lie algebra: $\mathcal{L}(G)$ which doesn't have an Abelian invariant subalgebra

$$\text{Ex: } \mathcal{L}(SO(4)) = \mathcal{L}(SU(2)) \oplus \mathcal{L}(SU(2))$$

Some useful geometric concepts

Killing metric: $g^{-1}dg \rightarrow$ Left-invariant 1-form

$g \rightarrow hg, h \in G$ but constant

$$g^{-1}dg \in T^*(M) \otimes \mathcal{L}(G)$$

$dg g^{-1} \rightarrow$ Right-invariant 1-form

Bi-invariant $\text{Tr}(\cdot)$

Bi-invariant $\text{Tr}(g^{-1}dg g^{-1}dg) = \text{Tr}(dg g^{-1} dg g^{-1})$

at 1 form $g dg = E_{\mu}^a(\varphi) T^a d\varphi^{\mu}$. $G \rightarrow$ compact Lie group $\text{tr}(T_a T_b) = \lambda \delta_{ab}$

$\varphi^{\mu} \rightarrow$ coordinates on $\mathcal{M} \rightarrow$ The manifold of G

$T^a \rightarrow$ generators in the defining rep

$$\text{Tr}(g^{-1}dg g^{-1}dg) = \underbrace{\begin{pmatrix} E_{\mu}^a & E_{\nu}^b \\ & \lambda \delta_{ab} \end{pmatrix}}_{g_{\mu\nu}(\varphi)} d\varphi^{\mu} d\varphi^{\nu} = g_{\mu\nu}(\varphi) d\varphi^{\mu} d\varphi^{\nu}$$

Some useful geometric concepts

Killing metric $g^{-1}dg \rightarrow$ Left-invariant 1-form

$$g \in G$$

$$g = e^{i\varphi \cdot T}$$

$$dg = g(iT_a dg_a)$$

$$g^{-1}dg = i T_a dg_a$$

$$g \rightarrow hg, h \in G \text{ but constant}$$

$$g^{-1}dg \in T^*(M) \otimes \underline{\mathcal{L}}(G)$$

$$dg g^{-1} \rightarrow \text{Right-invariant 1-form}$$

$$g \rightarrow gh$$

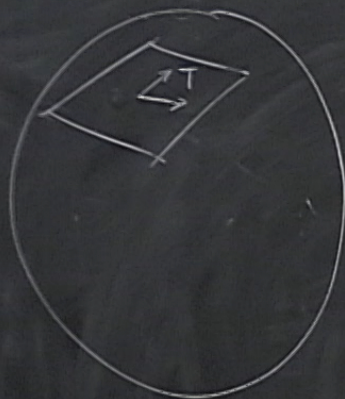
Bi-invariant

$$g dg = E_a^a$$

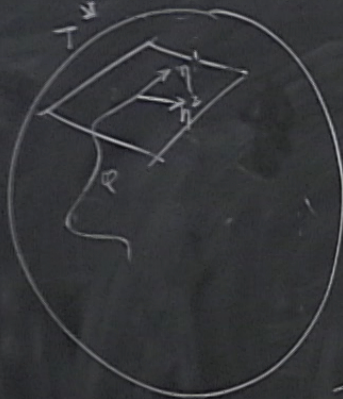
$$\varphi^a \rightarrow$$

$$T^a \rightarrow$$

$$\text{Tr}(g^{-1}dg)$$



$$E^a_m(\psi) d\psi^m = \eta^a$$



$$E_m^a(\varphi) d\varphi^m = \eta^a$$

$$\text{Tr}(T^a g dg^{-1}) = \lambda E_m^a(\varphi) d\varphi^m = i d\zeta^a$$

Some useful geometric concepts

Killing metric $g^{-1}dg \rightarrow$ Left-invariant 1-form

$$g \in G$$

$$g = e^{i\sigma \cdot \tau}$$

$$dg = g(iT_a dg_a)$$

$$g^{-1}dg = i \underset{=}{T_a} dg_a$$

$$g \rightarrow hg, h \in G \text{ but constant}$$

$$g^{-1}dg \in T^*(M) \otimes \underline{L(G)}$$

$$dg g^{-1} \rightarrow \text{Right-invariant 1-form}$$

$$g \rightarrow gh$$

Bi-invariant

$$g^{-1}dg =$$

$$\text{Tr} ($$

Transitive: If for any $p_1, p_2 \in M$ $\exists$ an element $g \in G$ st. $p_2 = \sigma(g)p_1$

Free: If $g \neq e$ there is no fixed point of the action. If $p \in M$ $\sigma(g)p = p$

