Title: Loop Models, Modular Invariance, and Three Dimensional Bosonization

Date: Jul 13, 2018 03:00 PM

URL: http://pirsa.org/18070046

Abstract: <p class="gmail-p2">Recently, a web of quantum field theory dualities was proposed linking several problems in the study of strongly correlated quantum critical points and phases in two spatial dimensions. These dualities follow from a relativistic flux attachment duality, which relates a Wilson-Fisher boson with a unit of attached flux to a free Dirac fermion. While several derivations of members of the web of dualities have been presented thus far, none explicitly involve the physics of flux attachment, which in relativistic systems affects both statistics and spin. We discuss how this can be achieved in models of relativistic current loops, where the concept of relativistic flux attachment can be made precise. In this context, we provide simple, explicit â€œderivationsâ€• of members of the web of dualities. We describe some implications of this work for relativistic composite fermion theories arising in condensed matter physics, as well as new possibilities for deriving additional dualities using these techniques.</p>

Loop Models, Modular Invariance, and Three Dimensional Bosonization

Hart Goldman

Department of Physics and Institute for Condensed Matter Theory, University of Illinois at Urbana-Champaign

Condensed Matter Seminar, Perimeter Institute
July 13, 2018

This talk is based primarily on

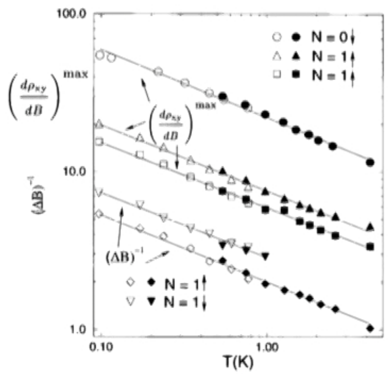
HG and Eduardo Fradkin, PRB 97, 195112 (2018).

See also work applying duality to problems with disorder + strong interactions:

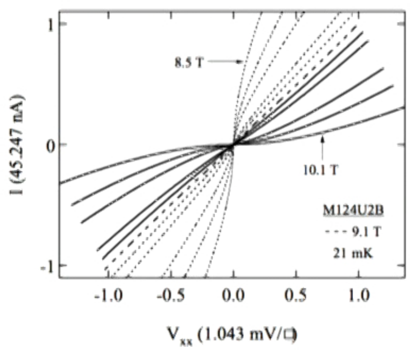
HG, Mike Mulligan, Sri Raghu, Gonzalo Torroba, and Max Zimet, PRB 96, 245140 (2017).

The mysterious world of strong interactions

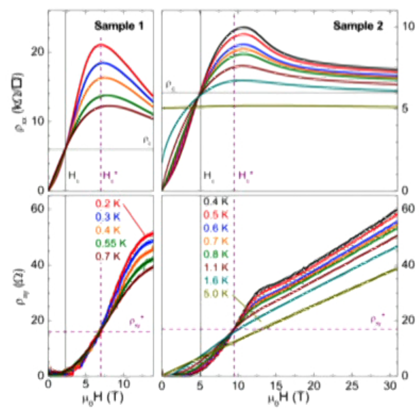
- Many mysterious phenomena observed in strongly coupled 2D quantum systems which seem **deeply interconnected.**
- **Even on their own, these phenomena are poorly understood**. Most available theoretical techniques are *perturbative*.



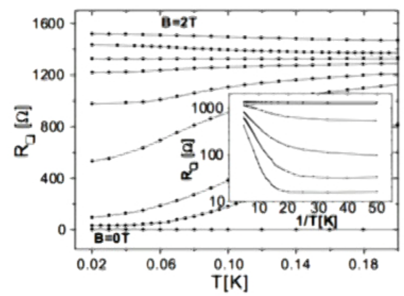
Superuniversality of QH plateau transitions. [Wei *et al.*, PRL (1988)]



Particle-hole symmetry in the half-filled Landau level. [Shahar *et al.*, Science (1996)]. (note that this plot is actually at $\nu = 1/4$)



Self-duality of the field-tuned SIT. [Breznay *et al.*, PNAS (2016)]



"Anomalous" 2D metallic phases. [Mason and Kapitulnik, PRL (1999)]

Duality

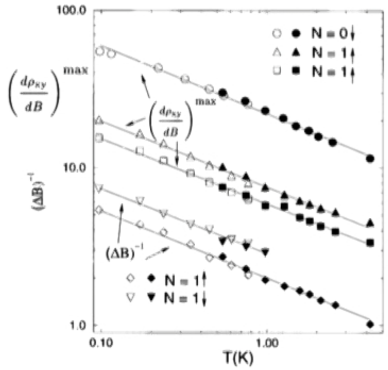
- **Duality** is the idea that two seemingly very different theories are one and the same.
  - *More precisely, we will say that two theories are dual if they are the same at long distances.*
- Duality is particularly useful when it relates a strongly interacting theory to a weakly interacting one, or when it relates a theory to itself.

