Title: Relativistic temperature gradients

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Abstract: <p>Despite being broadly accepted nowadays, temperature gradients in thermal equilibrium states continue to cause confusion, since they naively seem to contradict the laws of classical thermodynamics. In this talk, we will explore the physical meaning behind this concept, specifically discussing the role played by the university of free fall. We will show that temperature, just like time, is an observer dependent quantity and discuss why gravity is the only force capable of causing equilibrium thermal gradients without violating any of the laws of thermodynamics. We will also demonstrate that significant care and delicacy are necessary when extending Tolman's results to distinct classes of heat baths in stationary spacetimes.</p>

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Te Whare Wānanga o te Ūpoko o te Ika a Maui

Equilibrium temperature gradients in stationary spacetimes

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Outline

Equilibrium temperature gradients - static case

Temperature gradients in stationary spacetimes

Temperature distribution in a rotating frame

Some Kerr Black Hole examples

Equilibrium temperature gradients - static case

- Richard C. Tolman: *On the weight of heat and thermal equilibrium in General Relativity*[1].

  Static spherically symmetric spacetime:

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

  + perfect fluid: $T^{ab} = (\rho + p)V^a V^b + p\, g^{ab}$

- $\left(\nabla_a T^{ab} = 0\right) + \left(G^{ab} = 8\pi T^{ab}\right)$ + assumptions depending on the fluid $\rightarrow$ temperature gradient.

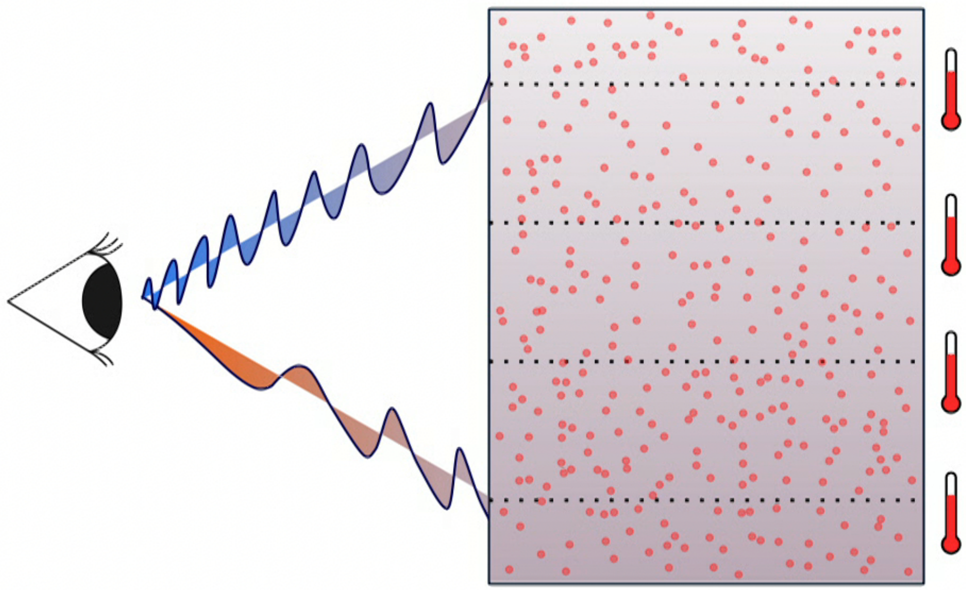
$$T(x) = T_0\sqrt{-g^{tt}}$$

[1] Tolman, R. C., Phys. Rev. 35, 904 (1930).

