

Title: PSI 2018/2019 - Lie Groups and Lie Algebras - Lecture 4

Date: Aug 14, 2018 10:30 AM

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Abstract:

Gockeler & Schücker — chapter 8

Weinberg - Vol I, chapter 3

G\"ockeler & Sch\"ucker — chapter 8

Weinberg - Vol I, Chapter 3

Last Time: Generators of $SO(n)$, $T^{ab} = -(T^{ab})^T$

$$T^{ab} = -T^{ba} \sim \frac{n(n-1)}{2}$$

Suppose that

$$T^{ab} = - (T^{ab})^T$$

Suppose that $\dim$ of $SO(n)$ is $n = 2l$

We can define l linearly independent matrices

$$H^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$H^l = \begin{pmatrix} & & & \\ & & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

$$H^2 = \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

$$[H_i, H_j] = 0 \quad \forall i, j$$

Def 4: Cartan Subalgebra (CSA): The maximal set of mutually commuting

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$\{H_i\} \rightarrow$ generate rotations in planes which do not share an axis

The dimension l of CSA is known as the rank of $\mathfrak{g}$

The vectors of a rep that the lie alg acts will be labelled λ

Consider $n = 2l + 1$

Consider $n = 2l + 1$, the rank is still l .

For every Lie algebra (arising from a compact Lie group) we can define the

Schur's Lemma:

Any matrix which commutes with all the matrices of an ^{irreducible} representation must be proportional to the identity.

So if $T_a^{(r)} \rightarrow$ generators in the irrep (r) then

$$T_a^{(r)} T_a^{(r)} = C_2(r) \mathbb{1}$$

$C_2(r) \rightarrow$ characterizes the whole irrep

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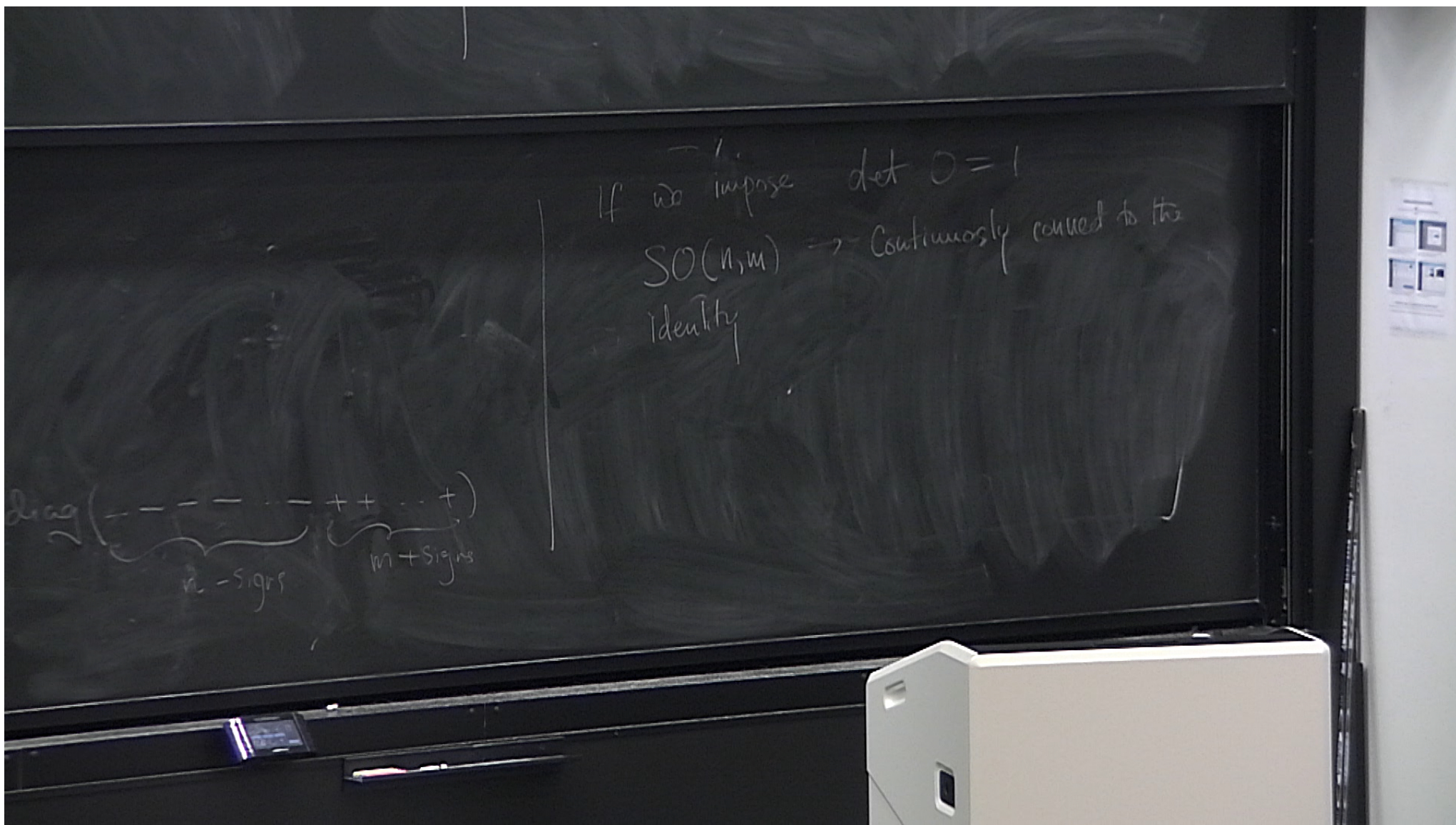
$SO(n, m)$ groups:

