Title: Towards synthetic Euclidean quantum field theory

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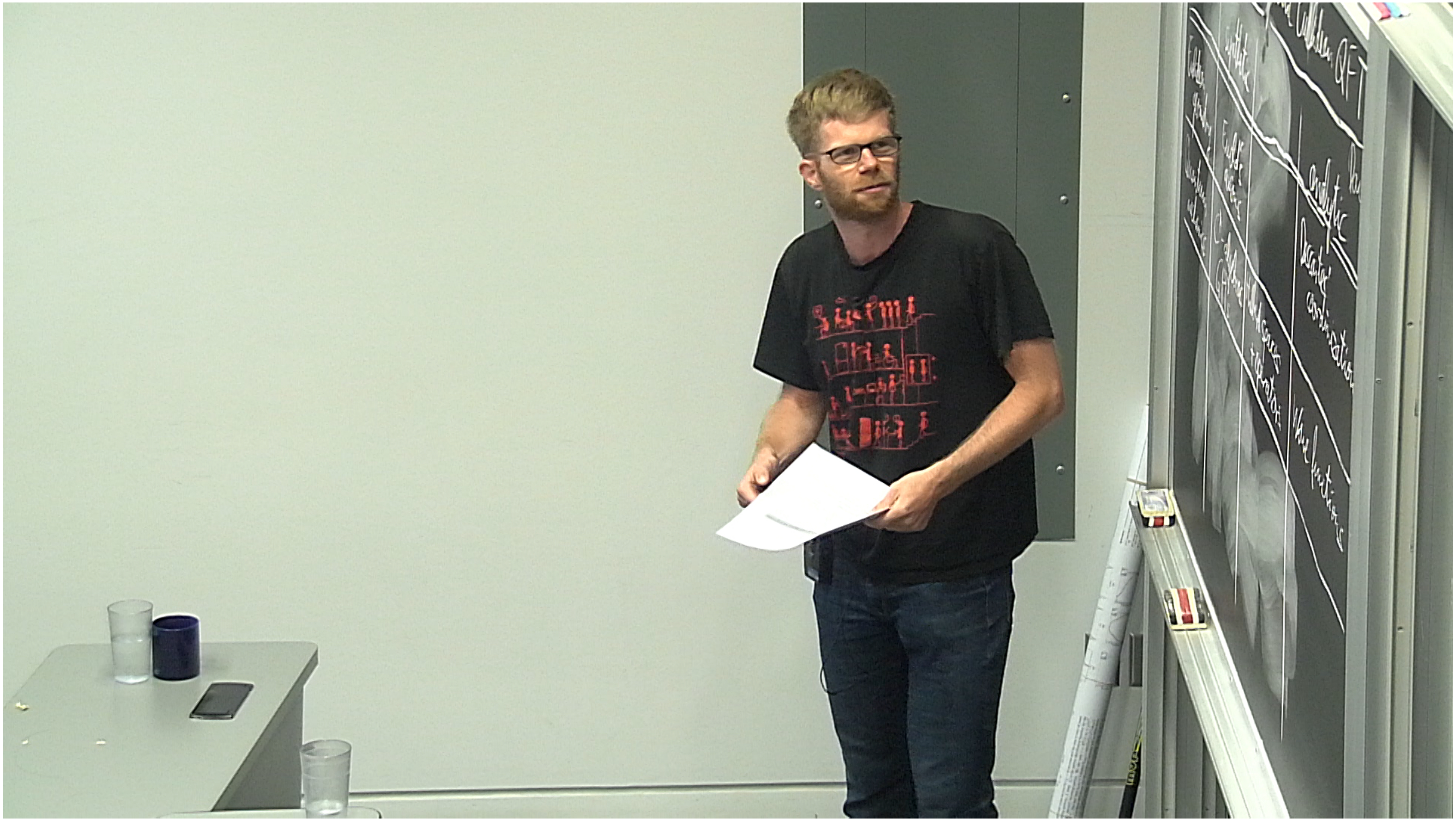
Abstract: In this status report on current work in progress, I will sketch a generalization of the temporal type theory introduced by Schultz and Spivak to a logic of space and spacetime. If one writes down a definition of probability space within this logic, one conjecturally obtains a notion whose semantics is precisely that of a Euclidean quantum field. I will sketch how to use the logic to reason about probabilities of events involving fields, sketch the relation to AQFT, and attempt to formulate the DLR equations within the logic.
Joint work with David Spivak

