Title: Cosmology Theory 3

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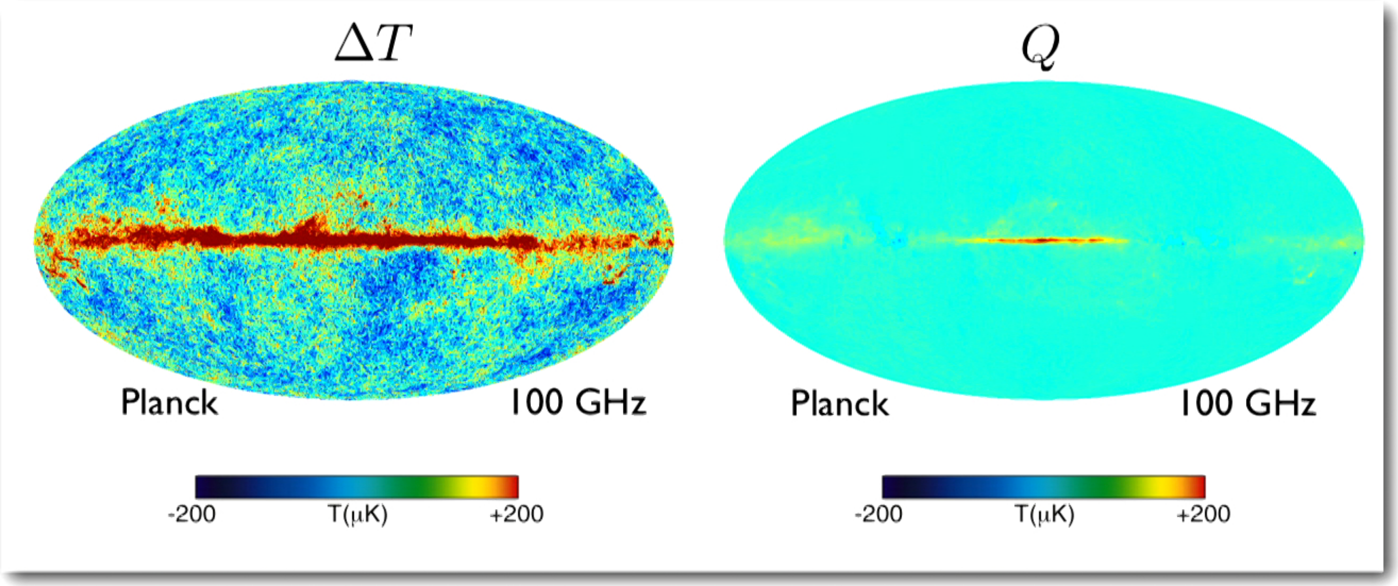
Abstract:

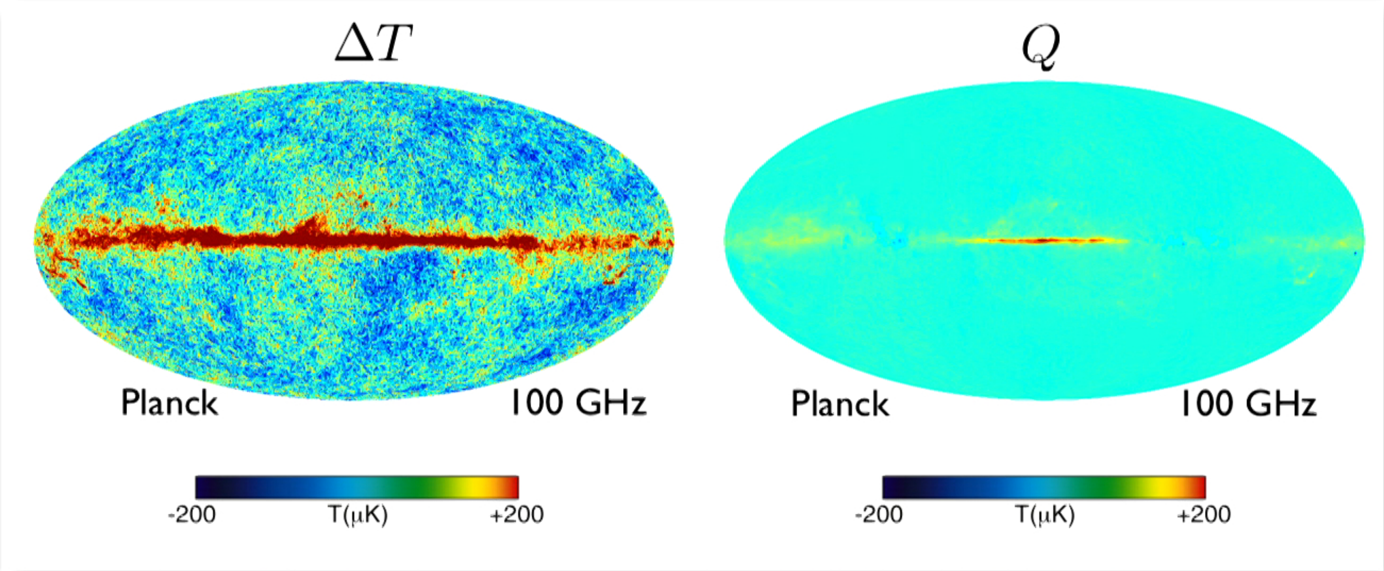
# Anisotropies

After monopole and dipole are removed, the microwave sky reveals small anisotropies.



# Anisotropies

After monopole and dipole are removed, the microwave sky reveals small anisotropies.



# Anisotropies

Only the correlation functions can be predicted by theory

$$\begin{array}{ll}
\langle \Delta T(\hat{n}) \Delta T(\hat{n}') \rangle & , \\
\langle \Delta T(\hat{n}) \left[ Q(\hat{n}') + iU(\hat{n}') \right] \rangle & , \\
\langle \left[ Q(\hat{n}) + iU(\hat{n}) \right] \left[ Q(\hat{n}') + iU(\hat{n}') \right] \rangle & , \\
\langle \left[ Q(\hat{n}) + iU(\hat{n}) \right] \left[ Q(\hat{n}') - iU(\hat{n}') \right] \rangle & .
\end{array}$$

as well as higher n-point functions

# Anisotropies

For data analysis and comparison with theory, it is more convenient to use multipole coefficients

$$a_{T,\ell m} = \int d^2\hat{n}\; Y_\ell^{m*}(\hat{n})\Delta T(\hat{n})$$

$$a_{P,\ell m} = \int d^2\hat{n}\; {}_2Y_\ell^{m*}(\hat{n})\left(Q(\hat{n}) + iU(\hat{n})\right)$$

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$$a_{P,\ell m} = \int d^2\hat{n}\ {}_2Y_\ell^{m*}(\hat{n}) \left(Q(\hat{n}) + iU(\hat{n})\right)$$

$$a_{E,\ell\, m} \equiv -(a_{P,\ell\, m} + a^*_{P,\ell\, -m})/2$$

$$a_{B,\ell\, m} \equiv i(a_{P,\ell\, m} - a^*_{P,\ell\, -m})/2$$

under parity $\quad a_{E,\ell\, m} \to (-1)^\ell a_{E,\ell\, m} \quad$ "gradient"

$\quad a_{B,\ell\, m} \to -(-1)^\ell a_{B,\ell\, m} \quad$ "curl"

# Anisotropies

The correlations are then encoded in the angular power spectra

$$\langle a_{T,\ell\, m} a^*_{T,\ell'\, m'} \rangle = C_{TT,\ell}\delta_{\ell\ell'}\delta_{mm'} \ ,$$

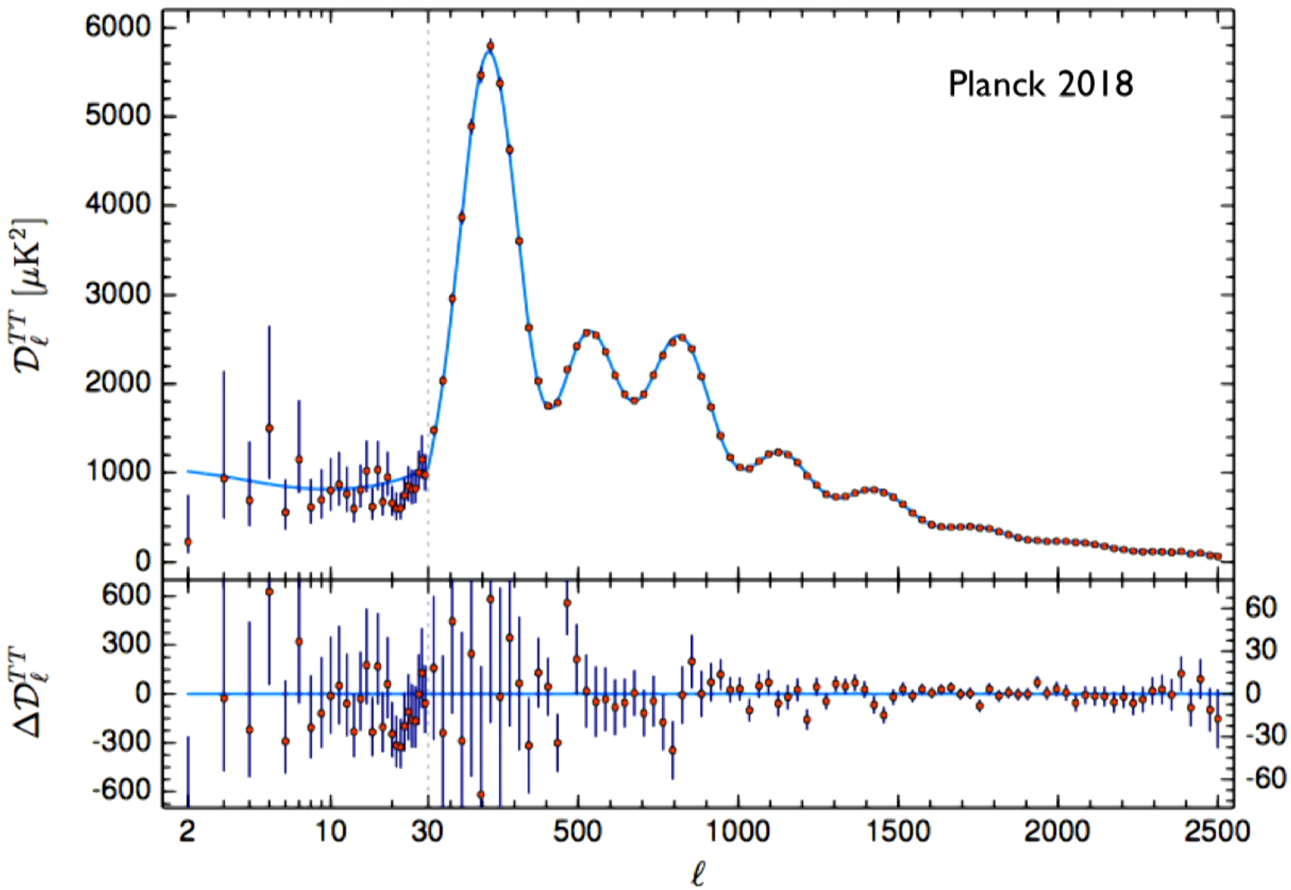
$$\langle a_{T,\ell\, m} a^*_{E,\ell'\, m'} \rangle = C_{TE,\ell}\delta_{\ell\ell'}\delta_{mm'} \ ,$$

$$\langle a_{E,\ell\, m} a^*_{E,\ell'\, m'} \rangle = C_{EE,\ell}\delta_{\ell\ell'}\delta_{mm'} \ ,$$

$$\langle a_{B,\ell\, m} a^*_{B,\ell'\, m'} \rangle = C_{BB,\ell}\delta_{\ell\ell'}\delta_{mm'} \ ,$$

For Gaussian fluctuations these contain all the information, for non-Gaussian fluctuations we would need higher n-point functions

# Anisotropies



# Newtonian warm-up

Equations of motion for fluid in the Newtonian theory

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla \phi$$

$$\nabla^2 \phi = 4\pi G \rho$$

Consider perturbations in a fluid at rest with constant density and pressure

$$\rho(t, \mathbf{x}) = \bar{\rho} + \delta\rho(t, \mathbf{x})$$

$$p(t, \mathbf{x}) = \bar{p} + \delta p(t, \mathbf{x})$$

$$\mathbf{v}(t, \mathbf{x}) = \delta\mathbf{v}(t, \mathbf{x})$$

# Newtonian warm-up

Equations of motion for perturbations to linear order

$$\frac{\partial \delta\rho}{\partial t} + \bar{\rho}\nabla \cdot \delta\mathbf{v} = 0$$
$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c_s^2}{\bar{\rho}}\nabla\delta\rho - \nabla\phi$$
$$\nabla^2\phi = 4\pi G\delta\rho$$

or combining the first two

$$\ddot{\delta}\rho - c_s^2\nabla^2\delta\rho - 4\pi G\bar{\rho}\delta\rho = 0$$

# Newtonian warm-up

Translational invariance suggests to Fourier transform

$$\ddot{\delta}\rho_{\mathbf{k}} + c_s^2 k^2 \delta\rho_{\mathbf{k}} - 4\pi G\bar{\rho}\delta\rho_{\mathbf{k}} = 0$$

so the dispersion relation is

$$\omega^2 = c_s^2 k^2 - 4\pi G\bar{\rho}$$

- Sound waves on small scales
- Instability to gravitational collapse on large scales (Jeans instability)

In FLRW, the growth becomes linear rather than exponential, but the basic picture remains.