Duality through history

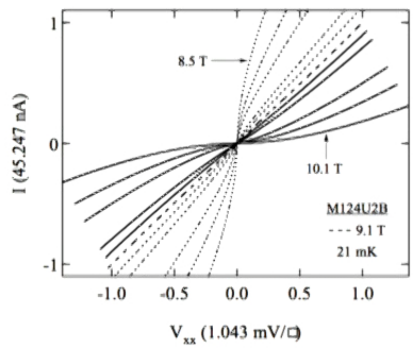
Duality has a long history in condensed matter and high energy physics:

- **EM duality** (electric charges $\longleftrightarrow$ monopoles)
  [Dirac, Proc. R. Soc., 1931]
- **Kramers-Wannier duality** of the 2D Ising model:
  order (low T) $\longleftrightarrow$ disorder (high T)
  [Kramers and Wannier, Phys. Rev. (1941)]
  - 3D: Ising $\longleftrightarrow$ $\mathbb{Z}_2$ lattice gauge theory [Wegner, J. Math. Phys. (1971)]
- **Boson-vortex duality**: 3D XY model $\longleftrightarrow$ Abelian Higgs model,
  insulator of particles (charge) $\longleftrightarrow$ condensate of vortices (flux).
  [Peskin, Ann. Phys. (1978)], [Thomas and Stone, Nuc. Phys. B (1978)],
  [Dasgupta and Halperin, PRL (1981)]
- **Flux attachment**: particles $\longleftrightarrow$ composite particles.
  [Wilczek, PRL (1982)]
- **Relativistic flux attachment: web of dualities in 2+1 dimensions** [Seiberg, Senthil, Wang, and Witten, Ann. Phys. (2016)], [Karch and Tong, PRX (2016)]

The mysterious world of strong interactions

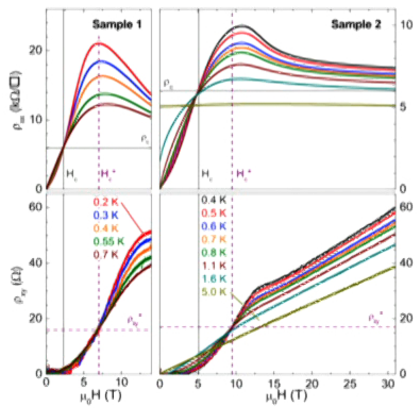


Superuniversality of QH plateau transitions.
[Wei *et al.*, PRL (1988)]

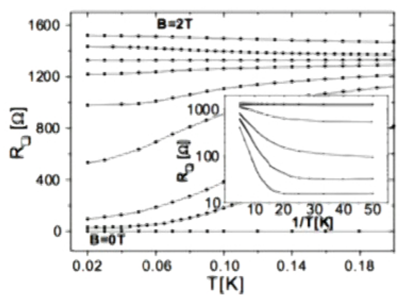


Particle-hole symmetry in the half-filled Landau level. [Shahar *et al.*, Science (1996)]

- Many of these problems have deep and intimate connections, e.g. particle-hole symmetry and self-duality. **Duality quantifies these connections and leverages them to make progress.**
- **We discuss a simple way of deriving the “web of dualities” using loop models**.
- Loop models are settings where relativistic flux attachment can be made precise.



Self-duality of the field-tuned SIT.
[Breznay *et al.*, PNAS (2016)]

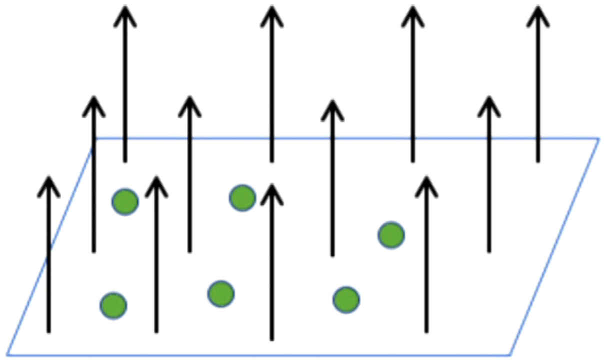


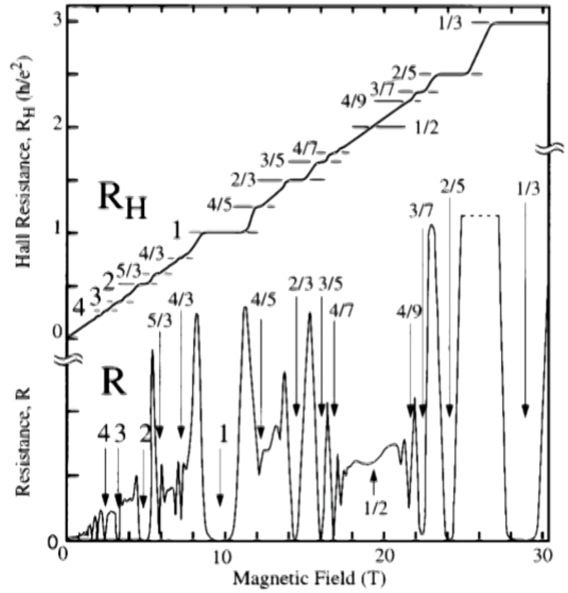
“Anomalous” 2D metallic phases.
[Mason and Kapitulnik, PRL (1999)]

