Title: Boundary contributions to (holographic) entanglement entropy

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Abstract: <p>The entanglement entropy, while being under the spotlight of theoretical physics for more than ten years now, remains very challenging to compute, even in free quantum field theories, and a number of issues are yet to be explored.</p>

<p>One such issues concerns boundary effects on entanglement entropy, which is important both for theoretical explorations of entanglement and for applications of entanglement entropy to lattice simulations, condensed matter systems, etc.. During this talk, I will show how the presence of spacetime boundaries affects the entanglement entropy, with emphasize on universal (boundary-induced) logarithmic terms, using field theoretic, lattice, and holographic methods. </p>

Boundary contributions to (holographic) entanglement entropy

Talk at Perimeter Institute

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CNRS – Université de Tours

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Motivations

Boundary effects as a common thread

- In condensed matter systems (quantum impurity models, etc.)
- In gauge/gravity (conformal boundary of AdS)
- In entanglement entropy (entangling surface)

Boundaries and entanglement entropy?

- EE is sensible to boundary effects (info. about BQFT)
- In odd d, universal log terms are pure boundary effects
- Most physical systems are confined to some boundaries
- Why not?!

Focus of the talk: the entangling surface **intersects** the boundary

Density operator formalism

For a **pure state** $|\Psi\rangle \in \mathcal{H}$ of a quantum system, the **density matrix** is

$$\rho = |\Psi\rangle\langle\Psi|, \qquad \mathrm{Tr}\,\rho = 1,$$

such that $\langle\mathcal{O}\rangle = \mathrm{Tr}(\mathcal{O}\rho)$.

Divide the quantum system into two **complementary** parts A and B.
Assume that the Hilbert space $\mathcal{H}$ associated to $|\psi\rangle \in \mathcal{H}$ factorizes

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

The **reduced density matrix** ρ_A for A is defined as

$$\rho_A = \mathrm{Tr}_B \rho$$

For a **pure state** $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, Schmidt decomposition implies that ρ_A and ρ_B have **same non-zero eigenvalues**.

Entanglement in quantum mechanics

Entangled state:

For a **bipartite** system in a **pure state** $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, if

$$\mathrm{Tr}_\sigma \rho_\sigma^2 = 1 \quad \Rightarrow \quad |\psi\rangle \text{ is separable, i.e. } |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$
$$\mathrm{Tr}_\sigma \rho_\sigma^2 < 1 \quad \Rightarrow \quad |\psi\rangle \text{ is \textbf{entangled}, i.e. } |\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

where $\sigma = \{A, B\}$.

A measure: the entanglement entropy

- **Von Neumann's** definition: $S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A)$
- Equivalent to the **replica formula**: $S_A = (1 - n\partial_n) \log \mathrm{Tr}_A(\rho_A^n)\Big|_{n=1}$

Some properties: $S_A \geq 0$, $S_A = S_B$ (pure state), SSA, etc.

Outline

Boundary contributions to entanglement entropy

- **QFT** results [CB,Solodukhin (2016)]
  - Heat kernel technology
  - Some explicit calculations
  - CFT vs BCFT: conformal anomaly and log term
- **Holographic** calculations [Astaneh,CB,Fursaev,Solodukhin (2017)]
  - AdS/BCFT
  - Logarithmic contribution
  - Free field $\mathcal{N} = 4$ supermultiplet

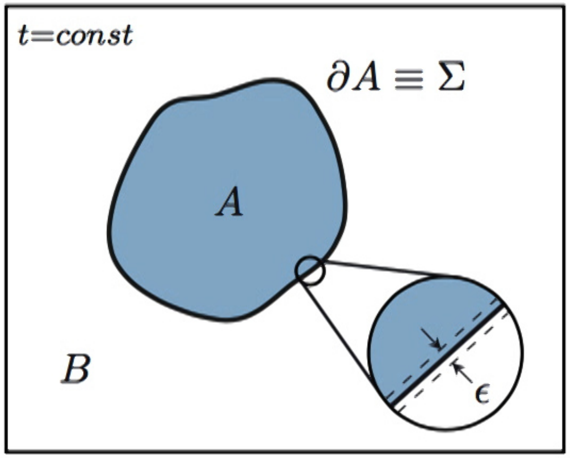
Summary, open questions and outlook

Entanglement entropy in QFT

- The density matrix ρ is evaluated on a **spatial hypersurface** (e.g. $t = 0$)
- An **entangling surface** Σ divides the space into two regions
- The entanglement entropy is identified with **von Neumann's entropy**:

$$S_{vN} = -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

- **Short-range correlations** across Σ $\to$ **UV divergences** $\to$ spatial cutoff ϵ