# General Relativity - Part II

We saw that the line element and stress tensor in the FLRW universe are described by

$$ds^2 = -dt^2 + a^2 d\vec{x}^2$$

$$T = \bar{\rho} dt^2 + a^2 \bar{p} d\vec{x}^2$$

To describe the anisotropies, we must consider small perturbations around the FLRW background

$$ds^2 = (-1 + h_{00})dt^2 + 2h_{0i}dtdx^i + (a^2\delta_{ij} + h_{ij})dx^i dx^j$$

$$T = (\bar{\rho} + \delta T_{00})dt^2 + 2\delta T_{0i}dtdx^i + (a^2\bar{p}\delta_{ij} + \delta T_{ij})dx^i dx^j$$

# General Relativity - Part II

Under an infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \epsilon^\mu(x)$$

the perturbations transform

$$\begin{aligned}
\Delta h_{00} &= -2\frac{\partial \epsilon_0}{\partial t} \\
\Delta h_{0i} &= -\frac{\partial \epsilon_i}{\partial t} - \frac{\partial \epsilon_0}{\partial x^i} + 2\frac{\dot{a}}{a}\epsilon_i \\
\Delta h_{ij} &= -\frac{\partial \epsilon_i}{\partial x^j} - \frac{\partial \epsilon_j}{\partial x^i} + 2a\dot{a}\epsilon_0
\end{aligned}$$

We can use and choice of coordinates (or gauge) that is convenient

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We can use and choice of coordinates (or gauge) that is convenient

# General Relativity - Part II

In synchronous gauge

$$ds^2 = -dt^2 + (a^2\delta_{ij} + h_{ij})dx^i dx^j$$

$$\delta T = \delta\rho\, dt^2 - 2(\bar{\rho} + \bar{p})\delta u_i dt dx^i + \left(a^2(\delta p\, \delta_{ij} + \pi_{ij}) + \bar{p}h_{ij}\right) dx^i dx^j$$

We can decompose the perturbations into scalar, vector, and tensor perturbations.

$$\delta u_i = \partial_i \delta u + \delta u_i^V$$

$$h_{ij} = a^2(A\delta_{ij} + \partial_i\partial_j B + \partial_i C_j^V + \partial_j C_i^V + h_{ij}^T)$$

$$\pi_{ij} = \partial_i\partial_j \pi^S + \partial_i \pi_j^V + \partial_j \pi_i^V + \pi_{ij}^T$$

# General Relativity - Part II

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Rotational invariance of the background imply that these do not mix and we can study one at a time.

# General Relativity - Part II

Equations of motion for scalar modes

Einstein equations

$$-\frac{\nabla^2}{a^2}A + 3\frac{\dot{a}}{a}\dot{A} + \frac{\dot{a}}{a}\nabla^2\dot{B} = 8\pi G\delta\rho \qquad \text{(00)}$$

$$\dot{A} = 8\pi G(\overline{\rho} + \overline{p})\delta u \qquad \text{(0i)}$$

$$\frac{1}{2}\frac{\nabla^2}{a^2}A - \ddot{A} - 3\frac{\dot{a}}{a}\dot{A} - \frac{1}{2}\nabla^2\ddot{B} - \frac{3}{2}\frac{\dot{a}}{a}\nabla^2\dot{B} = 8\pi G\delta p$$

(ij)

$$-A + a^2\ddot{B} + 3a\dot{a}\dot{B} = 16\pi G a^2 \pi^S$$

# General Relativity - Part II

Equations of motion for scalar modes

Energy and momentum conservation

$$\delta\dot{\rho} + \frac{3\dot{a}}{a}(\delta\rho + \delta p) + \frac{\nabla^2}{a^2}\left[(\overline{\rho} + \overline{p})\delta u + a\dot{a}\pi^S\right] + \frac{1}{2}(\overline{\rho} + \overline{p})\left(3\dot{A} + \nabla^2\dot{B}\right) = 0$$

$$\delta p + \nabla^2\pi^S + \frac{1}{a^3}\frac{\partial}{\partial t}\left[a^3(\overline{\rho} + \overline{p})\delta u\right] = 0$$

Similarly for vectors and tensors

# Equations of motion

Consider the universe at a time early enough for rapid thermalization, not so early that other degrees of freedom appear in the plasma

$$6 \times 10^6 K < T < 10^9 K$$

In $\Lambda$CDM

$e^-$ $p$ $\mathrm{He}$

$\gamma$ dark matter

$\nu$ cosmological constant

# Equations of motion

How do we describe the various components?

Electrons and protons elastically scatter very efficiently. They can be described as one "baryon" fluid.

For cold dark matter a "hydrodynamic" description is also sufficient because it is extremely non-relativistic, i.e. "dust".

Neutrinos free-stream, leading to anisotropic stress. They are usually described by a Boltzmann hierarchy.

If we are interested in the polarization of photons we have to keep track of it and describe them by a Boltzmann hierarchy.

# Equations of motion

Toy example:

Perturbations in a thermal gas of massless particles

Instead of keeping track of the trajectories of all particles, we will describe it by the phase space density

$$n(\vec{x}, \vec{p}, t) \equiv \sum_r \delta(\vec{x} - \vec{x}_r(t))\delta(\vec{p} - \vec{p}_r(t))$$

Since

$$\frac{d\vec{x}_r}{dt} = \hat{p}_r \quad \text{and} \quad \frac{d\vec{p}_r}{dt} = 0$$

it satisfies a collisionless Boltzmann equation

$$\frac{\partial n}{\partial t} = -\hat{p} \cdot \nabla n$$

$$D_\ell = \frac{\ell(\ell+1)}{2\pi} C_\ell$$

$$\frac{\partial n}{\partial t} = \sum_r -\frac{dx_r}{dt}\frac{\partial}{\partial x^i}\,\delta(\vec{x}-\vec{x}_r(t))\,\delta(\vec{p}-p_r(t))$$

$$=$$

# Equations of motion

Toy example:

Temperature perturbations are related to intensity perturbations by

$$\Delta I_\nu(\hat{n}) = \left.\frac{d\bar{I}_\nu}{dT}\right|_{T_0} \Delta T(\hat{n})$$

A differential measurement sensitive to all frequencies probes

$$\int_0^\infty d\nu \Delta I_\nu(\hat{n}) = \frac{4\Delta T(\hat{n})}{T_0} \int_0^\infty d\nu \bar{I}_\nu$$

This makes it natural to define the "temperature" anisotropy

$$\Delta_T(\vec{x},\hat{p}) = \frac{1}{\bar{\bar{I}}} \int \frac{p^3 dp}{(2\pi)^3} \delta n(\vec{x}, p\,\hat{p})$$

# Equations of motion

Toy example:

It satisfies

$$\frac{\partial \Delta_T(\vec{x},\hat{p},t)}{\partial t} + \hat{p}\cdot\nabla\Delta_T(\vec{x},\hat{p},t) = 0$$

Translational invariance suggests to look for solutions

$$\Delta_T(\vec{x},\hat{p},t) = \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})\Delta_T(q,\mu,t)e^{i\vec{q}\cdot\vec{x}}$$

$$\hat{q}\cdot\hat{p}$$

$$\frac{\partial \Delta_T(q,\mu,t)}{\partial t} + iq\mu\Delta_T(q,\mu,t) = 0$$

(Of course, the solution to this equation is trivial, but let's keep going)

# Equations of motion

Toy example:

The temperature anisotropies at the origin at some time $t_0$ are

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{4}\Delta_T(\vec{x}=0, -\hat{n}, t_0)$$

if we expand

$$\Delta_T(q,\mu,t_0) = \sum_\ell (-i)^\ell (2\ell+1) P_\ell(\mu)\Delta_{T,\ell}(q,t_0)$$

we find the multipole coefficients

$$a_{T,\ell\, m} = \pi i^\ell \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q}) Y^*_{\ell\, m}(\hat{q})\Delta_{T,\ell}(q,t_0)$$

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# Equations of motion

Toy example:

This suggests to derive equations directly for

$$\Delta_{T,\ell}(q,t_0)$$

These equations are called the Boltzmann hierarchy

In our toy example

$$\dot{\Delta}_{T,\ell}(q,t) + \frac{q}{2\ell+1}\left[(\ell+1)\Delta_{T,\ell+1}(q,t) - \ell\Delta_{T,\ell-1}(q,t)\right] = 0$$

Analogous equations can be derived for the polarization anisotropy.

# Equations of motion

## Beyond the toy example

For interacting particles one finds

$$\frac{\partial \Delta_T(q,\mu,t)}{\partial t} + iq\mu\Delta_T(q,\mu,t) = -\omega\Delta_T(q,\mu,t) + \omega F\left[\Delta_{T,0}(q,t), \Delta_{T,2}(q,t), t\right]$$

with formal solution

$$\Delta_T(q,\mu,t) = \Delta_T(q,\mu,t_i)e^{-iq\mu(t-t_i)}e^{-\omega(t-t_i)} + \omega \int_{t_i}^{t} dt e^{-iq\mu(t-t')}e^{-\omega(t-t')}F[\Delta_{T,0}(q,t), \Delta_{T,2}(q,t), t]$$

Since only low multipoles appear in the collision terms, one can solve a truncation of the hierarchy and obtain the higher multipoles through this "line-of-sight integration"

# Equations of motion

Beyond the toy example

The same derivation generalizes to a general spacetime

In this case define the phase space density

$$n(x^i, p_i, t) \equiv \sum_r \delta(x^i - x^i_r(t))\delta(p_i - p_{i\,r}(t))$$

The definition of momentum and the geodesic equation imply

$$\frac{dx^i}{dt} = \frac{p^i}{p^0} \qquad\qquad \frac{dp_i}{dt} = \frac{p^k p^l}{2p^0}\frac{\partial g_{kl}}{\partial x^i}$$

and
$$\frac{\partial n}{\partial t} + \frac{p^k}{p^0}\frac{\partial n}{\partial x^k} + \frac{1}{2}\frac{p^k p^l}{p^0}\frac{\partial g^{kl}}{\partial x^m}\frac{\partial n}{\partial p_m} = C$$

Derivation of the Boltzmann hierarchy as before but more tedious.

# Equations of motion

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and
$$\frac{\partial n}{\partial t} + \frac{p^k}{p^0}\frac{\partial n}{\partial x^k} + \frac{1}{2}\frac{p^k p^l}{p^0}\frac{\partial g^{kl}}{\partial x^m}\frac{\partial n}{\partial p_m} = C$$

Derivation of the Boltzmann hierarchy as before but more tedious.

# Equations of motion

Photons

$$\dot{\Delta}^{(S)}_{T,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{T,\ell+1}(q,t) - \ell\Delta^{(S)}_{T,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{T,\ell}(q,t) - 2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$
$$+\omega_c\Delta^{(S)}_{T,0}\delta_{\ell,0} + \frac{1}{10}\omega_c\Pi\delta_{\ell,2} - \frac{4}{3}\frac{q}{a}\omega_c\delta u_{b\,q}\delta_{\ell,1}$$

$$\dot{\Delta}^{(S)}_{P,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{P,\ell+1}(q,t) - \ell\Delta^{(S)}_{P,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{P,\ell}(q,t) + \frac{1}{2}\omega_c(t)\Pi(q,t)\left(\delta_{\ell,0} + \frac{1}{5}\delta_{\ell,2}\right)$$

with source function

$$\Pi = \Delta^{(S)}_{P,0} + \Delta^{(S)}_{T,2} + \Delta^{(S)}_{P,2}$$

# Equations of motion

Photons

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with source function

$$\Pi = \Delta^{(S)}_{P,0} + \Delta^{(S)}_{T,2} + \Delta^{(S)}_{P,2}$$

Polarization sourced by temperature quadrupole

# Equations of motion

Photons

The components of the stress tensor can be written as

$$
\begin{aligned}
\delta\rho_{\gamma\,q} &= \bar{\rho}_\gamma \Delta^{(S)}_{T,0}\,, \\
\delta p_{\gamma\,q} &= \frac{\bar{\rho}_\gamma}{3}\left(\Delta^{(S)}_{T,0} + \Delta^{(S)}_{T,2}\right)\,, \\
\delta u_{\gamma\,q} &= -\frac{3}{4}\frac{a}{q}\Delta^{(S)}_{T,1}\,, \\
q^2\pi^S_{\gamma\,q} &= \bar{\rho}_\gamma \Delta^{(S)}_{T,2}\,.
\end{aligned}
$$

At early times when Compton scattering is efficient

$$\Delta_{T,\ell} \to 0 \quad \text{for} \quad \ell \geq 2$$

$$\Delta_{P,\ell} \to 0$$

The Boltzmann hierarchy reduces to the equations of hydrodynamics

# Equations of motion

Photons

$$\dot{\Delta}^{(S)}_{T,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{T,\ell+1}(q,t) - \ell\Delta^{(S)}_{T,\ell-1}(q,t)\right]$$
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$$+\omega_c\Delta^{(S)}_{T,0}\delta_{\ell,0} + \frac{1}{10}\omega_c\Pi\delta_{\ell,2} - \frac{4}{3}\frac{q}{a}\omega_c\delta u_{b\,q}\delta_{\ell,1}$$