Roadmap

I. Motivating relativistic flux attachment and the "web of dualities": the example of the half-filled Landau Level
II. Introduction to loop models
III. Flux attachment in loop models, modular invariance (or lack thereof), and the emergence of "fractional spin"
IV. From 3D bosonization to a web of loop model dualities

The half-filled Landau level



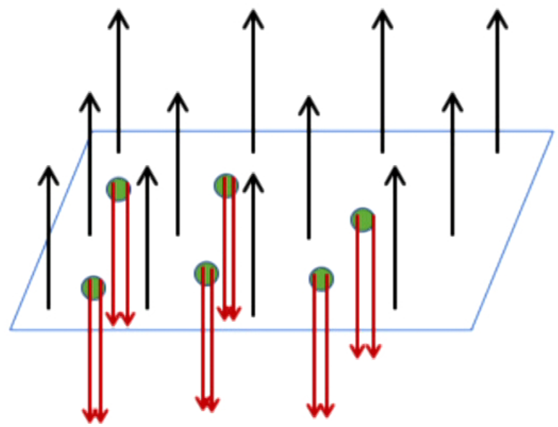


[Stormer, RMP (1999)]

$$H = \sum_{a}^{N_e} \frac{1}{2m} (\mathbf{p}_a + e\mathbf{A}_a)^2 + \sum_{a,b}^{N_e} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$

$$\nu = 2\pi \frac{\rho_e}{B} = \frac{1}{2}$$

Flux attachment: a non-relativistic duality



Original electrons $+ B$

$\updownarrow$

Composite particles $+ \frac{1}{4\pi n} ada + B_{\text{eff}} = B - 2\pi\, n\, \rho_e$

Composite fermions: n even, Composite bosons: n odd

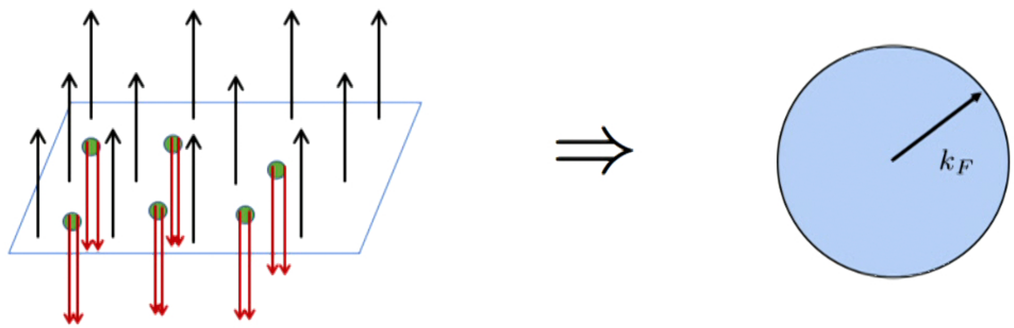
Statistical periodicity: $n \mapsto n + 2$ a symmetry of the partition function

Jain sequences: FQHE of electrons $\longleftrightarrow$ IQHE of composite fermions

[Wilczek, PRL (1982)], [Jain, PRL (1989)], [López and Fradkin, PRB (1991)], [Zhang, Hansson, and Kivelson, PRL (1989)]

The Halperin-Lee-Read theory of $\nu = 1/2$

The Halperin-Lee-Read Theory (HLR): Attach two flux quanta to each electron, so $B_{\text{eff}} = 0$. Leads to a Fermi sea of composite fermions (CFs) strongly interacting with a Chern-Simons gauge field. [Halperin, Lee, and Read, PRB (1993)]

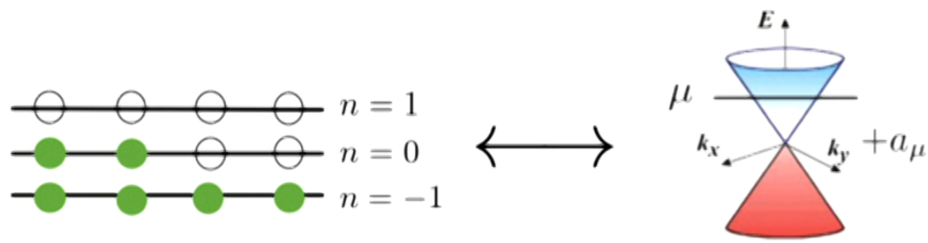


$$+\frac{1}{8\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$$

- HLR is a very successful "phenomenological theory." Composite Fermi surface is observed in quantum oscillations near $\nu = 1/2$.
- IR divergences everywhere!
- **HLR is not particle-hole ($\mathbf{PH}$) symmetric.**
  - $\mathbf{PH}$: exchanges empty and filled states, i.e. $\nu \mapsto 1 - \nu$
  - $\mathbf{PH}$ is a symmetry of LLL Hamiltonian, fixes $\sigma_{xy} = e^2/2h$. Seen experimentally.