General structure of entanglement entropy:

$$S(\Sigma) = \gamma \frac{A(\Sigma)}{\epsilon^{d-2}} + \frac{s_{d-4}}{\epsilon^{d-4}} + \cdots + \begin{cases} s_{log} \ln \frac{\epsilon}{\mu} & d \text{ even} \\ s_0 & d \text{ odd} \end{cases}$$

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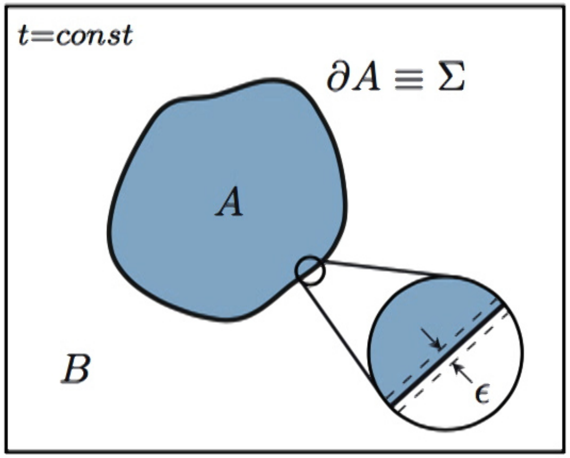


Compute the entanglement entropy via the replica trick:

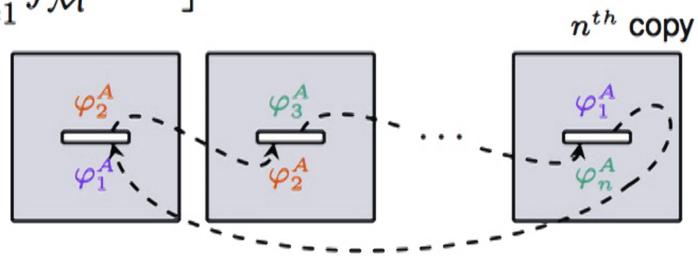
$$S(\Sigma) = (1 - n\partial_n) \ln \mathrm{Tr}_A(\rho_A^n)\Big|_{n=1}$$

Replica method

Take n copies of $\rho_A \to (\rho_A)^n$ and trace over d.o.f. in A:

$$\mathrm{Tr}_A(\rho_A^n) = \left[\prod_{i=1}^{n}\int_{\mathcal{M}} \mathcal{D}\varphi_i^A\right] \langle\varphi_1^A|\rho_A|\varphi_2^A\rangle\langle\varphi_2^A|\rho_A|\varphi_3^A\rangle\cdots\langle\varphi_n^A|\rho_A|\varphi_1^A\rangle$$

$$= \frac{1}{Z^n}$$



$$= \frac{1}{Z^n}\int_{\mathcal{M}_n} \mathcal{D}\varphi\, e^{-S_E(\varphi)} \equiv \frac{Z(n)}{Z^n}$$

- $\mathcal{M}_n$ is the n-fold branched cover over the Euclidean space $\mathcal{M}$
- $Z(n)$ is the partition function on $\mathcal{M}_n$

Entanglement entropy: $S(\Sigma) = (1 - n\partial_n)\ln Z(n)\Big|_{n=1}$

How to compute $Z(n)$?

At the **one-loop** level, the partition function $Z(n)$ can be expressed as a **functional determinant** of a Laplace-type operator $\Delta^{(j)}$ that describes the field theory:

$$\ln Z(n) = -\frac{(-1)^{2j}}{2} \ln \det \Delta^{(j)} = \frac{(-1)^{2j}}{2} \int_{\epsilon^2}^{\infty} \frac{ds}{s} \, \mathrm{Tr} K_n^{(j)}(s)$$

The **kernel** $K_n^{(j)}$ of the heat operator $e^{-s\Delta^{(j)}}$,

$$K_n^{(j)}(s, x, x') = \langle x' | e^{-s\Delta^{(j)}} | x \rangle \, , \qquad s > 0, \quad x, x' \in \mathcal{M}_n$$

is the solution of the **heat equation**

$$(\partial_s + \Delta^{(j)}) K_n^{(j)}(s, x, x') = 0$$

with the initial condition $K_n^{(j)}(s = 0, x, x') = \delta(x - x')/\sqrt{g}$.

