Title: Quantum axiomatics Ã la carte

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Abstract: The past decade or so has produced a handful of derivations, or reconstructions, of finite-dimensional quantum mechanics from various packages of operational and/or information-theoretic principles. I will present a selection of these principles --- including symmetry postulates, dilational assumptions, and versions of Hardy's subspace axiom --- in a common framework, and indicate several ways, some familiar and some new, in which these can be combined to yield either standard complex QM (with or without SSRs) or broader theories embracing formally real Jordan algebras.

Quantum Axiomatics a la Carte

Alex Wilce
Susquehanna University

Foundations of QM, Perimeter Institute,
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OUTLINE:

- I *Probabilistic models (with examples)*
- II *An easy route towards QM, and some axioms*
- III *Processes, symmetries, and more axioms*
- IV *Probabilistic theories; even more axioms*
- V *Five axiomatizations of QM*
- VI *Conclusions, etc.*

Color-coding

definitions in blue

potential axioms in green

Test spaces

In discrete classical probability theory, a *probabilistic model* is a pair (E, μ): E a set of outcomes, μ a probability weight on E.

Obvious generalization: Allow both E and μ to vary. Start with E:

A **test space**: a collection

$$\mathcal{M} = \{E, F, ...\}$$

of (outcome-sets of) possible experiments, *tests*, etc. Mathematically, $\mathcal{M}$ is just a hypergraph.

Remarks: Idea due to C. H. Randall (1928-1987) and D. J. Foulis (1930-2018). Original (better?) term: *manual*. Also called *contextuality scenarios* in some more recent literature.

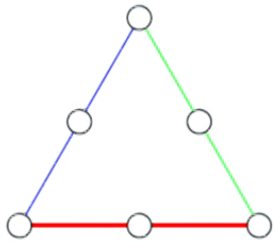
Test spaces

Let $X := \bigcup \mathcal{M}$, i.e., space of *all* outcomes. A **probability weight** on $\mathcal{M}$:

$$\alpha : X \to [0,1] \text{ with } \sum_{x \in E} \alpha(x) = 1 \; \forall E \in \mathcal{M}.$$

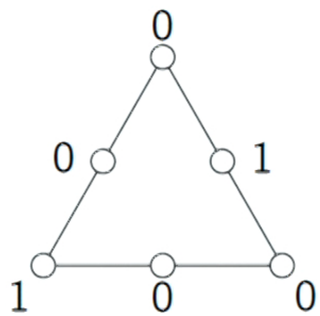
Remarks:
(i) probability weights are *non-contextual*
(ii) Contextuality easy to handle if desired.
(iii) Set of all probability weights on $\mathcal{M}$ a convex subset of $[0,1]^X$; closed if all tests are finite.

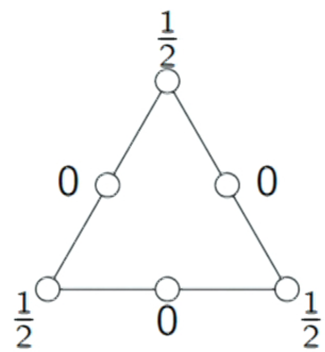
Triangular Example (rather weird)



X = nodes; $\mathcal{M}$ = sides

Sample probability weights:





Note: Both of these are pure!

(General) Probabilistic models

Generalizing classical definition:

A **probabilistic model** (or just *model*): a pair $A = (\mathcal{M}, \Omega)$,

- $\mathcal{M} =: \mathcal{M}(A)$ a test space,
- $\Omega =: \Omega(A)$ a *closed, convex* set of probability weights on $\mathcal{M}$ (the **state space** of A).

Remark: (i) Such models are easy to build and manipulate.
(ii) Easy to add more structure (topological, group-theoretic, etc.) if desired.

Standing assumption: $\Omega(A)$ always finite-dimensional.

Classical and Quantum Examples

Simple **classical model**: $A = (\{E\}, \Delta(E))$ — one test, all probability weights.