Toward relativistic composite fermions

- Son noticed that the $\nu = 1/2$ problem for non-relativistic fermions can be mapped to the analogous (massless) Dirac fermion problem in the LLL limit [Son, PRX (2015)].



- **Conjecture: a relativistic CF duality**

$$\text{a free Dirac} + B \quad \longleftrightarrow \quad \text{QED}_3 + \mu$$

$$i\bar{\Psi}\not{D}_A\Psi - \frac{1}{8\pi}AdA \quad \longleftrightarrow \quad i\bar{\psi}\not{D}_a\psi + \frac{1}{4\pi}adA - \frac{1}{8\pi}AdA + \cdots$$

A_μ: background EM field, $D_B = \partial - iB$, $AdB = \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda$

- **fermions** $\leftrightarrow$ **vortices:** charge $\leftrightarrow$ flux, $J^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda/4\pi$, $\rho_{CF} \propto B$
- **Son's duality is useful:** explains $\nu = 1/2$ in a $\mathbf{PH}$ symmetric way, relates a strongly interacting theory to a free theory.

Can we do *even* better?

- Son's duality is an example of a relativistic field theory duality which is useful in condensed matter physics. **Can we relate it to any others?**
- Son's duality is a fermionic version of **boson-vortex duality**: 3D XY model $\longleftrightarrow$ Abelian Higgs model. [Peskin, Ann. Phys. (1978)], [Thomas and Stone, Nuc. Phys. B (1978)], [Dasgupta and Halperin, PRL (1981)]

$$|D_A\phi|^2 - r|\phi|^2 - |\phi|^4 \longleftrightarrow |D_a\tilde{\phi}|^2 - \tilde{r}|\tilde{\phi}|^2 - |\tilde{\phi}|^4 + \frac{1}{2\pi}adA + \cdots ,$$

$$\mathrm{sgn}(r) = -\,\mathrm{sgn}(\tilde{r}),\; J^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda/2\pi$$

condensed phase $\longleftrightarrow$ gapped phase

Can we do *even* better?

- **Can we deepen the connection to bosonic theories?**
  Non-relativistic flux attachment:
  fermions + odd # of flux quanta $\longleftrightarrow$ composite bosons.
  - $\Rightarrow$ QH plateau transitions are SITs of composite bosons [Kivelson, Lee, and Zhang, PRB (1992)].
  - $\Rightarrow$ Can explain the experimental connections discussed at the beginning.
- Can this story work for Dirac fermions?

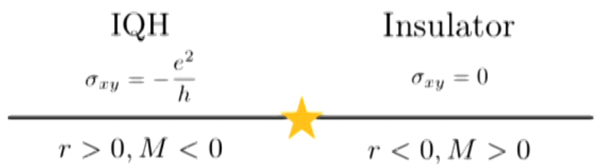
A web of dualities and condensed matter physics

Conjecture:

a free Dirac fermion $\longleftrightarrow$ a Wilson-Fisher boson + a unit of flux

$$\bar{\psi}(i\not{D}_A - M)\psi - \frac{1}{8\pi}AdA \longleftrightarrow |D_a\phi|^2 - r|\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}Ada$$

a_μ: emergent gauge field, A_μ: background EM field, $\mathrm{sgn}(r) = -\,\mathrm{sgn}(M)$



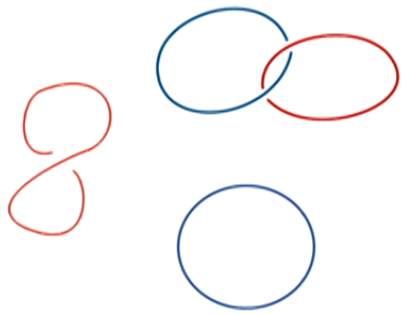
- This "2+1D Bosonization" duality **implies a whole web of dualities, among them the boson-vortex and fermion-vortex dualities.** [Seiberg, Senthil, Wang, and Witten, Ann. Phys. (2016)], [Karch and Tong, PRX (2016)]
- Critical point: fermions have $\mathbf{PH}$ symmetry, bosons are "self-dual."

Can the web of dualities be "derived"?

