Title: Analytical techniques for finding optimal quantum measurements

Date: Jul 11, 2018 04:00 PM

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Abstract: <p>For many optimal measurement problems of interest, the problem may be re-cast as a semi-definite program, for which efficient numerical techniques are available. Nevertheless, numerical solutions give limited insight into more general instances of the problem, and further, analytical solutions may be desirable when an optimised measurement appears as a sub-problem in a larger problem of interest. I will discuss analytical techniques for finding optimal measurements for state discrimination with minimum error and present applications to studying the gap between the theoretically optimal measurement and simpler, experimentally achievable schemes for bi-partite measurement problems.</p>

Sarah Croke
School of Physics and Astronomy,
University of Glasgow
Perimeter Institute Quantum Information Seminar
11/07/2018

ANALYTICAL TECHNIQUES FOR FINDING OPTIMAL QUANTUM MEASUREMENTS

The theme of the talk: When are simple experimental strategies almost as good as globally optimal measurements? Conversely, when is the advantage enough to merit the additional experimental difficulty?

Work with:

Robin Blume-Kohout

Graeme Weir

Steve Barnett

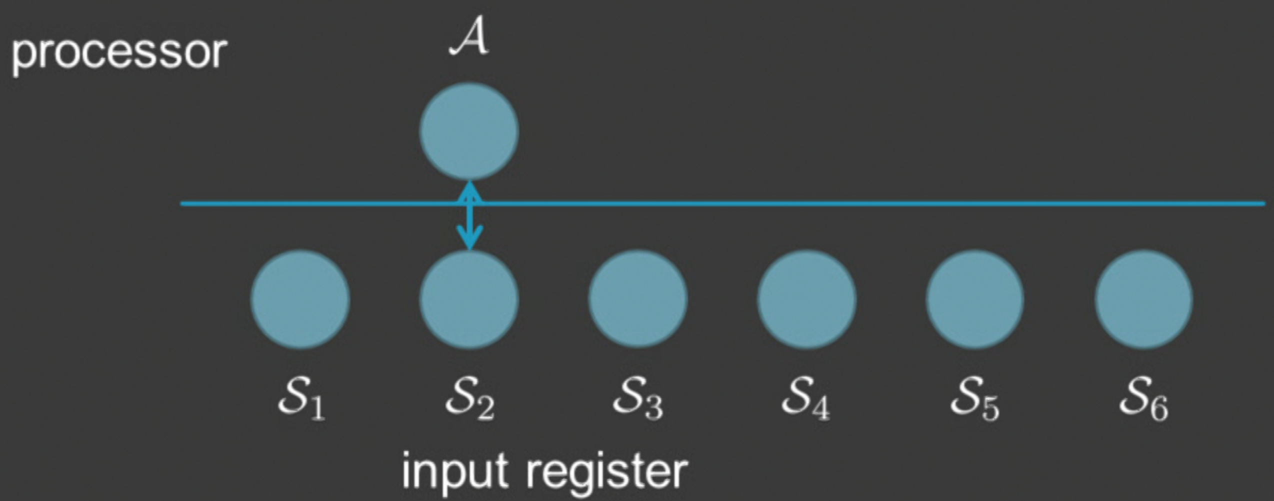
Outline

- Motivation: quantum data gathering.
- Simple examples of measurement tasks for which a quantum probe does not help (very much)
- Analytical techniques 1: minimum error conditions for sequential measurements.
  - Example: Domino states
- Analytical techniques 2: qubit state discrimination.
  - Example: Trine states with arbitrary priors
- Example: sequential measurement with imperfect detectors.
- Summary

Motivation


processor $\mathcal{A}$
$\mathcal{S}_1$ $\mathcal{S}_2$ $\mathcal{S}_3$ $\mathcal{S}_4$ $\mathcal{S}_5$ $\mathcal{S}_6$
input register

Motivation

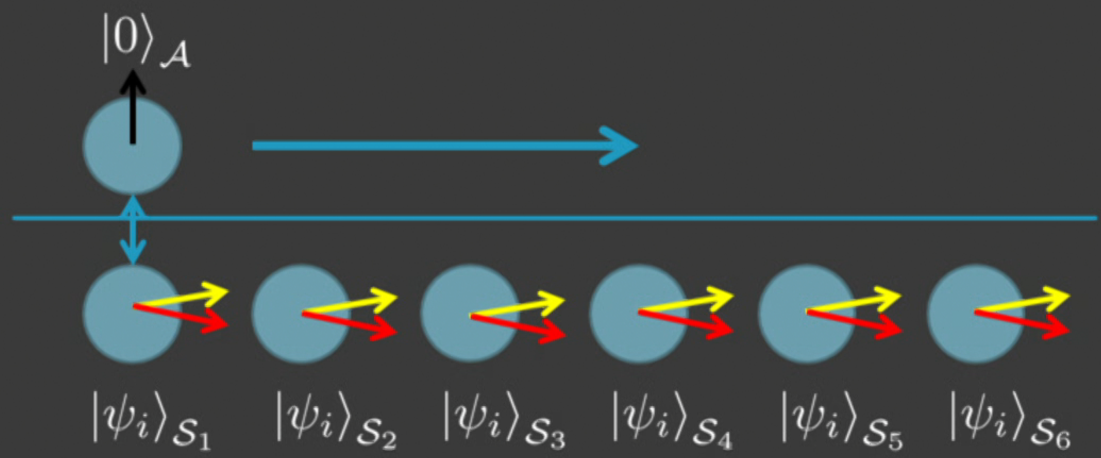

processor
$\mathcal{A}$
$\mathcal{S}_1$
$\mathcal{S}_2$
$\mathcal{S}_3$
$\mathcal{S}_4$
$\mathcal{S}_5$
$\mathcal{S}_6$
input register

Motivation

- Similarity to other schemes, e.g. ancilla driven quantum computation, in which an ancilla is used to drive evolution of information in a register.
- Structural similarity to collision models for open systems.

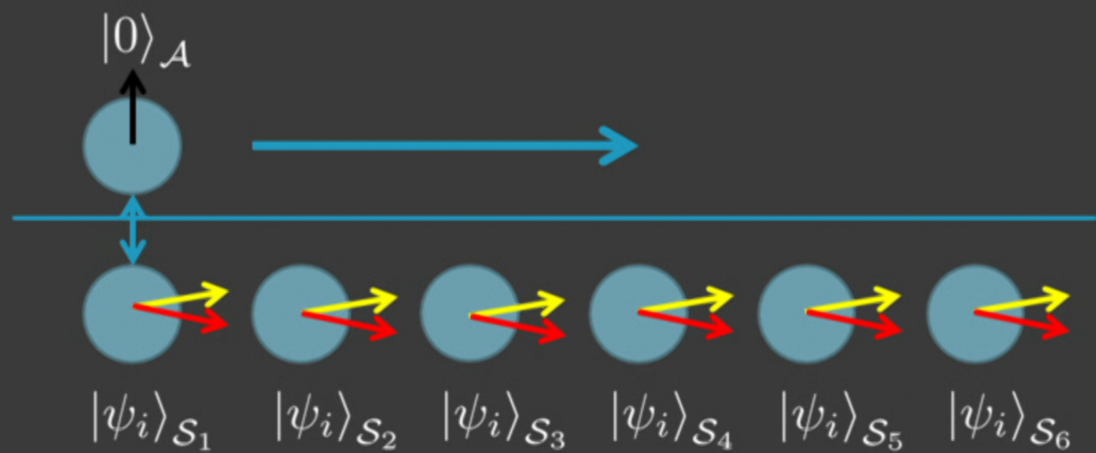
Ancilla-driven quantum computation: J. Anders, D. K. L. Oi, E. Kashefi, D. E. Browne, and E. Andersson, *PRA* **82**, 020301(R) (2010)

Example: 2 states



$$
\begin{aligned}
|\psi_1\rangle &= \cos\theta\,|0\rangle + \sin\theta\,|1\rangle\,, \\
|\psi_2\rangle &= \cos\theta\,|0\rangle - \sin\theta\,|1\rangle
\end{aligned}
$$