$SO(n) \rightarrow$ preserving the Euclidean metric

$$O^T \mathbb{1} O = \mathbb{1}$$

Analogously for $\mathbb{R}^{n, m}$ which have metric $\eta_{\mu\nu} = \text{diag}(\underbrace{- \dots -}_{n \text{ signs}}, \underbrace{+ \dots +}_m)$

$$O(n, m) = \{ O \mid O^T \eta O = \eta \}$$



There are no finite dimensional Unitary representation of non-compact Lie groups.

Lorentz group $O(1,3)$

$SO(1,3)$ is a subgroup. Proper, orthochronous Lorentz transformations

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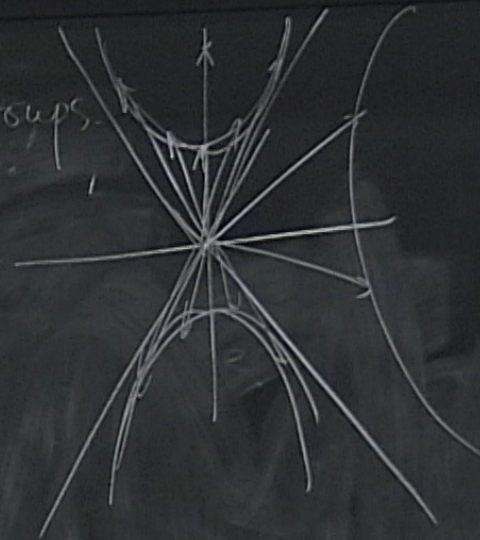
Lorentz group $O(1,3)$

$SO(1,3)$ is a subgroup. Proper, orthochronous Lorentz transformations

Proper $\rightarrow \det O = 1$

Orthochronous $O_{00} > 0$

irreducible, non-trivial
 unitary representation of non-compact Lie groups



orthochronous Lorentz transformations

$O(1,3) \rightarrow$ contains

$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \sim \text{Parity}, T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \sim \text{Time inversion}$

Consider $n = 2\ell + 1$

Isometry group of AdS_n

AdS_n is defined by a hyperboloid in $\mathbb{R}^{2, n-1}$

$$-x_0^2 - x_1^2 + x_2^2 + \dots + x_n^2 = -R^2$$

↖ 'radius' of AdS_n

↖ Invariant under $\text{SO}(2, n-1)$

Conformal Field Theory in $(n-1)$ dim flat space-time has S

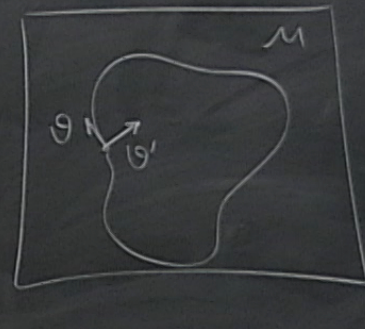
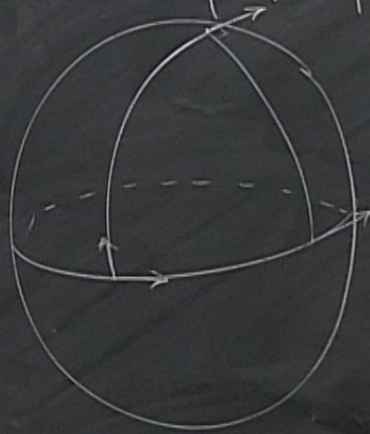
(+---) dS_4 hyperboloid in $\mathbb{R}^{1,4}$