- Already several "derivations":
  - Euclidean lattice models [Chen, Son, Wang, and Raghu, PRL (2018)]
  - Wire constructions [Mross, Alicea, and Motrunich, PRX (2017)]
  - Deformations of supersymmetric dualities [Kachru, Mulligan, Torroba, and Wang, PRL (2017)]
- However, **none directly implement "relativistic flux attachment" using the full 2+1D Chern-Simons term.**
- **We show how this can be done using loop models.**
- *Disclaimer: Our derivations are carried out in a phase, but near criticality. We cannot prove duality of the critical points using this technique.*

What is a loop model?

A loop model consists of variables J_μ satisfying $\partial_\mu J^\mu = 0$. J_μ represents a configuration of closed worldlines of particles.



$$Z = \int \mathcal{D}J \, \delta(\partial_\mu J^\mu) \, e^{iS[J]}$$

- Can be defined on a lattice or in the continuum.
- *Any* gapped field theory can be written in this form.

Example: boson-vortex duality

Many familiar theories can be recast as loop models. For example, boson-vortex duality can be written as a duality of loop models:

	3D XY		Abelian Higgs
loop model action:	$\frac{J^2}{2K}$	$\longleftrightarrow$	$\tilde{J}_\mu a^\mu - \frac{1}{(2\pi)^2 K}(\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda)^2$

- charge $\leftrightarrow$ flux: $J^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda / 2\pi$
- two phases: loops are small or loops condense
- condensation of $J \leftrightarrow \tilde{J}$ loops are small and gapped and vice versa
- In field theory language, the duality is

$$|\partial\phi|^2 - r|\phi|^2 - |\phi|^4 \longleftrightarrow |D_a\tilde{\phi}|^2 - \tilde{r}|\tilde{\phi}|^2 - |\tilde{\phi}|^4 - \frac{1}{4g^2} f_{\mu\nu}f^{\mu\nu}$$

where $\text{sgn}(r) = -\,\text{sgn}(\tilde{r})$.

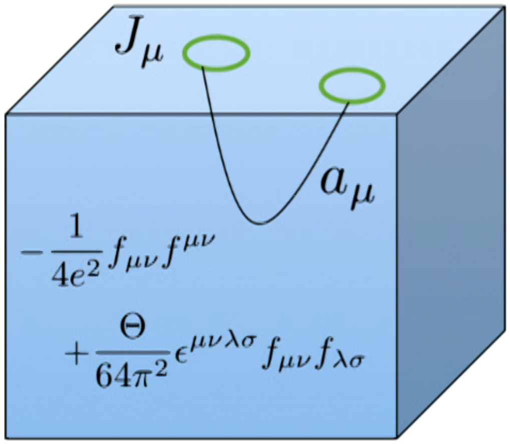
A self-dual loop model

We consider models originally studied by [Fradkin and Kivelson, Nuc. Phys. B (1996)]. Start with *hard core* bosonic loops J_μ + a gauge field a_μ,

$$J_\mu a^\mu - \frac{1}{4e^2} f_{\mu\nu} \frac{i}{\sqrt{\partial^2}} f^{\mu\nu} + \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \cdots$$

$\downarrow$ integrate out a_μ

$$\frac{g^2}{2} J_\mu \frac{i}{\sqrt{\partial^2}} J^\mu + \theta\, \epsilon^{\mu\nu\lambda} J_\mu \frac{\partial_\nu}{\partial^2} J_\lambda + \cdots$$



- Let $\tau = \frac{\theta}{\pi} + i\frac{g^2}{2\pi}$, $\tilde{\tau} = k + i\frac{2\pi}{e^2}$. Then $\tau = -\frac{1}{\tilde{\tau}}$.
- First term: **long-ranged interactions.**
- Second term: **linking number** $\theta\Phi[J]$. Manifestation of flux attachment.
- Critical points should be Wilson-Fisher + a_μ.
- Boson-vortex: $\tau \mapsto -\frac{1}{\tau} \Rightarrow$ **"Self-duality"!**

Modular Invariance

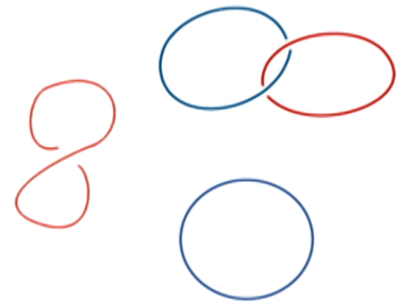
- In the original work of Fradkin and Kivelson, the loop model partition functions introduced in the previous slide were expected to display *invariance* under
  - **Boson-vortex duality** $\mathcal{S}_{\mathrm{FK}} : \tau \mapsto -\frac{1}{\tau}$
  - **Periodicity** $\mathcal{T}_{\mathrm{FK}} : \tau \mapsto \tau + 1$ $(\theta \mapsto \theta + \pi)$

  Together, $\mathcal{S}_{\mathrm{FK}}$ and $\mathcal{T}_{\mathrm{FK}}$ generate the modular group, $\mathrm{PSL}(2, \mathbb{Z})$.
- Invariance under $\mathrm{PSL}(2, \mathbb{Z})$ implies the existence of *superuniversal* families of fixed points sharing transport properties and critical exponents.
- However, invariance under periodicity requires a choice of regularization which becomes inconsistent as one approaches criticality: it dispenses with spin!