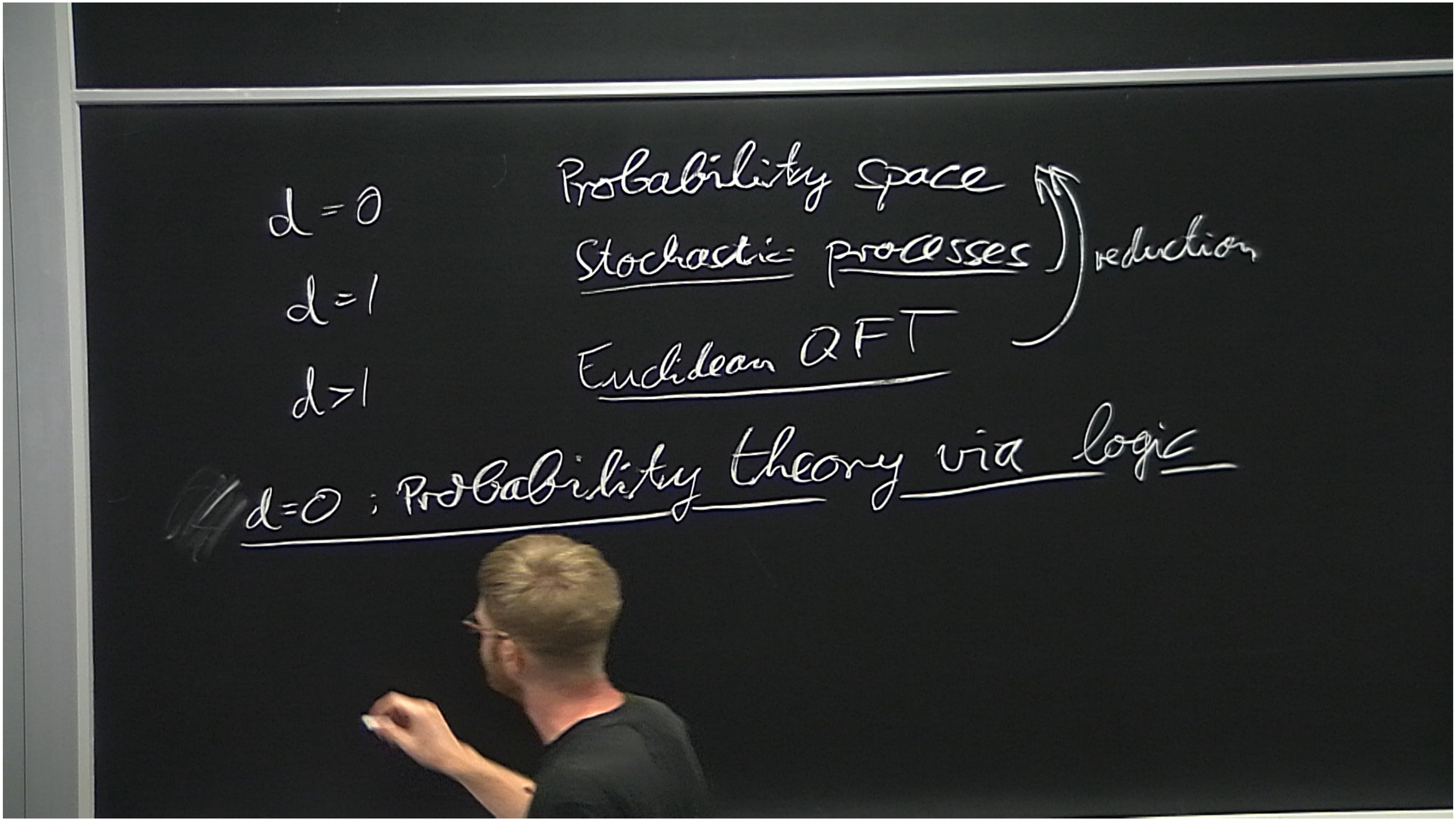

analytic
Wave functions


TOWARDS synthetic Euclidean QFT
synthetic
analytic
C*-algebras
Hilbert spaces + operators
Wave functions
Euclidean QFT
??
Gibbs measures,
DLR equations

TOWARDS synthetic Euclidean QFT

David Spivak

	synthetic		analytic
Euclidean geometry	Euclid's axioms		Descartes' coordinization
Quantum mechanics	C^*-algebras GPTs	Hilbert spaces + operators	Wave functions
Euclidean QFT	??	regularity structures	Gibbs measures, DLR equations



d=0 Probability space
d=1 Stochastic processes
d>1 Euclidean QFT
reduction
d=0 : Probability theory via logic