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with source function

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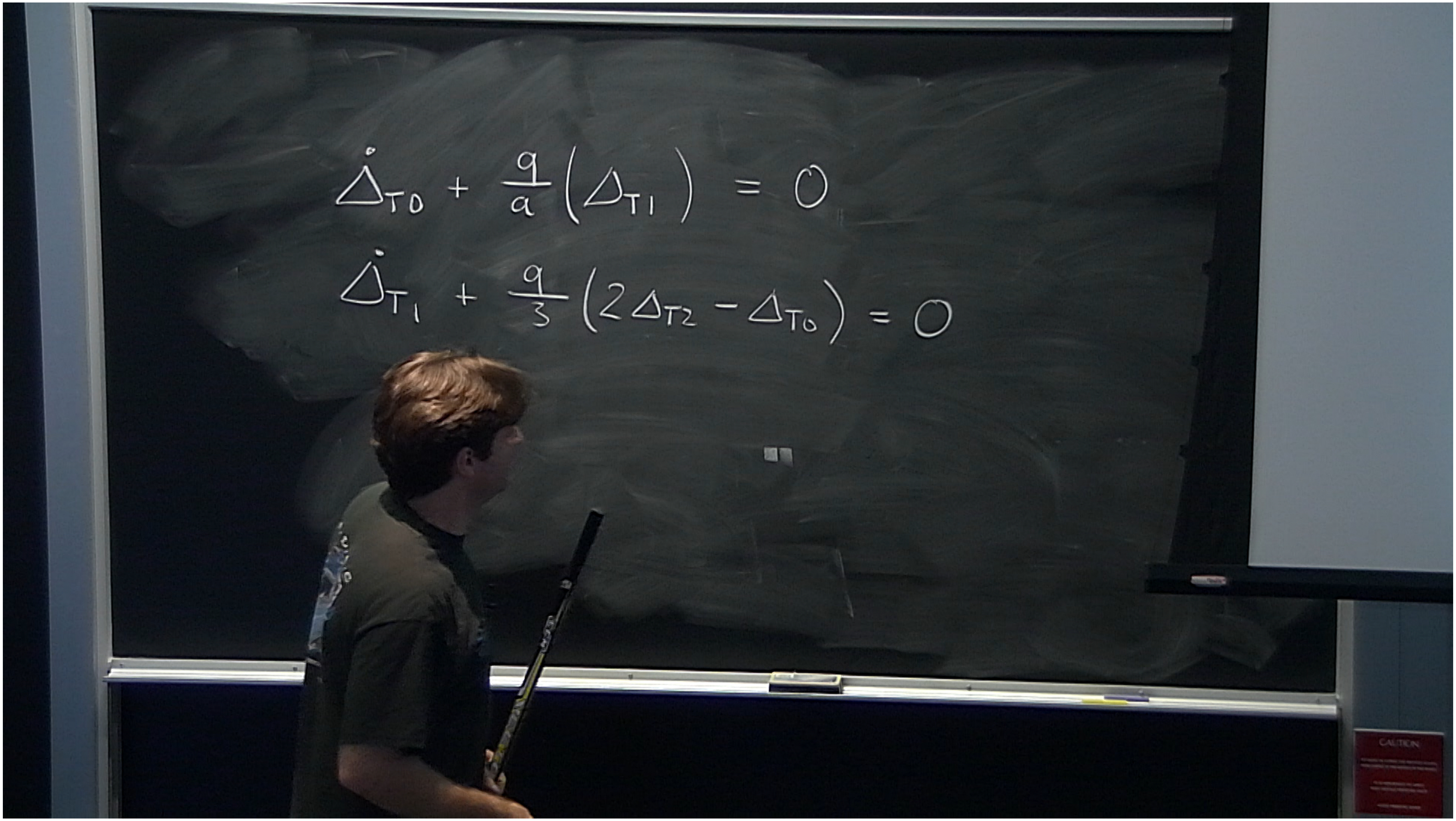
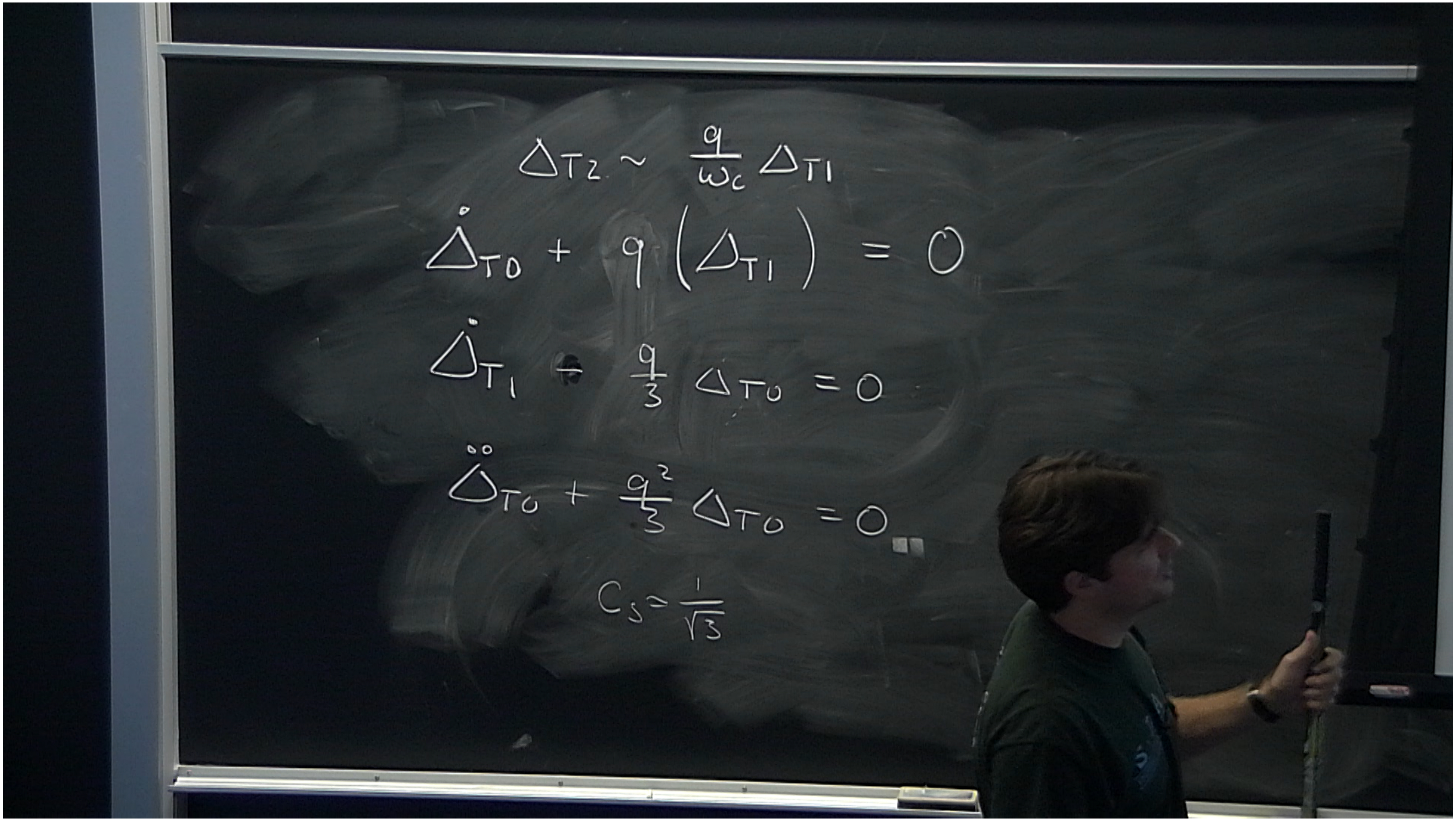
Polarization sourced by temperature quadrupole


$\dot{\Delta}_{T0} + \frac{q}{a}\left(\Delta_{T1}\right) = 0$
$\dot{\Delta}_{T1} + \frac{q}{3}\left(2\Delta_{T2} - \Delta_{T0}\right) = 0$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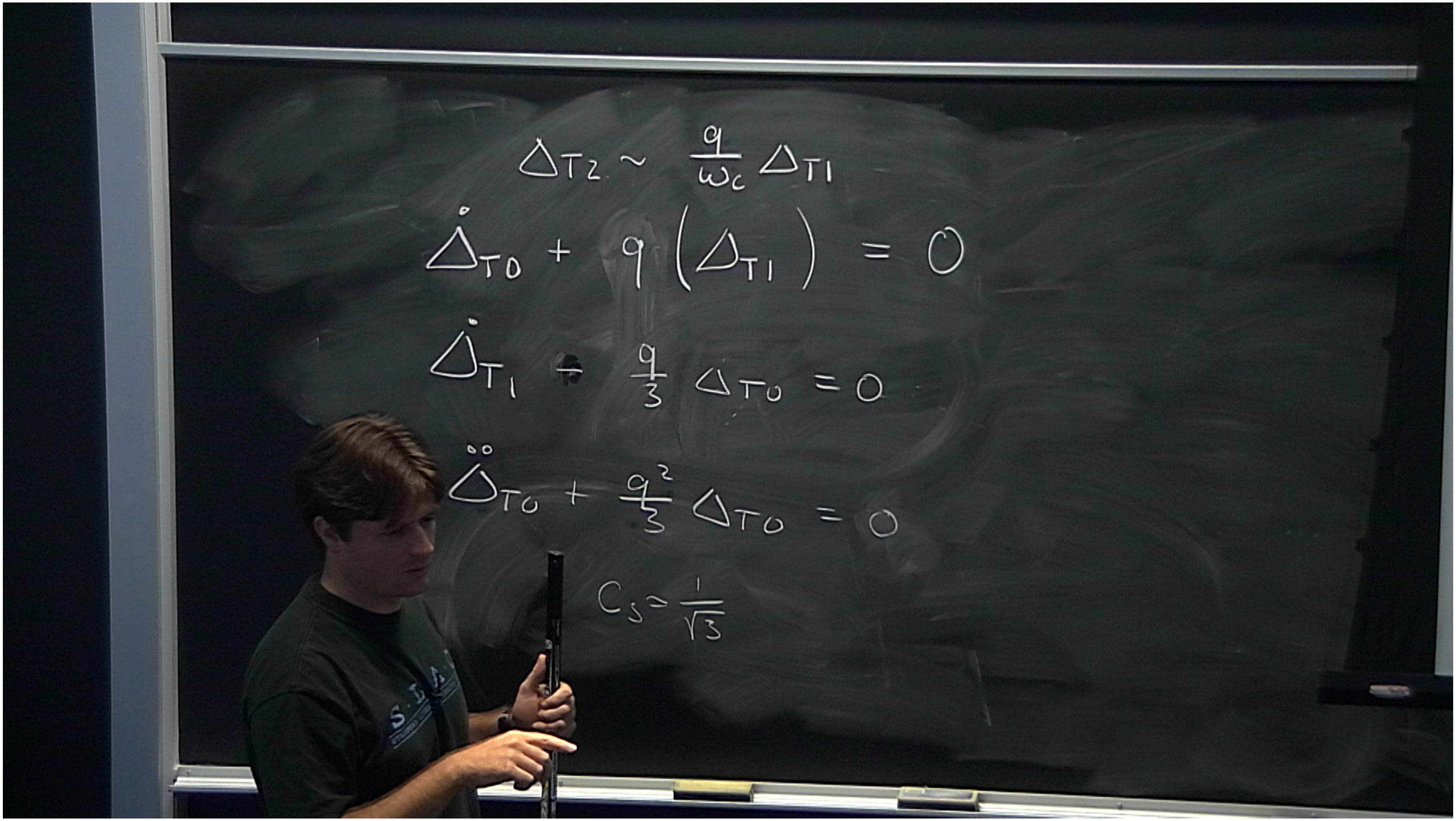
$\dot{\Delta}_{T0} + q\left(\Delta_{T1}\right) = 0$
$\dot{\Delta}_{T1} - \frac{q}{3} \Delta_{T0} = 0$
$\ddot{\Delta}_{T0} + \frac{q^2}{3} \Delta_{T0} = 0$
$c_s = \frac{1}{\sqrt{3}}$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\delta u_{\gamma\, q} &= -\frac{3}{4}\frac{a}{q}\Delta^{(S)}_{T,1}\,, \\
q^2 \pi^S_{\gamma\, q} &= \bar{\rho}_\gamma \Delta^{(S)}_{T,2}\,.
\end{aligned}
$$

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$\Delta_{P,\ell} \to 0$

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# Equations of motion

## Neutrinos

$$\dot{\Delta}^{(S)}_{\nu,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{\nu,\ell+1}(q,t) - \ell\Delta^{(S)}_{\nu,\ell-1}(q,t)\right] = \\ -2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$

## Baryons

### Energy conservation

$$\delta\dot{\rho}_{b\,q} + \frac{3\dot{a}}{a}\delta\rho_{b\,q} - \frac{q^2}{a^2}\bar{\rho}_b\delta u_{b\,q} + \frac{1}{2}\bar{\rho}_b\left(3\dot{A}_q - q^2\dot{B}_q\right) = 0$$

### Momentum conservation

$$\delta\dot{u}_{b\,q} + \frac{4}{3}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b}\omega_c(t)\left(\delta u_{b\,q} + \frac{3}{4}\frac{a}{q}\Delta^{(S)}_{T,1}(q,t)\right) = 0$$

# Equations of motion

Dark Matter

$$\delta\dot{\rho}_{c\,q} + \frac{3\dot{a}}{a}\delta\rho_{c\,q} + \frac{1}{2}\overline{\rho}_{c\,q}\left(3\dot{A}_q - q^2\dot{B}_q\right) = 0$$