Toward Dirac fermions: introducing fractional spin

Focus on the **linking number.** It can be written as

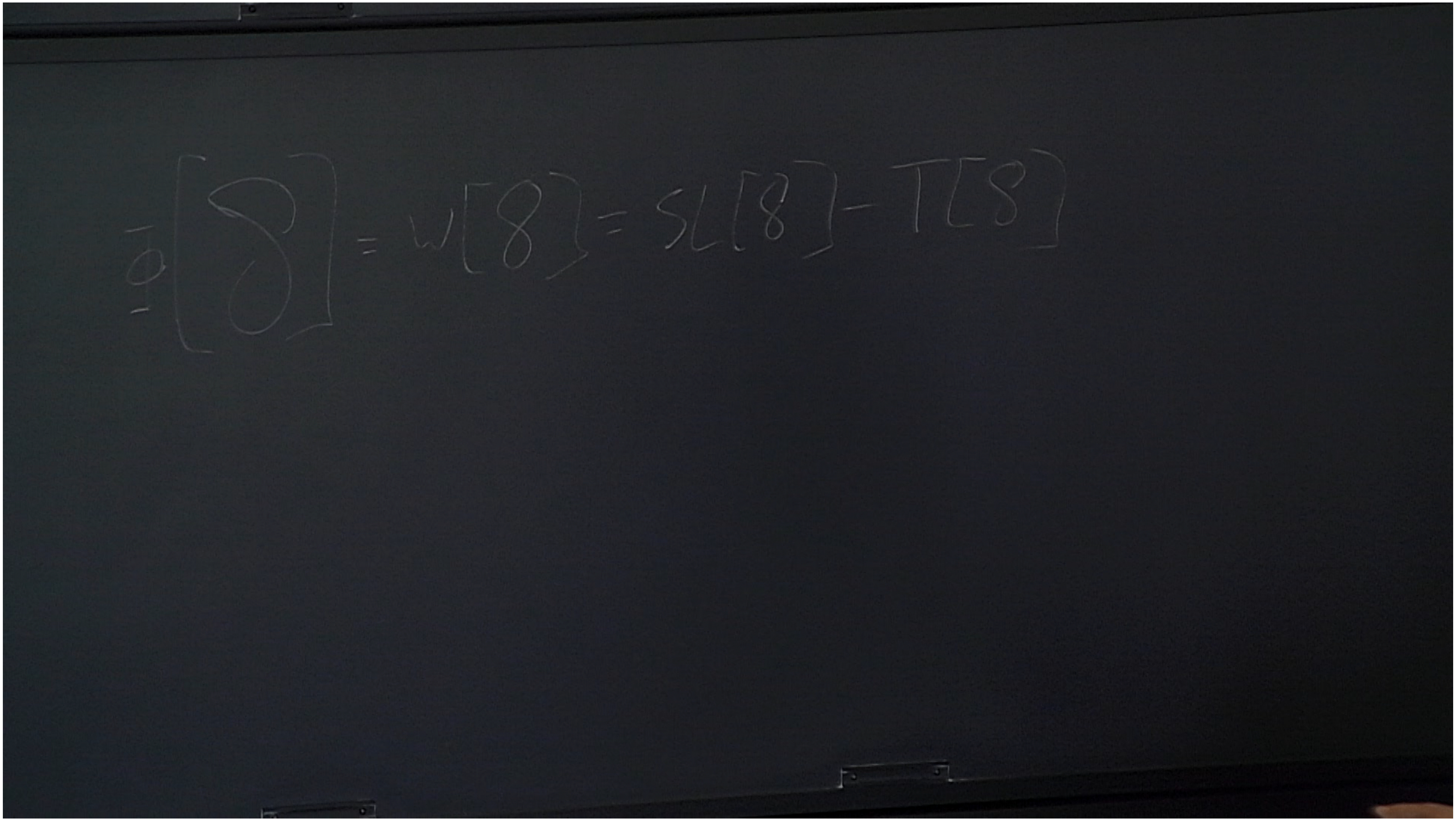
$$\Phi[J] = \frac{1}{4\pi} \int d^3x d^3y \epsilon^{\mu\nu\lambda} J_\mu(x) \frac{x_\nu - y_\nu}{|x-y|^3} J_\lambda(y)$$



