Title: PSI 2018/2019 - Lie Groups and Lie Algebras - Lecture 2

Date: Aug 09, 2018 10:30 AM

URL: http://pirsa.org/18080004

Abstract:

Lagrange's theorem. For any finite groups the order of proper sub

$$H \subset G, \; n\,O(H) = O(G)$$

If $O(G) =$ prime $\Rightarrow$ G is simple.

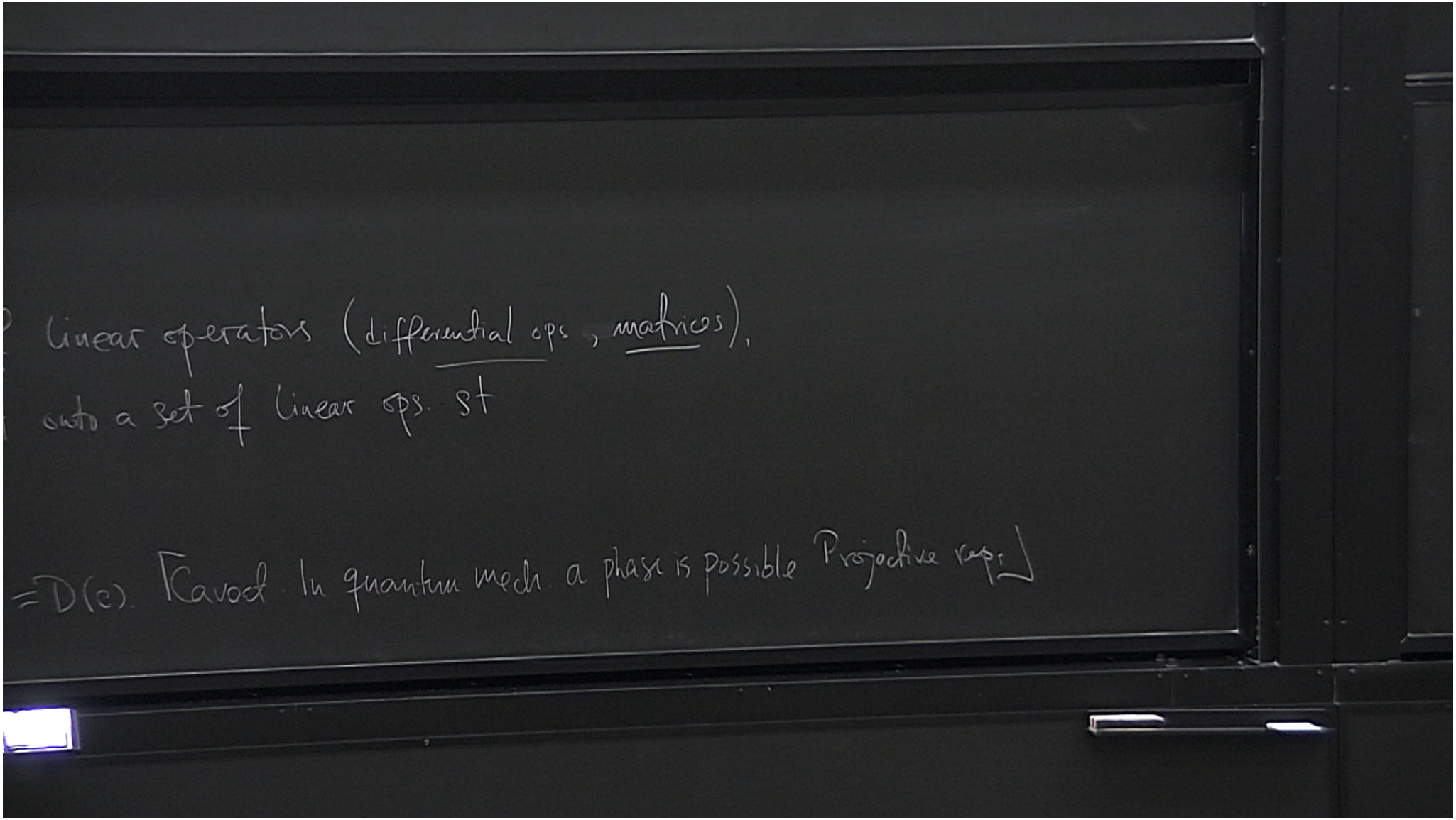
Basic defns of Representation Theory

In interested mainly in representations of groups in terms of linear operat

Defn: The representation of G is a map D of the elements of G onto a set of

$D(e) = \mathbb{1}$, where $\mathbb{1}$ is identity op acting on the vector space.