Example: 2 states



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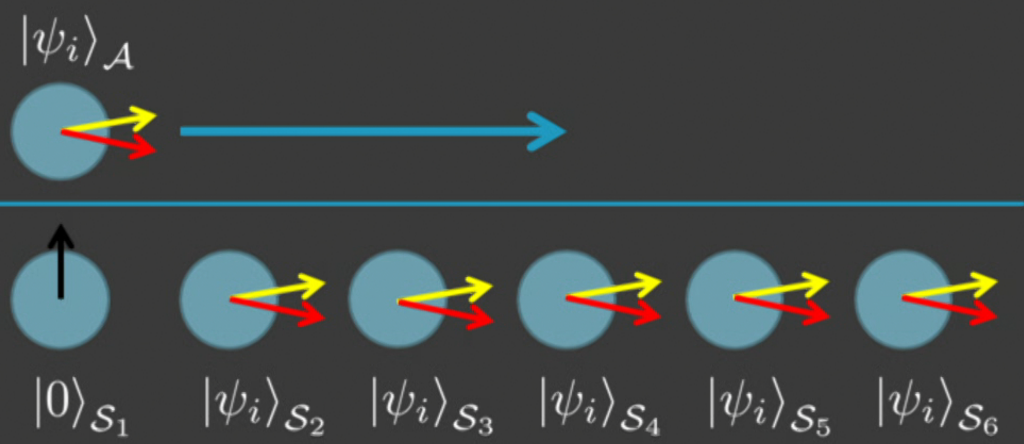

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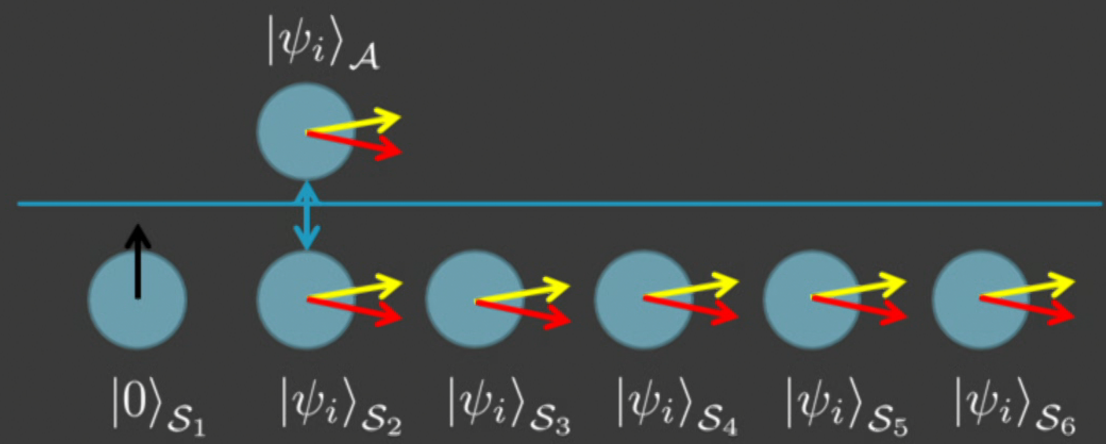
$$|0\rangle_{\mathcal{A}}\,|\psi_i\rangle_{\mathcal{S}_1}\,|\psi_i\rangle^{\otimes(N-1)} \rightarrow |\psi_i\rangle_{\mathcal{A}}\,|0\rangle_{\mathcal{S}_1}\,|\psi_i\rangle^{\otimes(N-1)}$$

Example: 2 states



$$
\begin{aligned}
|\psi_i\rangle_{\mathcal{A}} |\psi_i\rangle_{\mathcal{S}_2} &= \cos^2\theta\,|00\rangle + \sin^2\theta\,|11\rangle \pm \cos\theta\sin\theta(|01\rangle + |10\rangle) \\
&= \cos\theta_2\,|0\rangle_{\mathcal{AS}_2} \pm \sin\theta_2\,|1\rangle_{\mathcal{AS}_2}
\end{aligned}
$$

Need U_2 s. t.:

$$
\begin{aligned}
U_2\,|0\rangle_{\mathcal{AS}_2} &= |0\rangle_{\mathcal{A}}\,|0\rangle_{\mathcal{S}_2} \\
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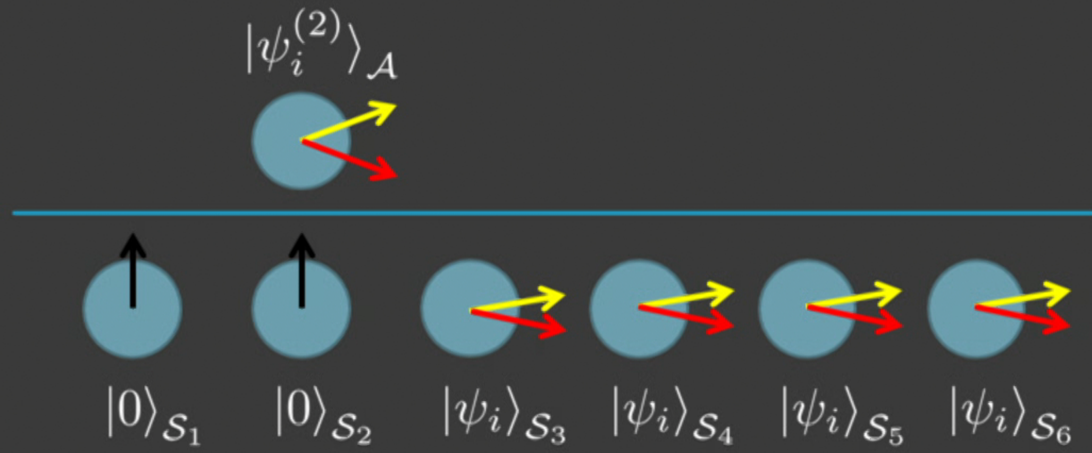
Example: 2 states



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&\to (\cos\theta_2\,|0\rangle_{\mathcal{A}} \pm \sin\theta_2\,|1\rangle_{\mathcal{A}})\,|0\rangle_{\mathcal{S}_2}
\end{aligned}
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Need U_2 s. t.:
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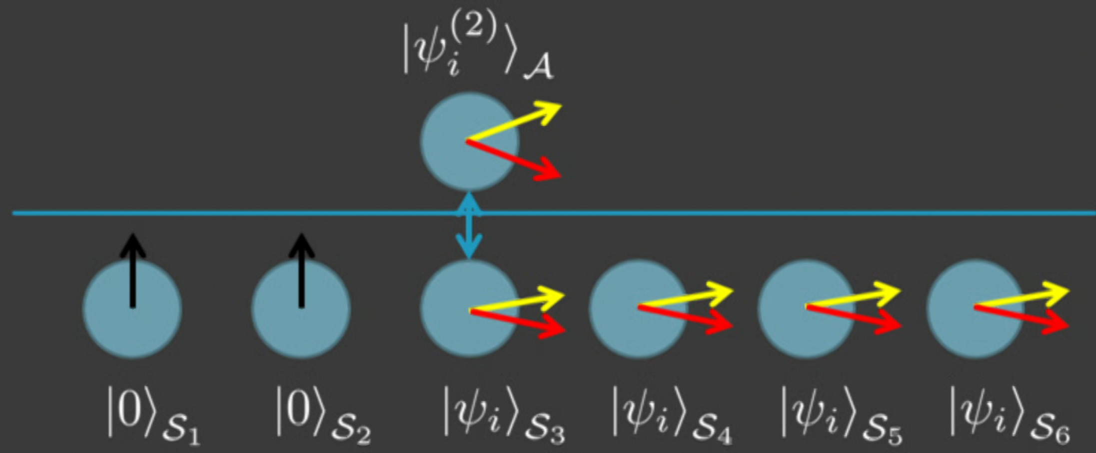
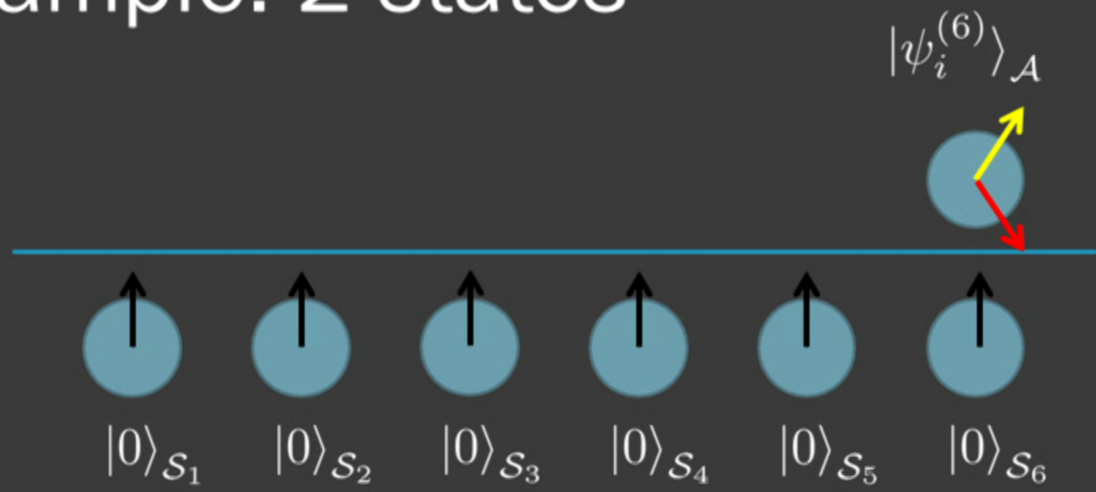