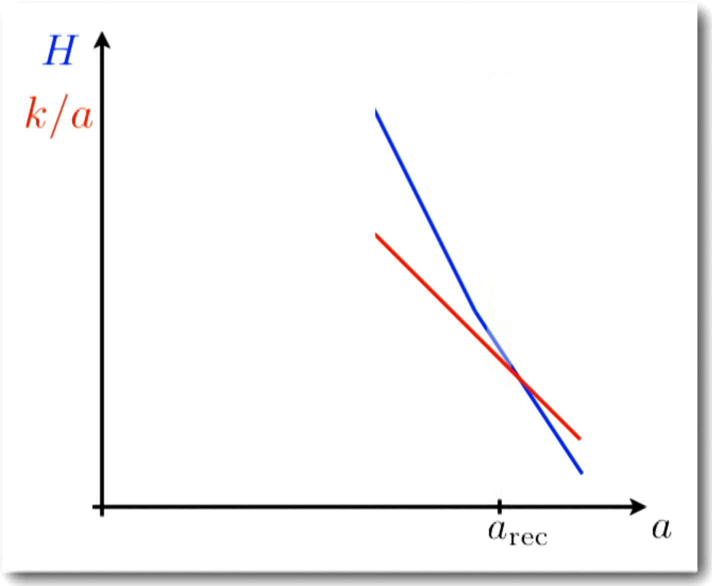
Scalar metric perturbations

$$\frac{q^2}{a^2}A_q + \frac{\dot{a}}{a}\left(3\dot{A}_q - q^2\dot{B}_q\right) = 8\pi G\left(\delta\rho_{q\,b} + \delta\rho_{q\,c} + \overline{\rho}_\gamma\Delta^{(S)}_{T,0} + \overline{\rho}_\nu\Delta^{(S)}_{\nu,0}\right)$$

$$\dot{A}_q = 8\pi G\left(\overline{\rho}_b\delta u_{b\,q} - \frac{a}{q}\overline{\rho}_\gamma\Delta^{(S)}_{T,1}(q,t) - \frac{a}{q}\overline{\rho}_\nu\Delta^{(S)}_{\nu,1}(q,t)\right)$$

# Initial Conditions

What remains is the choice of initial conditions




All modes are "outside the horizon" at early times.

$$\frac{q}{a} \ll H$$

# Initial Conditions

At early times the Boltzmann hierarchy for photons reduces to the equations of hydrodynamics

This suggests we can look for a solution of the form

$$\Delta_{T,0}^{(S)} = \Delta_{\nu,0}^{(S)} = \frac{4}{3}\frac{\delta\rho_c}{\bar{\rho}_c} = \frac{4}{3}\frac{\delta\rho_b}{\bar{\rho}_b} \equiv \Delta_0^{(S)}$$

$$\Delta_{\nu,1}^{(S)} \propto \Delta_{T,1}^{(S)} = -\frac{4}{3}\frac{q}{a}\delta u_{b\,q} \equiv \Delta_1^{(S)}$$

These are adiabatic initial conditions

# Initial Conditions

These are the equations and initial conditions used by the Boltzmann codes such as CAMB or CLASS.

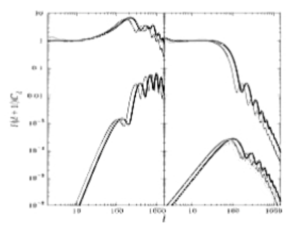
With the solution at hand, one computes

$$a^{(S)}_{T,\ell m} = \pi T_0 i^{\ell} \int d^3q \ \alpha(\mathbf{q}) Y_{\ell}^{m*}(\hat{q}) \Delta^{(S)}_{T,\ell}(q,t_0)$$

or directly

$$C^{(S)}_{TT,\ell} = \pi^2 T_0^2 \int q^2 dq \ \left| \Delta^{(S)}_{T,\ell}(q,t_0) \right|^2$$

similarly for polarization and tensor contribution



**Code for Anisotropies in the Microwave Background**

by Antony Lewis and Anthony Challinor

**CLASS**

the Cosmic Linear Anisotropy Solving System

# Cosmological Parameters

By performing a likelihood analysis, one arrives at

| Parameter | *Planck* alone |
|---|---|
| $\Omega_b h^2$ . . . . . . . . . . . | $0.02237 \pm 0.00015$ |
| $\Omega_c h^2$ . . . . . . . . . . . | $0.1200 \pm 0.0012$ |
| $100\theta_{MC}$ . . . . . . . . . | $1.04092 \pm 0.00031$ |
| $\tau$ . . . . . . . . . . . . . . | $0.0544 \pm 0.0073$ |
| $\ln(10^{10} A_s)$ . . . . . . | $3.044 \pm 0.014$ |
| $n_s$ . . . . . . . . . . . . . | $0.9649 \pm 0.0042$ |

**(Planck Collaboration 1807.06205)**

# More on temperature anisotropies

Recall that the temperature anisotropy is given by

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{4}\Delta_T(\vec{x}=0, -\hat{n}, t_0)$$

where $\Delta_T(\vec{x}, \hat{p}, t_0)$ satisfies a Boltzmann equation.

We looked for solutions of the form

$$\Delta_T(\vec{x}, \hat{p}, t) = \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})\Delta_T(q, \mu, t)e^{i\vec{q}\cdot\vec{x}}$$

and expanded $\Delta_T(q, \mu, t)$ in terms of Legendre polynomials

$$\Delta_T(q, \mu, t) = \sum_\ell (-i)^\ell (2\ell+1)P_\ell(\mu)\Delta_{T,\ell}(q, t)$$

to arrive at the Boltzmann hierarchy.

# More on temperature anisotropies

For scalar perturbations

$$\dot{\Delta}^{(S)}_{T,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{T,\ell+1}(q,t) - \ell\Delta^{(S)}_{T,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{T,\ell}(q,t) - 2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$
$$+\omega_c\Delta^{(S)}_{T,0}\delta_{\ell,0} + \frac{1}{10}\omega_c\Pi\delta_{\ell,2} - \frac{4}{3}\frac{q}{a}\omega_c\delta u_{b\,q}\delta_{\ell,1}$$

$$\dot{\Delta}^{(S)}_{P,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{P,\ell+1}(q,t) - \ell\Delta^{(S)}_{P,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{P,\ell}(q,t) + \frac{1}{2}\omega_c(t)\Pi(q,t)\left(\delta_{\ell,0} + \frac{1}{5}\delta_{\ell,2}\right)$$

Let's undo the last step and consider the equation satisfied by $\Delta^{(S)}_T(q,\mu,t)$

# More on temperature anisotropies

$$\dot{\Delta}_T^{(S)}(q,\mu,t) + i\frac{q\mu}{a(t)}\Delta_T^{(S)}(q,\mu,t) = -\omega_c(t)\Delta_T^{(S)}(q,\mu,t)$$
$$+\omega_c\Delta_{T,0}^{(S)}(q,t) - \frac{1}{2}\omega_c P_2(\mu)\Pi(q,t)$$
$$+\frac{4iq\mu}{a(t)}\omega_c(t)\delta u_{B\,q}(t) - 2\dot{A}_q(t) + 2q^2\mu^2\dot{B}_q(t)$$

$$\dot{\Delta}_P^{(S)}(q,\mu,t) + i\frac{q\mu}{a(t)}\Delta_P^{(S)}(q,\mu,t) = -\omega_c(t)\Delta_P^{(S)}(q,\mu,t)$$
$$+\frac{3}{4}\omega_c(t)(1-\mu^2)\Pi(q,t)$$

with source function

$$\Pi = \Delta_{P,0}^{(S)} + \Delta_{T,2}^{(S)} + \Delta_{P,2}^{(S)}$$

# More on temperature anisotropies

The formal solution obtained by line-of-sight integration

$$\Delta_T^{(S)}(q,\mu,t_0) = \int_{t_1}^{t_0} dt\ \exp\left[-iq\mu\int_t^{t_0}\frac{dt'}{a(t')} - \int_t^{t_0} dt'\omega_c(t')\right]$$
$$\times\left\{\omega_c\left[\Delta_{T,0}^{(S)} - \frac{1}{2}P_2(\mu)\Pi(q,t) - 2a^2(t)\ddot{B}_q(t) - 2a(t)\dot{a}(t)\dot{B}_q(t)\right.\right.$$
$$\left. + 4i\mu q\left(\delta u_q(t)/a(t) + a(t)\dot{B}_q(t)/2\right)\right]$$
$$\left. - \frac{d}{dt}\left(2A_q(t) + 2a^2(t)\ddot{B}_q(t) + 2a(t)\dot{a}(t)\dot{B}_q(t)\right)\right\}$$

shows that the temperature perturbations consist of two contributions

$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)} = \left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)}_{LSS} + \left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)}_{ISW}$$

# More on temperature anisotropies

$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)}_{LSS} = \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})$$

$$\times \int_{t_1}^{t_0} dt\ \exp\left[-iq\mu \int_t^{t_0} \frac{dt'}{a(t')}\right] \exp\left[-\int_t^{t_0} dt'\omega_c(t')\right] \omega_c(t)$$

$$\times \left[\frac{1}{4}\Delta^{(S)}_{T,0}(q,t) - \frac{1}{8}P_2(\mu)\Pi(q,t) - \frac{1}{2}a^2(t)\ddot{B}_q(t) - \frac{1}{2}a(t)\dot{a}(t)\dot{B}_q(t)\right.$$

$$\left. + i\mu q\left(\delta u_q(t)/a(t) + a(t)\dot{B}_q(t)/2\right)\right]$$

# More on temperature anisotropies

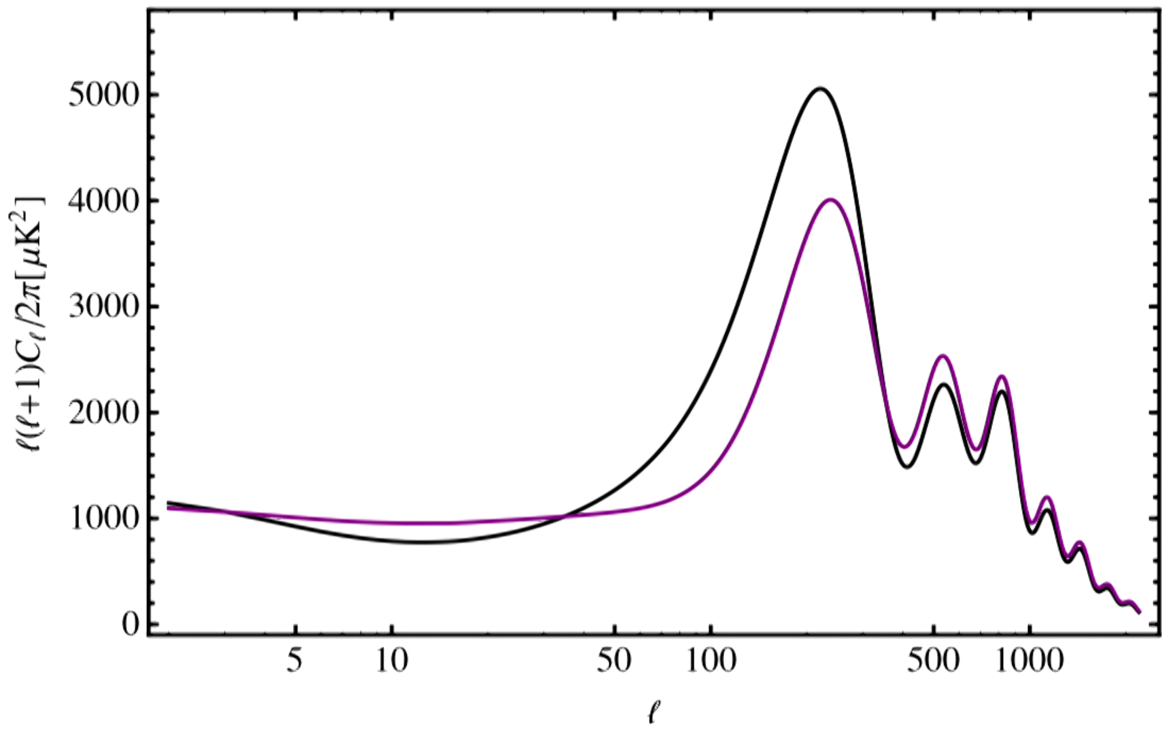
$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)}_{LSS} = \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})$$

Last scattering probability

$$\times \int_{t_1}^{t_0} dt \;\exp\left[-iq\mu \int_t^{t_0} \frac{dt'}{a(t')}\right] \exp\left[-\int_t^{t_0} dt' \omega_c(t')\right] \omega_c(t)$$

$$\times \left[\frac{1}{4}\Delta^{(S)}_{T,0}(q,t) - \frac{1}{8}P_2(\mu)\Pi(q,t) - \frac{1}{2}a^2(t)\ddot{B}_q(t) - \frac{1}{2}a(t)\dot{a}(t)\dot{B}_q(t) + i\mu q\left(\delta u_q(t)/a(t) + a(t)\dot{B}_q(t)/2\right)\right]$$

# More on temperature anisotropies



# More on temperature anisotropies

$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)}_{LSS} = \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})$$