- **Separate linked loops:** $\Phi \in 2\mathbb{Z}$ is a topological invariant. Counts the number of exchanges. *Fradkin-Kivelson regularization only has these kinds of processes.*
- **Self-linking:** Φ not automatically a topological invariant. *Depends on regularization:*
  - **Point-splitting:** Split loop into a ribbon $\Rightarrow$ topological, but ambiguous.
  - **Fractional spin:** Not topological, but unambiguous.
  - **Non-topological part is a Berry phase ("fractional spin" term):**

$$T = \frac{1}{2\pi} \int_0^L ds \int_0^1 du\, \hat{e} \cdot (\partial_s \hat{e} \times \partial_u \hat{e})$$

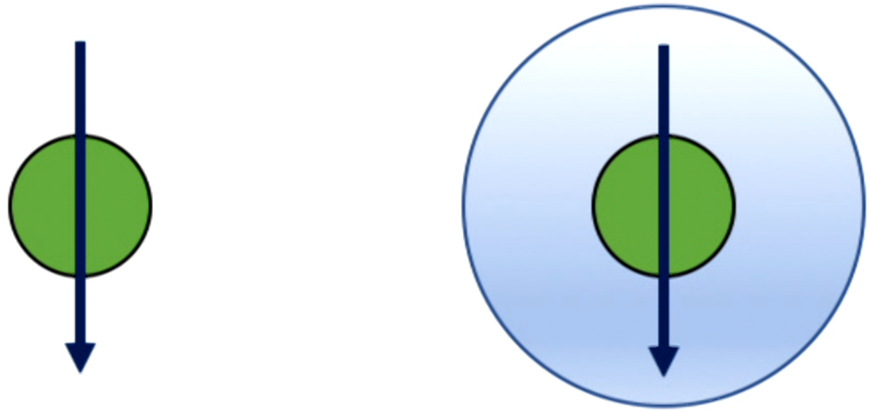
$\hat{e}$: tangent vector to the loop. **Presence of T breaks periodicity**.



Emergence of spin

Point-splitting is inconsistent in relativistic theories, as we might expect from the spin-statistics theorem. In particular, it is impossible with Maxwell terms present, which spread out the attached flux [Hansson, Karlhede, and Roček, Phys. Lett. B (1989)]. Such terms are necessary as criticality is approached.

⇒ **Our loop models have emergent spin!**



2+1D bosonization from fractional spin

[Polyakov, Mod. Phys. Lett. A (1988)] showed that **fractional spin converts a gapped scalar boson + a unit of flux to a gapped Dirac fermion at long distances.** Leads to a loop model representation of the Dirac functional determinant with $\theta = \pm\pi$

$$\det[i\partial\!\!\!/ - M] = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] + i\pi\, \mathrm{sgn}(M)\Phi[J]}$$

$L[J]$: sum of lengths of loops in configuration J.

- **Why the $\mathrm{sgn}(M)$?** Polyakov's derivation can only be carried out directly on the gapped variables. When the matter loops condense, the derivation is done for the gapped vortices, which see $\theta = +\pi$.
- Polyakov's derivation had some problems, however:
  - Bosons were not hard-core, meaning that their world-lines could cross. **The linking number $\Phi[J]$ is ill defined in such cases.** The bosons must therefore be Wilson-Fisher.
  - **Did not include coupling to background fields.** Can the loop model representation accommodate the parity anomaly?

2+1D bosonization from fractional spin

Background gauge fields can be added straightforwardly on the bosonic side, which is automatically gauge invariant with no anomaly. Indeed, we show

$$\det[i\not{D}_A - M]e^{-i\int_x AdA/8\pi} = \int \mathcal{D}J\,\delta(\partial_\mu J^\mu)\,e^{-|m|L[J]+iS_{\text{fermion}}[J,A]-i\int_x AdA/8\pi}$$

$$\boxed{S_{\text{fermion}}[J,A;M] = \int_x A_\mu J^\mu + \text{sgn}(M)\,\pi\,\Phi[J] + \frac{\text{sgn}(M)}{8\pi}\int_x AdA + \cdots}$$

In field theory language, this duality is

$$\bar{\Psi}(i\not{D}_A - M)\Psi - \frac{1}{8\pi}AdA \longleftrightarrow |D_a\phi|^2 - r|\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA$$

where $\text{sgn}(r) = -\text{sgn}(M)$.

Loop models as duality tools

We can now derive a web of dualities of loop models! Our strategy is:

1. Start with a proposed duality and **write down a bosonic loop model representation** of each side using the 2+1D bosonization duality.
2. Use path integral manipulations to **equate the two loop model partition functions** in the phase where the loops are gapped.
3. **Match each side of the critical point** by invoking particle-vortex duality to describe the condensed phase of the matter loops as a gapped phase of vortex loops.