Example: 2 states

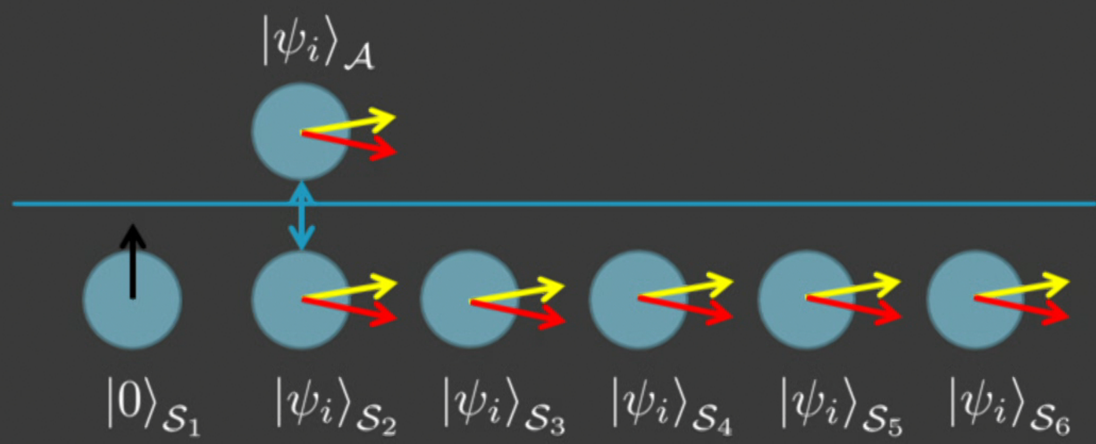


$$U_5 \, |\psi_i^{(5)}\rangle_{\mathcal{A}} \, |\psi_i\rangle_{\mathcal{S}_6} = |\psi_i^{(6)}\rangle_{\mathcal{A}} \, |0\rangle_{\mathcal{S}_6}$$

Conceptual advantages

- Generalizations to more general (entangled, unknown,…) input states.
- Provides an explicit means of performing joint measurements.
- Probe system must be at least as big as the subspace in which quantum information is stored: quantifies resources needed for joint measurement.
- Restores symmetry between classical and quantum data gathering

Example: 2 states



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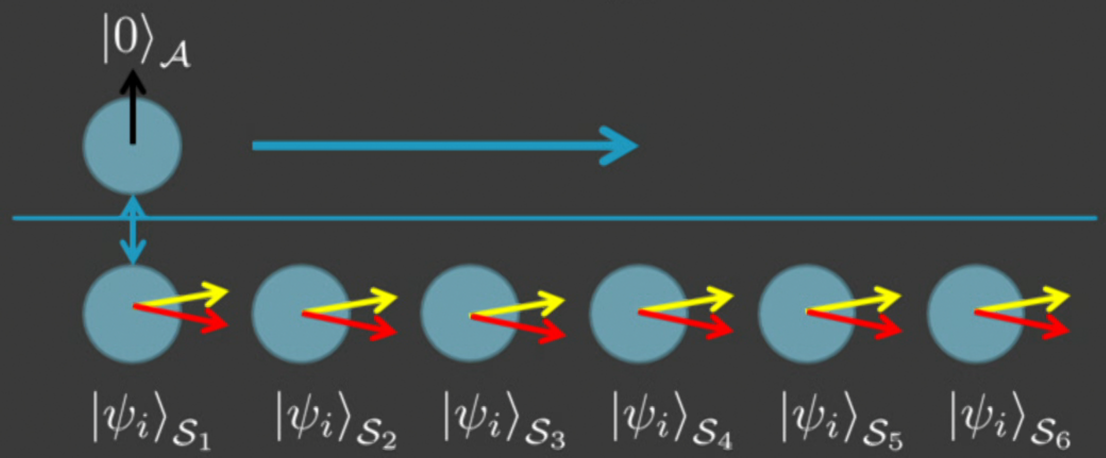
Practical advantages?



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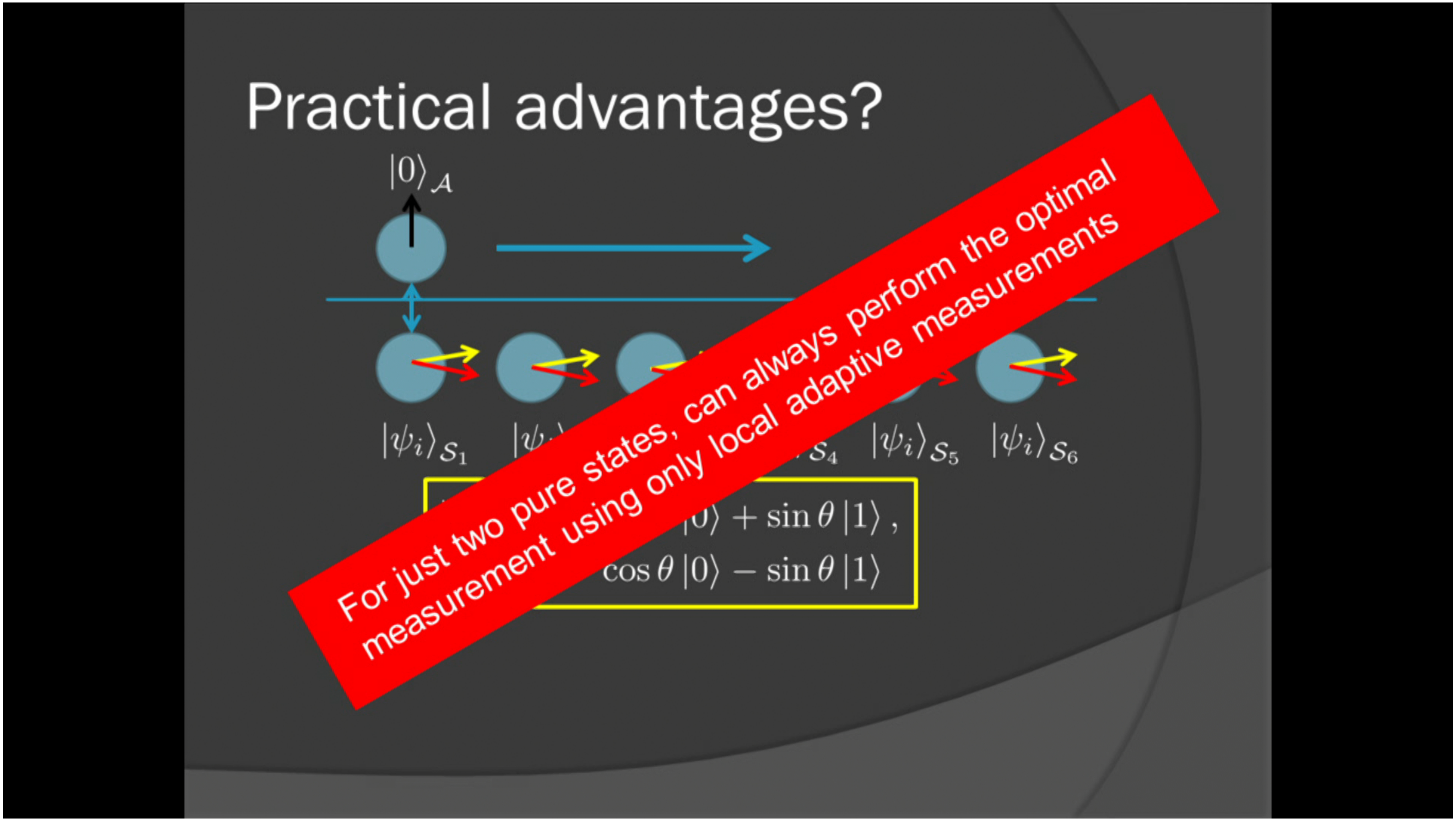