$d=0$

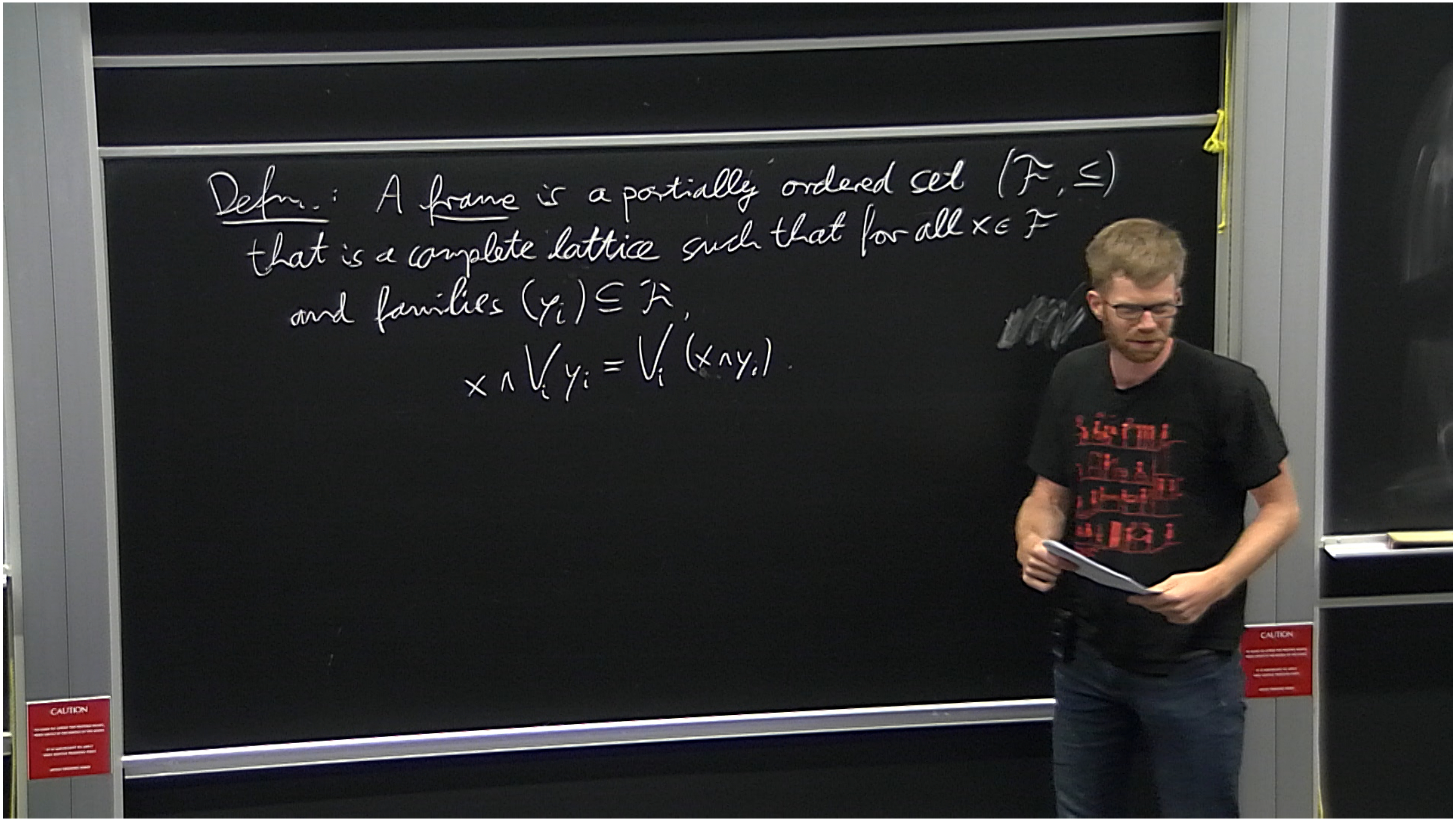
$d=1$ Stoc... processes } reduction

$d>1$ Euclidean QFT

$d=0$; Probability theory via logic

Prob theory assigns numbers to logical propositions:

e.g.: P[" It will rain 2mrw"]