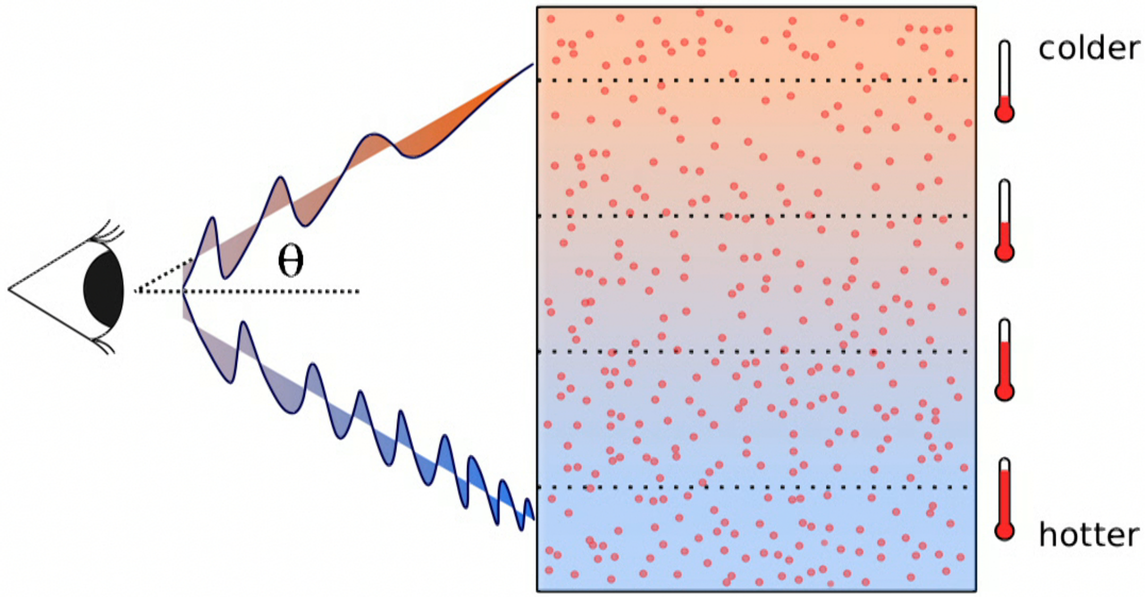
What are the physical consequences?

- Locally measured temperatures $T(x)$ have a small non-zero spatial gradient $\left(T(x) = T_0\,\sqrt{-g^{tt}}\,\right)$ for states in thermodynamic equilibrium.
- However, due to gravitational redshift, observers in the fluid's rest frame will see a constant temperature (T_0).
- Distinct observers will measure different values for T_0:
  The temperature of a fluid in thermal equilibrium is observer dependent.

Constant local temperatures



Relativistic thermal equilibrium



Temperature gradients in stationary spacetimes

In 1949 Buchdahl[2] extended Tolman's results to stationary spacetimes

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

for fluids moving along the Killing vector $K^a = (1,0,0,0)$.

Since then, people have been using the well known result obtained by him:

$$T(x) = \frac{T_0}{||K||}.$$

[2] H. A. Buchdahl, Phys. Rev. **76** (1949) 427.

Temperature gradients in stationary spacetimes

Consider the relativistic Euler equation:

$$(\rho + p)A_a = -(\delta_a{}^b + V_a V^b)\nabla_b p.$$

For a photon gas [$\rho = aT^4$, $p = (a/3)T^4$] this simplifies to

$$A_a = -(\delta_a{}^b + V_a V^b)\nabla_b \ln T.$$

At thermal equilibrium $V^b \nabla_b T = 0$, so

$$A_a = -\nabla_a \ln T.$$

This is a *general relation* between the fluid's 4-acceleration and its temperature gradients for any fluid in equilibrium in a stationary spacetime[3].

[3] R. Tolman and P. Ehrenfest, Phys. Rev. **36** (1930) no.12, 1791.

Temperature gradients in stationary spacetimes

Tolman's results can easily be recovered and extended

For any *static spacetime* with its metric in the block-diagonal form,

$$ds^2 = g_{tt}dt^2 + g_{ij}dx^i dx^j,$$

the 4-acceleration of the observers "at rest", $V^a \propto (1,0,0,0)$, is:

$$A_a = \nabla_a \ln \sqrt{-g_{tt}}.$$

Leading to

$$T(x) = T_0 \sqrt{-g^{tt}}$$

Temperature gradients in stationary spacetimes

Buchdahl's 1949 results can also be recovered and extended

Assume a fluid in a static or stationary spacetime following the integral curves an *arbitrary timelike* Killing vector, as in

$$V^a = \hat{K}^a = \frac{K^a}{||K||},$$

then the fluid 4-acceleration can be easily computed to be

$$A_a = \nabla_a \ln ||K||,$$

leading to

$$T(x) = \frac{T_0}{||K||}.$$

Temperature gradients in stationary spacetimes

Fluid following a normal flow

The other "natural" option is to take the fluid to follow a normal flow. Explicitly,