Heat kernel technique

Relating $K_n(s)$ on $\mathcal{M}_n$ to $K(s)$ on $\mathcal{M}$:

- **Sommerfeld's formula:** Σ at $t = 0 = x,\ (t,x) \to (r, \phi = \phi + 2\pi n)$

$$K_n(s,r,r',\phi,\phi') = K(s,r,r',\phi,\phi') + \frac{i}{4\pi n}\int_\gamma dz\ \cot\frac{z}{2n}\,K(s,r,r',\phi+z,\phi')$$

- **Heat kernel expansion:**

$$\mathrm{Tr}K_n(s\to 0) \simeq \sum_{p=0} a_p(n)\, s^{(p-d)/2}$$

with $\quad a_{2k}(n) = n a_{2k} + (1-n)\, a^{\Sigma}_{2k} + \mathcal{O}((1-n)^2)$

- a^{Σ}_p depend on the **geometry** of the entangling surface Σ
- $a_p = 0$ for p **odd** on manifold **without boundary**

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  - a^{Σ}_p depend on the geometry of the entangling surface Σ
  - However, $a_p \neq 0$ **for p odd on manifold with boundary!**

An example: half space

Half space: a scalar field in flat space, Σ is a $(d-2)$–plan

- The corresponding heat kernel on $\mathbb{R}^d$ is

$$K(s, \mathbf{x}, \mathbf{x}') = \frac{1}{(4\pi s)^{d/2}} e^{-\frac{1}{4s}(\mathbf{x}-\mathbf{x}')^2}$$

- Sommerfeld's formula yields

$$\mathrm{Tr} K_{\mathcal{M}_n}(s) = \frac{n V(\mathbb{R}^d)}{(4\pi s)^{d/2}} + \frac{1}{12}\left(\frac{1}{n} - n\right)\frac{A(\Sigma)}{(4\pi s)^{\frac{d-2}{2}}}$$

- The entanglement entropy is then

$$S_d(\Sigma) = \frac{A(\Sigma)}{6(4\pi)^{(d-2)/2}(d-2)\epsilon^{d-2}}$$

With boundary: the general setup

We work in d-dimensional **flat** spacetime $\mathcal{M}$ with boundary $\partial\mathcal{M}$

- Cartesian coordinates $X^\mu = (\tau, x, y, z_{i\,,\,i=1,..,d-3})$.
- The **entangling surface** Σ : $\tau = 0$ and $x = 0$
- The boundary $\partial\mathcal{M}$ is a collection of **hyperplanes**.
- Σ **intersects** the boundary $\partial\mathcal{M}$ **orthogonally** at $\mathcal{P} = \Sigma \cap \partial\mathcal{M}$
- **Boundary conditions:** Neumann, Dirichlet, Robin.

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The heat kernel on $\mathbb{R}^d$ for a free massive scalar field is

$$K_\infty(s, \mathbf{X}, \mathbf{X}') = \frac{e^{-m^2 s}}{(4\pi s)^{d/2}} e^{-\frac{1}{4s}(\mathbf{X}-\mathbf{X}')^2}$$

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- **Boundary conditions:** Neumann, Dirichlet, Robin.

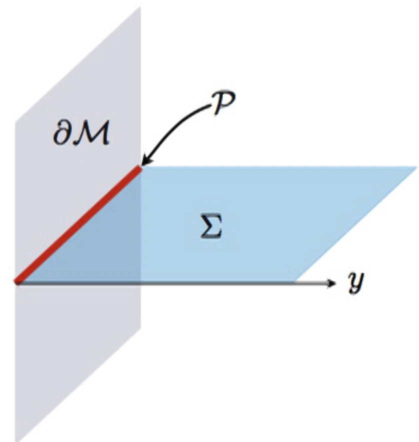
The entropy in **infinite** spacetime with Σ being a codimension 2 plan is

$$S_d(\Sigma) = \frac{A(\Sigma)}{6(4\pi)^{(d-2)/2}\epsilon^{d-2}} \sum_{k=0}^{[\frac{d-2}{2}]} \frac{(-1)^k m^{2k} \epsilon^{2k}}{k!(d-2k-2)}$$

Single plane boundary

Boundary conditions at $y = 0$:

- Neumann: $\partial_y K^{(N)}\Big|_{y=0} = 0$
- Dirichlet: $K^{(D)}\Big|_{y=0} = 0$
- Robin: $(\partial_y - h)K^{(h)}\Big|_{y=0} = 0$