Practical advantages?
For just two pure states, can always perform the optimal measurement using only local adaptive measurements

- For any two bi-partite orthogonal states there is a decomposition of the form:

$$|\Psi_0\rangle = \sum_i \sqrt{p_i}|i\rangle_A|\eta_i\rangle_B$$

$$|\Psi_1\rangle = \sum_i \sqrt{q_i}|i\rangle_A|\eta_i^\perp\rangle_B$$

Walgate et al, *PRL* **85**, 4972 (2000)

- Thus, these can be distinguished by only local measurements.
- Can show that if information is encoded in a superposition of just two states (i.e. in a qubit subspace), can perform any allowed POVM using only local adaptive measurements.

- For any two bi-partite orthogonal states there is a decomposition of the form:

$$|\Psi_0\rangle = \sum_i \sqrt{p_i}|i\rangle_A|\eta_i\rangle_B$$

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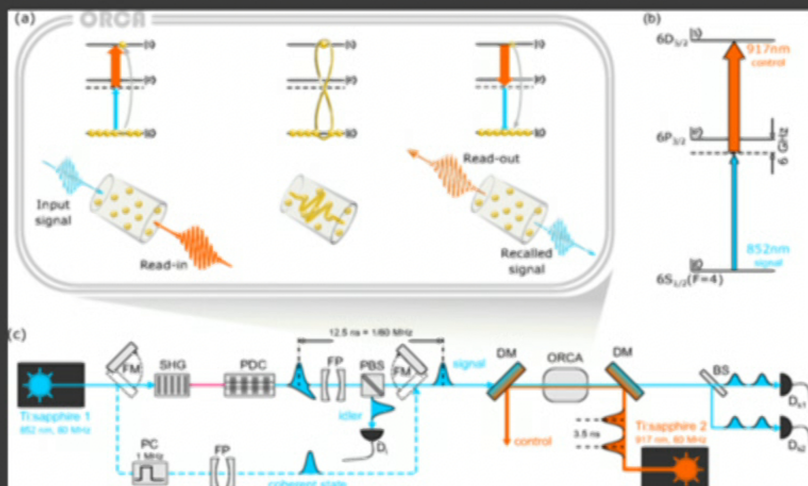
..., *PRL* **85**, 4972 (2000)

- Thus, these can be ... by only local measurements...
- Can show ...ation is encoded in a sup... ...st two states (i.e. in a qubit sub... ...an perform any allowed POVM using only ... adaptive measurements.

A single qubit probe does not help at all!

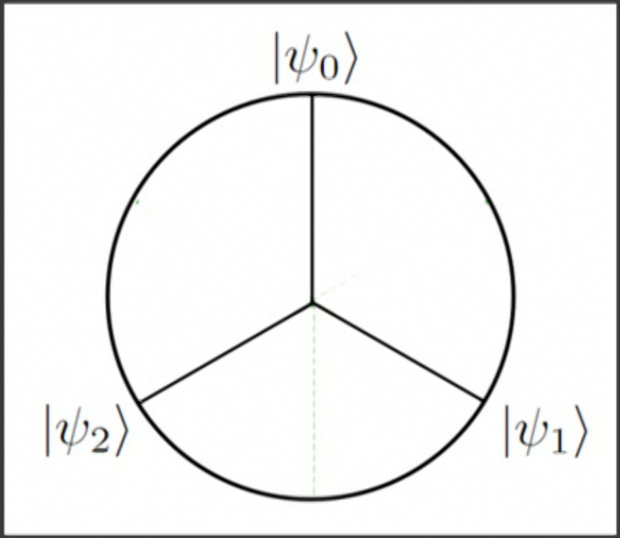
What operations are easy to perform in a lab?



To compare joint strategies against simple local strategies, we do not allow entanglement or quantum memory: allow local measurement, with feed-forward of measurement results (one-way LOCC).

Image from: KT Kaczmarek *et al*, PRA **97**, 042316 (2018)

Next simplest example:



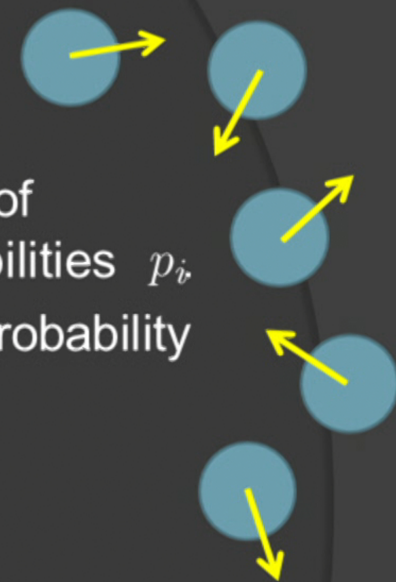
- Trine states: three qubit states with symmetry.
- Given two copies of these states, with equal probabilities, the optimal joint measurement performs strictly better than any local measurement.
- The gap is small:

$$\mathrm{P}_{\mathrm{corr}}^{\mathrm{joint}} = 0.971$$

$$\mathrm{P}_{\mathrm{corr}}^{\mathrm{seq}} = 0.933$$

A. Peres and W.K. Wootters, *PRL* **66**, 1119 (1991)
E. Chitambar and M.-H. Hsieh, *PRA* **88**, 020302(R) (2013)

Minimum error state discrimination



- A system is prepared in one of a number of possible states ρ_i with known prior probabilities p_i.
- In a measurement with POVM $\{\pi_i\}$, the probability of correctly determining the state is

$$\mathrm{P_{corr}} = \sum_j p_j \mathrm{Tr}\left(\rho_j \pi_j\right)$$

- It is known that the measurement minimising the error satisfies the conditions:

$$\sum_i p_i \rho_i \pi_i - p_j \rho_j \geq 0, \qquad \pi_i (p_i \rho_i - p_j \rho_j) \pi_j = 0.$$

AS Holevo, J. Multivariate Analysis **3**, 337 (1973);
HP Yuen, RS Kennedy, and M Lax, IEEE Trans. Inf. Theory **IT-21**, 125 (1975);
CW Helstrom, *Quantum Detection and Estimation Theory* (New York: Academic, 1976)

Sketch of proof

- If $\{\pi_i\}$ is an optimal measurement, then by definition

$$\mathrm{Tr}\left(\sum_i p_i \rho_i \pi_i\right) \geq \mathrm{Tr}\left(\sum_j p_j \rho_j \pi'_j\right)$$