Defn.: A frame is a partially ordered set $(\mathcal{F}, \leq)$
that is a complete lattice such that for all $x \in \mathcal{F}$
and families $(y_i) \subseteq \mathcal{F}$,
$x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i)$.
CAUTION
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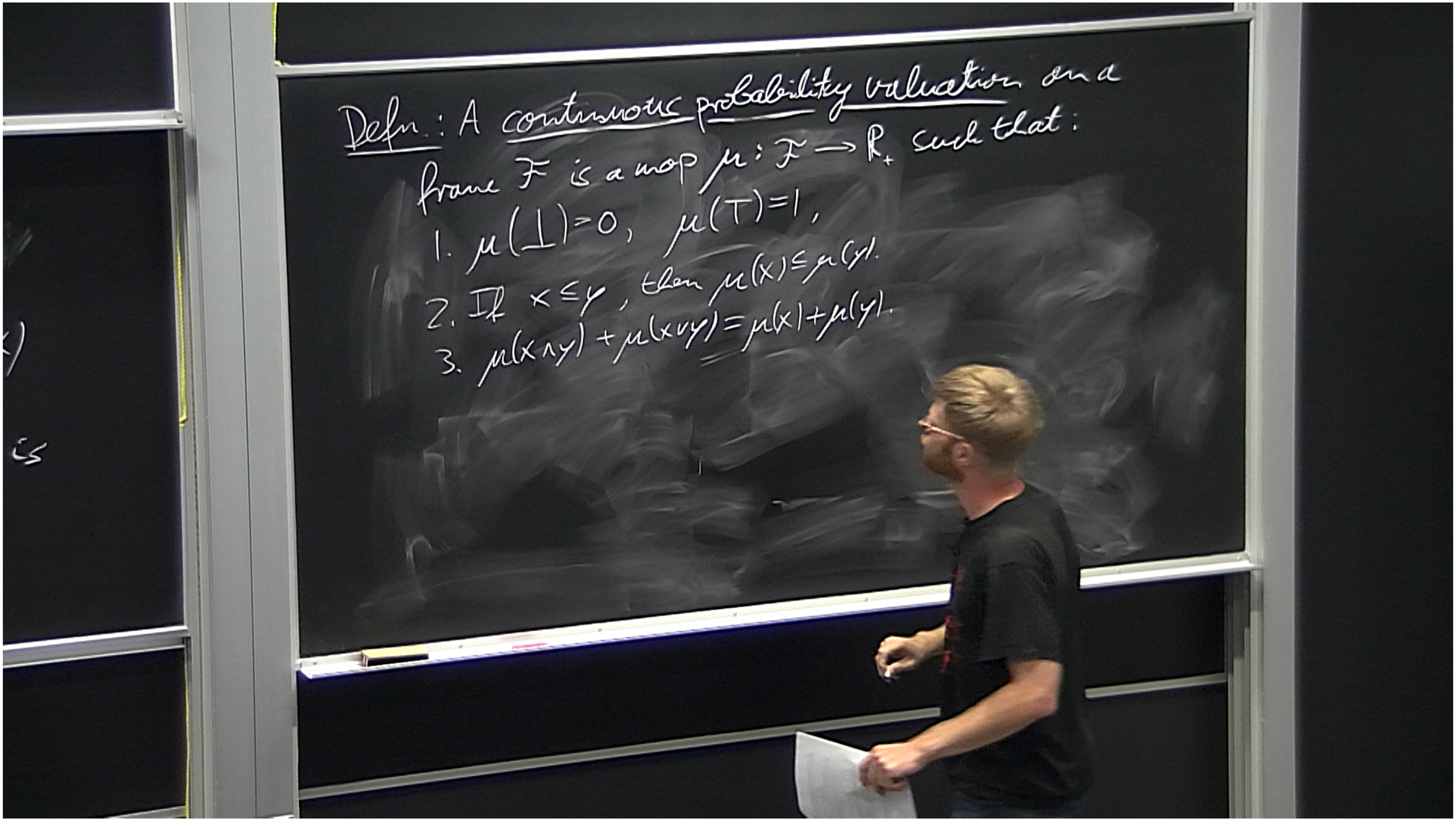
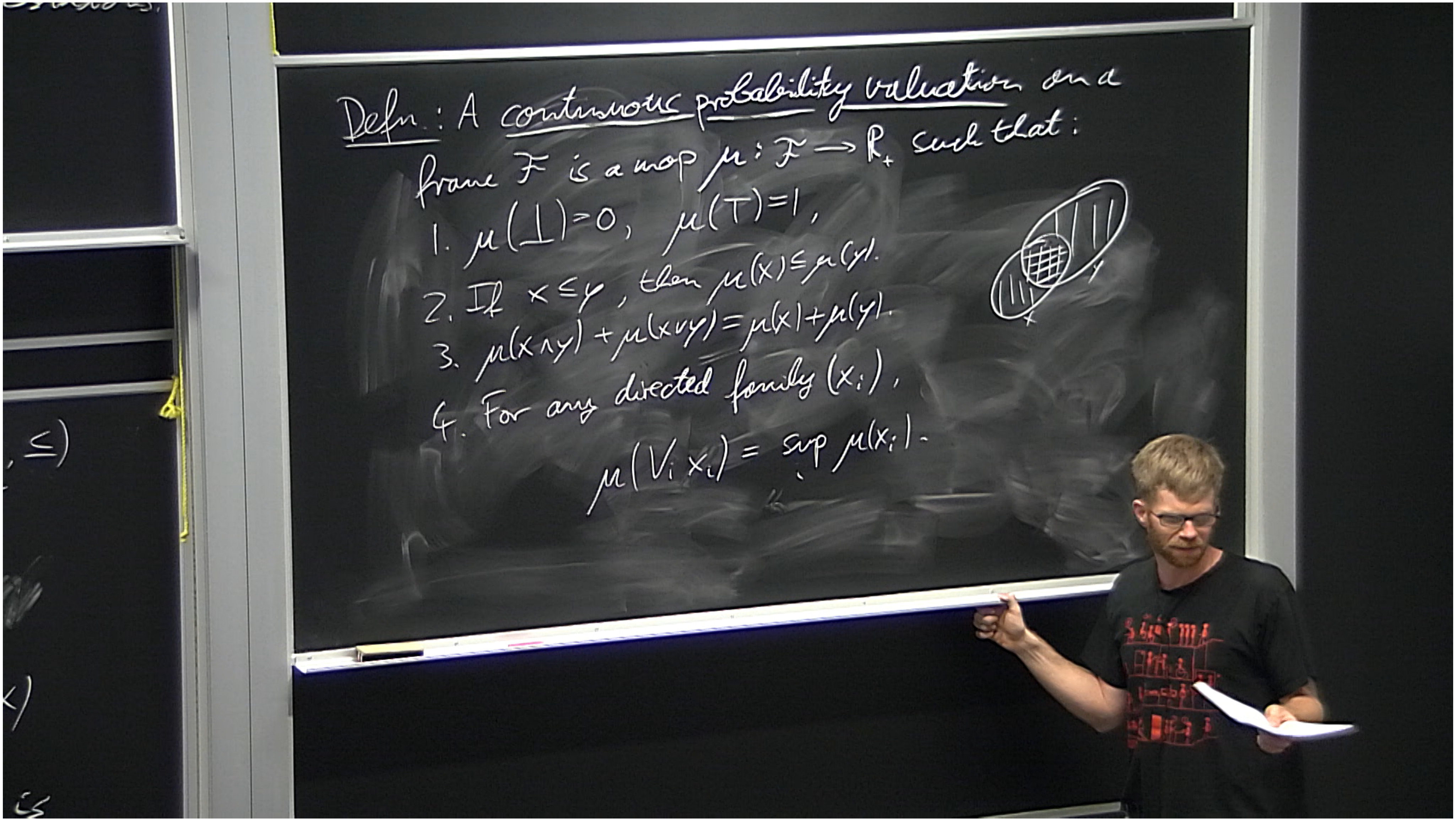
Ex.: If $(X, \mathcal{O}(X))$ is a topological space, then $\mathcal{O}(X)$ is a frame.