Simple **quantum model**: For a (f.d.) Hilbert space $\mathcal{H}$, let

- $\mathcal{M}(\mathcal{H})$ = set of ONBs for $\mathcal{H}$;
- $\Omega(\mathcal{H})$ = all probability weights states of the form

$$\alpha(x) = \langle Wx, x \rangle,$$

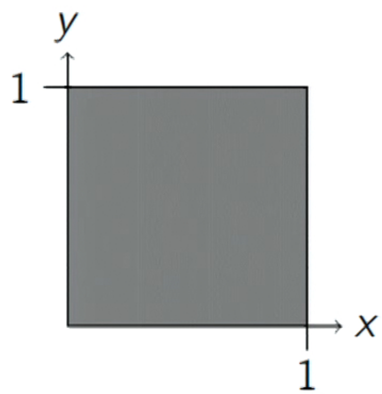
W a density operator on $\mathcal{H}$ (= all prob. weights, if $\dim \mathcal{H} > 2$).

Two-bit examples

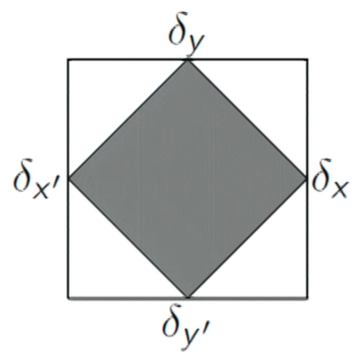
The **square bit** B and **diamond bit** B' have the same test space:

$$\mathcal{M}(B) = \mathcal{M}(B') = \{\{x, x'\}, \{y, y'\}\}$$

but different state spaces:



$\Omega(A)$ = all prob weights on $\mathcal{M}(A)$



$\Omega(A')$

Some properties of probabilistic models

A probabilistic model A is

- **uniform** iff all tests $E \in \mathcal{M}(A)$ have a common size, say $|E| = n$ (the **rank** of A)
- **sharp** iff $\forall x \in X(A)\ \exists! \delta_x \in \Omega(A)$ with $\delta_x(x) = 1$;
- **spectral** iff sharp and, $\forall \boldsymbol{\alpha} \in \Omega(A)$, $\exists E \in \mathcal{M}(A)$ with

$$\boldsymbol{\alpha} = \sum_{x \in E} \alpha(x)\delta_x.$$

Triangle, square bit $\rightarrow$ uniform, not sharp.
Diamond bit $\rightarrow$ uniform and sharp, not spectral.
Classical and quantum models $\rightarrow$ uniform, sharp, spectral.

The spaces $\mathbf{V}(A)$ and $\mathbf{E}(A)$

$\mathbf{V}(A) :=$ span of $\Omega(A)$ in $\mathbb{R}^{X(A)}$, with positive cone

$$\mathbf{V}(A)_+ := \{\ t\boldsymbol{\alpha} \,|\, \boldsymbol{\alpha} \in \Omega,\ t \geq 0\ \}$$

Effects are elements $a \in \mathbf{V}(A)^*$ with $0 \leq a(\alpha) \leq 1\ \forall \boldsymbol{\alpha} \in \Omega(A)$. ("mathematically possible" measurement-outcomes). *Note* that

$$\widehat{x}(\boldsymbol{\alpha}) \ := \ \boldsymbol{\alpha}(x)$$

is an effect for all $x \in X(A)$. For convenience, from now on identify x with $\widehat{x}$, so that $X \subseteq \mathbf{V}(A)^*$.

Also useful to define $\mathbf{E}(A) := \mathbf{V}(A)^*$, but ordered by

$$\mathbf{E}(A)_+ := \left\{ \sum_{i=1}^{k} t_i x_i \,\middle|\, x_i \in X(A),\ t_i \geq 0 \right\}$$

No-Restriction Hypotheses

We always have a *unit effect* $u_A(\alpha) \equiv 1$ on $\Omega(A)$. An **observable** is a set of effects $a_1, ..., a_n$ with $\sum_i a_i = u_A$. Thus, each test $E \in \mathcal{M}(A)$ is an observable.

A mathematically attractive assumption:

No Restriction Hypothesis: Every effect $a \in \mathbf{E}(A)_+$ (or even in $V(A)^*$) is physically accessible measurement outcome.

A weaker assumption of a similar kind:

NR Hypothesis for measurements: If $a_1, ..., a_n \in \mathbf{E}(A)_+$ with $a_1 + \cdots + a_n = u_A$, then $\{a_1, ..., a_n\}$ is a physically accessible measurement.

Both are true in QM, but neither is easy to motivate!