Transitive: If for any $p_1, p_2 \in M$ $\exists$ an element $g \in G$ st. $p_2 = \sigma(g, p_1)$
 # Free: If $g \neq e$ there is no fixed point of the action. If $p \in M$ $\sigma(g, p) \neq p$
 Ex: The action $SO(3)$ on $\mathbb{R}^3$ is not transitive. If $|x| \neq |x'|$ where $x, x' \in \mathbb{R}^3$

Ex: $SO(2) \times SO(2) \times SO(2) \times \dots$

Coset Manifold

Suppose $H \subset G$

Consider $G/H = \{gh \mid h \in H\}$

$gh \rightarrow$ coset for the h

Coset Manifold

Suppose $H \subset G$

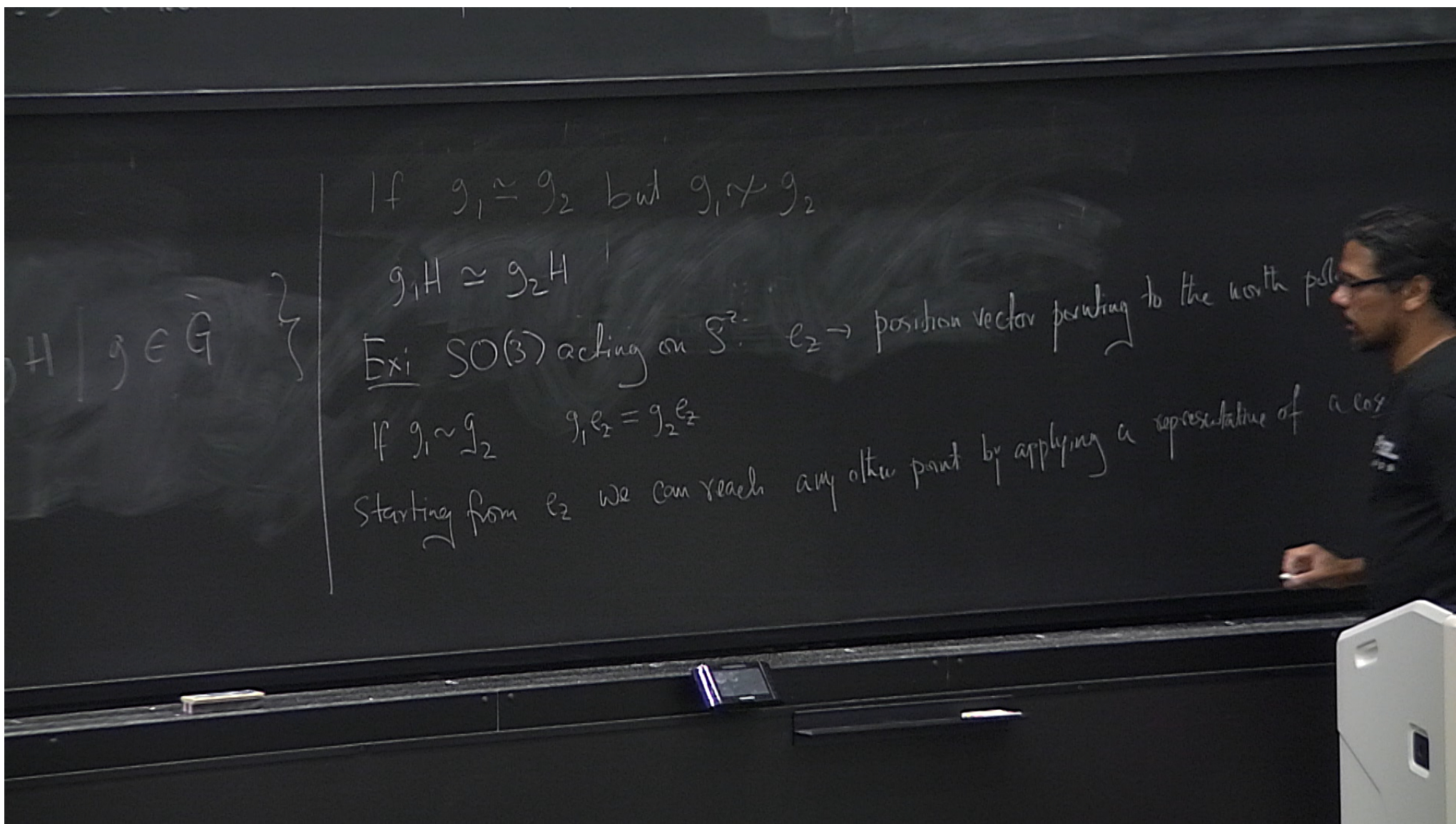
Consider $G/H = \{gH \mid g \in G\}$

↑
Coset space

$$g_1 = g_2 h$$

$$g_1 \sim g_2$$

$$|f \quad g_1 \sim g_2 \quad |$$



If $g_1 \sim g_2$ but $g_1 \neq g_2$

$$g_1 H \approx g_2 H$$

Ex: $SO(3)$ acting on S^2 . $e_z \rightarrow$ position vector pointing to pole.