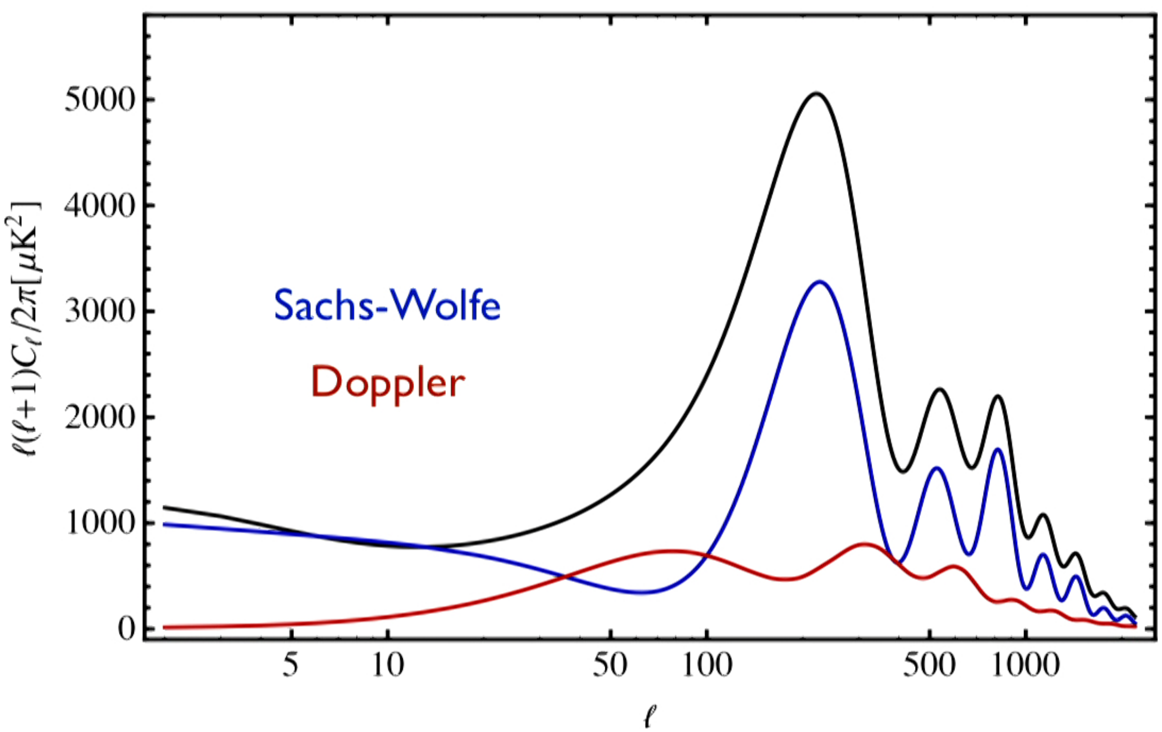
$$\times \int_{t_1}^{t_0} dt\ \exp\left[-iq\mu\int_t^{t_0}\frac{dt'}{a(t')}\right]\exp\left[-\int_t^{t_0}dt'\omega_c(t')\right]\omega_c(t)$$

$$\times \left[\frac{1}{4}\Delta^{(S)}_{T,0}(q,t) - \frac{1}{8}P_2(\mu)\Pi(q,t) - \frac{1}{2}a^2(t)\ddot{B}_q(t) - \frac{1}{2}a(t)\dot{a}(t)\dot{B}_q(t)\right.$$

$$\left. + i\mu q\left(\delta u_q(t)/a(t) + a(t)\dot{B}_q(t)/2\right)\right]$$

Intrinsic density fluctuation and
gravitational redshifting

# More on temperature anisotropies



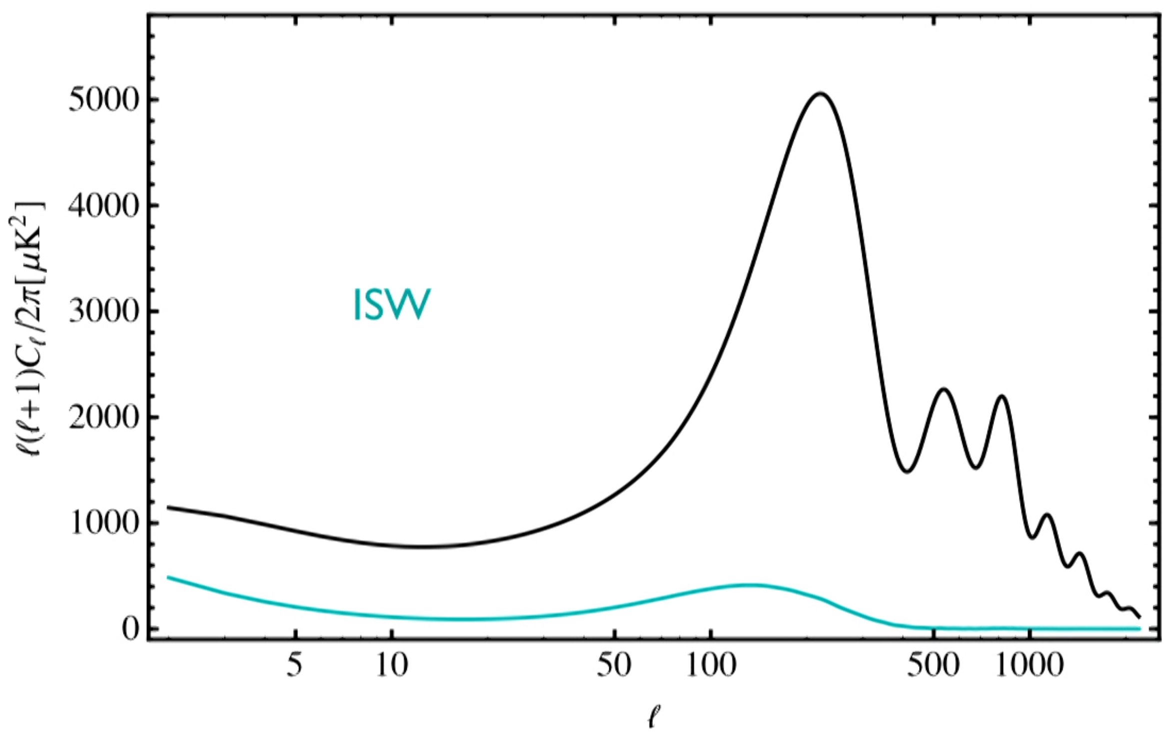
# More on temperature anisotropies

Integrated Sachs-Wolfe effect

$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)}_{ISW} = -\frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})$$
$$\times \int_{t_1}^{t_0} dt\ \exp\left[-iq\mu \int_t^{t_0} \frac{dt'}{a(t')}\right] \exp\left[-\int_t^{t_0} dt' \omega_c(t')\right]$$
$$\times \frac{d}{dt}\Big(A_q(t) + a^2(t)\ddot{B}_q(t) + a(t)\dot{a}(t)\dot{B}_q(t)\Big)$$

This contribution can be generated even in the absence of free electrons.

# More on temperature anisotropies



# More on temperature anisotropies

During matter domination the gravitational potential does not evolve

$$\frac{d}{dt}\left(A_q(t) + a^2(t)\ddot{B}_q(t) + a(t)\dot{a}(t)\dot{B}_q(t)\right) = 0$$

The integrated Sachs-Wolfe effect has two contributions

early contribution:

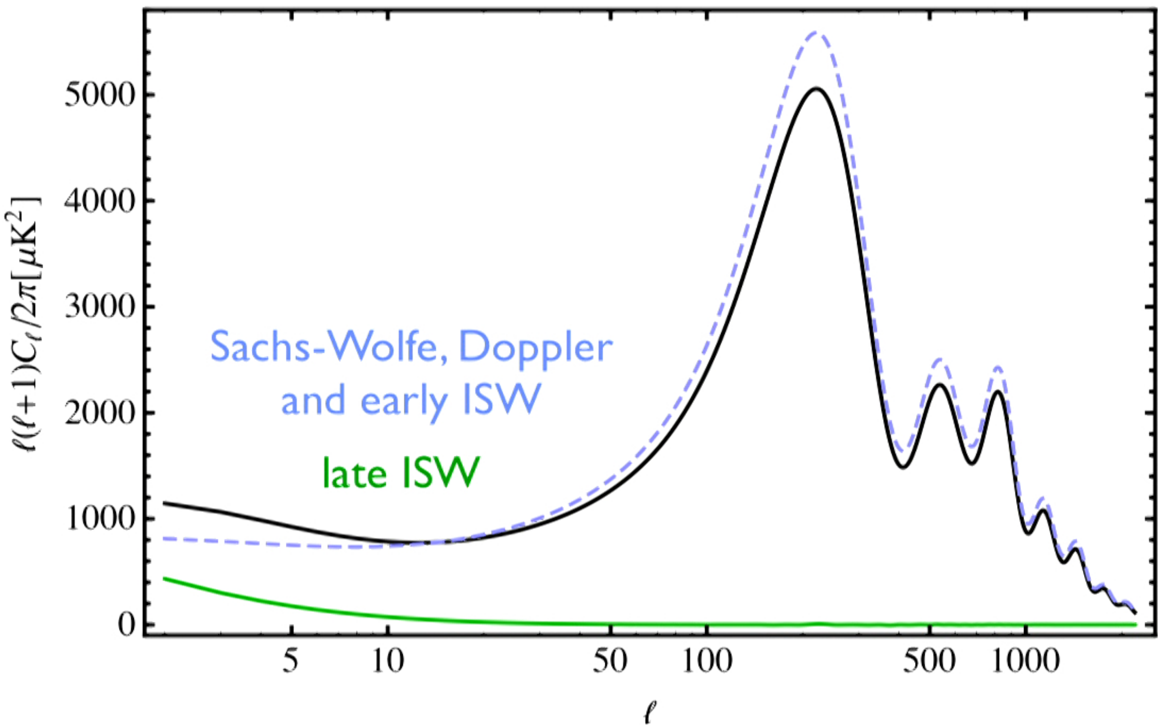
During recombination radiation is not yet completely negligible.

late contribution:

At late times dark energy becomes important

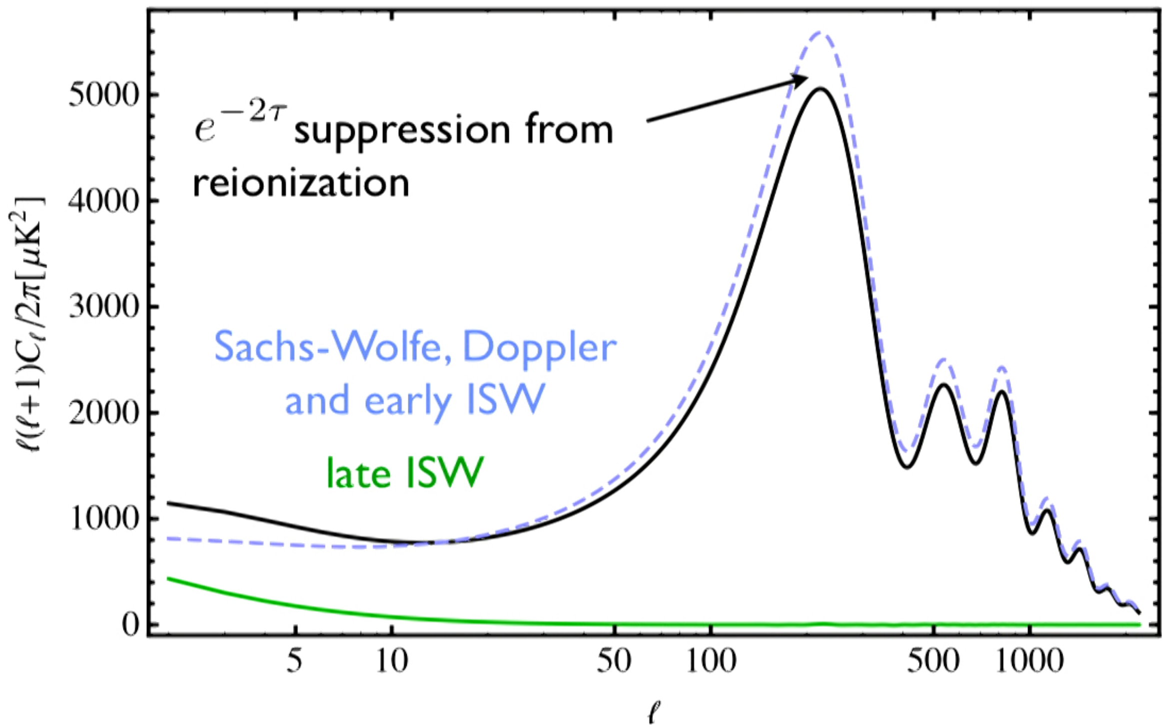
# More on temperature anisotropies

Recombination vs late time contributions



# More on temperature anisotropies

Recombination vs late time contributions



# More on temperature anisotropies

Much of this can be understood analytically. Let us focus on the dominant Sachs-Wolfe and Doppler contributions