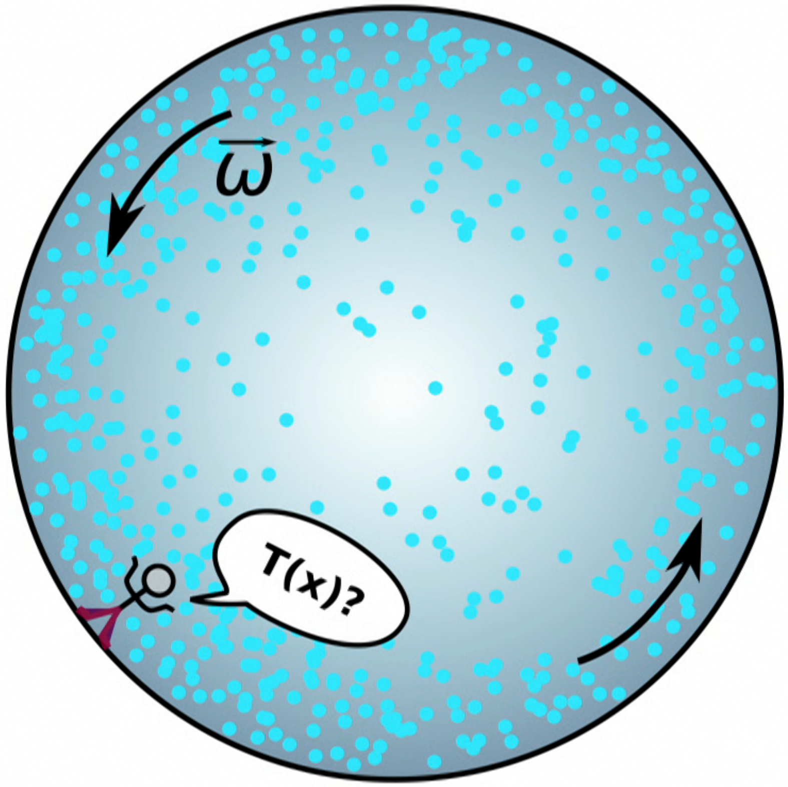
$$\hat{N}_a = -\frac{\nabla_a t}{||\nabla t||}; \qquad \hat{N}^a = V^a = -\frac{\nabla^a t}{||\nabla t||} = \frac{(1; v^i)}{N},$$

where N comes from the ADM-like decomposition of the metric,

$$ds^2 = -N^2 dt^2 + h_{ij}\,(dx^i - v^i\,dt)\,(dx^j - v^j\,dt),$$

and satisfies $||\nabla t|| = \sqrt{-g^{tt}} = N^{-1}$.

What is the temperature distribution in a rotating frame?



The rotating frame temperature distribution

The metric seen by the rotating comoving observers is given by:

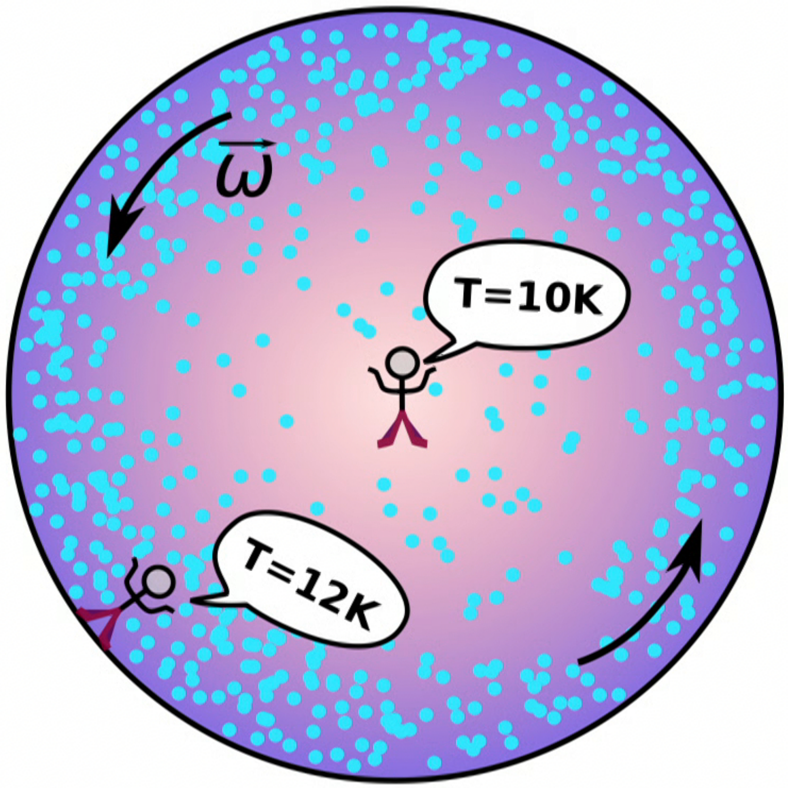
$$ds^2 = -dt^2 + dr^2 + r^2(d\phi - \omega dt)^2 + dz^2.$$