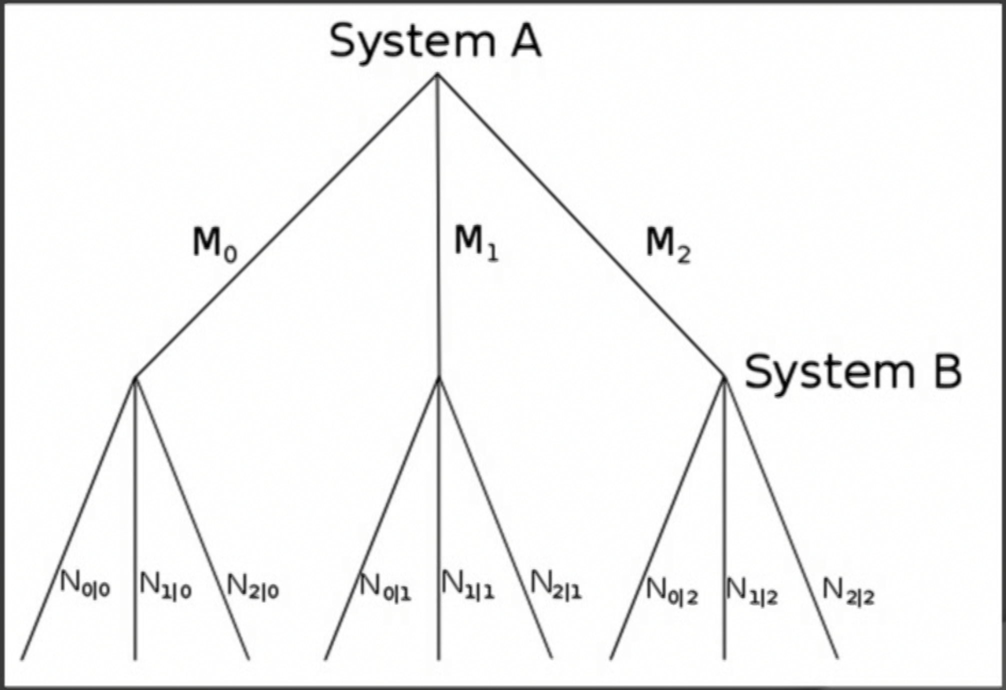
for all allowed measurements $\{\pi'_j\}$.

- Thus, using $\sum_j \pi'_j = I$:

$$\sum_j \mathrm{Tr}\left(\left(\sum_i p_i \rho_i \pi_i - p_j \rho_j\right) \pi'_j\right) \geq 0$$

SM Barnett and S Croke, J. Phys. A **42**, 062001 (2009).

1. Sequential measurement of bi-partite systems



$$\pi_i^{AB} = \sum_j M_i^A \otimes N_{i|j}^B$$

Minimum error conditions

- Wish to optimise: $$\mathrm{P_{corr}} = \sum_{ij} p_i \mathrm{Tr}_{AB}(\rho_i^{AB} M_j^A \otimes N_{i|j}^B).$$
- Necessary conditions:

$$\sum_i p_i \mathrm{Tr}_A(\rho_i^{AB} M_j) N_{i|j} - p_k \mathrm{Tr}_A(\rho_k^{AB} M_j) \geqslant 0,$$

$$\sum_{i,j} p_i \mathrm{Tr}_B(\rho_i^{AB} N_{i|j}) M_j - \sum_i p_i \mathrm{Tr}_B(\rho_i^{AB} N_{i|k}) \geqslant 0,$$

- A necessary and sufficient condition:

$$\sum_{i,j} p_i \mathrm{Tr}_B(\rho_i^{AB} N_{i|j}) M_j - \sum_k p_k \mathrm{Tr}_B(\rho_k^{AB} \widetilde{N}_k) \geqslant 0,$$

S Croke, SM Barnett, and G Weir, PRA **95**, 052308 (2017).

Example: the domino states

$$
\begin{aligned}
|\psi_{00}\rangle &= |0\rangle|0-1\rangle, \\
|\psi_{01}\rangle &= |0\rangle|0+1\rangle, \\
|\psi_{02}\rangle &= |0-1\rangle|2\rangle, \\
|\psi_{10}\rangle &= |1+2\rangle|0\rangle, \\
|\psi_{11}\rangle &= |1\rangle|1\rangle, \\
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CH Bennett *et* al, PRA **59**, 1070 (1999).

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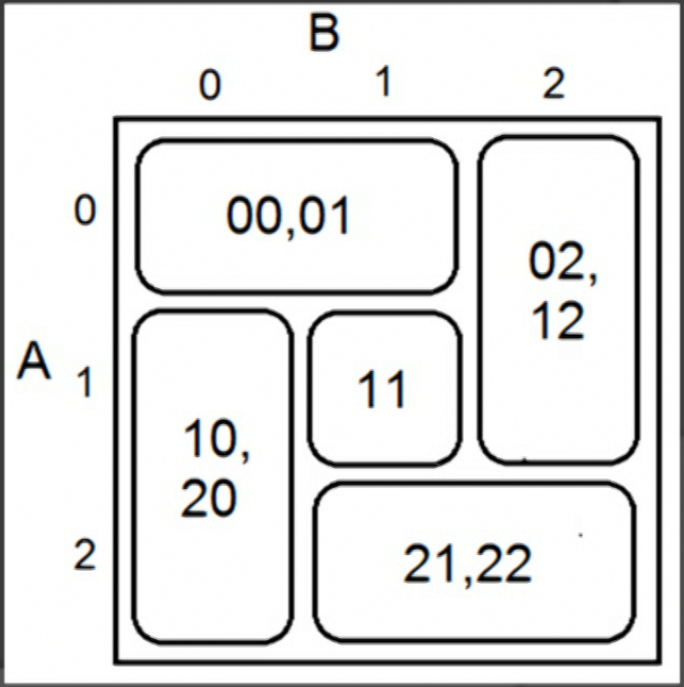

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Why are the domino states surprising?

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\end{aligned}
$$

These are perfectly distinguishable on B alone

Use the measurement on A to (try to) determine to which subset the state belongs.

Probability of correctly determining the state is 84%.

This is the best that can be done without joint measurement and/or a quantum memory.

S Croke and SM Barnett, PRA **95**, 012337 (2017).

Why are the domino states surprising?

A. They have no entanglement, but joint measurements are needed to distinguish them.

Yes, but the probability of error of any local scheme is reeeeeally small (10^{-8})…

A2. Simple local measurements do pretty badly…

AM Childs, D Leung, L Mancinska, and M Ozols, CMP **323**, 121 (2013).

Why are the domino states surprising?

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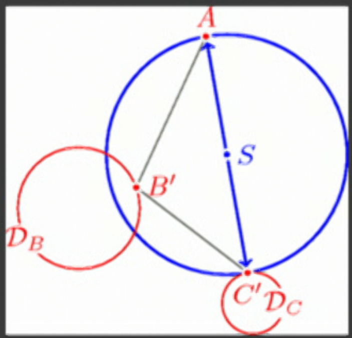
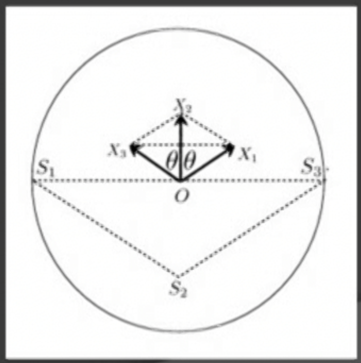
2. Qubit state discrimination

- Define the Lagrange operator $\Gamma = \sum_i p_i \rho_i \pi_i$, this is unique for a given set of states.
- In terms of this, the necessary and sufficient condition for a measurement to be optimal is: $\Gamma - p_j \rho_j \geq 0.$
- Geometric methods:





$$\Gamma = p_j \rho_j + r_j \sigma_j$$

ME Deconinck and BM Terhal, PRA **80**, 062304 (2010);
J.Bae and WY Hwang, PRA **87**, 012334 (2013); J Bae, NJP **15**, 073037 (2013).