Joint States

A (non-signaling) **joint state** on A and B is a mapping

$$\omega : X(A) \times X(B) \to [0,1]$$

with

(a) $(E,F) \in \mathcal{M}(A) \times \mathcal{M}(B) \implies \sum_{(x,y)\in E\times F} \omega(x,y) = 1;$

(b) $x \in X(A),\ y \in X(B) \implies$

$$\omega(x\ \cdot\ ) \in \mathbf{V}_+(B) \text{ and } \omega(\ \cdot\ y) \in \mathbf{V}_+(A)$$

Condition (b) implies ω has well-defined **marginal and conditional states**:

$$\omega_1(x) := \sum_{y\in F} \omega(\cdot, y) \in \Omega(A) \quad \text{and} \quad \omega_{2|x}(y) := \frac{\omega(x,y)}{\omega_1(x)} \in \Omega(B);$$

similarly for $\omega_2(y), \omega_{1|y}$.

Joint States

Marginal and conditional states are related by a
bf Law of total probability: $\forall\ E \in \mathcal{M}(A),\ F \in \mathcal{M}(B)$,

$$\omega_2 = \sum_{x \in E} \omega_1(x)\omega_{2|x} \quad \text{and} \quad \omega_1 = \sum_{y \in F} \omega_2(y)\omega_{1|y}$$

Important observation: Every joint state extends to a unique positive linear mapping

$$\widehat{\omega} : \mathbf{E}(A) \to \mathbf{V}(B),$$

such that $\widehat{\omega}(x)(y) = \omega(x, y)\ \ \forall x \in X(A),\ y \in X(B)$. If $\widehat{\omega}$ is an order-isomorphism, call ω an **isomorphism state**.

II. An easy route towards QM

Euclidean Jordan algebras as ordered vector spaces

Let $\mathbf{E}$ be a f.d. ordered real vector space with positive cone $\mathbf{E}_+$ and with an inner product $\langle\, , \,\rangle$. $\mathbf{E}$ is

- *self-dual* iff $\langle a, b\rangle \geq 0 \; \forall b \in \mathbf{E}_+$ iff $a \in \mathbf{E}_+$.
- *homogeneous* iff group of order-automorphisms of $\mathbf{E}$ is transitive on the *interior* of $\mathbf{E}_+$.

Koecher-Vingerg Theorem [1957/1961]: *$\mathbf{E}$ is HSD $\Leftrightarrow$ $\mathbf{E}$ a euclidean (=formally real) Jordan algebra with $\mathbf{E}_+ = \{a^2 | a \in \mathbf{E}\}$*

Jordan-von Neumann-Wigner Classification [1932]: *Formally real Jordan algebras = direct sums of self-adjoint parts of $M_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $M_3(\mathbb{O})$, or "spin factors" V_n ("bit" with state space an n-ball.)*

Self-duality in QM

$\mathcal{H}$ a complex Hilbert space, $\dim(\mathcal{H}) = n$. Let $\mathbf{E} = \mathcal{L}_h(\mathcal{H})$ with $\mathbf{E}_+$ $=$ cone of positive operators. This is SD w.r.t.

$$\langle a, b \rangle := \frac{1}{n}\mathrm{Tr}(ab).$$

Note that $\langle \quad \rangle = \frac{1}{n}\mathrm{Tr}$ is a *bipartite state*: if

$$\mathbf{\Psi} = \frac{1}{\sqrt{n}} \sum_{x \in E} x \otimes \overline{x} \in \mathcal{H} \otimes \overline{\mathcal{H}},$$

E any ONB for $\mathcal{H}$, then $\langle (a \otimes \overline{b}), \Psi, \Psi \rangle \ = \ \frac{1}{n}\mathrm{Tr}(ab)$.

So $\mathbf{\Psi}$ perfectly, and uniformly correlates every ONB of $\mathcal{H}$ with its counterpart in $\overline{\mathcal{H}}$: $|\langle \mathbf{\Psi}, x \otimes \overline{y} \rangle|^2 = \frac{1}{n}$ if $x = y$, 0 if $x \perp y$. $\mathbf{\Psi}$ *is uniquely defined by this feature.*

Conjugate Models

Let A be uniform, with rank n. A **conjugate** for A: a model $\overline{A}$ plus an isomorphism $\gamma_A : A \simeq \overline{A}$ taking $x \in X(A)$ to $\overline{x} := \gamma_A(x) \in X(\overline{A})$, and a joint state η_A on A and $\overline{A}$ such that

(a) $\eta(x, \overline{y}) = \eta(y, \overline{x})$ and

(b) $\eta_A(x, \overline{x}) = \frac{1}{n}\ \forall x \in X(A)$.