The co-rotating gas will follow trajectories of the Killing field $K^a = (1,0,0,0)$, with $||K|| = \sqrt{1-\omega^2 r^2}$ and $V^a = K^a/||K^a||$.

Applying this result to the stationary temperature gradients, we obtain:

$$T(x) = \frac{T_*}{||K||} = \frac{T_*}{\sqrt{1-\omega^2 r^2}}.$$

What happens for a non-inertial observer?



What is the blackbody spectrum seen by a comoving observer?

With V_e^a the 4-velocity of the emitter (thermal bath) and V_o^a the 4-velocity of the internal (comoving) observer, the redshift is given by

$$1 + z = \frac{(g_{ab} V_e^a k^b)_e}{(g_{ab} V_o^a k^b)_o} = \frac{\nu_e}{\nu_o}.$$

What is the blackbody spectrum seen by a comoving observer?

With some calculations we obtain:

$$\nu_e = \gamma_e \left(-1 + \frac{\omega r_e r_o \sin\theta}{\delta t} \right),$$

and

$$\nu_0 = \gamma_o \left(-1 + \frac{\omega r_e r_o \sin\theta}{\delta t} \right).$$

Consequently

$$1 + z = \frac{\gamma_e}{\gamma_o} = \sqrt{\frac{1 - \omega^2 r_o^2}{1 - \omega^2 r_e^2}}.$$

(It factorizes!)

What is the black-body spectrum seen by a comoving observer?

Given that $\nu_e/\nu_o = 1 + z$, we have

$$\nu_o = \frac{\nu_e}{1+z} = \frac{\nu_*}{\sqrt{1-\omega^2 r_e^2}} \sqrt{\frac{1-\omega^2 r_e^2}{1-\omega^2 r_o^2}} = \frac{\nu_*}{\sqrt{1-\omega^2 r_0^2}},$$

so the temperature seen by the observer is:

$$T(x_o) = \frac{T_*}{\sqrt{1-\omega^2 r_o^2}},$$

which is exactly the equilibrium temperature at the observer's location.

Some Kerr Black Hole examples

Kerr observers - Normal flow in Doran coordinates

Let us choose the Doran coordinate system:

$$
\begin{aligned}
ds^2 &= dt^2 - \left(\frac{\rho}{\sqrt{r^2+a^2}}dr + \alpha(dt - a\sin^2\theta\, d\phi)\right)^2 \\
&\quad -\rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta\, d\phi^2,
\end{aligned}
$$

$$
\alpha = \rho^{-1}\sqrt{2mr}, \quad \rho^2 = r^2 + a^2 cos^2\theta
$$

In these coordinates the normal flow is

$$
\hat{N}_a = -\nabla_a t = (-1; 0,0,0)_a.
$$

We have $||\nabla t|| = N^{-1} = 1$, implying $A = 0$. That is, our "reference fluid" is now in free-fall and we deduce $T(x) = (\text{constant})$.

Kerr observers - Killing flows

In the Boyer-Lindquist coordinate system

$$
\begin{aligned}
ds^2 &= -\left[1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right] dt^2 - \frac{4mra\sin^2\theta}{r^2 + a^2\cos^2\theta}\, dt\, d\phi \\
&+ \left[\frac{r^2 + a^2\cos^2\theta}{r^2 - 2mr + a^2}\right] dr^2 + (r^2 + a^2\cos^2\theta)\, d\theta^2 \\
&+ \left[r^2 + a^2 + \frac{2mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right] \sin^2\theta\, d\phi^2 .
\end{aligned}
$$

we have the "natural" timelike Killing vector (1,0,0,0) plus the rotational Killing vector (0,0,0,1). Let us look at some interesting cases.