Heat kernel for Neumann and Dirichlet:

$$K^{N(D)}(s, y, y') = K_\infty(s, y, y') \pm K_\infty(s, -y, y')$$

Heat kernel for Robin:

$$K^{(h)}(s, y, y') = K^{(N)}(s, y, y') - 2h\, e^{h(y+y')} \int_{y+y'}^{\infty} d\sigma\, e^{-h\sigma} K(s, \sigma)$$

Single plane boundary: Neumann/Dirichlet

In the entanglement entropy appears a contribution from $\mathcal{P}$:

$$S_d^{N(D)}(\Sigma, \mathcal{P}) \;=\; S_d(\Sigma) \;\pm\; S_d(\mathcal{P})$$

with boundary part

$$S_d(\mathcal{P}) \;=\; \frac{A(\mathcal{P})}{24(4\pi)^{(d-3)/2}\epsilon^{d-3}} \sum_{k=0}^{\left[\frac{d-3}{2}\right]} \frac{(-1)^k m^{2k}\epsilon^{2k}}{k!(d-2k-3)}$$

Always a logarithmic term either due to $S_d(\Sigma)$ in even dimension or due to $S_d(\mathcal{P})$ in odd dimension.

$$S_3(\mathcal{P}) = -\frac{1}{24}\ln(\epsilon m)\,, \qquad S_4(\mathcal{P}) = \frac{A(\mathcal{P})}{48\epsilon\sqrt{\pi}},$$

$$S_5(\mathcal{P}) = \frac{A(\mathcal{P})}{192\pi}\left(\frac{1}{\epsilon^2} + 2m^2\ln(\epsilon m)\right)$$

Single plane boundary: Robin

Entanglement entropy:

$$S_d^{(h)}(\Sigma, \mathcal{P}) = S_d^{(N)}(\Sigma) - \frac{A(\mathcal{P})}{24(4\pi)^{(d-3)/2}} \int_{\epsilon^2}^{\infty} ds \, \frac{e^{-sm^2}}{s^{(d-1)/2}} F(h\sqrt{s})$$

where $F(h\sqrt{s}) = 1 + (\Phi(h\sqrt{s}) - 1)e^{h^2 s} \simeq \frac{2h}{\sqrt{\pi}}\sqrt{s} + \mathcal{O}(s)$.

New logarithmic terms in any d:

$$S_4^{(h)}(\Sigma, \mathcal{P}) = S_4^{(N)}(\Sigma, \mathcal{P}) + \frac{A(\mathcal{P})}{12\pi} h \ln \epsilon + \mathcal{O}(h^2)$$

$$S_5^{(h)}(\Sigma, \mathcal{P}) = S_5^{(N)}(\Sigma, \mathcal{P}) - \frac{A(\mathcal{P})}{48\pi} \left(\frac{2}{\sqrt{\pi}} \frac{h}{\epsilon} + h^2 \ln \epsilon + \mathcal{O}(h^3) \right)$$

Some inequality and limits

Inequality: $S_d^{(N)} > S_d^{(h)} > S_d^{(D)}$ $\qquad h > 0$

Limits:

$$\lim_{h\to 0} S_d^{(h)} = S_d^{(N)}$$
$$\lim_{h\to +\infty} S_d^{(h)} = S_d^{(D)} \qquad (h\epsilon \to \infty)$$

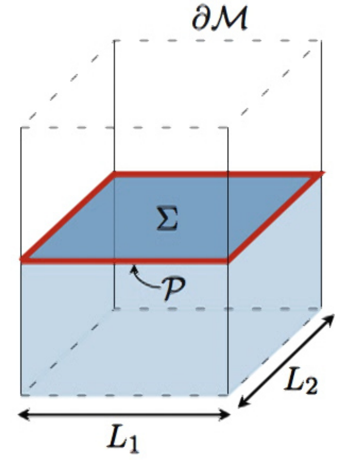
The h**-flow interpolates** between Neumann and Dirichlet phases:

$$\partial_h F(h\sqrt{s}) > 0 \quad \Rightarrow \quad \partial_h S_d^{(h)}(\Sigma, \mathcal{P}) < 0$$

$\Rightarrow$ **Monotonic decreasing** from Neumann to Dirichlet.

Open question: a relation to boundary RG flow?