Lemma: *If A is sharp, spectral, and has a conjugate, then*

$$\langle a, b \rangle := \eta_A(a, \overline{b})$$

is a self-dualizing inner product on $\mathbf{E}(A)$.

See `arXiv:1606.09306` for the easy proof.

Why spectrality?

A joint state $\omega \in \Omega(AB)$ **correlating** iff $\exists E \in \mathcal{M}(A)$, $F \in \mathcal{M}(B)$, and partial bijection $f \subseteq E \times F$ such that

$$\omega(x,y) > 0 \Leftrightarrow (x,y) \in f.$$

Lemma: *A sharp and $\omega \in \Omega(AB)$, correlating $\Rightarrow \omega_1$ spectral.*

This suggests the

Correlation Principle: Every state is the marginal of a correlating joint state.

So: CP $\Rightarrow$ spectrality.

Processes and dual processes

A **process** from A to B is represented by a positive linear mapping

$$\phi : \mathbf{V}(A) \to \mathbf{V}(B) \quad \text{with} \quad u_B(\phi(\boldsymbol{\alpha})) \leq 1 \ \forall \boldsymbol{\alpha} \in \Omega(A).$$

($p = u_B(\tau(\boldsymbol{\alpha})) =$ probability for the process to "fail" on input state $\boldsymbol{\alpha}$.) Equivalently, the dual process

$$\phi^* : \mathbf{V}(B)^* \to \mathbf{V}(A)^*$$

given by $\phi^*(b) := b \circ \phi$, takes effects to effects (so that $\phi^*(u_B) \leq u_A$.

Remarks: (i) Not every such map needs to count as a physical process; (ii) A dual process need not preserve $\mathcal{M}(A)$, or even $\mathbf{E}(A)_+$!

Processes

We can enrich our notion of a model by equipping A with a designated semigroup $\mathrm{Proc}(A)$ of processes.

A process $\phi \in \mathrm{Proc}(A)$ is **p-reversible** iff there exists $\psi \in \mathrm{Proc}(A)$ with $\psi \circ \phi = p\mathrm{id}_A$, where $0 < p \leq 1$.

Think of p as the probability with which ψ undoes ϕ. Implies ϕ invertible as a linear map, with positive inverse. If $p = 1$, $\psi = \phi^{-1}$ and ϕ is simply reversible.

A **symmetry** of A: a p-reversible process g such that g^* maps $\mathcal{M}(A)$ onto $\mathcal{M}(A)$.

This implies g is reversible with $p = 1$. Let $G(A)$ = set of symmetries, and note it's a group.

Symmetry Principles

QM suggests looking at models in which $\mathcal{M}(A)$ is very homogeneous under $G(A)$:

Call a model A

(a) **symmetric** iff $G(A)$ acts transitively on outcomes;

(b) **fully symmetric** iff for every bijection $f : E \to F$, $E, F \in \mathcal{M}(A)$, $\exists g \in G(A)$ with $gx = f(x)\ \forall x \in E$;

All of the examples above *except* the triangle are fully symmetric.

Symmetry Principles

Alternatively, one can impose homogeneity conditions on $\Omega(A)$ [MU, BMU]:

A list $\alpha_1, ..., \alpha_k$ of states is *sharply distinguishable* iff there exist effects $a_1, ..., a_n$ with $\sum_i a_i \leq u$ and $\alpha_i(x_j) = \delta_{i,j}$.

A model A is

(a) **bit-symmetric** iff for all sharply distinguishable pairs (α_1, α_2) and (β_1, β_2), there exists $g \in G(A)$ with $g\alpha_i = \beta_i$;

(b) **strongly symmetric** iff for all maximal sharply distinguishable sets of states $\alpha_1, .., \alpha_n$ and $\beta_1, ..., \beta_n$, $\exists g \in G(A)$ with $g(\alpha_i) = \beta_i$ for all i

Remark: In some sense, these depend on NR/NR for measurements.