Kerr observers - Killing flows

- Setting $\Omega \to \Omega_H$, the angular velocity of the horizon, we have the Killing vector $(1{,}0{,}0{,}\Omega_H)$, which gives us

$$T(x) = \frac{T_0}{\sqrt{N^2 - h_{\phi\phi}(v_\phi - \Omega_H)^2}}.$$

In this situation the Killing vector has a norm $||(1{,}0{,}0{,}\Omega_H)||$ which is zero *at the horizon* — not *at the ergosurface*.

However, its norm also vanishes in the asymptotic region, near $r \sin\theta \approx 1/\Omega_H$. [5]

[5](Which is not a "problem". The same thing happens for a rotating coordinate system in flat Minkowski space.)

Kerr observers - ZAMO flow

Still in the Boyer-Lindquist coordinate system, the normal flow is given by:

$$\hat{N}_a = -\frac{\nabla_a t}{||\nabla t||} = \frac{(-1;0,0,0)}{\sqrt{-g^{tt}}} = \sqrt{-g_{tt} + \frac{g_{t\phi}^2}{g_{\phi\phi}}}\,(-1;0,0,0).$$

The corresponding flow vector in terms of the time translation and axial Killing vectors is

$$V^a = \hat{N}^a = \frac{[K_T]^a + \varpi [K_\phi]^a}{||K_T + \varpi K_\phi||}, \quad \varpi = -g_{t\phi}/g_{tt}.$$

Note that this is *not* a Killing vector since ϖ is not a constant.

Kerr observers - ZAMO flow

Since this is still a special case of a normal flow, we obtain

$$T(x) = T_0\,||\nabla t|| = \frac{T_0}{N} = T_0\sqrt{-g^{tt}}.$$

In terms of these coordinates and the free parameters m and a,

$$T(x) = T_0\,\sqrt{1 + \frac{2mr\,(r^2+a^2)}{(a^2-2\,mr+r^2)\;(r^2+a^2\cos^2\theta)}}.$$

Noticing that $(a^2 - 2\,mr + r^2) = 0$ defines the event horizon,

$$T(x) = T_0\,\sqrt{1 + \frac{2mr\,(r^2+a^2)}{(r-r_+)(r-r_-)\,(r^2+a^2\cos^2\theta)}}.$$

Kerr observers - ZAMO flow

$$T(x) = T_0 \sqrt{1 + \frac{2mr\,(r^2 + a^2)}{(r - r_+)(r - r_-)\,(r^2 + a^2 \cos^2\theta)}}$$

For this particular ZAMO gradient flow the redshifted temperature is well behaved from just above the horizon all the way out to spatial infinity with

$$T(x) \to T_0 \quad \text{for} \quad r \to \infty$$

and diverging only at the event horizon.

Summary

- Tolman temperature gradients depend *both* on the spacetime *and* on the choice of 4-velocity for the heat bath of interest.
- For a heat bath that follows the trajectories of *any* timelike Killing vector

$$T(x) = \frac{T_0}{||K||}.$$

- For a suitably chosen normal flow,

$$T(x) = \frac{T_0}{N}.$$

- For more general cases, $A_a = -\nabla_a \ln T$.

Additional references

J. Santiago and M. Visser,
Tolman temperature gradients in a gravitational field;
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https://doi.org/10.1142/S021827181846001X

J. Santiago and M. Visser,
Tolman-like temperature gradients in stationary spacetimes;
arXiv:1807.02915 [gr-qc].

Summary

- Tolman temperature gradients depend *both* on the spacetime *and* on the choice of 4-velocity for the heat bath of interest.
- For a heat bath that follows the trajectories of *any* timelike Killing vector
$$T(x) = \frac{T_0}{||K||}.$$
- For a suitably chosen normal flow,
$$T(x) = \frac{T_0}{N}.$$
- For more general cases, $A_a = -\nabla_a \ln T$.