group multiplication is respected if $a, b, c \in G$ st $ab = c$, $D(a)D(b) = D(c)$ [Cavoet


linear operators (differential ops, matrices),
onto a set of linear ops. st
=D(c). [Caveat In quantum mech a phase is possible Projective rep:]

Ex: 1. $C_4 = \{a, a^2, a^3, a^4 = e\}$

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a) = \frac{1}{2}\begin{pmatrix} 1+e^{i\pi/2} & -1+e^{i\pi/2} \\ -1+e^{i\pi/2} & 1+e^{i\pi/2} \end{pmatrix}$$

$$D(b) = [D(a)]^2 \quad , \quad D(c) = [D(a)]^3$$

2 $D(e) = 1 = D(a) = D(b) = D(c)$

→ Trivial representation.

Defn: Faithful Rep. If D is a rep of G is st every element $g \in G$ is mapped to a distinct element $D(g)$ then D is a faithful rep.

Otherwise → Unfaithful

Def^n Equivalent reps: Two reps D & D' are said to be equivalent

$$D'(g) = S D(g) S^{-1} \quad \forall g \in G$$

Ex: $D'(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $D'(a) = \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & 1 \end{pmatrix}$

$D'(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $D'(c) = \begin{pmatrix} e^{3\pi i/2} & 0 \\ 0 & 1 \end{pmatrix}$