$$SO(3)/SO(2)$$

If $g_1 \sim g_2$ $g_1 e_z = g_2 e_z$

Starting from e_z we can reach any other point by applying a rotation. Each coset corresponds to a unique point on S^2 .

Finite dim rep of $SU(2)$

$$[J_a, J_b] = i\epsilon_{abc} J_c$$

$J_{\pm} = J_1 \pm iJ_2 \sim$ Ladder op

$$J_+^\dagger = J_-, \quad J_-^\dagger = J_+$$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_+, J_-] = 2J_3$$

Rank $\mathcal{L}(SU(2)) = 1$. CSA = $\{J_3\}$

Label vectors $J_3|m\rangle = m|m\rangle$

$|m\rangle \rightarrow$ a vector m which $\mathcal{L}(SU(2))$ acts

$m \rightarrow$ Mot assuming this is an integer.

$$SA = \{J_3\}$$

$$= m|m\rangle$$

which $L(SU(2))$ acts
this is an integer.

$$J_3|m\rangle = m|m\rangle$$

$$\begin{aligned} J_3(J_+|m\rangle) &= J_+J_3|m\rangle + J_+|m\rangle \\ &= J_+m|m\rangle + J_+|m\rangle \\ &= (m+1)(J_+|m\rangle) \end{aligned}$$

$$J_3(J_-|m\rangle) = (m-1)(J_-|m\rangle)$$

$$SA = \{J_3\}$$

$$= m|m\rangle$$

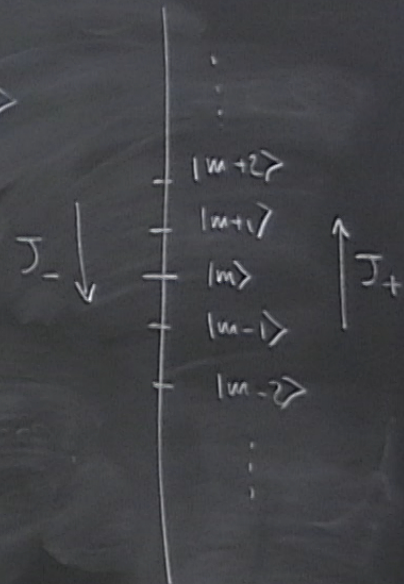
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is an integer.

$$J_3|m\rangle = m|m\rangle$$

$$\begin{aligned} J_3(J_+|m\rangle) &= J_+J_3|m\rangle + J_+|m\rangle \\ &= J_+(m|m\rangle + |m\rangle) \\ &= (m+1)(J_+|m\rangle) \end{aligned}$$

$$J_3(J_-|m\rangle) = (m-1)(J_-|m\rangle)$$

$$\text{Assume } \langle m|m\rangle = 1$$



$$[J_+, J_-] = 2J_z$$

$$J_+ |m\rangle = C_{m+1} |m+1\rangle$$

$$J_- |m\rangle = D_{m-1} |m-1\rangle$$

$C_m, D_m \rightarrow$ to be determined

$$\begin{aligned} \langle m+1 | J_+ | m \rangle &= \langle m+1 | C_{m+1} | m+1 \rangle \\ &= C_{m+1} \end{aligned}$$

$$J_- |m+1\rangle = D_m |m\rangle$$

$$\langle m+1 | J_+ = D_m^* \langle m |$$

$$\begin{aligned} \langle m+1 | J_+ | m \rangle &= D_m^* \langle m | m \rangle \\ &= D_m^* \end{aligned}$$