Qubit state discrimination: algebraic approach

- Γ is enough to determine the probability of success $\mathrm{P}_{\mathrm{corr}} = \mathrm{Tr}(\Gamma)$ and the optimal measurement, via

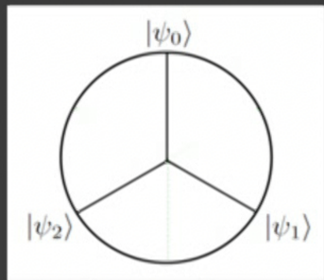
$$(\Gamma - p_j \rho_j)\, \pi_j = 0$$

Key to algebraic method: if π_j is non-zero, and ρ_j is pure $\rho_j = |\psi_j\rangle\langle\psi_j|$, then $p_j \langle\psi_j|\Gamma^{-1}|\psi_j\rangle = 1$

- Sketch of method:
  - Check if “no measurement” solution is optimal.
  - Check if two outcome solution is optimal.
  - Search for three outcome solutions.
  - Search for four outcome solutions.

G Weir, SM Barnett, and S Croke, PRA **96**, 022312 (2017).

Example: Trine states with arbitrary priors

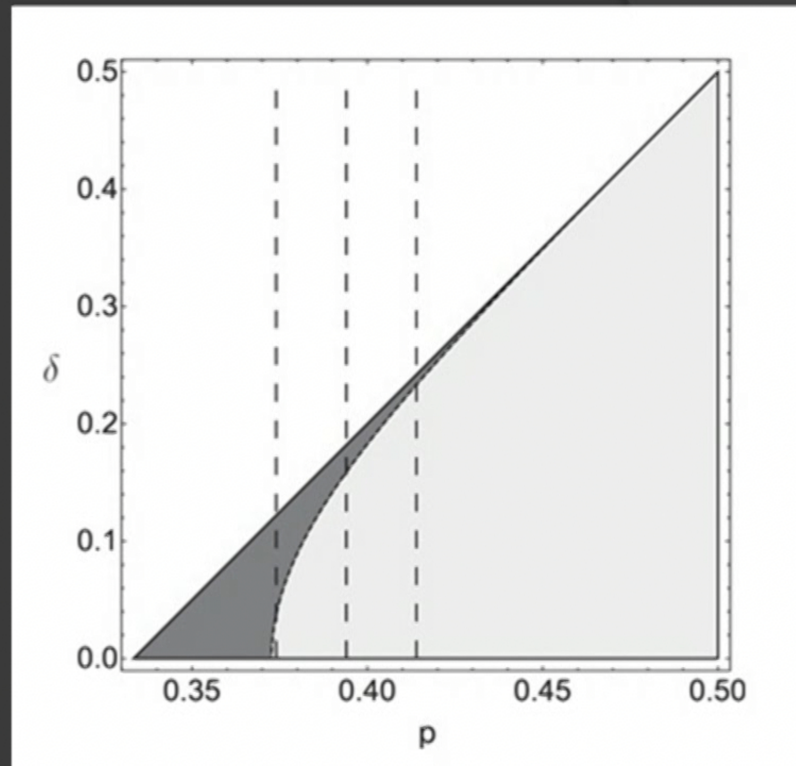


$$p_0 \geq p_1 \geq p_2$$

$$\begin{aligned} p_0 &= p + \delta \\ p_1 &= p - \delta \\ p_2 &= 1 - 2p \end{aligned}$$

$$P_{2\text{-el}} = \frac{1}{2}\left(p_0 + p_1 + \sqrt{p_0^2 + p_0 p_1 + p_1^2}\right)$$

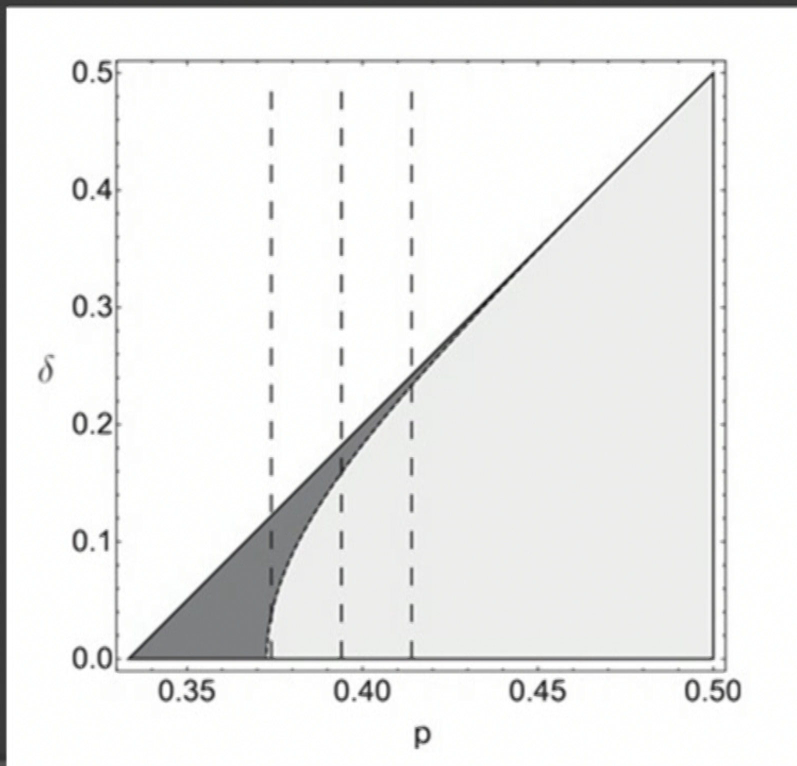
$$P_{3\text{-el}} = \frac{2(p_0 p_1 + p_0 p_2 + p_1 p_2)}{2 - \left(\frac{p_0 p_1}{p_2} + \frac{p_0 p_2}{p_1} + \frac{p_1 p_2}{p_0}\right)}$$

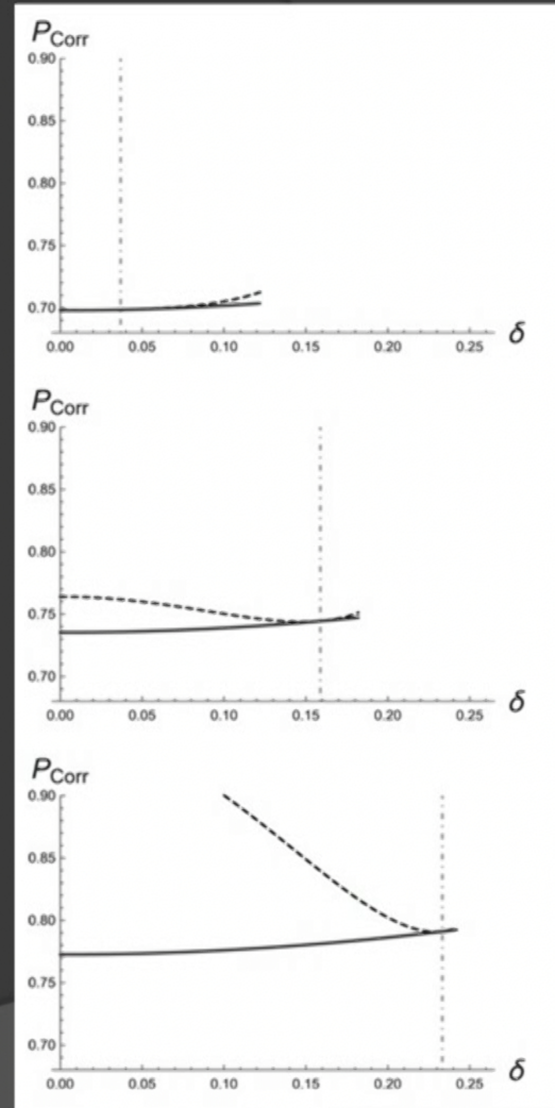


Dark grey: region in which 3 outcome measurement is optimal.
Light grey: 2 outcome measurement is optimal.

G Weir, C Hughes, SM Barnett, and S Croke, Q Sci. Technol. **3**, 035003 (2018).