D & D' are said to be <u>equivalent</u> if $\exists\ S$ st

$\forall g \in G$

$$D'(g) = S D(g) S^{-1}$$

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & 1 \end{pmatrix}$$

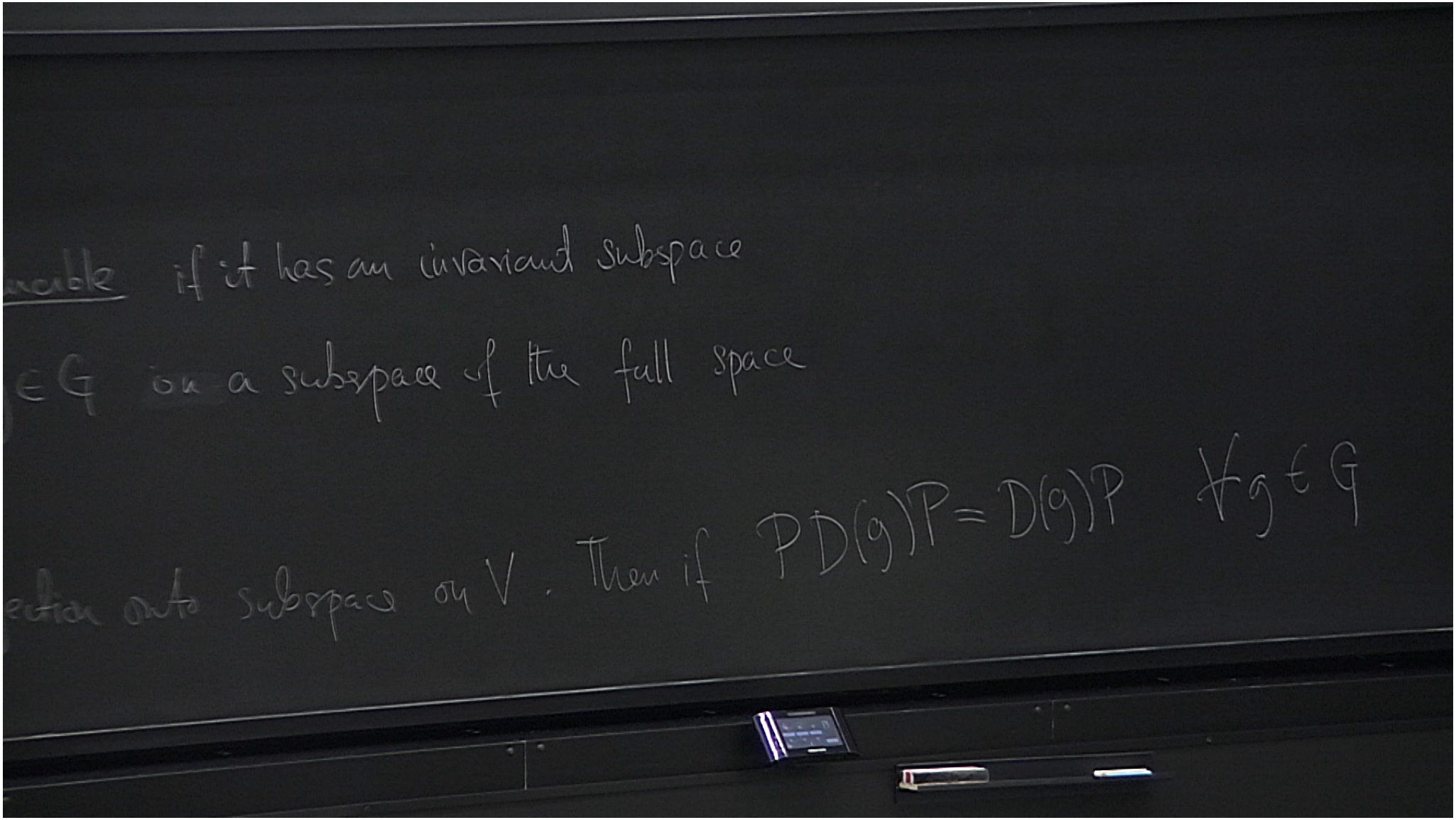
☆ Irreducible representation

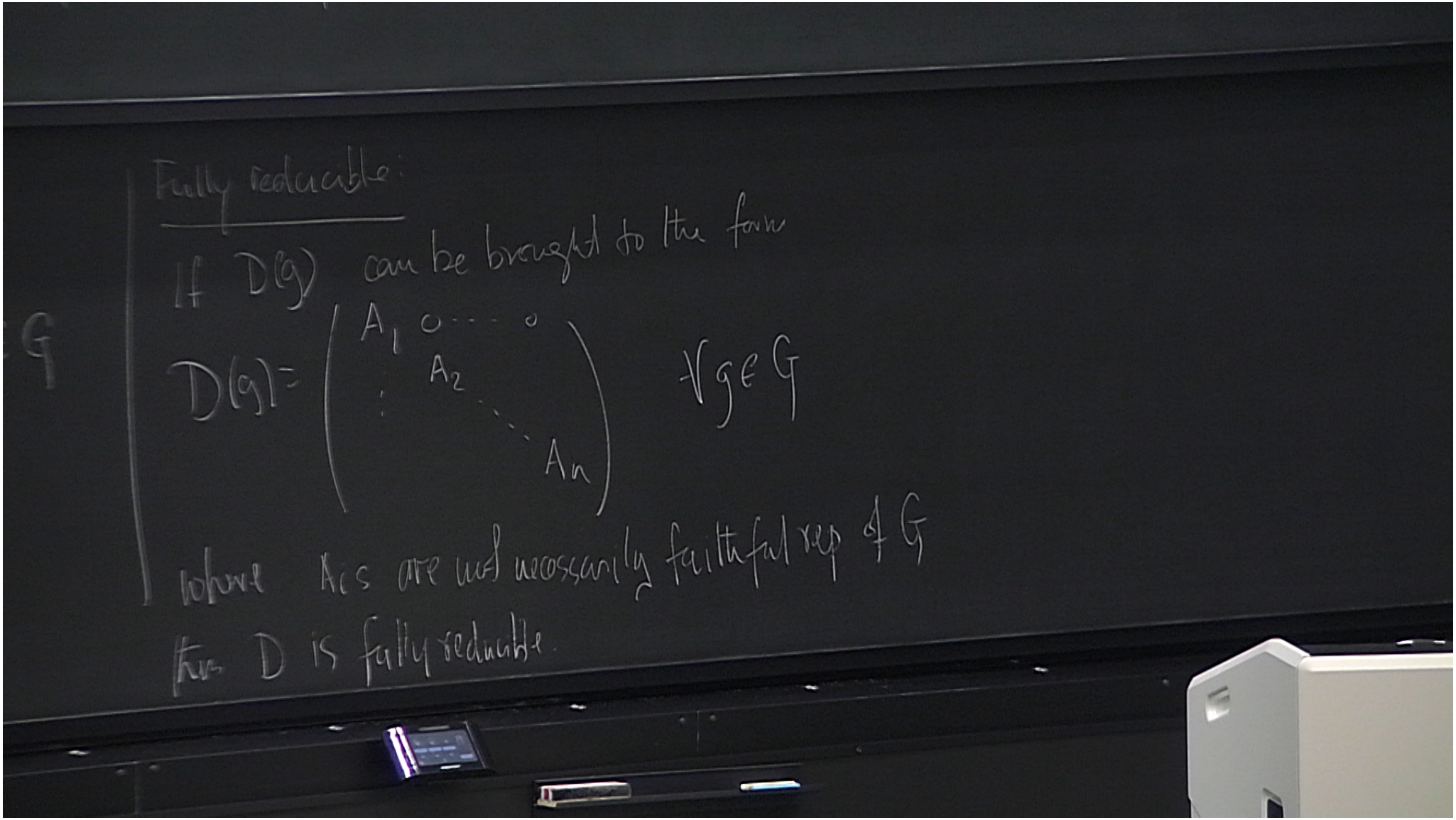
A rep D is said to be reducible if it has an invariant subspace