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 - X_4^2 = R^2$$



Conformal Field Theory in $(n-1)$ dim flat spacetime has

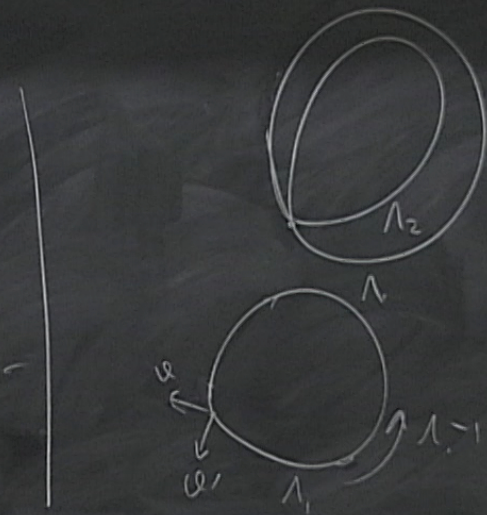
Holonomy Groups



g , if
 $O(u)$

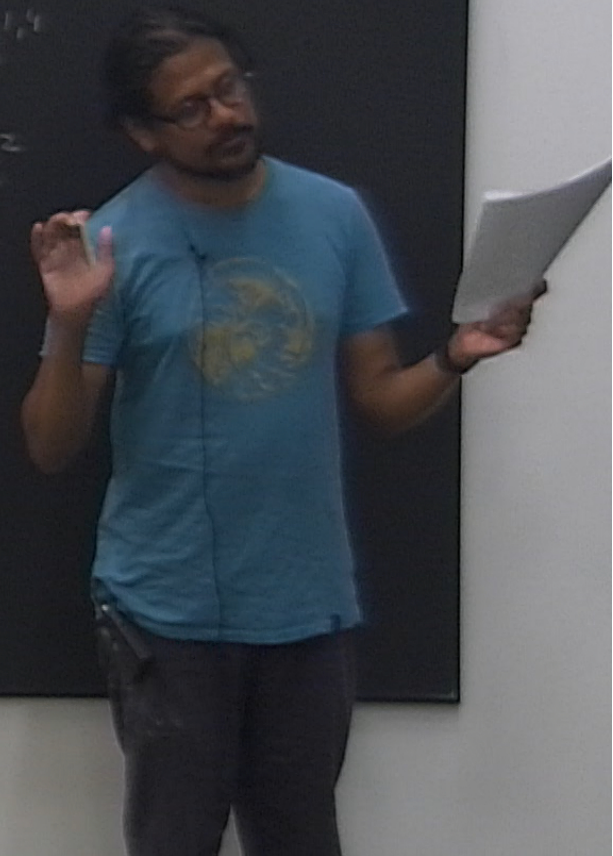
has $SO(2, n-1)$.

ϑ , if $|\vartheta'| = |\vartheta|$
 $D(u)$



Restricted holonomy group

$(+---)$
 $-+++$
 $\mathbb{S}_q^?$ hyperboloid in $\mathbb{R}^{1,q}$
 $-X_0^2 + X_1^2 + X_2^2 + X_3^2 - X_4^2 = R^2$



Special Unitary Groups $SU(n)$:

Gauge 'Symmetry' group of Standard model: $SU(3) \times SU(2) \times U(1)$

Accidental global $U(1)$ symmetries Baryon # $\neq$ lepton # cons

$SU(n)$ consists of $n \times n$ complex matrices U which satisfy

$$U^\dagger = U^{-1}$$

$\times SU(2) \times U(1)$

lepton # conservation

$$U^\dagger U = U U^\dagger = \mathbb{1}$$

$$\neq \det U = +1$$

If we didn't impose $\det U = 1$, we get $U(1)$ group

Dimension of $SU(n)$:

$U \rightarrow$ as a complex matrix has $2n^2$ real parameters

$$U_{ji}^* U_{jk} = \delta_{ik}$$

If 1 set $i=k \Rightarrow n$ real constraints.