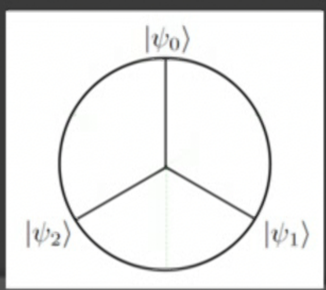
Comparison of two and three outcome strategies

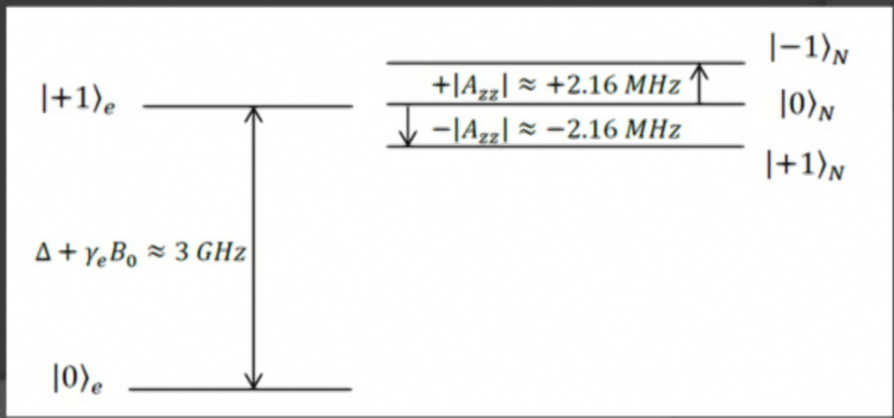




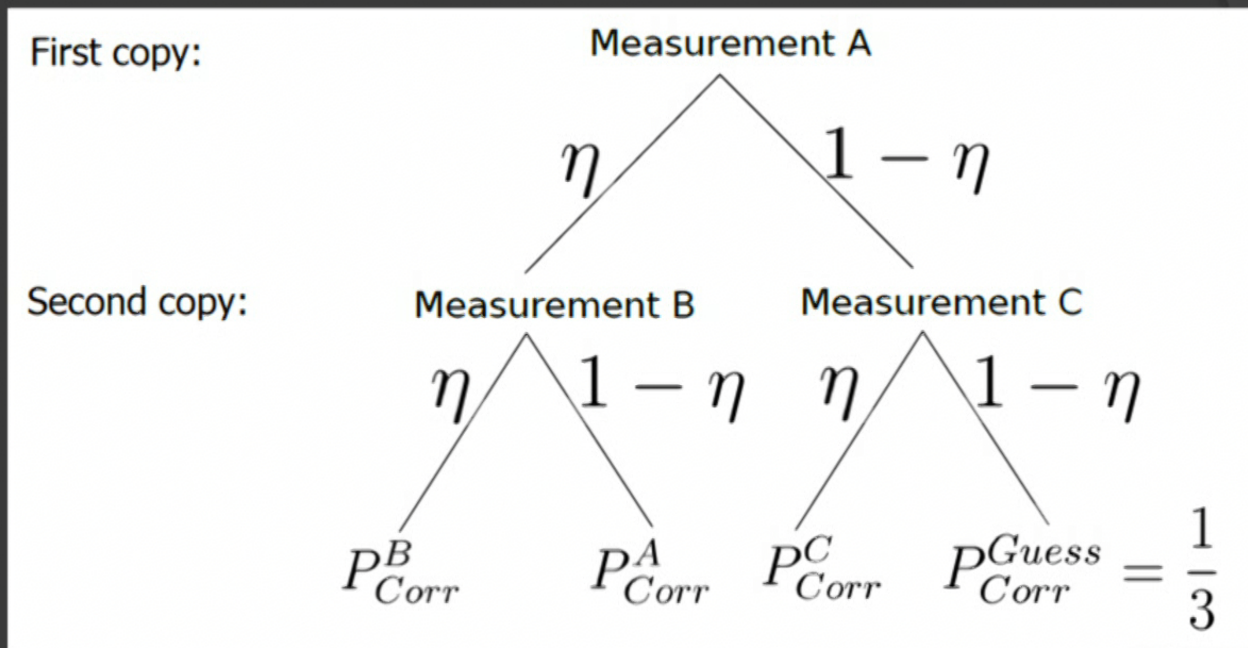
Example: sequential measurement with imperfect detectors

- With imperfect detectors, need both detectors to fire to complete optimal adaptive measurement.
- More important to fully optimise for those cases in which the detector does work.
- Increased advantage for joint measurement?
- Motivated by discussion of demonstration in NV centre.





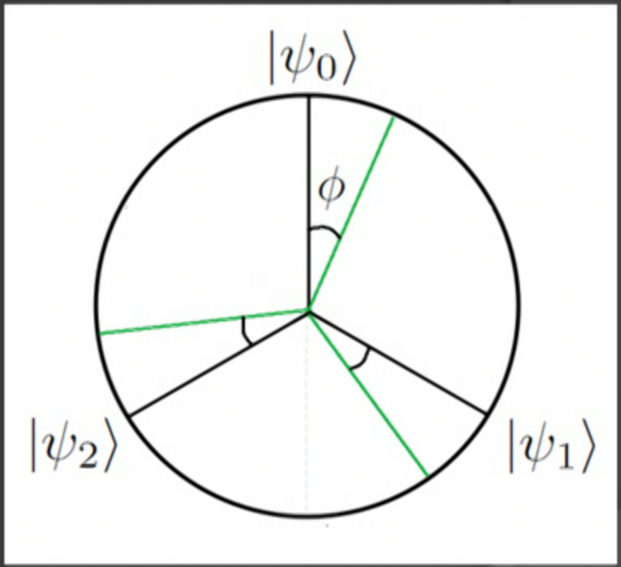
Imperfect detector efficiency



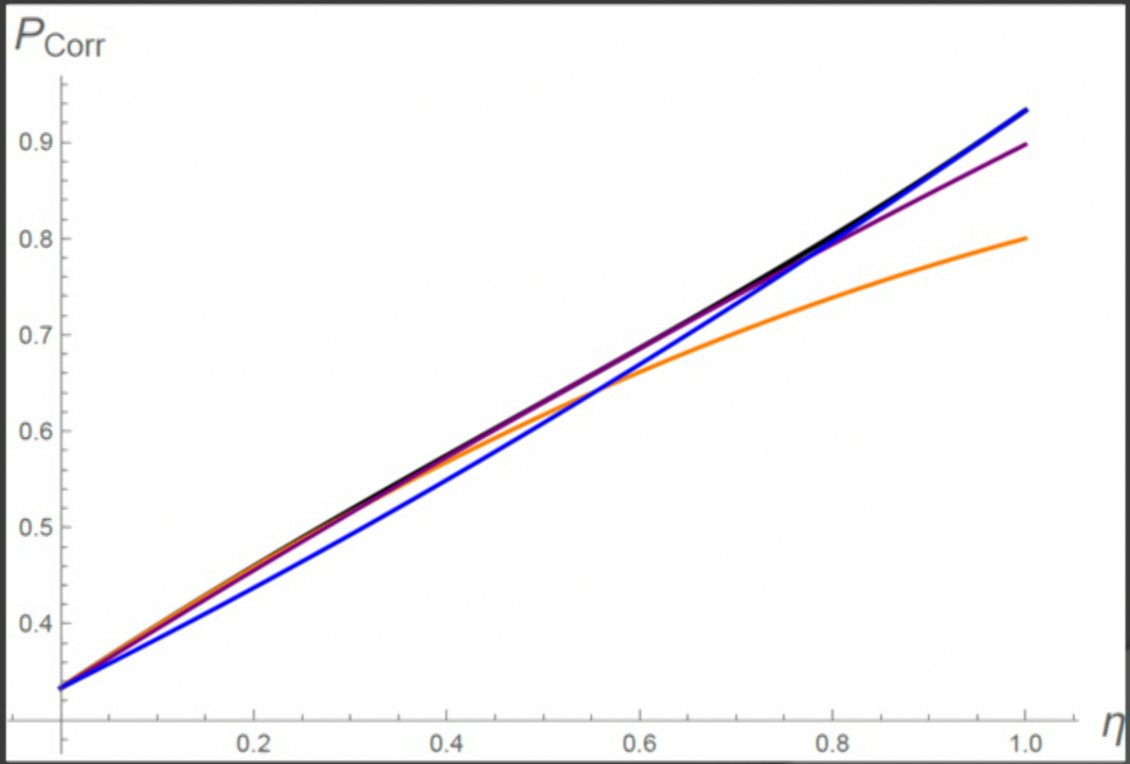
$$\mathrm{P}^{\mathrm{seq}}_{\mathrm{corr}} = \eta^2 \mathrm{P}^{\mathrm{B}}_{\mathrm{corr}} + \eta(1-\eta)\left(\mathrm{P}^{\mathrm{A}}_{\mathrm{corr}} + \mathrm{P}^{\mathrm{C}}_{\mathrm{corr}}\right) + \frac{1}{3}(1-\eta)^2$$