$$C_{m+1} = D_m^*$$

$$J_- |m\rangle = C_m^* |m-1\rangle$$

Generic $|m\rangle$

$$\langle m | (J_+ J_- - J_- J_+) | m \rangle = 2 \langle m | J_z | m \rangle$$

$$\boxed{|c_m|^2 - |c_{m+1}|^2 = 2m}$$

$$|c_{j-1}|^2 = |c_j|^2 + 2(j-1)$$

$$= 2j + 2(j-1)$$

$$= 2(2j-1)$$

$$|c_{j-2}|^2 = |c_{j-1}|^2 + 2(j-2)$$

$$= 2(3j-1-2)$$

$$|c_{j-s}|^2 = 2((s+1)j - 1 - 2 - \dots - s)$$

$$= 2((s+1)j - \frac{1}{2}s(s+1))$$

$$= (s+1)(2j-s)$$

$s \rightarrow$ positive
integer

$$|C_j| = 2j$$

$$j^2 + 2(j-2)$$

$$3j-1-2$$

$$2((s+1)j-1-2-\dots-s)$$

$$2((s+1)j - \frac{1}{2}s(s+1))$$

$$(s+1)(2j-s)$$

$$C_{j-s} = \sqrt{(s+1)(2j-s)}$$

$$2j = s$$

$$j = \frac{s}{2}$$

$m \rightarrow$ also an integer

$$\frac{1}{2} \text{ "}$$

Ex: $SO(2) \subset SO(3)$. $SO(2)$ that leaves the north pole invariant is the little group of north pole

$$J_+ |m\rangle = C_{m+1} |m+1\rangle$$

$$= \sqrt{(j-m)(j+m+1)} |m+1\rangle$$

$$J_- |m\rangle = \underbrace{\sqrt{(j+m)(j-m+1)}}_{C_m^*} |m-1\rangle$$

When you apply J_+ enough number of times $m=j$

$$= C_{m+1} |m+1\rangle$$

When you apply J_+ enough number of times $m=j+1$

$$= \sqrt{(j-m)(j+m+1)} |m+1\rangle$$

In total $(2j+1)$ steps in the ladder

$$= \sqrt{(j+m)(j-m+1)} |m-1\rangle \quad 0 \xleftarrow{J_+} |j\rangle \xrightleftharpoons[J_-]{J_+} |j-1\rangle \xrightleftharpoons[J_-]{J_+} |j-2\rangle \dots \xrightleftharpoons[J_-]{J_+} |-j+1\rangle \xrightleftharpoons[J_-]{J_+} |-j\rangle \xrightarrow{J_-} 0$$

$$C_m^*$$

$$J_a J_a |m\rangle = j(j+1) |m\rangle \quad \forall m$$

$J_a J_a \rightarrow$ Casimir invariant

the fixed point of the action of H .

$SO(2) \subset SO(3)$, $SO(2)$ that leaves the north pole invariant is the little group of $\vec{p}$

$$\langle m | T_a | n \rangle$$

Find the matrices for $j=2$ rep.
 $| -2 \rangle, | -1 \rangle, | 0 \rangle, | +1 \rangle, | +2 \rangle$

$SU(2)$ tensors $\Leftarrow$ Young tableaux

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

$$2 \otimes 2 = 3 \oplus 1$$

2 of $SU(2)$

$$\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2$$

$$[T_a, T_b] = i \epsilon_{abc} T_c$$

$$(-T_a^*) \equiv \tilde{T}_a$$

$$[\tilde{T}_a, \tilde{T}_b] = i \epsilon_{abc} \tilde{T}_c$$

2 of $SU(2)$

$$\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2$$

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$$[\tilde{T}_a, \tilde{T}_b] = i \epsilon_{abc} \tilde{T}_c$$

$\tilde{T}_a \rightarrow$ conjugate rep (anti-fundamental)

However for $SU(2)$

$$\tilde{T}_a = C T_a C^{-1}$$

$$C = i\sigma_2$$

For $SU(2)$ $\psi_a^* \rightarrow 2^*$

$$2 \equiv 2^*$$

$SU(n)$, $n \geq 3$ $n \neq n^*$

If $g_1 \simeq g_2$ but $g_1 \neq g_2$

$$S^2 = SO(3)/SO(2)$$

$$SO(3)/SO(2)$$

1. Fundamental rep: 2

Under $SU(2)$ $\psi_a \rightarrow U_a^b \psi_b$

Invariant tensors

1. ϵ_{ab} , ϵ^{ab}

$$\epsilon_{ab} \rightarrow U_a^{a'} U_b^{b'} \epsilon_{a'b'} = (\det U) \epsilon_{ab} \\ = \epsilon_{ab}$$

$$2^*: \psi^a = \psi_a^*$$

Under a $SU(2)$ transformation

$$\psi^a \rightarrow (U^\dagger)_{a'}^a \psi^{a'}$$

$$\epsilon^{ab} \rightarrow (U^\dagger)_{a'}^a (U^\dagger)_{b'}^b \epsilon^{a'b'}$$

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