If the action of $D(g)$, $\forall g \in G$ on a subspace of the full space

remains in the subspace.

In other words, P is projection onto subspace on V. Then if PD


if it has an invariant subspace
on a subspace of the full space
Then if $PD(g)P = D(g)P \quad \forall g \in G$


Fully reducible:
If D(g) can be brought to the form
$D(g) = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ & A_2 & & \\ \vdots & & \ddots & \\ & & & A_n \end{pmatrix}$ $\forall g \in G$
where A_is are not necessarily faithful rep of G
then D is fully reducible.

In practical terms, there exists some S st

$$SD(g)S^{-1} = \begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & 0 \\ \vdots & 0 & D_3(g) \\ & \cdots & \ddots \end{pmatrix} \quad \forall g \in G$$

Decomposable

where $D_i(g)$ form reps of G

Fully

If

D(

whe

the

$\in G$

Fully reducible:

If $D(g)$ can be brought to the form

$$D(g) = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ & A_2 & & \\ \vdots & & \ddots & \\ & & & A_n \end{pmatrix} \quad \forall g \in G$$

where A_is are not necessarily faithful rep of G

then D is fully reducible.

Groups & QM

Due to a theorem by Wigner 1931:

Theorem: The group of symmetries of quantum system is represented by

on the Hilbert space of the theory. [Weinberg vol I, QTF]

Outline of proof: $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$. Under symmetry x formation $|\psi_1\rangle$

$|\psi_2\rangle$

ented by linear, unitary or antilinear, antiunitary operators acting

QTF

$|\langle\psi_2|U^\dagger U|\psi_1\rangle|^2 = |\langle\psi_2|\psi_1\rangle|^2$

$U \to$ unitary $\quad \langle\psi_2|U^\dagger U|\psi_1\rangle = \langle\psi_1|\psi_2\rangle$

$U \to$ antiunitary $\quad \langle\psi_2|U^\dagger U|\psi_1\rangle = \langle\psi_1|\psi_2\rangle^*$