Optimisation: two copies of the trine states

- Optimal measurement should have the same symmetry as the states.
- If first measurement fails, left with one copy of the trine states, with equal probabilities: optimal measurement well known, succeeds with probability 2/3.
- If first measurement succeeds, update the probabilities of each state, and perform optimal measurement for new priors.

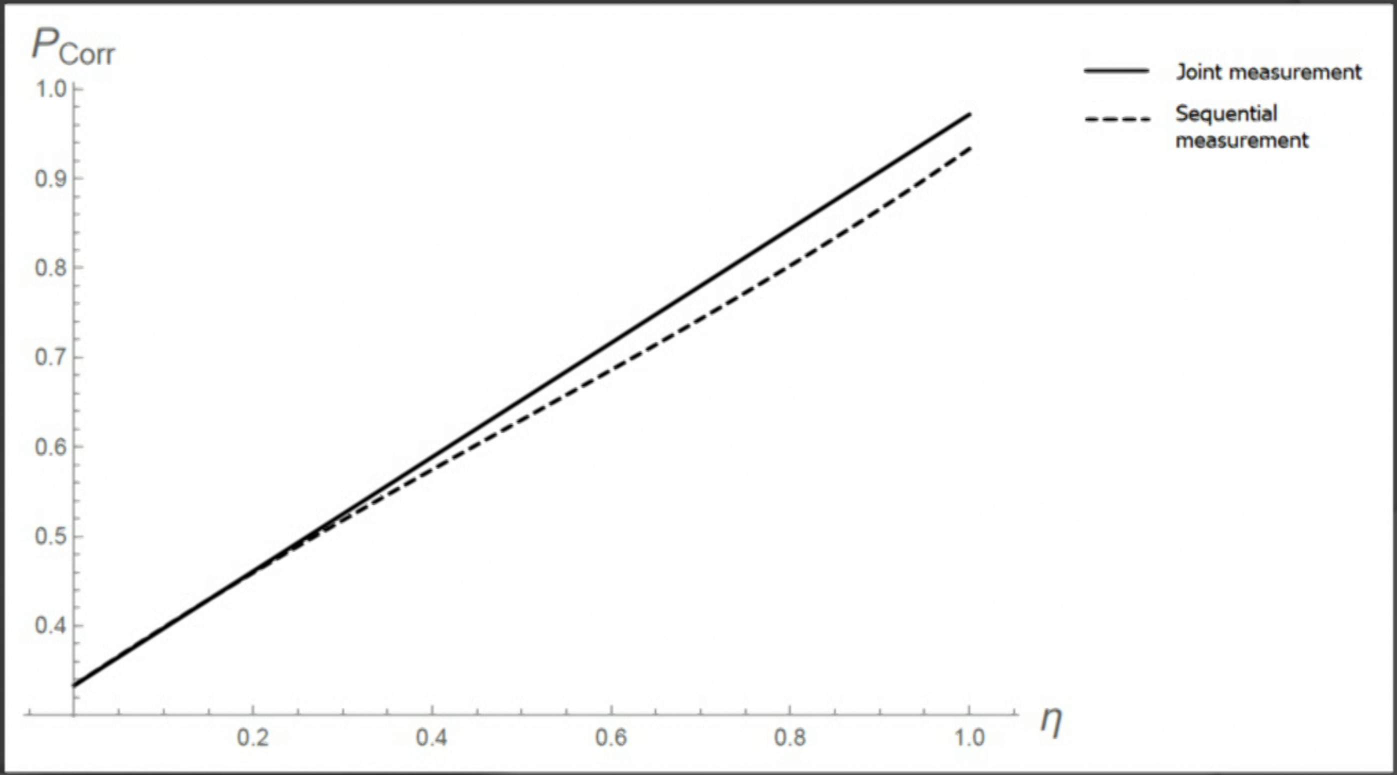


Results: sequential measurement strategies

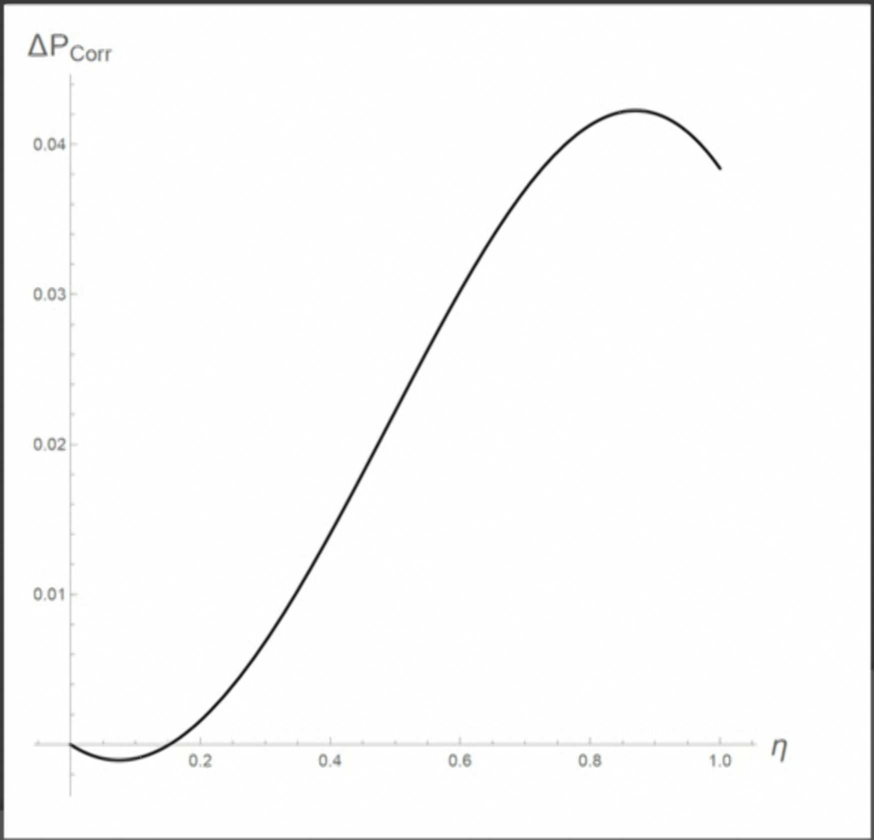


Black line shows optimal sequential strategy, coloured lines show strategies for Φ=0 (orange),π/6 (purple), π/3 (blue).

Results: comparison with joint measurement



Results: comparison with joint measurement



In conclusion...

- It is relatively challenging to find simple examples of measurement tasks for which a small quantum measuring apparatus provides an advantage over simpler techniques.
- Sequential measurements: known optimal measurements assign states to a subset in the first step, and discriminate between states in a subset in the second step.
- Qubit state discrimination example: trine states with arbitrary priors – simple von Neumann measurement is optimal for most of the parameter space.
- With imperfect detector efficiencies, the advantage of joint measurements over simpler schemes is further decreased.
- Sometimes simple procedures are good enough!

Thank you for your attention!