Rectangular boundary



Boundary conditions:

- Neumann-Neumann
- Dirichlet-Dirichlet
- Mixed (Neumann-Dirichlet)

Using Cartesian coordinates:

- $\mathcal{M}$ **factorizes** as $\mathcal{M} = \mathbb{R}^2 \times_{i=1}^{d-2} \Omega_i$ with $\Omega_i = (0, L_i)$.
- The associated heat kernel for the scalar Laplacian does too:

$$K^{N(D)}(s,z,z') = \sum_{k\in\mathbb{Z}} K_\infty(s, 2Lk + z, z') \pm K_\infty(s, 2Lk - z, z')$$

$$K^{mixed}(s,z,z') = \sum_{k\in\mathbb{Z}} (-1)^k \Big(K_\infty(s, 2Lk + z, z') - K_\infty(s, 2Lk - z, z') \Big)$$

EE with rectangular boundary

The entanglement entropy is found to be [CB, (2017)]

$$S_d(\Sigma, \mathcal{P}) = \frac{1}{12(4\pi)^{\frac{d-2}{2}}} \int_{\epsilon^2}^{\infty} \frac{ds}{s^{(d-1)/2}} \prod_{i=1}^{d-2} \left(L_i\, \theta_{(i)}(e^{-L_i^2/s}) + \mathcal{B}_i \sqrt{\pi s} \right)$$

where

$$\mathcal{B}_i = \begin{cases} \pm 1 & \text{Neumann (Dirichlet)} \\ 0 & \text{Mixed} \end{cases}$$

$$\theta_{(i)}(z) = \begin{cases} \theta_3(z) & \text{Neumann/Dirichlet} \\ \theta_4(z) & \text{Mixed} \end{cases}$$

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Hierarchy of UV terms

Taking the limit $\epsilon \to 0$, one finds the EE to be

$$S_d(\Sigma, \mathcal{P}) = s_{d-2}\frac{A(\Sigma)}{\epsilon^{d-2}} + s_{d-3}\frac{\mathcal{P}_{d-3}}{\epsilon^{d-3}} + \cdots + s_1\frac{\mathcal{P}_1}{\epsilon} + s_{log}^{(d)}\ln\epsilon + s_0$$

where $A(\Sigma) = L_1 L_2 \cdots L_{d-2}$ is the area of Σ and

$$s_{log}^{(d)} = \frac{(-1)^{d-1}}{6 \times 2^{2(d-2)}}\mathcal{P}_0$$

The terms $\mathcal{P}_n$ read

$$\mathcal{P}_{d-2-p} = \frac{2^p}{p!(d-2-p)!}\sum_{\sigma}\mathcal{B}_{\sigma_1}\cdots\mathcal{B}_{\sigma_p}L_{\sigma_{p+1}}\cdots L_{\sigma_{d-2}}\,,$$

where the sum extends over all permutations of $\{1, \cdots, d-2\}$.

The boundary terms $\mathcal{P}_n$

The coefficients $\mathcal{P}_n$ have a simple interpretation in the case where the combination of BC is **pure Neumann or Dirichlet** ($L_i = L$):

$$\mathcal{P}_n = 2^{d-2-n} \binom{d-2}{n} L^n$$

$\longrightarrow$ $\mathcal{P}_n$ is the "**n–area**" of Σ (a $(d-2)$-square), defined at $\Sigma \cap \partial\mathcal{M}$

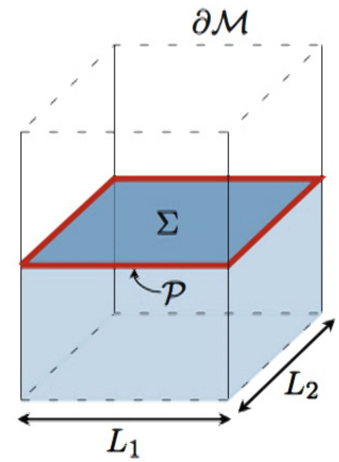
Example in $d = 3$: [see also Fursaev, Solodukhin (2016)]

- Σ is a line of length L
- $\mathcal{P}_0 = 2 \to$ 2 intersections with $\partial\mathcal{M}$ (for only 1 boundary: $\mathcal{P}_0 = 1$)

Example in $d = 4$: [see also Fursaev (2006)]

- Σ is a square of sides L
- 4 edges (1–faces) $\to \mathcal{P}_1 = 4L$ is the perimeter of Σ
- 4 vertices (0–faces) $\to \mathcal{P}_0 = 4$ is the number of vertices

Rectangular boundary



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Conformal anomalies: CFT vs BCFT

For a spacetime **without** boundary:

- $\langle T_\mu{}^\mu \rangle^{d=2n} = \mathrm{a}_d E_d + \sum_i b_i I_i$
- $\langle T_\mu{}^\mu \rangle^{d=2n+1} = 0$

E_d = Euler density
I_i = Weyl invariants

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For a spacetime **with** boundary:

- $\langle T_\mu{}^\mu \rangle^{d=2n} = \mathrm{a}_d E_d + \sum_i b_i I_i + \delta_{\partial\mathcal{M}_d} \sum_j c_j \widehat{I}_j$
- $\langle T_\mu{}^\mu \rangle^{d=2n+1} = \delta_{\partial\mathcal{M}_d} \left(\mathrm{a}_d \widehat{E}_d + \sum_j c_j \widehat{I}_j \right)$

$\widehat{E}_d$ = boundary Euler dens.
$\widehat{I}_j$ = boundary Weyl inv.

[Dowker, Schofield, (1990)]
[Fursaev (2015)]
[Herzog, Huang, Jensen (2016)]
[Moss, Poletti, (1994)]
[Solodukhin (2016)]
[Herzog, Huang (2017)]

Universal logarithmic term in the EE

For a generic CFT$_d$ on a **curved** spacetime, there appears a **universal** logarithmic term in the entanglement entropy:

$$S_d(\Sigma) = \gamma \frac{A(\Sigma)}{\epsilon^{d-2}} + \cdots + s_{log}^{(d)} \log \epsilon + \cdots$$

$s_{log}^{(d)}$ is expressed in terms of the heat coefficient $a_d(n)$, which is related to the **integrated conformal anomaly** $\int_{\mathcal{M}_d} \langle T_\mu{}^\mu \rangle = \eta a_d$:

$$\begin{aligned} s_{log}^{(d)} &= \eta \Big(n a_d - n \partial_n a_d(n) \Big) \Big|_{n=1} \\ &= \int_{\mathcal{M}_d} \langle T_\mu{}^\mu \rangle - \partial_n \int_{\mathcal{M}_{d,n}} \langle T_\mu{}^\mu \rangle \Big|_{n=1} \end{aligned}$$

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$$\begin{aligned} s_{log}^{(d)} &= \eta \Big(n a_d - n \partial_n a_d(n) \Big) \Big|_{n=1} \\ &= \int_{\mathcal{M}_d} \langle T_\mu{}^\mu \rangle - \partial_n \int_{\mathcal{M}_{d,n}} \langle T_\mu{}^\mu \rangle \Big|_{n=1} \end{aligned}$$

$\Rightarrow$ s_{log} vanishes for **even** dimensional CFTs.

$\Rightarrow$ s_{log} does **not** necessarily vanish for **BCFTs** for **any** d!

Logarithmic term in BCFT$_3$

The anomaly in $d = 3$ is a **pure boundary term**:

$$\int_{\mathcal{M}_3} \langle T \rangle = -\frac{a}{384\pi} \int_{\partial\mathcal{M}_3} \hat{R} + \frac{q}{256\pi} \int_{\partial\mathcal{M}_3} \mathrm{Tr}\, \hat{k}^2$$

In flat space with planar boundaries and an entangling surface that **intersects** these boundaries, the entanglement entropy is

$$S(\Sigma) = \alpha \frac{L}{\epsilon} - s_{log} \ln \frac{L}{\epsilon} + \cdots$$

In the case where $\Sigma \perp \partial\mathcal{M}$, the logarithmic term reads

$$s_{log} = \frac{a}{24} n_p$$

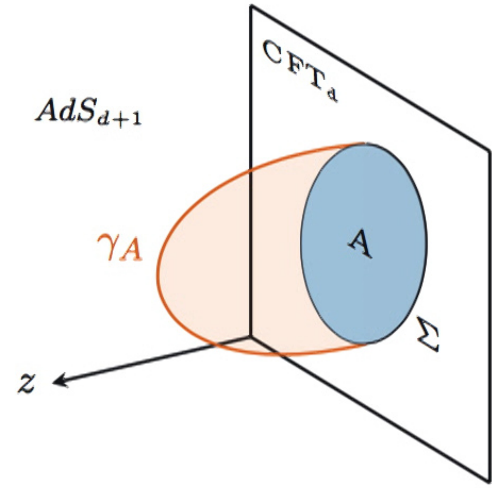
- Proportional to the charge a
- Proportional to the number of intersection(s) n_p between Σ and $\partial\mathcal{M}$

[Fursaev,Solodukhin (2016); CB,Solodukhin (2016)]