If $i \neq k$, we only need to consider $i > j$

$n(n-1)$ real constraints

$2n^2$
 $=$
 $\# \det$

Dimension of $SU(n)$:

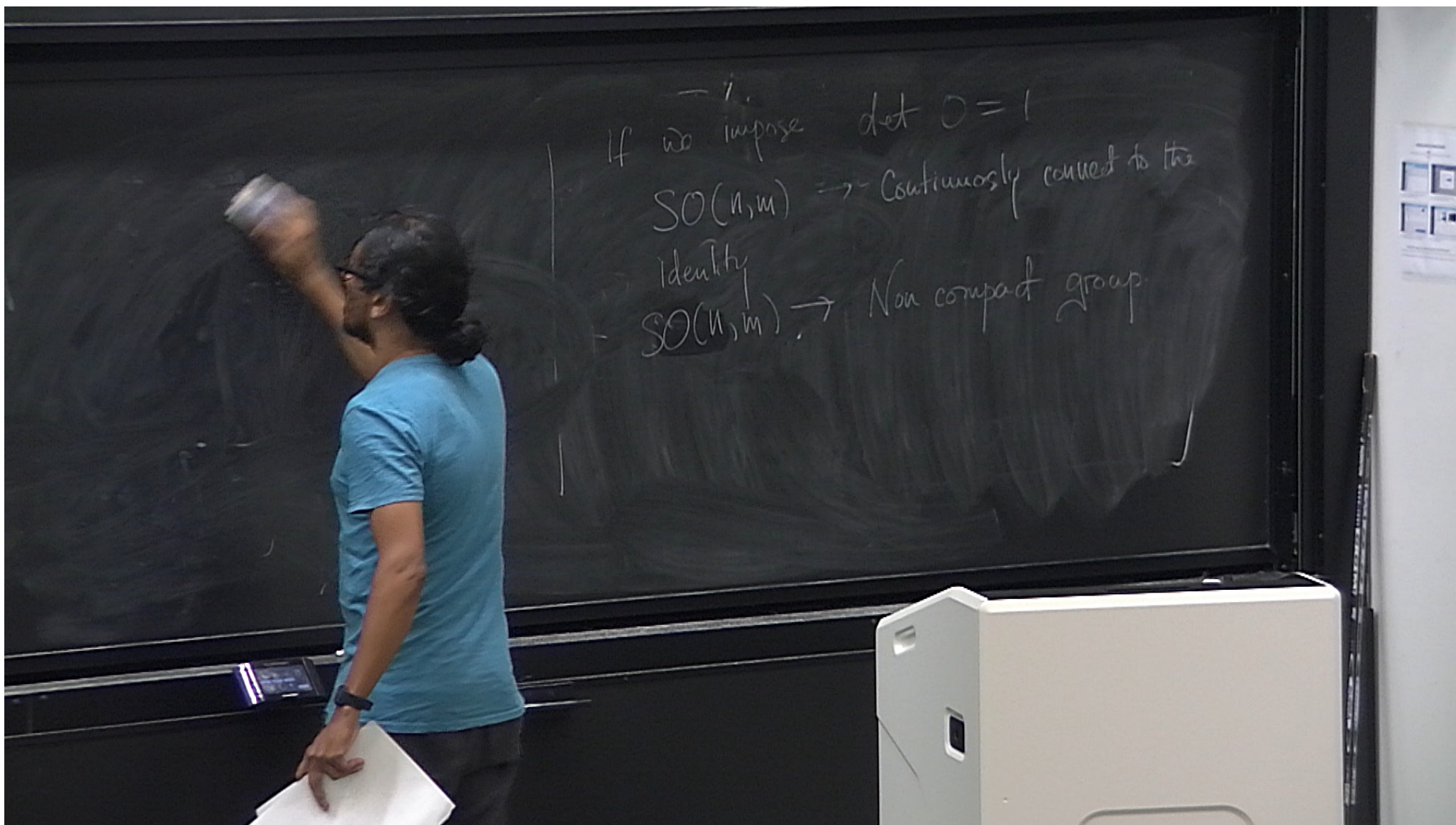
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$n(n-1)$ real constraints



$\det U = 1$ eliminates

$$U = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \sim U(1) \text{ subgroup}$$

$$\det U = e^{i\theta n}$$

$$\theta = \frac{2\pi k}{n}, k \in \mathbb{Z}$$

$$\mathbb{1}^k$$

$$\begin{pmatrix} e^{\frac{2\pi i}{n} k} & \\ & e^{-\frac{2\pi i}{n} k} \end{pmatrix}$$