tion $|\psi_1\rangle \to U|\psi_1\rangle$

$|\psi_2\rangle \to U|\psi_2\rangle$

$|\psi_2\rangle \rightarrow$

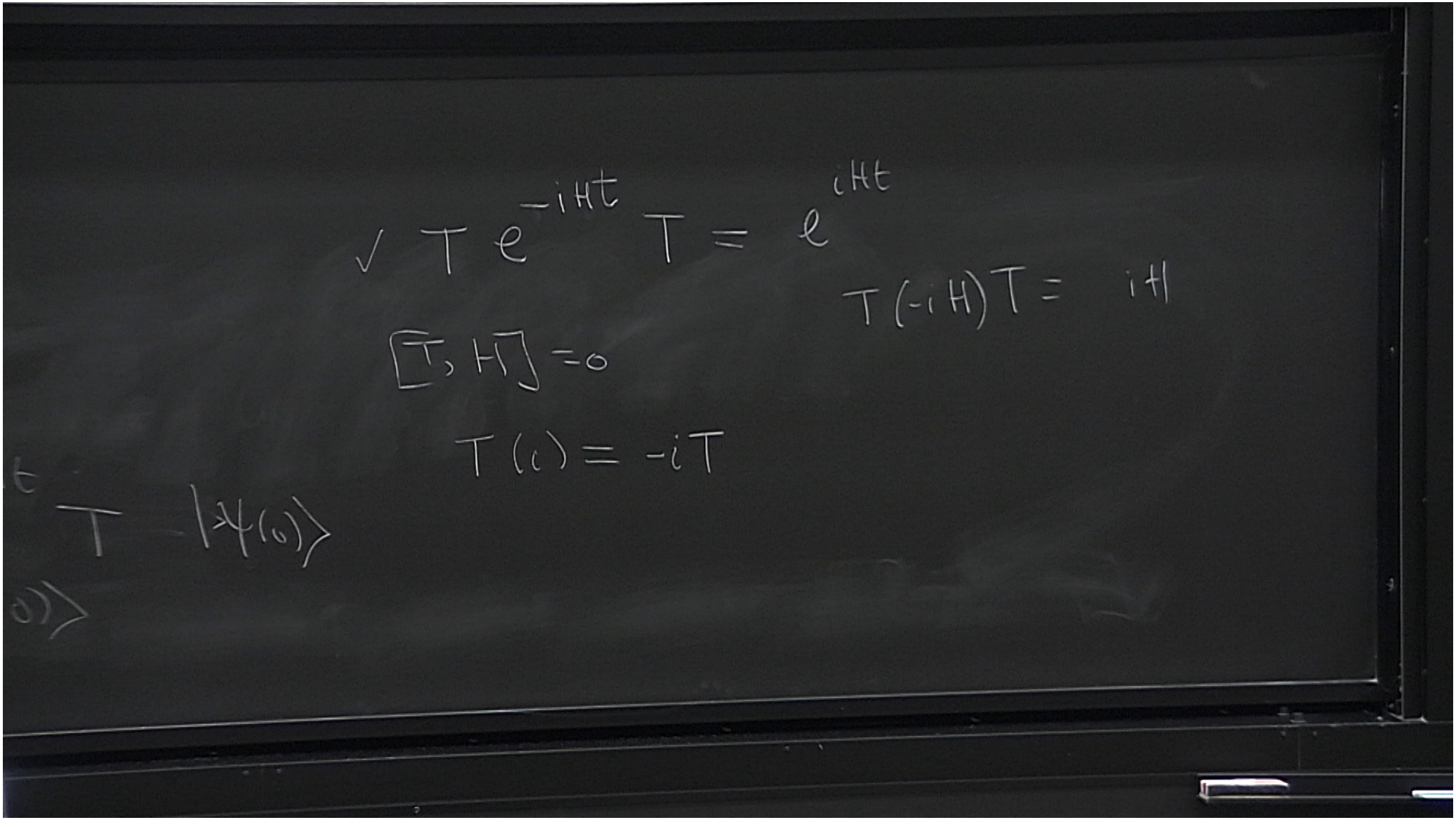
Antilinear: $U(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha^* U|\psi\rangle + \beta^* U|\varphi\rangle$

Ex $[T, H] = 0$

$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$

$\boxed{T|\psi(t)\rangle = |\psi(-t)\rangle} = T e^{-iHt} \overset{=T}{T^{-1}T}|\psi(0)\rangle = T e^{-iHt} T |\psi(0)\rangle$

$|\psi(-t)\rangle = e^{iHt}|\psi(0)\rangle$ | $|\psi(-t)\rangle = T e^{-iHt} T|\psi(0)\rangle$


$T e^{-iHt} T = e^{iHt}$
$T(-iH)T = iH$
$[T, H] = 0$
$T(t) = -iT$
$T \; |\psi(0)\rangle$

Degeneracy & Groups

$G \rightarrow$ group of symm of Quantal system.