Assigning probabilities to logical propositions is given by:


Defn.: A continuous probability valuation on a
frame $\mathcal{F}$ is a map $\mu : \mathcal{F} \to \mathbb{R}_+$ such that:
1. $\mu(\bot)=0$, $\mu(\top)=1$,
2. If $x \leq y$, then $\mu(x) \leq \mu(y)$.
3. $\mu(x \wedge y) + \mu(x \vee y) = \mu(x) + \mu(y)$.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3. $\mu(x \wedge y) + \mu(x \vee y) = \mu(x) + \mu(y)$.
4. For any directed family (x_i),
$\mu(\bigvee_i x_i) = \sup_i \mu(x_i)$.
x
y

Thm.: (Alvarez-Manilla '02) If X is regular, then the CPVs on $\mathcal{O}(X)$ are in bijection with regular τ-smooth Borel prob measures on X.

$\Rightarrow$ For me, $\mu : \mathcal{O}(X) \to \mathbb{R}_+$ a CPV
= probability space

Topos theory as the logic of space

What are possible truth values of "It is raining"?

$\rightarrow$ Answer is a function of space and time!

The truth is a function $S \rightarrow \{\perp, \top\}$

Equivalently, the truth value is the collection of points at which it is raining.

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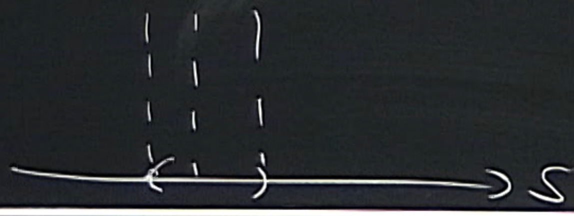
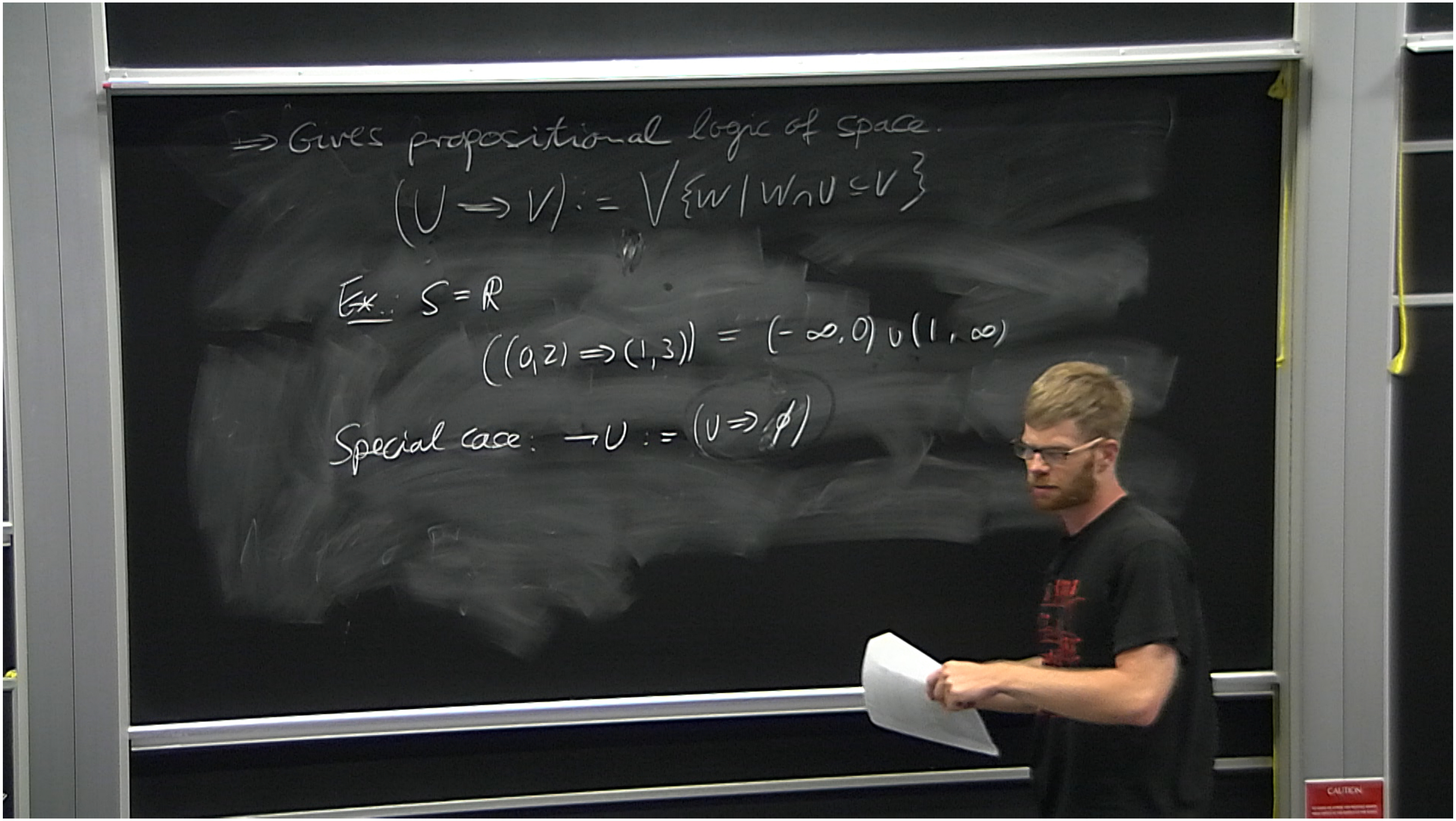
$\Rightarrow$ The set of truth values is the frame of opens $\mathcal{O}(S)$.