Mueller-Ududec Theorem

The following really remarkable result gives another route to self-duality:

Theorem [Mueller and Ududec, 2010] *If* $\Omega(A)$ *is bit-symmetric, then* $\mathbf{V}(A)$ *carries a self-dualizing inner product.*

See (`arXiv:arXiv:1110.3516`) for the beautiful proof.

Why Homogeneity?

We still need to motivate homogeneity. We'll see several ways to do so, but here are two easy ones:

- Just **take it as an axiom**! (All nonsingular states are "alike")
- Adopt the **Iso-dilation principle**[1] Every state is the marginal of an isomorphism state

Another approach involves the concept of a *filter*:

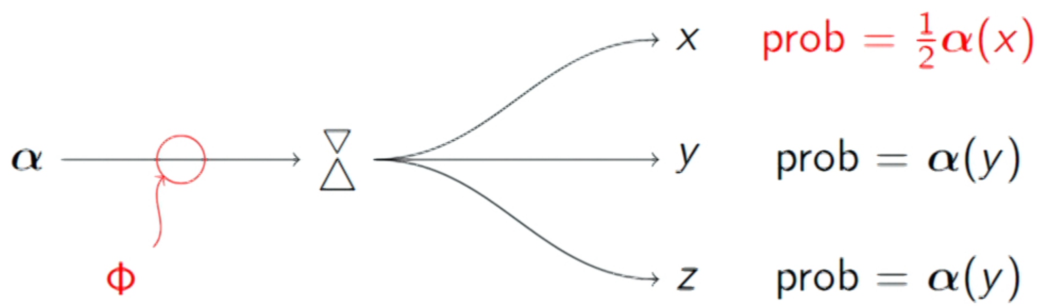
[1] H. Barnum, C. P. E. Gaebler, AW, arXiv:0912.5532

Filters and Homogeneity

A **filter** for $E \in \mathcal{M}(A)$: a process $\Phi : \mathbf{V}(A) \to \mathbf{V}(A)$ such that $\forall x \in E \;\; \exists t_x \geq 0$ with

$$\Phi(\boldsymbol{\alpha})(x) = t_x \boldsymbol{\alpha}(x)$$

for all $\alpha \in \Omega(A)$.



Example: For W a density operator on $\mathcal{H}$, $\Phi : a \mapsto W^{1/2} a W^{1/2}$ is a filter for any eigenbasis of W, reversible iff W is nonsingular.

Lemma: *Let A be sharp, spectral. Then TAE:*

(a) *A has arbitrary reversible filters*

(b) *$\mathbf{V}(A)$ is homogeneous*

So if A is also self-dual, $\mathbf{E}(A) \simeq \mathbf{V}(A)$ has a euclidean Jordan structure. (One can also show that then $X(A)$ is the set of all minimal idempotents in $\mathbf{E}$, and $\mathcal{M}(A)$ is the set of Jordan frames, i.e., A is a *Jordan model*. See `arXiv: 1206.2897`.)

Theory-wide axioms

A probabilistic theory: a *class*, $\mathcal{C}$, of models — maybe a category, with designated processes $\mathcal{C}(A, B)$. More sparingly, a class of models equipped with semigroups $\text{Proc}(A)$

All axioms, principles, etc. considered to this point have been "single-system" [BMU]. But some widely used assumptions depend the entire theory:

(1) An *isomorphism principle*,

(2) the *subspace axiom*

(3) *monoidality*, and, related to this, the PP and the principle of *local tomography*

Let's review them in turn.

Subspace postulates

For $x \in X(A)$, let

$$\mathcal{M}_x(A) = \{E \setminus \{x\} \mid E \in \mathcal{M}(A)\} \text{ and } \Omega_x(A) = \{\alpha \in \Omega(A) \mid \alpha(x) = 0\}.$$

States in $\Omega_x(A)$ can be regarded as states on $\mathcal{M}_x(A)$. Let

$$A_x = (\mathcal{M}_x(A), \Omega_x(A)).$$

Subspace Postulate: For every $A \in \mathcal{C}$, and every $x \in X(A)$, A_x also belongs to $\mathcal{C}$. Any symmetry $g \in G(A_x)$ extends to some $g_1 \in G(A)$ with $g_1(x) = x$.

Equally plausible:

Strong subspace postulate (SSP): For every $A \in \mathcal{C}$, and every $x \in X(A)$, A_x also belongs to $\mathcal{C}$. Any process $\phi \in \text{Proc}(A_x)$ extends to some $\phi_1 \in \text{Proc}(A)$ with $\phi_1^*(x) = x$.