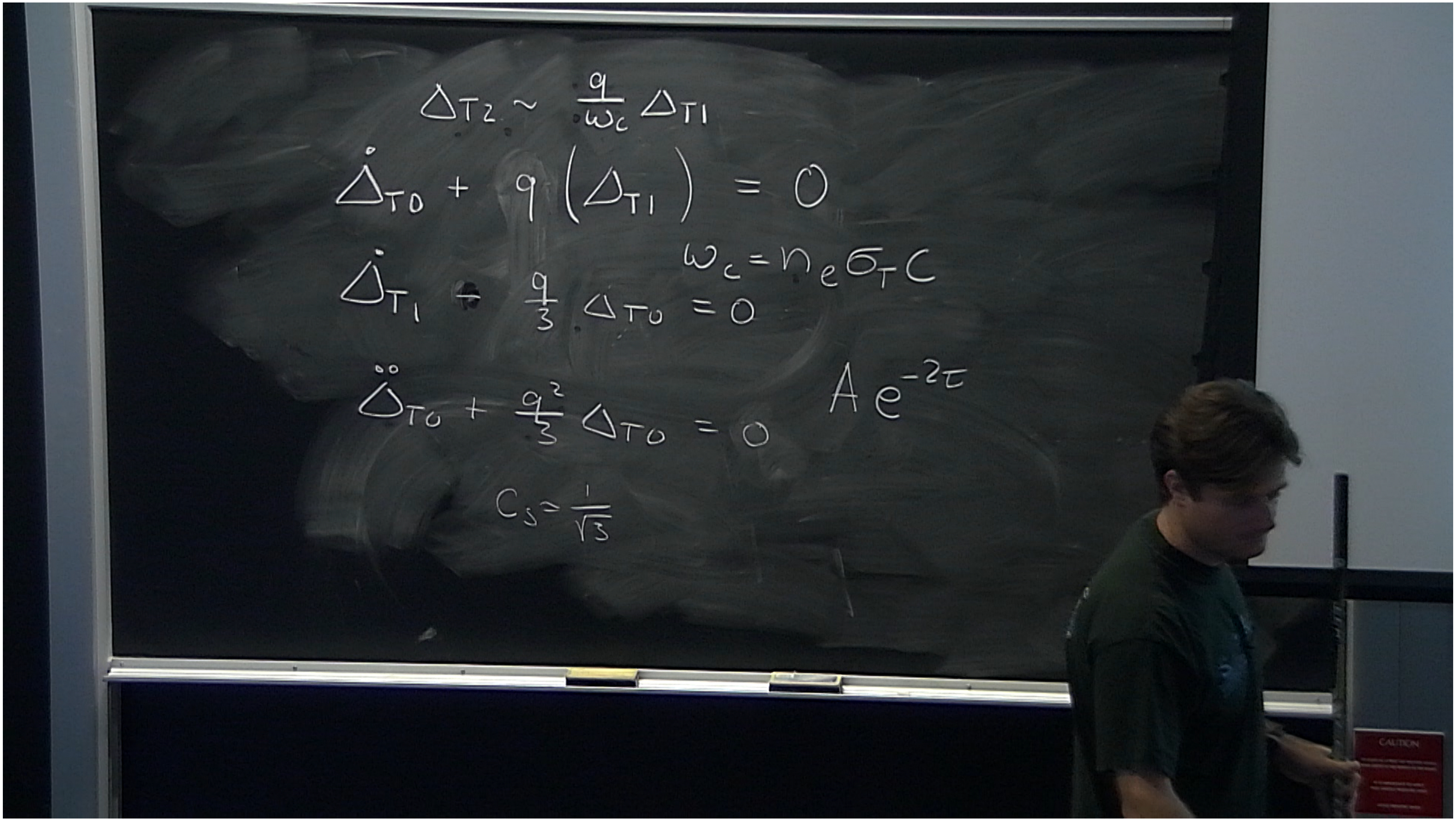
$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)}_{LSS} = \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})$$

$$\times \int_{t_1}^{t_0} dt\ \exp\left[-iq\mu\int_t^{t_0}\frac{dt'}{a(t')}\right]\exp\left[-\int_t^{t_0}dt'\omega_c(t')\right]\omega_c(t)$$

$$\times\left[\frac{1}{4}\Delta^{(S)}_{T,0}(q,t) - \frac{1}{8}P_2(\mu)\Pi(q,t) - \frac{1}{2}a^2(t)\ddot{B}_q(t) - \frac{1}{2}a(t)\dot{a}(t)\dot{B}_q(t)\right.$$

$$\left. + i\mu q\left(\delta u_q(t)/a(t) + a(t)\dot{B}_q(t)/2\right)\right]$$

and as a first approximation set $P(t) \approx \delta(t - t_L)$.


$\Delta_{T2} \sim \frac{q}{\omega_c}\Delta_{T1}$
$\dot{\Delta}_{T0} + q\left(\Delta_{T1}\right) = 0$
$\omega_c = n_e \sigma_T c$
$\dot{\Delta}_{T1} - \frac{q}{3}\Delta_{T0} = 0$
$\ddot{\Delta}_{T0} + \frac{q^2}{3}\Delta_{T0} = 0$
$Ae^{-2\tau}$
$c_s = \frac{1}{\sqrt{3}}$

# More on temperature anisotropies

After neglecting contributions from polarization and anisotropic stress

$$\left(\frac{\Delta T(\hat{n})}{T_0}\right)^{(S)}_{LSS} = \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})e^{i\vec{q}\cdot\hat{n}r_L}$$

$$\times \left[\frac{1}{4}\Delta^{(S)}_{T,0}(q,t_L) - \frac{1}{2}a^2(t_L)\ddot{B}_q(t_L) - \frac{1}{2}a(t_L)\dot{a}(t_L)\dot{B}_q(t_L)\right.$$

$$\left. + i\mu q\left(\delta u_q(t_L)/a(t_L) + a(t_L)\dot{B}_q(t_L)/2\right)\right]$$

# More on temperature anisotropies

The multipole coefficients are

$$a^{(S)}_{T,\ell\, m} = 4\pi i^{\ell} \int \frac{d^3q}{(2\pi)^3} \alpha(\vec{q}) Y^*_{\ell\, m}(\hat{q})$$
$$\times \left[ \left( \frac{1}{4} \Delta^{(S)}_{T,0}(q,t_L) - \frac{1}{2} a^2(t_L)\ddot{B}_q(t_L) - \frac{1}{2} a(t_L)\dot{a}(t_L)\dot{B}_q(t_L)) \right) j_\ell(qr_L) \right.$$
$$\left. + iq \left( \delta u_q(t_L)/a(t_L) + a(t_L)\dot{B}_q(t_L)/2 \right) j'_\ell(qr_L) \right]$$

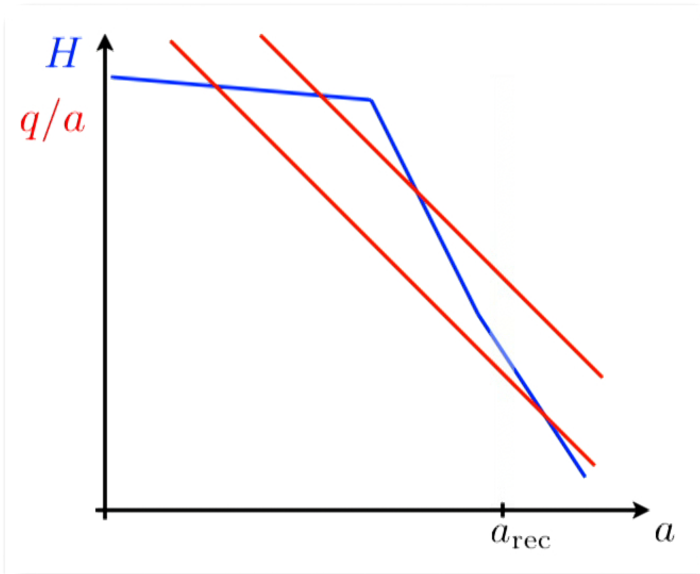
The behavior of the spherical Bessel functions for $\ell \gg 1$ implies that the dominant contributions arises from wave numbers

$$qr_L \approx \ell$$

# More on temperature anisotropies

For the adiabatic solution, modes are frozen outside the horizon. So the behavior of modes will be very different for

$$\frac{q}{a_L H_L} < 1 \quad \text{or} \quad \frac{q}{a_L H_L} > 1$$



# More on temperature anisotropies

Where does the transition happen?

$$\frac{q}{a_L H_L} = \frac{\ell}{a_L r_L H_L} \approx \frac{\ell}{60}$$

$\ell < 60$ contribution predominantly from modes still frozen during recombination

$\ell > 60$ contribution predominantly from modes inside the horizon during recombination

# More on temperature anisotropies

For the frozen long modes we can write the multipole coefficients in terms of the curvature perturbation

$$a^{(S)}_{T,\ell m} \approx 4\pi i^{\ell} \int \frac{d^3q}{(2\pi)^3} \mathcal{R}(\vec{q}) Y^*_{\ell m}(\hat{q}) \left[ -\frac{1}{5} j_{\ell}(q r_L) \right]$$

and for a scale-invariant* primordial power spectrum

$$\frac{\ell(\ell+1)C_\ell}{2\pi} = \frac{T_0^2}{25} \Delta^2_{\mathcal{R}}$$

This is sometimes referred to as the Sachs-Wolfe plateau

(*) it can also be evaluated for the LCDM power law spectrum

# More on temperature anisotropies

At leading order, the Boltzmann hierarchy reduces to the hydrodynamics, and the solutions are sound waves.
The Sachs-Wolfe contribution takes the form

$$a_{T,\ell m}^{(S)} = 4\pi i^{\ell} \int \frac{d^3q}{(2\pi)^3} \mathcal{R}(\vec{q}) Y_{\ell m}^{*}(\hat{q})$$

$$\times \left[ \frac{3}{5} \mathcal{T}(q) R_L - \frac{1}{(1+R_L)^{1/4}} \cos(q r_s) \right] j_\ell(q r_L)$$

with

$$R = \frac{3}{4} \frac{\rho_b}{\rho_\gamma}$$ baryon loading

$$r_s = \int_0^{t_L} \frac{dt}{a(t)\sqrt{3(1+R(t))}}$$ (comoving) sound horizon

$\mathcal{T}(q)$ transfer function

# More on temperature anisotropies

There are two effects we have ignored in this approximation.

1. The solutions oscillate around last scattering and the finite width of the last scattering surface leads to damping.

2. The mean free path of the photons becomes comparable to the momentum of the modes for large $q$ which leads to Silk damping.

$$a^{(S)}_{T,\ell\, m} = 4\pi i^{\ell} \int \frac{d^3 q}{(2\pi)^3} \mathcal{R}(\vec{q}) Y^{*}_{\ell\, m}(\hat{q})$$

$$\times \left[ \frac{3}{5} \mathcal{T}(q) R_L - \frac{e^{-\int_0^{t_L} \Gamma(q,t) dt}}{(1+R_L)^{1/4}} \cos(q r_s) \right] j_\ell(q r_L)$$

# More on temperature anisotropies

Including the Doppler contribution

$$a^{(S)}_{T,\ell m} = 4\pi i^\ell \int \frac{d^3q}{(2\pi)^3} \mathcal{R}(\vec{q}) Y^*_{\ell m}(\hat{q})$$
$$\times \left\{ \left[ \frac{3}{5}\mathcal{T}(q) R_L - \frac{e^{-\int_0^{t_L}\Gamma(q,t)dt}}{(1+R_L)^{1/4}} \cos(qr_s) \right] j_\ell(qr_L) \right.$$
$$\left. - \left[ \frac{\sqrt{3}e^{-\int_0^{t_L}\Gamma(q,t)dt}}{(1+R_L)^{3/4}} \sin(qr_s) \right] j'_\ell(qr_L) \right\}$$

- Since the integral is dominated by $q \approx \ell/r_L$, the peak positions are set by $\theta = r_s/r_L$, which e.g. probes curvature.

- Since $R \propto \Omega_b$ the relative height of the peaks is a sensitive probe of the baryon abundance.

- The damping scale probes the mean free path of the photons and thus, for example, the Helium abundance.

# From eV to Inflation

$$C^{(S)}_{XX,\ell} = 4\pi T_0^2 \int \frac{dk}{k} \Delta^2_{\mathcal{R}}(k) \left| \int_0^{\tau_0} d\tau S^{(S)}_X(k,\tau) j_\ell(k(\tau_0 - \tau)) \right|^2$$

# More on temperature anisotropies

Including the Doppler contribution

$$a^{(S)}_{T,\ell m} = 4\pi i^\ell \int \frac{d^3q}{(2\pi)^3} \mathcal{R}(\vec{q}) Y^*_{\ell m}(\hat{q})$$
$$\times \left\{ \left[ \frac{3}{5}\mathcal{T}(q) R_L - \frac{e^{-\int_0^{t_L}\Gamma(q,t)dt}}{(1+R_L)^{1/4}} \cos(qr_s) \right] j_\ell(qr_L) \right.$$
$$\left. - \left[ \frac{\sqrt{3} e^{-\int_0^{t_L}\Gamma(q,t)dt}}{(1+R_L)^{3/4}} \sin(qr_s) \right] j'_\ell(qr_L) \right\}$$

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# From eV to Inflation

Initial Conditions

Late time evolution

$$C^{(S)}_{XX,\ell} = 4\pi T_0^2 \int \frac{dk}{k}\Delta^2_{\mathcal{R}}(k) \left| \int_0^{\tau_0} d\tau\, S^{(S)}_X(k,\tau)\, j_\ell(k(\tau_0-\tau)) \right|^2$$

Physics of Recombination

Geometry

# From eV to Inflation

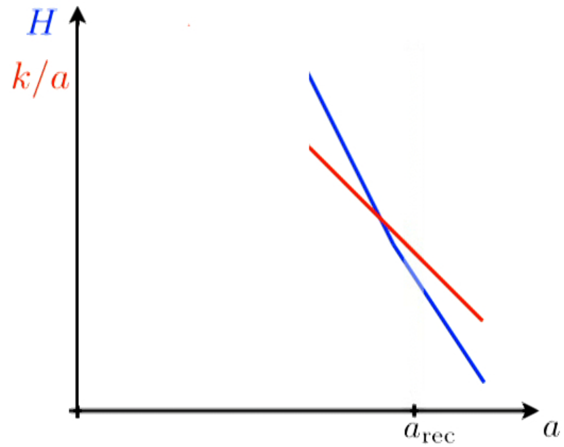
So far, these are initial conditions for the system of equations that governs the evolution of the universe from around few keV to the present

In this limit, the system has 5 solutions that do not decay, one "adiabatic" solution and 4 "isocurvature" solutions. (Bucher, Moodley, Turok 1999)

Experimentally, only the adiabatic solution seems excited for which $\mathcal{R}$ is constant.

# From eV to Inflation

What generated these perturbations?




To generate the perturbations causally, they cannot have been outside the horizon very early on, requiring a phase with

$$\frac{d}{dt}\left(\frac{k}{a|H|}\right) < 0 \qquad \text{(inflation or bounce)}$$

# Inflation

Inflation, a phase of nearly exponential expansion, was proposed as a solution to the horizon, flatness, monopole problem.