⇒ Gives propositional logic of space.
$(U \Rightarrow V) := \bigvee\{W \mid W \cap U \subseteq V\}$
Ex.: $S = \mathbb{R}$
$((0,2) \Rightarrow (1,3)) = (-\infty, 0) \cup (1, \infty)$
Special case: $\neg U := (U \Rightarrow \emptyset)$

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$$(U \Rightarrow V) \wedge (V \Rightarrow W) \Rightarrow (U \Rightarrow W)$$

$\Rightarrow$ Gives propositional logic of space.

$$(U \Rightarrow V) := \bigvee \{ W \mid W \cap U \subseteq V \}$$

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$$(U \Rightarrow V) \wedge (V \Rightarrow W) \Rightarrow (U \Rightarrow W)$$

Topos theory: the above logic + quantifiers

From now on, I will work in this logic.

Defn.: A *topological space* is a set X together with a subset $\mathcal{O}(X) \subseteq \Omega^X$ s.t.:

truth values

1. $\emptyset, X \in \mathcal{O}(X)$
2. $\mathcal{O}(X)$ closed under binary intersections + arbitrary unions.

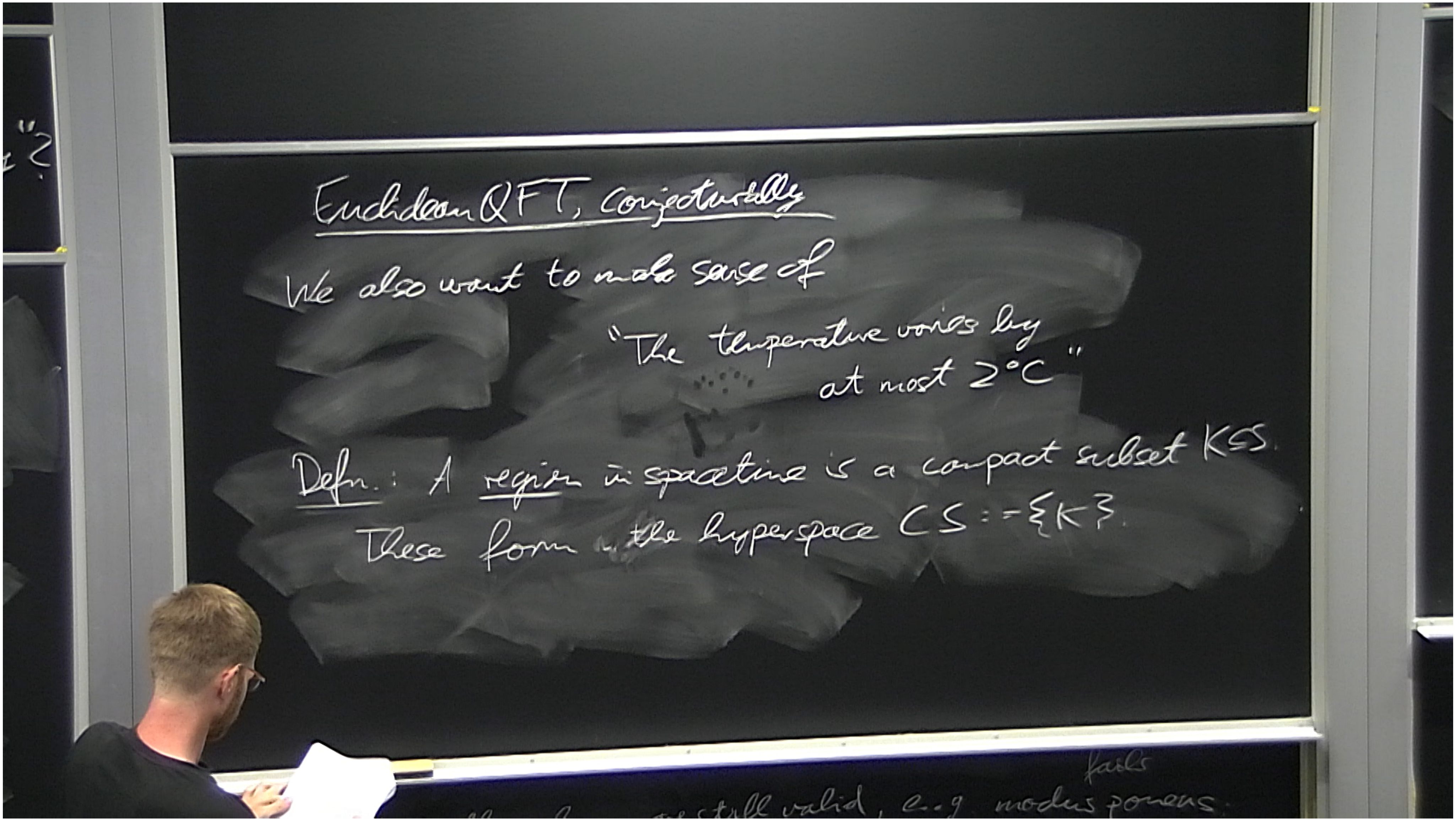
Topos theory as the logic of space

Def.: A CPV on a space X is ... (same as before)

Prop.: If S is a discrete space, then the internal prob spaces are in correspondence with families of prob spaces $(X_s, \mathcal{O}(X)_s, \mu_s)_{s \in S}$

$\Longrightarrow$ No correlations

$\Longrightarrow$ NOT what we want!