Subspace postulates

By induction on rank, one has

Lemma: *Let $\mathcal{C}$ satisfy the SP, and suppose every A is uniform, with $X(A)$ compact. Then*

(a) *Every A is spectral (in particular: sharp).*

(b) *If every $A \in \mathcal{C}$ is symmetric, then every $A \in \mathcal{C}$ is fully symmetric.*

Moreover, if $\mathcal{C}$ satisfies the SSP and every A is symmetric, then A has arbitrary reversible filters.

Proposition: *Suppose $\mathcal{C}$ is a theory in which every A is uniform, with $X(A)$ compact. If $\mathcal{C}$ satisies the SSP and every A is symmetric, then A is homogeneous. If every A has a conjugate, A is also self-dual.*

Monoidality

A theory $\mathcal{C}$ is **monoidal** iff it supplies, for every pair of models A, B in $\mathcal{C}$, a *composite model* AB and, in particular, a joint state space $\Omega(AB)$. (Strong version of this: require $\mathcal{C}$ to be a symmetric monoidal category).

Purification Principle [CDP]: Let $\mathcal{C}$ be monoidal. For every model A in $\mathcal{C}$, every state of A is the marginal of a pure state of some $\Omega(AB)$, unique up to a reversible transformation on the purifying system B.

Remark: If $\mathbf{V}(A)$ is irreducible, isomorphism states are pure [BGW]. In this case, Iso-dilation implies the purification postulate.

Local tomography

A monoidal theory $\mathcal{C}$ is **locally tomographic** (LT) iff states in $\Omega(AB)$ are distinguishable by joint tests $E \times F$, where $E \in \mathcal{M}(A)$ and $F \in \mathcal{M}(B)$, for all A, B in $\mathcal{C}$.

Equivalently: $\mathbf{V}(AB) \simeq \mathbf{V}(A) \otimes \mathbf{V}(B)$ as a vector space, i.e.,

$$\dim(\mathbf{V}(AB)) = \dim(\mathbf{V}(A))\dim(\mathbf{V}(B)).$$

QM over $\mathbb{C}$ satisfies LT, real/quaternionic QM do not. LT often invoked to rule out the latter two.

Theorem [BW]: *If $\mathcal{C}$ is a locally tomographic theory consisting of Jordan models, and contains the qubit, it is a subtheory of QM over $\mathbb{C}$.*

Two involving conjugates

A) For all systems A in $\mathcal{C}$,

(1) A is sharp,

(2) A has a conjugate,

(3) A satisfies the Correlation Principle (CP)

(4) A has arbitrary reversible filters

B) $\mathcal{C}$ satisfies SSP, and for all systems $A \in \mathcal{C}$,

(1) A is sharp,

(2) A has a conjugate,

(3) A is symmetric, with $X(A)$ compact;

Theorem : *(A)* $\longrightarrow$ *Jordan algebras (EJAs). (B)* $\longrightarrow$ irreducible *EJAs.*

Another route to EJAs

(C) [BMU] Every system satisfies

(1) NR for measurements

(2) Spectrality

(3) Strong symmetry

(4) Existence of an energy observable (not covered here)

(1)-(3) lead to EJAs. (1)-(4) single out standard QM without SSRs.

The Classics

(D) [MM] $\mathcal{C}$ satisfies

(1) Subspace Principle

(2) Isomorphism

(3) NR for bits

(4) Local tomography

(E) [CDP] $\mathcal{C}$ satisfies

(1) State-discrimination (not discussed here)

(2) Ideal Compression (implies SSP)

(3) Pure conditioning ($\widehat{\omega}$ preserves pure states)

(4) Purification Principle ((1) - (4) imply CP and existence of conjugates)

(4) Local tomography

Evidently, many other choices are possible!

A conclusion, a question, and a speculation

(1) While there's more to do, it's pretty clear one can give a unified and streamlined account of all the main reconstructions, steering a course towards/through EJAs.

(2) Can one obtain the JNW classification directly, without appeal to the KV theorem and Jordan structure?

(3) That one can so freely "mix and match" of axioms and arrive more or less the same place suggests (to me, at least right now) that the probabilistic apparatus of QM arises more from methodological than from physical constraints.

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