$D \rightarrow$ Rep. $[D(g), H] = 0$

$H|\psi\rangle = E|\psi\rangle$

Let the subspace of $\mathcal{H}$ spanned by

$D(g)|\psi\rangle$, $\forall\, g \in G$

be given by

$\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ → Linearly independent set

$H|\psi_i\rangle = E|\psi_i\rangle$

$$D(g)\,|\psi_i\rangle = \sum_j |\psi_j\rangle\langle\psi_j|D(g)|\psi_i\rangle$$

$$= \sum_j |\psi_j\rangle\, D(g)_{ji}$$

$D(g)_{ji} \to n \times n$ irrep of G

early independent set

Lie Groups & Lie Algebras

Sophus Lee . The geometry of solution space differential equations

Continuous groups → Elements are parametrized by set of continuous

$U(1) \rightarrow e^{i\theta} \quad 0 < \theta \leq 2\pi$

$U(1) \cong S^1$

solution space differential equations.

are parametrized by set of continuous parameters.

Def: Lie group A Lie group G is a group which is also a manifold.

It is equipped with multiplication $\cdot : G \times G \to G$ with the usual properties

G includes a identity element and for each element there is a unique inverse.

Suppose that the group elements are parametrized by $\underline{\phi} = (\phi_1, \ldots, \phi$

We write generic element $g(\underline{\phi})$ and $e = g(\underline{\phi}=0)$

The number independent parameters is dimension of the Lie group.

We concentrate on Matrix group → defined by some matrix eqn.

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element $g(\underline{\Phi})$ and $e = g(\underline{\Phi}=0)$

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atrix group → defined by some matrix eqn:

Relationship between Lie Groups & Lie algebra

Consider a spin 1/2 particle

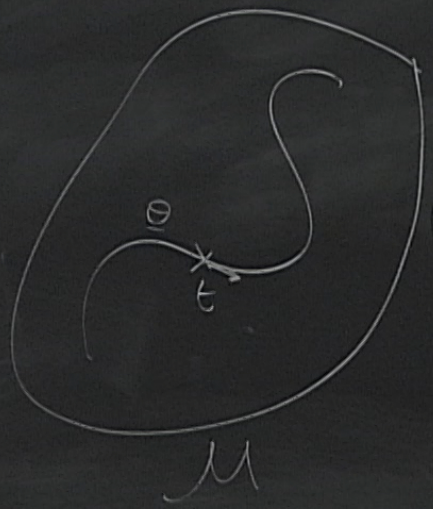
$$|\underline{\psi}\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$$

Rotation by θ around $\hat{n}$

$$e^{\frac{i}{2}\theta\,\hat{n}\cdot\underline{\sigma}}\,|\underline{\psi}\rangle$$

$$\underline{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$$

Tangent vector Suppose $\underline{\theta}(t) = \{\theta_1, \theta_2, \theta_3\}(t)$



Tangent vector to $\underline{\theta}(t)$

Coordinate rep tangent vec

$$\underline{v}(t) = \frac{d\underline{\theta}(t)}{dt}$$

$$e^{i\frac{\theta_3}{2}\sigma_3} = \cos\frac{\theta_3}{2} + i\sigma_3 \sin\frac{\theta_3}{2}$$

$$\left[e^{i\frac{\theta_3}{2}\sigma_3}, e^{i\frac{\theta_2}{2}\sigma_2}\right] = 0$$