Euclidean QFT, conjecturally
We also want to make sense of
"The temperature varies by at most 2°C"
Defn.: A region in spacetime is a compact subset K⊆S.
These form the hyperspace CS := {K}.
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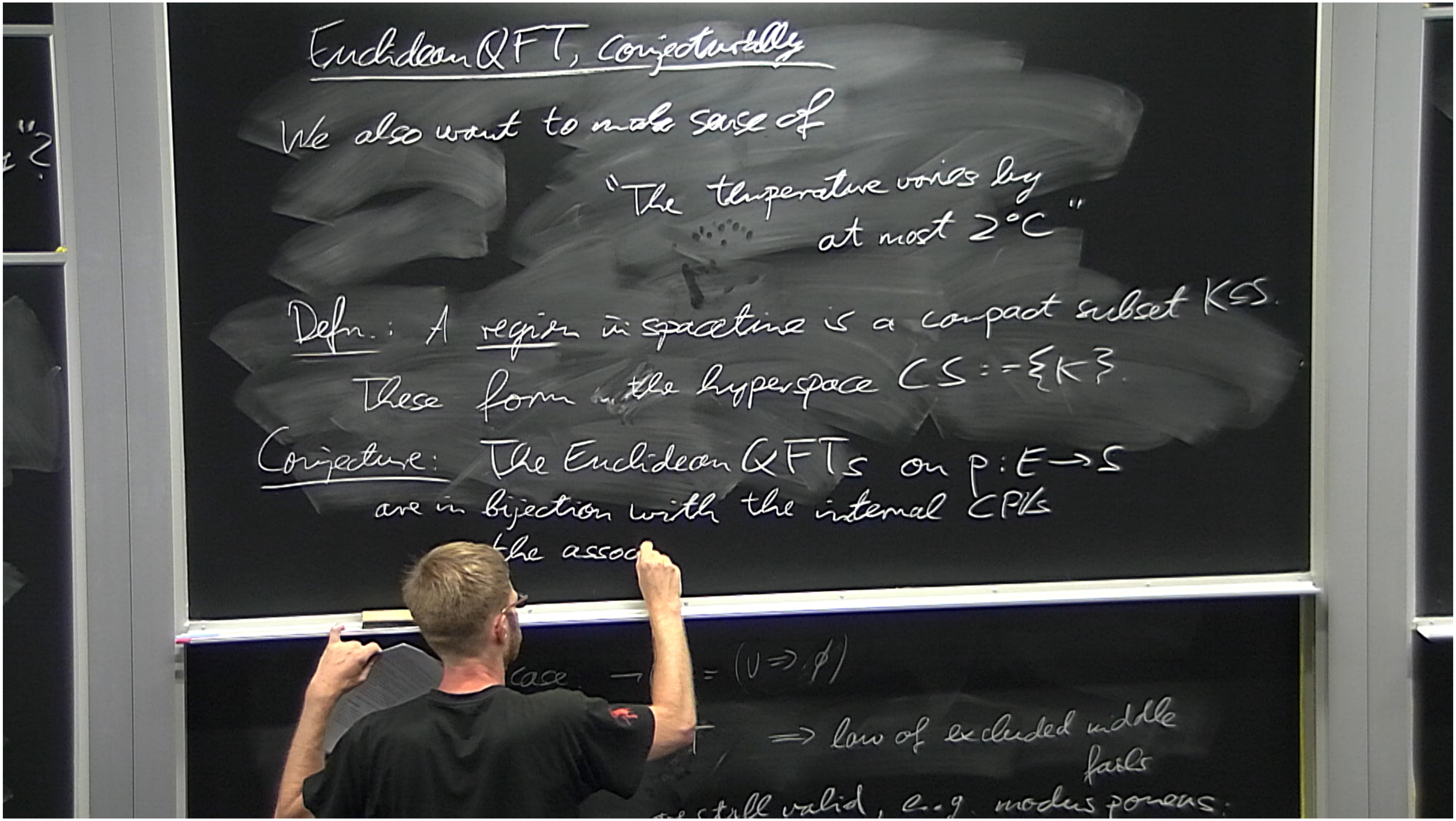
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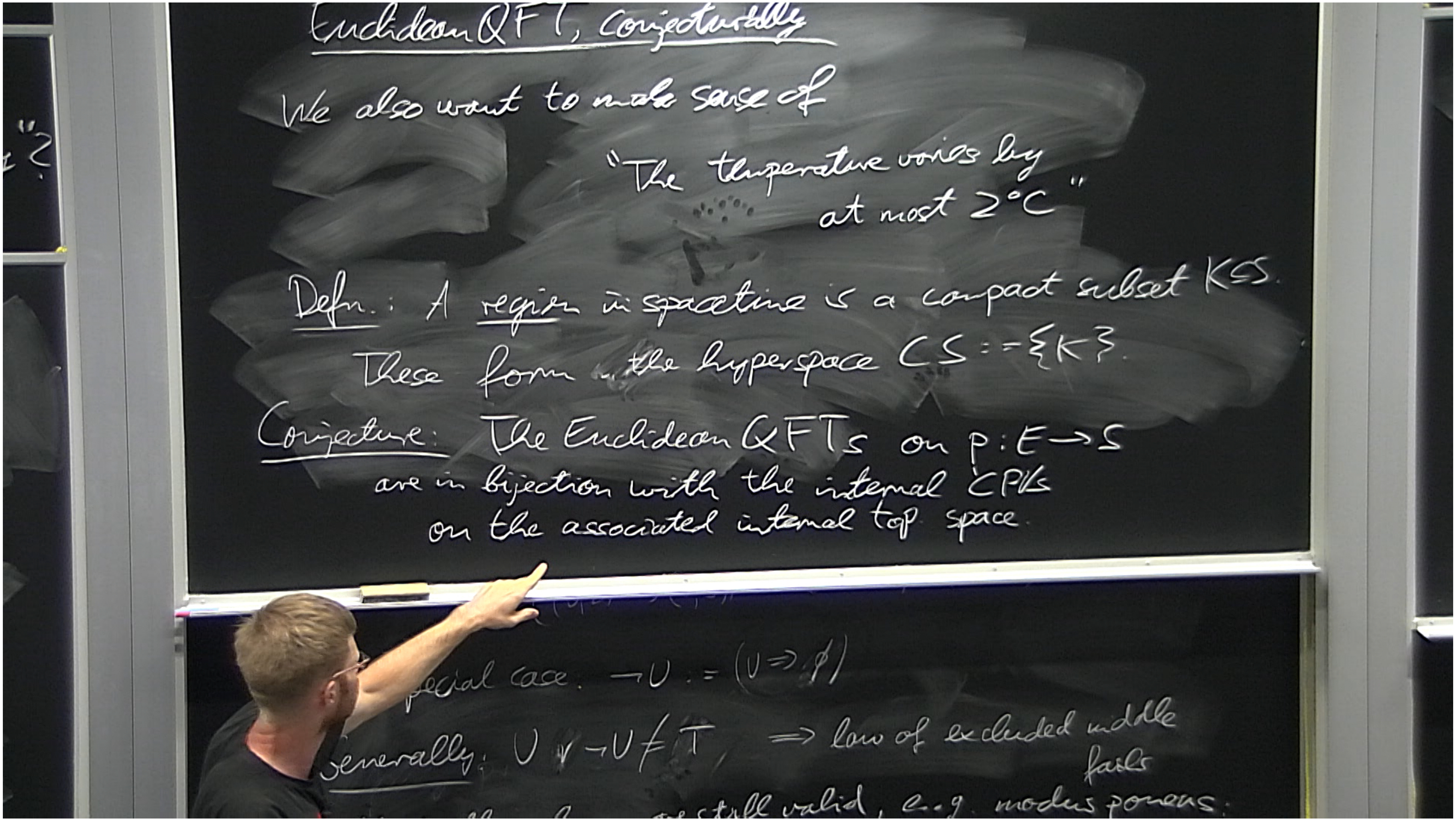
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