Horizon problem

$$d_h = a_L \int_{t_i}^{t_\mathrm{r}} \frac{dt}{a_i \exp(H(t-t_i))} + a_L \int_{t_\mathrm{r}}^{t_L} \frac{dt}{a(t)}$$

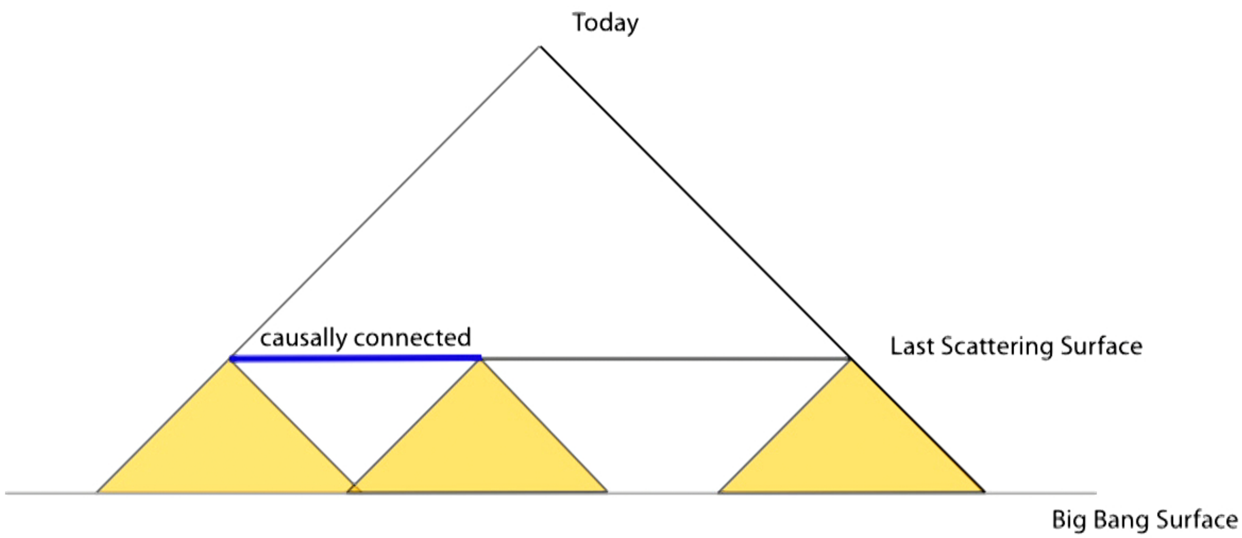
This becomes

$$d_h = \frac{a_L}{a_0}\frac{a_0}{a_\mathrm{r}}\frac{e^{\mathcal{N}}}{H} + d_h^{\mathrm{BB}} \approx \frac{1+z_\mathrm{r}}{1+z_L}\frac{e^{\mathcal{N}}}{H}$$

and $d_h > d_A$ for sufficiently large $\mathcal{N} = H(t_\mathrm{r} - t_i)$ .

# Inflation

Inflation and the horizon problem



# Inflation

We know that vacuum energy gives rise to exponential expansion, but we need this period to end. So we need a clock

$$S = \int \sqrt{-g} d^4x \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

In FLRW

$$S = \int dt d^3x \, a^3(t) \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2a^2} (\nabla \phi)^2 - V(\phi) \right]$$

with equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + V'(\phi) = 0$$

# Inflation

For a homogeneous field in and FLRW spacetime the equation of motion becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

For the field equations we also need the energy density and pressure

$$\rho = \dot{\phi}^2 + V(\phi)$$

$$p = \dot{\phi}^2 - V(\phi)$$

So the equations of motion can be taken as

$$H^2 = \frac{8\pi G}{3}\rho$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

# Inflation

Recall that we are interested in nearly exponential expansion or nearly constant $H$

$$\left|\frac{\dot{H}}{H^2}\right| \ll 1$$

With

$$\dot{H} = -4\pi G\dot{\phi}^2$$

This becomes

$$\frac{\dot{\phi}^2}{\dot{\phi}^2 + V(\phi)} \ll 1$$

or

$$\dot{\phi}^2 \ll V(\phi)$$

In this case $p \approx -\rho$ as desired.

# Inflation

For a homogeneous field in and FLRW spacetime the equation of motion becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

For the field equations we also need the energy density and pressure

$$\rho = \dot{\phi}^2 + V(\phi)$$

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So the equations of motion can be taken as

$$H^2 = \frac{8\pi G}{3}\rho$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

# Inflation

If we want this to be an extended period, we also want

$$\left|\frac{\ddot{H}}{2\dot{H}H}\right| = \left|\frac{\ddot{\phi}}{\dot{\phi}H}\right| \ll 1$$

The equations of motion are then

$$H^2 = \frac{8\pi G}{3}V(\phi)$$

$$3H\dot{\phi} + V'(\phi) = 0$$

This is referred to as single-field slow-roll inflation.

# Inflation

Slow roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2} \qquad \delta = \frac{\ddot{H}}{2\dot{H}H}$$

It is also convenient to introduce

$$\epsilon_V = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$$
$$\eta_V = \frac{1}{8\pi G}\frac{V''}{V}$$

In the slow-roll approximation

$$\epsilon \approx \epsilon_V \qquad \delta \approx -\eta_V - \epsilon_V$$

# Inflation

The inflaton is a quantum field and fluctuates

The claim is that the quantum fluctuations in this field are the source of primordial perturbations

$$\mathcal{R}_q = -H\frac{\delta\phi_q}{\dot{\phi}}$$

To compute the spectrum, we canonically quantize

The quadratic action for the fluctuations schematically is

$$S = \int dt d^3x\, a^3(t)\left[\frac{1}{2}\delta\dot{\phi}^2 - \frac{1}{2a^2}(\nabla\delta\phi)^2 - \frac{1}{2}V''(\phi)\delta\phi^2\right]$$

$$\Delta_{T2} \sim \frac{q}{\omega_c}\Delta_{T1}$$

$$\dot{\Delta}_{T0} + q\left(\Delta_{T1}\right) = 0$$

$$\omega_c = n_e \sigma_T c$$

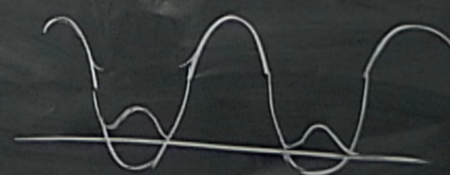
$$\dot{\Delta}_{T1} - \frac{q}{3}\Delta_{T0} = 0$$

$$A e^{-2\tau}$$

$$\ddot{\Delta}_{T0} + \frac{q^2}{3}\Delta_{T0} = 0$$

$$c_s = \frac{1}{\sqrt{3}}$$

$$R = \delta u$$



# Inflation

As usual, we expand the field in creation annihilation operators

$$\delta\phi(t,\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3}\ \left[\delta\phi_q(t) e^{i\mathbf{q}\cdot\mathbf{x}} a(\mathbf{q}) + h.c.\right]$$

so that the mode functions obey

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \frac{q^2}{a^2}\delta\phi \approx 0$$

Oscillatory at early times, constant at late times.

# Inflation

For

$$\left[a(\mathbf{q}), a^\dagger(\mathbf{q}')\right] = (2\pi)^3\delta(\mathbf{q}-\mathbf{q}')$$

the field then obeys canonical commutation relations if at early times the positive frequency modes approach

$$\delta\phi_q(t) \to \frac{1}{a(t)\sqrt{2q}} \exp\left[-iq\int_{t_*}^{t}\frac{dt'}{a(t')}\right]$$

and it is typically assumed that the modes are in the Bunch-Davies state

$$a(\mathbf{q})|0\rangle = 0$$

# Inflation

Ignoring slow-roll corrections, we can give the mode function in terms of elementary functions

$$\delta\phi_q(t) = \frac{H}{\sqrt{2q}}\left(\frac{i}{q} + \frac{1}{aH}\right)e^{iq/aH}$$

This has the correct limit at early times, and approaches

$$\delta\phi_q(t) \to \frac{iH}{\sqrt{2}q^{3/2}}$$

at late times so

$$|\mathcal{R}_q|^2 \to \frac{H^2}{\dot{\phi}^2}\frac{H^2}{2q^3}$$

# Inflation

This formula remains correct if we keep the slow-roll corrections provided we evaluate it at horizon crossing

$$|\mathcal{R}_q|^2 \to \left. \frac{H^2}{\dot{\phi}^2} \frac{H^2}{2q^3} \right|_{q=aH}$$

Frequently one uses $\Delta^2_{\mathcal{R}}(q)$ defined by

$$|\mathcal{R}_q|^2 = 2\pi^2 \frac{\Delta^2_{\mathcal{R}}(q)}{q^3}$$

Explicitly in terms of slow-roll parameters

$$\Delta^2_{\mathcal{R}}(q) = \Delta^2_{\mathcal{R}}(q_*) \left( \frac{q}{q_*} \right)^{n_s - 1}$$

with

$$n_s = 1 - 4\epsilon_* - 2\delta_*$$

# B-mode search

Just like the inflaton, the graviton fluctuates. The corresponding power spectrum is

$$\Delta_h^2(k) = \frac{2H^2(t_k)}{\pi^2}$$

A measurement of the tensor contribution would provide a direct measurement of the expansion rate of the universe during inflation, as well as the energy scale

$$V_{\rm inf}^{1/4} = 1.06 \times 10^{16}\, GeV \left(\frac{r}{0.01}\right)^{1/4}$$

with $\quad r = \dfrac{\Delta_h^2}{\Delta_{\mathcal{R}}^2}$

# B-mode search

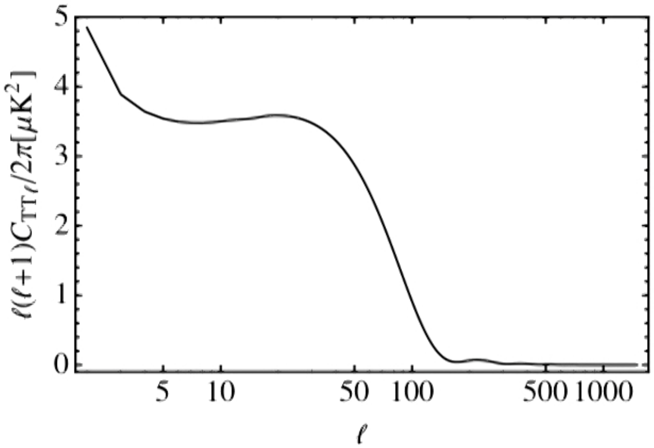
In addition to the density perturbations, inflation also predicts a nearly scale invariant spectrum of gravitational waves

$$\dot{\tilde{\Delta}}^{(T)}_{T,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\tilde{\Delta}^{(T)}_{T,\ell+1}(q,t) - \ell\tilde{\Delta}^{(T)}_{T,\ell-1}(q,t)\right]$$
$$= \left(-2\dot{\mathcal{D}}_q(t) + \omega_c(t)\Psi(q,t)\right)\,\delta_{\ell,0} - \omega_c(t)\tilde{\Delta}^{(T)}_{T,\ell}(q,t)$$
$$\dot{\tilde{\Delta}}^{(T)}_{P,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\tilde{\Delta}^{(T)}_{P,\ell+1}(q,t) - \ell\tilde{\Delta}^{(T)}_{P,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Psi(q,t)\,\delta_{\ell,0} - \omega_c(t)\tilde{\Delta}^{(T)}_{P,\ell}(q,t)$$

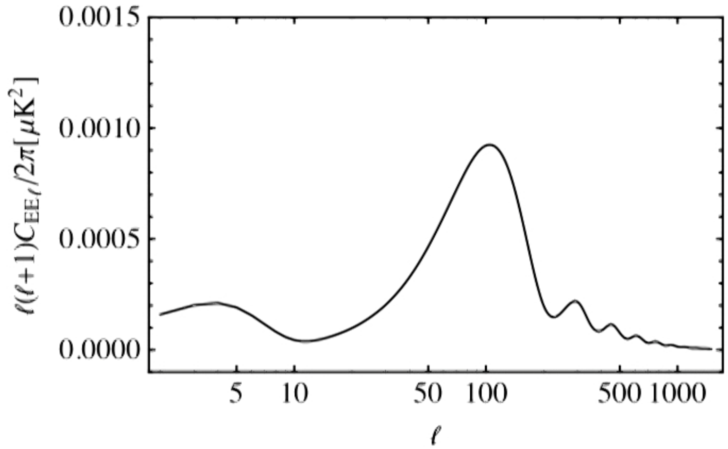
with

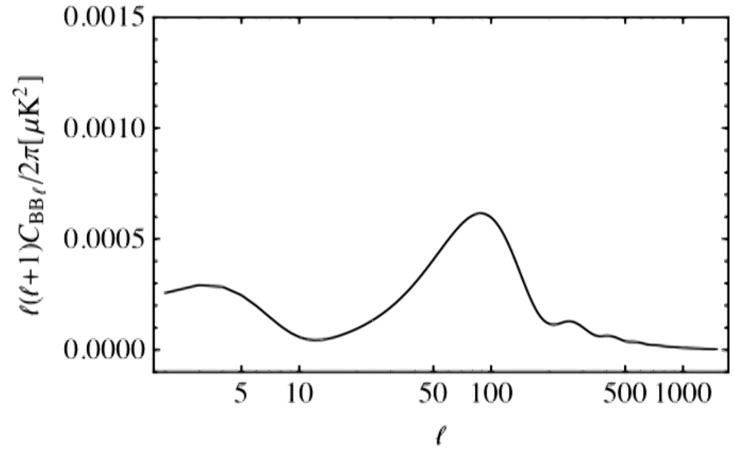
$$\Psi(q,t) = \frac{1}{10}\tilde{\Delta}^{(T)}_{T,0}(q,t) + \frac{1}{7}\tilde{\Delta}^{(T)}_{T,2}(q,t) + \frac{3}{70}\tilde{\Delta}^{(T)}_{T,4}(q,t)$$
$$-\frac{3}{5}\tilde{\Delta}^{(T)}_{P,0}(q,t) + \frac{6}{7}\tilde{\Delta}^{(T)}_{P,2}(q,t) - \frac{3}{70}\tilde{\Delta}^{(T)}_{P,4}(q,t)$$