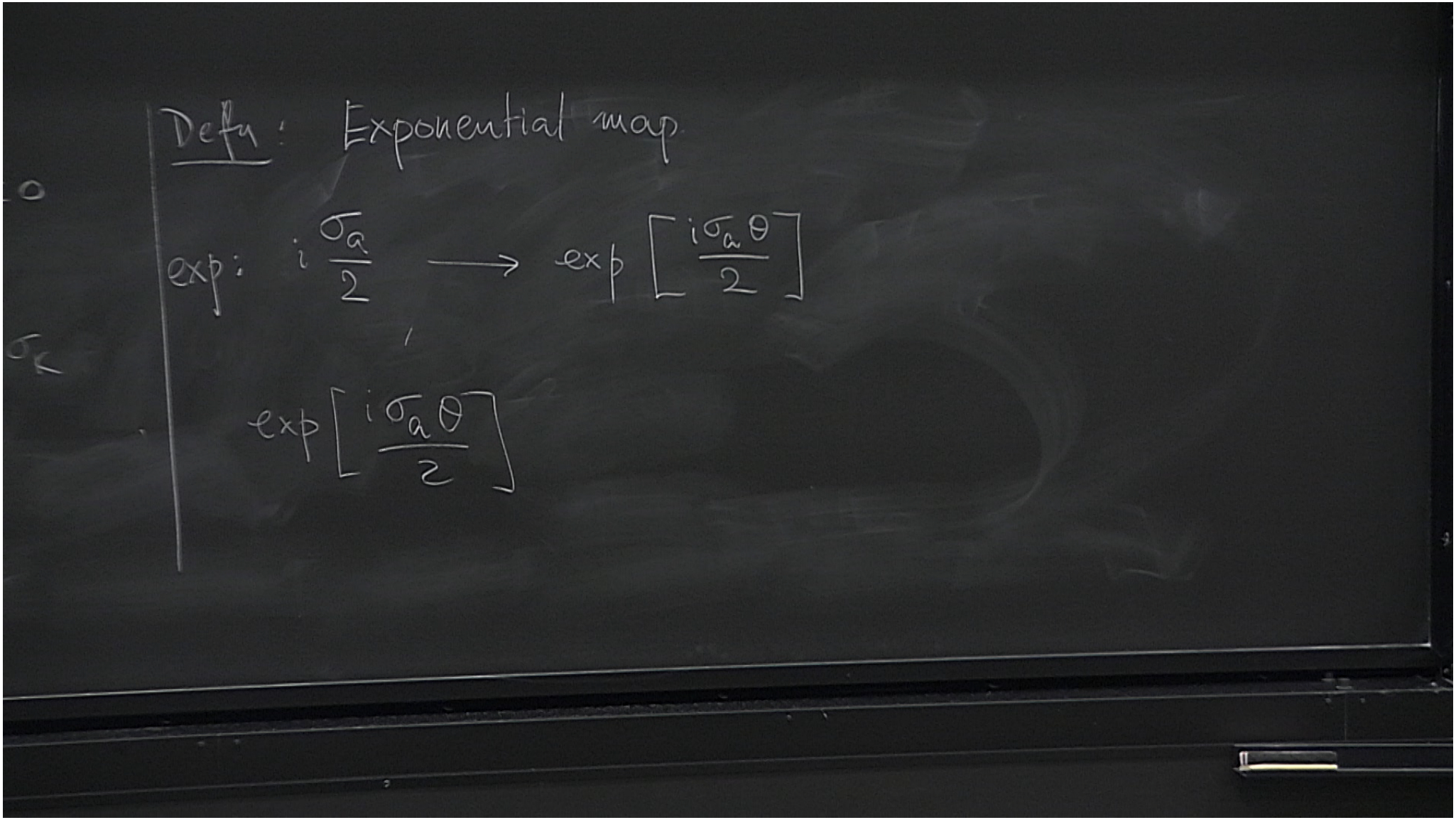
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pauli don't commute

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

$$\Rightarrow e^{\frac{1}{2}\theta\hat{n}\cdot\underline{\sigma}} \rightarrow \text{non-Abelian group}$$


Defn: Exponential map
exp: $i\frac{\sigma_a}{2} \longrightarrow \exp\left[\frac{i\sigma_a\theta}{2}\right]$
$\exp\left[\frac{i\sigma_a\theta}{2}\right]$

Defn: Exponential map

$$\exp: i\frac{\sigma_a}{2} \longrightarrow \exp\left[\frac{i\sigma_a\theta}{2}\right]$$

$\exp\left[\frac{i\sigma_a\theta}{2}\right]$ ~ form an Abelian Subgroup

$e^{i\sigma_1\theta_1/2} e^{i\sigma\theta_2/2} = e^{i\sigma_1(\theta_1+\theta_2)/2}$ | $e^{i\theta\sigma/2} e^{-i\theta\sigma/2} = \mathbb{1}$