2+1D bosonization from fractional spin

Background gauge fields can be added straightforwardly on the bosonic side, which is automatically gauge invariant with no anomaly. Indeed, we show

$$\det[i\not{D}_A - M] e^{-i\int_x AdA/8\pi} = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] + iS_{\text{fermion}}[J,A] - i\int_x AdA/8\pi}$$

$$\boxed{S_{\text{fermion}}[J, A; M] = \int_x A_\mu J^\mu + \text{sgn}(M)\,\pi\,\Phi[J] + \frac{\text{sgn}(M)}{8\pi}\int_x AdA + \cdots}$$

In field theory language, this duality is

$$\bar{\Psi}(i\not{D}_A - M)\Psi - \frac{1}{8\pi}AdA \longleftrightarrow |D_a\phi|^2 - r|\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA$$

where $\text{sgn}(r) = -\text{sgn}(M)$.

Example: Son's fermion-vortex duality

$$\bar{\Psi}(i\not{D}_A - M)\Psi - \frac{1}{8\pi}AdA \longleftrightarrow \bar{\psi}(i\not{D}_a - M')\psi + \frac{1}{8\pi}ada - \frac{1}{2\pi}bda + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA$$

$$M > 0 \quad \downarrow \qquad\qquad \downarrow \quad M' < 0$$

$$\int_x A_\mu J^\mu + \pi\,\Phi[J] \;\longleftarrow\; -\pi\,\Phi[J] + \int_x \left(J_\mu a^\mu - \frac{1}{2\pi}bda + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA \right)$$

- Argument for $(M < 0, M' > 0)$ phase is analogous.
- Note: a_μ, b_μ can be integrated out consistently with flux quantization.

More general bosonization dualities

What is the fermionic dual of

$$J_\mu a^\mu - \frac{1}{4e_\phi^2} f_{\mu\nu} \frac{i}{\sqrt{\partial^2}} f_{\mu\nu} + \frac{k_\phi}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \cdots \quad ?$$

Let $\tau_\phi = k_\phi + i\frac{2\pi}{e_\phi^2}$. This theory is dual to a theory of fermions ψ coupled to an emergent gauge field b_μ characterized by $\tau_\psi = 2k_\psi + i\frac{4\pi}{e_\psi^2}$, where

$$\tau_\psi = \frac{\tau_\phi - 1}{\tau_\phi + 1}$$

Means deforming k_ϕ (or θ) doesn't lead to particles with a strange spin, but a new, strongly interacting theory!

Summary

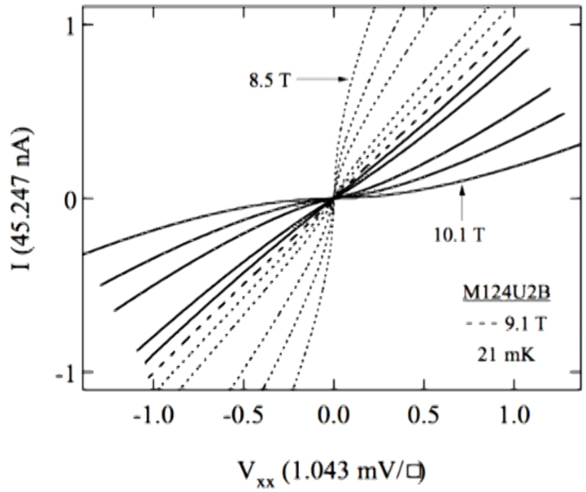
- We considered a family of relativistic loop models in 2+1D coupled to Chern-Simons gauge fields.
- By showing that relativistic flux attachment transmutes both statistics and spin in these loop models, we derived members of the recent web of field theory dualities, including Son's fermion-vortex duality.
- This is the first derivation of the web of dualities which implements relativistic flux attachment in a simple and transparent way.

More dualities from loop models?

- The dualities we derived are for a single species of matter field and Abelian gauge fields.
- The web of dualities extends to multiple matter fields and non-Abelian gauge groups [Aharony, JHEP (2016)]. Such theories have been proposed to describe deconfined quantum critical points [Wang *et al.*, PRX (2017)] and as examples of "superuniversal" plateau transitions [Hui, Kim, Mulligan, 1712.04942].
- There are also interesting proposals for bounds on the number of matter species for which boson-fermion duality is possible [Komargodski and Seiberg, JHEP (2017)].
- **Loop model derivations of these dualities should be possible.** Requires detailed analysis of the short-ranged interactions, development of *interacting models of non-Abelian anyon loops.*

A closing puzzle

- We saw that relativistic flux attachment transmutes statistics and spin, breaking periodicity.
- In HLR, each of the $\nu = 1/2n$ states is connected by periodicity of non-relativistic flux attachment. **This conclusion does not obviously extend to Son's Dirac CFs.**
- Experiments indicate an emergent $\mathbf{PH}$-like "reflection" symmetry at $\nu = 1/4$ which *is not* apparent in the LLL Hamiltonian.



[Shahar *et al.*, Science (1996)]