# B-mode search

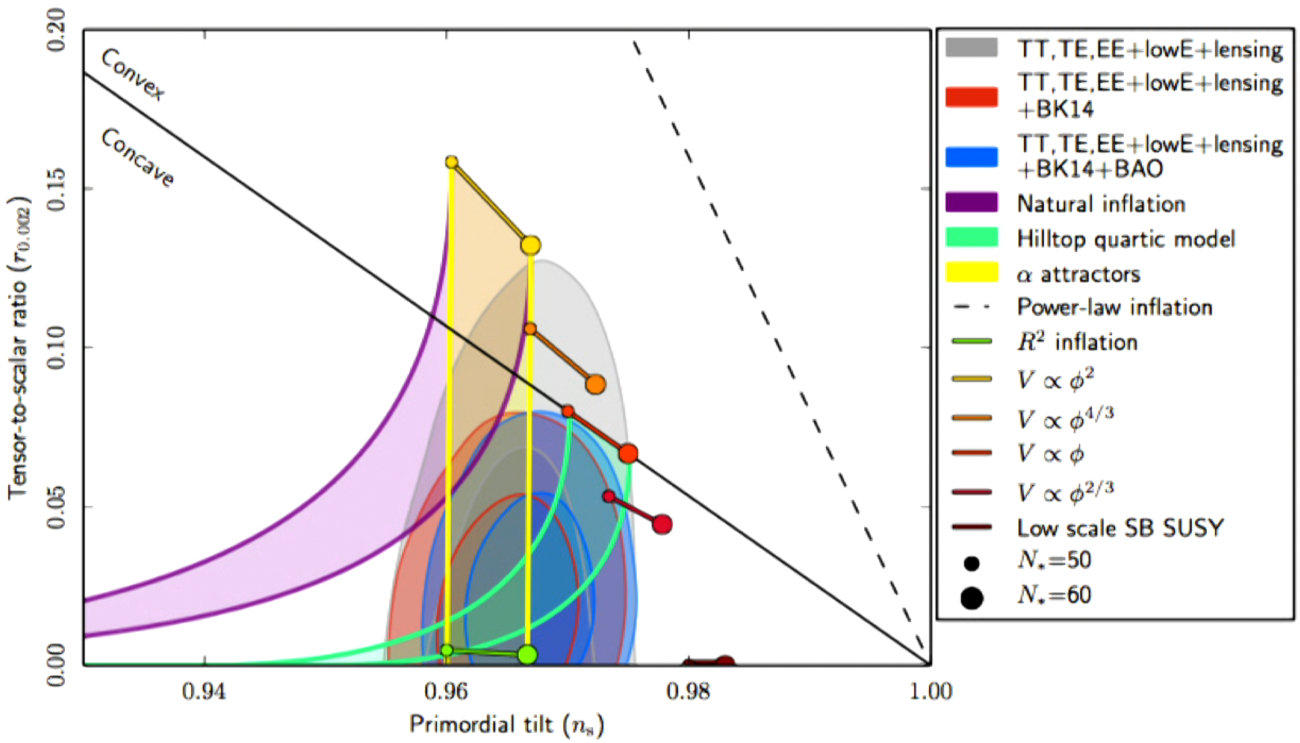



In addition to TT, TE, EE, primordial gravitational waves generate BB



# B-mode search




$r < 0.064$ at 95% CL

# B-mode search

Stage III: now-2020

# B-mode search

## Stage III.5: soon-2020

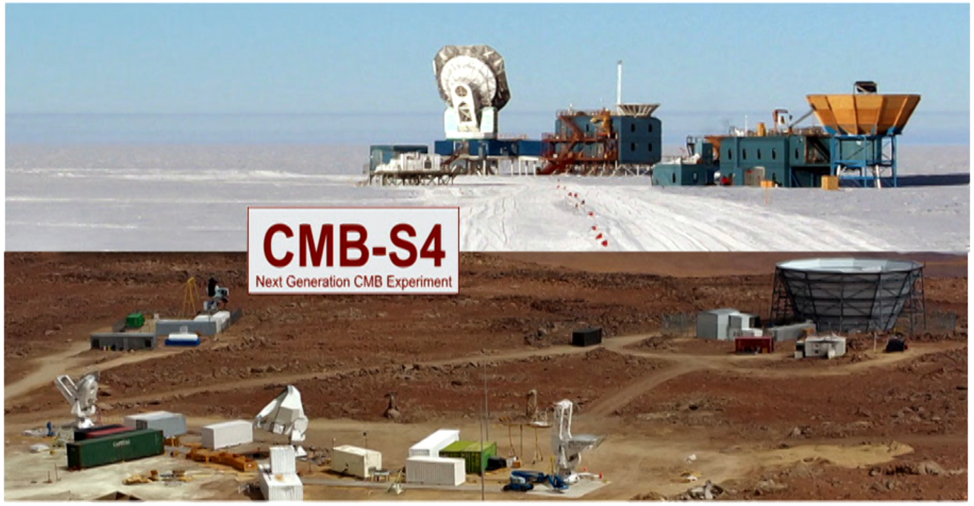
http://simonsobservatory.org

- A five year, $45M+ program to pursue key Cosmic Microwave Background science targets, and advance technology and infrastructure in preparation for CMB-S4.
- Merger of the ACT and POLARBEAR/Simons Array teams.
- Tentative plans include:
  - Major site infrastructure
  - Technology development (detectors, optics, cameras)
  - Demonstration of new high throughput telescopes.
  - CMB-S4 class receivers with partially filled focal planes.
  - Data analysis


BERKELEY LAB · CONICYT · HEISING-SIMONS FOUNDATION · NSF · Penn · PRINCETON UNIVERSITY · SIMONS FOUNDATION

# B-mode search

Stage IV: 2020-2030



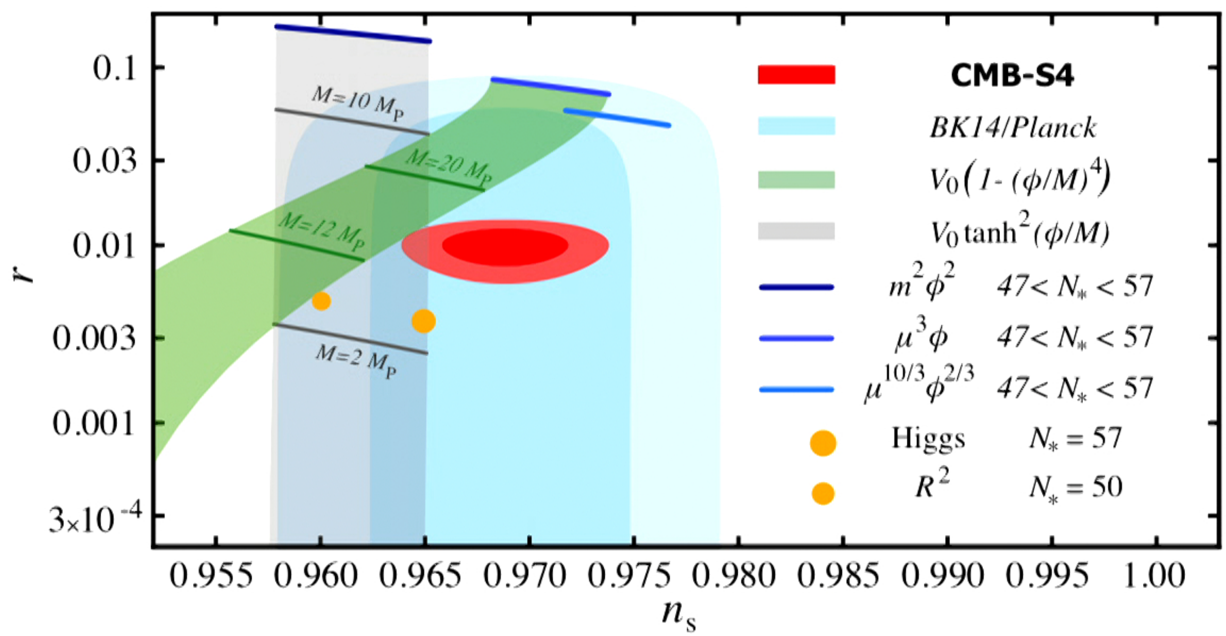

Potentially Space Missions

LiteBIRD, PIXIE

# B-mode search

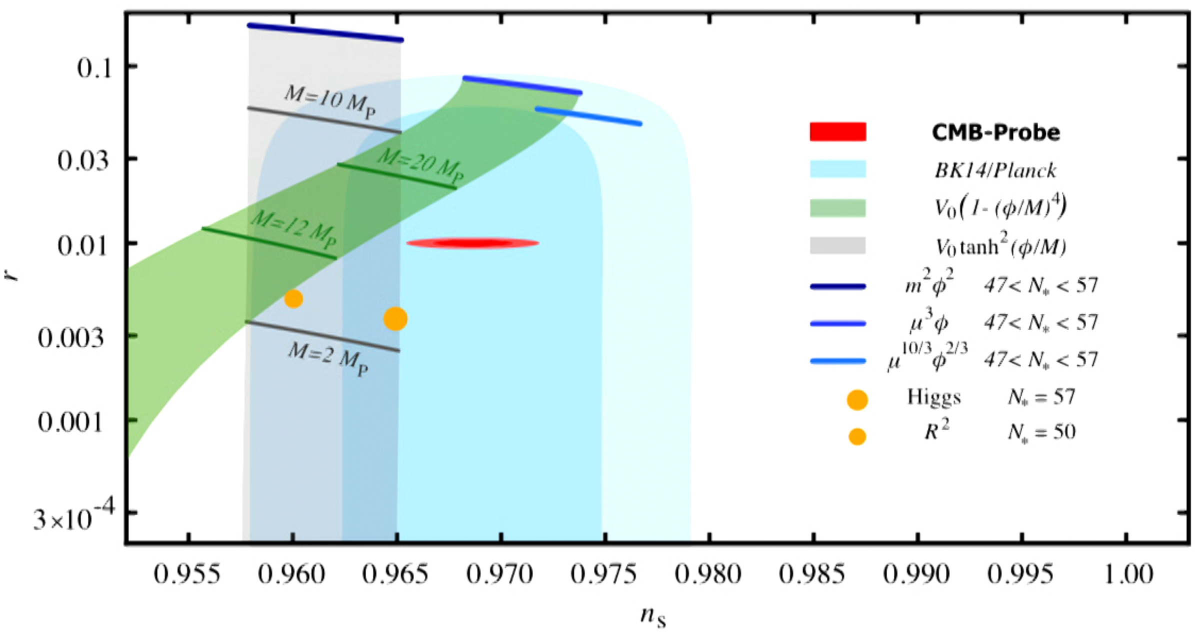
CMB-S4 would detect r=0.01 at high significance

CMB-S4 Science Book (http://www.cmbs4.org)



# B-mode search

Potential of a future space mission

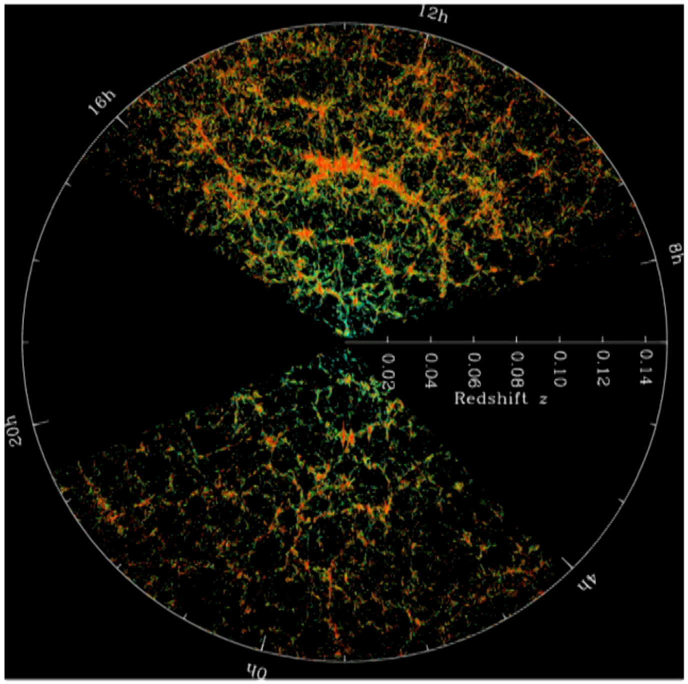


# Beyond B-modes

- The CMB provides a unique opportunity to study the physics of the universe
    - at the time when Hydrogen forms
    - through lensing at much later times
    - and at much earlier times when the perturbations were generated
- Polarization measurements on small scales can provide tight constraints on light relics, neutrinos, ...

# Beyond the CMB

- Galaxy redshift surveys similarly provide a wealth of information about our universe



# Beyond the CMB

- Some of the biggest questions remain
  - How was the baryon asymmetry generated?
  - What is dark matter?
  - What is dark energy?
  - How were the primordial fluctuations generated?
- Many new experiments are beginning to take data and will perhaps shed more light

Thank you