Generators

Generators:

$$U = \exp[i \sum_a \lambda_a T_a], \quad \lambda_a \rightarrow \text{real } n^2-1 \text{ parameters.}$$

$$U^\dagger = U^{-1} \Rightarrow T_a^\dagger = T_a$$

$$\text{Tr } T_a = 0$$

$$\begin{aligned} \det U &= 1 \\ \det U &= \exp[\text{Tr}(i \sum_a \lambda_a T_a)] \\ 1 &= \exp[\text{Tr}(i \sum_a \lambda_a T_a)] \end{aligned}$$

$$U = e^{i\xi_a T_a} = 1 + i\xi_a T_a + \frac{(i\xi_a T_a)^2}{2!} + \dots$$

$$U^\dagger = 1 + (-i\xi_a T_a)^\dagger + \dots$$

$$= e^{-i\xi_a T_a^\dagger}$$

$$UU^\dagger = e^{i(\xi_a T_a) - i(\xi_a T_a)} = e^0 = \mathbb{1}$$

$SU(2)$

$$T_a = \frac{\sigma_a}{2}$$

Hermitian & traceless

$$H_1 = \text{diag}(1, 0, \dots, 0, -1)$$

$$H_2 = \text{diag}(0, 1, 0, \dots, 0, -1)$$

$$\vdots$$
$$H_{n-1} = \text{diag}(0, 0, \dots, 1, -1)$$

$n(n-1)/2$ remain

Hermitian $\nleftrightarrow$ traceless

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$$\vdots$$
$$H_{n-1} = \text{diag}(0, 0, \dots, 1, -1)$$

$(n-1)$ mutually commuting generators
 $(SU(n)) = (n^2 - 1)$

$n(n-1)$ remaining

$\Rightarrow$ half to be real $\nleftrightarrow$ symm.
and non-diagonal
 $\Rightarrow$ other half pure i and
antisymmetric.

Defn: Simple Lie Group : A simple Lie group is a connected non-abelian Lie group with no connected normal subgroups.

Note: The excluded normal subgroups are Lie groups. A simple Lie group is a connected non-abelian Lie group with no connected normal subgroups.

Simple Lie algebra: $\tilde{\mathfrak{g}} = \mathcal{L}(G)$, $\tilde{\mathfrak{k}} \subset \tilde{\mathfrak{g}}$, if $\tilde{\mathfrak{k}}$ is a proper ideal of $\tilde{\mathfrak{g}}$.

included normal subgroups are Lie groups. A simple Lie group may have discrete

$$\tilde{g} = L(G), \tilde{k} \in \tilde{g}, \tilde{K} \rightarrow \text{invariant}^{\text{subalg}} \text{ if } [\tilde{K}, \tilde{g}] = \tilde{K} \text{ or } 0$$

group of Standard model: $SU(3) \times SU(2) \times U(1)$
 global $U(1)$ symmetries Baryon # $\neq$ lepton conservation
 U
 of $n \times n$ complex matrices which satisfy
 U^{-1}

$\neq \det U = +$
 If we didn't impos

$\mathfrak{g} \subset \mathfrak{gl}(V)$, $K \subset \mathfrak{g}$, $K \rightarrow \text{Invariant}$ if $[K, \mathfrak{g}] \subseteq K$

We say that a Lie alg. is simple if it has no nontrivial invariant subalgebra

A simple Lie group gives rise a simple Lie alg. $\nleftrightarrow$ vice versa.

$$(F^a)_{bc}$$

Suppose $\mathfrak{L}(G)$ is simple but the adjoint rep $\{F_a\}$ is reducible

$$F_a = \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix}$$

Diagram illustrating the adjoint representation F_a as a 2x2 block matrix. The top-left block is labeled $\tilde{a}b$ and the bottom-right block is labeled $\bar{a}\bar{b}$. The off-diagonal blocks are zero.

out if $[K, g] \subseteq K$ or 0

non-trivial invariant subalgebra.

$$(F^a)_{bc} = \begin{cases} F^a_{bc} \neq 0 \\ F^a_{\tilde{b}\tilde{c}} \neq 0 \\ F^a_{\bar{b}\tilde{c}} = 0 \end{cases}$$

reducible

$$F^a_{\bar{b}\tilde{c}} = -i f a \bar{b} \tilde{c} = +i f \bar{b} a \tilde{c} = -(F^{\bar{b}})_{a\tilde{c}} \rightarrow 0 \text{ if } a \neq \bar{a}$$

$$(F^{\tilde{a}})_{\bar{b}\tilde{c}} = 0$$

$$(F^{\bar{a}})_{\tilde{b}\tilde{c}} = 0$$

$$\{F^{\vec{a}}\} = \{F^{\tilde{a}}, F^{\bar{a}}\}$$