Holographic entanglement entropy

Holographic formula (RT 06)

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$



- γ_A **minimal** co-dim 2 surface :

$$\begin{cases} \partial\gamma_A = \partial A \equiv \Sigma \\ \gamma_A \text{ and } A \text{ homologous} \end{cases}$$

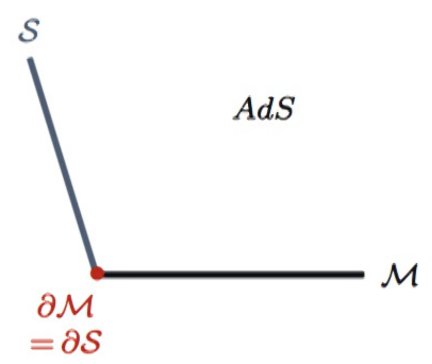
- Reproduces the **area law** divergence (ϵ : UV cutoff, $z > \epsilon$)

$$S_A \propto \frac{\text{Area}(\Sigma)}{\epsilon^{d-2}} + \cdots$$

What is the holographic dual to a BCFT?

Holographic picture:

- The bulk $\mathcal{B}$ is AdS_{d+1} and has a conformal boundary $\mathcal{M}$
- $\mathcal{M}$ has a boundary by itself, $\partial\mathcal{M}$
- $\partial\mathcal{M}$ is extended into the bulk, forming a surface (brane) $\mathcal{S}$
- Then $\partial\mathcal{B} = \mathcal{M} \cup \mathcal{S}$ such that $\partial\mathcal{M} = \partial\mathcal{S}$



What is the holographic dual to a BCFT?

Takayanagi's prescription: [Takayanagi (2011)]

- The gravitational action for holographic BCFT is

$$I_T = -\frac{1}{16\pi G}\int_{\mathcal{B}}\sqrt{g}\,(R-2\Lambda) - \frac{1}{8\pi G}\int_{\mathcal{M}}\sqrt{h}\,K - \frac{1}{8\pi G}\int_{\mathcal{S}}\sqrt{\gamma}\,(K+T)$$

- Varying the action with respect to γ_{ij}, the **shape** of $\mathcal{S}$ is determined by

$$K_{ij}-\gamma_{ij}K = T_{ij} \qquad \text{(for } T = const,\ T_{ij} = \gamma_{ij}T)$$

- In this picture, the brane $\mathcal{S}$ is dynamical in the sense that it **backreacts** with the bulk spacetime. (though for $\partial\mathcal{M}$ flat/ball, no backreaction)

  $\Rightarrow$ For arbitrary $\partial\mathcal{M}$, we need perturbation theory

  $\rightarrow$ Technically difficult to find solutions in general

What is the holographic dual to a BCFT?

Restricted Takayanagi's prescription:

[Astaneh,Solodukhin (2017); Chu,Miao,Guo (2017)]

- In more general situations, one can restrict the previous tensorial equations to a single **scalar equation** imposed on K of $\mathcal{S}$,

$$K = -\frac{d}{d-1}T, \qquad (T \equiv (d-1)\tanh m)$$

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Minimal surface prescription: [Astaneh,Solodukhin (2017)]

- $\mathcal{S}$ is described by the embedding functions $x^\mu = x^\mu(y_i)$. The gravitational action is modified by adding a boundary volume term

$$I_{min} = -\frac{1}{16\pi G}\int_{\mathcal{B}}\sqrt{g}\,(R-2\Lambda) - \frac{1}{8\pi G}\int_{\mathcal{M}}\sqrt{h}\,K - \frac{1}{8\pi G}\int_{\mathcal{S}}\sqrt{\gamma}\,\lambda$$

- Varying the action with respect to x^μ yields the **minimality** condition

$$K = 0 \quad \text{on } \mathcal{S}$$

Holographic entanglement entropy

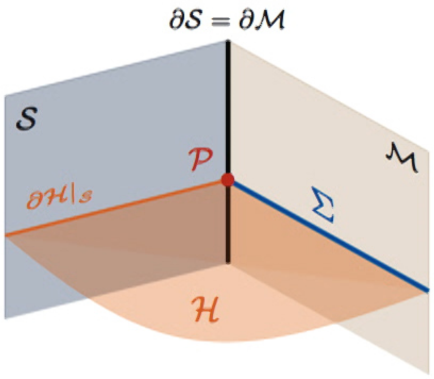
Holographic entanglement entropy:

- $\partial\mathcal{M}$ is **extended** into the AdS$_5$ bulk as $\mathcal{S}$
- $\mathcal{H}$ is the **minimal** surface anchored on Σ:

$$\partial\mathcal{H} = \partial\mathcal{H}|_{\mathcal{S}} \cup \Sigma$$

- **Holographic formula:**

$$S_{HEE}(\Sigma, \mathcal{P}) = \frac{A(\mathcal{H})}{4G_N}$$



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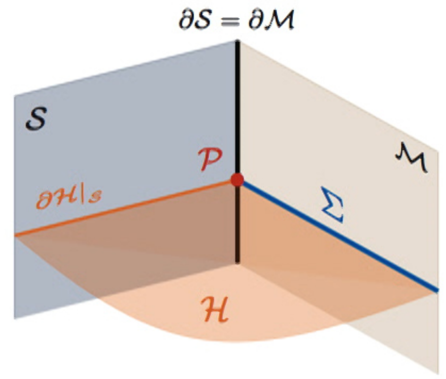
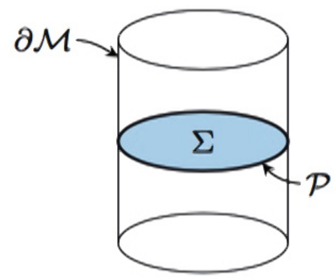


The $BCFT_4$:

- $\mathcal{M}$ is flat
- $\partial\mathcal{M}$ is cylindrical
- Σ **intersects** $\partial\mathcal{M}$ **orthogonally** at $\mathcal{P}$



Analog study for corner contribution in $AdS_4/BCFT_3$ [Seminara, Sisti, Tonni (2017), (2018)]

Logarithmic contribution in the HEE

With the **minimal surface prescription**, $K = 0$ (i.e. $T = m = 0$):

$$S_{log}^{(HEE)}(\Sigma, \mathcal{P}) = \frac{N^2}{8\pi}\left(\left[\int_\Sigma R_\Sigma + 2\int_{\mathcal{P}} k_p\right] + \int_\Sigma \mathrm{Tr}\,\hat{k}_i^2 - 2\int_{\mathcal{P}} \hat{k}_{\mu\nu} v^\mu v^\nu\right) \ln \epsilon$$

k_p: extrinsic curvature of $\mathcal{P}$
$\hat{k}_{\mu\nu}$, $(\hat{k}_i)_{\mu\nu}$: traceless part of extrinsic curvature tensors of $\partial\mathcal{M}$, Σ
v^μ: unit vector tangent to $\mathcal{P}$

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- The presence of boundaries **breaks** supersymmetry
- Some part of it can still be **preserved** with appropriate BC
- A question:

*For which boundary conditions are we computing the EE using the holographic **minimal surface** prescription?*

⇒ The answer is in the next slide..

Free field $\mathcal{N} = 4$ supermultiplet

For the $\mathcal{N} = 4$, $SU(N)$ SYM at **weak coupling** with BC which preserve **1/2 of supersymmetry**:

$$s_{log}^{(SYM)}(\Sigma, \mathcal{P}) = \frac{N^2 - 1}{8\pi}\left(\left[\int_\Sigma R_\Sigma + 2\int_{\mathcal{P}} k_p\right] + \int_\Sigma \mathrm{Tr}\,\hat{k}_i^2 - 2\int_{\mathcal{P}} \hat{k}_{\mu\nu} v^\mu v^\nu\right)$$

- **Agrees** in the large N limit with the holographic calculation for the **minimal surface prescription!**
- **Consistent** with the results for the **boundary conformal anomaly** [Astaneh,Solodukhin (2017)]

Summary

Boundary effects in entanglement entropy:

- New terms defined at $\mathcal{P}$ and dependent on boundary conditions
- Monotonicity of h-flow between Neumann and Dirichlet phases
- For Neumann and Dirichlet BC: log term proportional to $\#$ of vertices of Σ (in $d = 3$ to the $\#$ of intersection(s) between Σ and $\partial\mathcal{M}$) as well as boundary charges

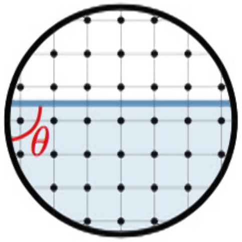
Entanglement entropy in AdS/BCFT correspondence:

- Boundary terms can be computed holographically
- Minimal surface prescription $\Leftrightarrow$ 1/2 SUSY preserving BC
  - boundary terms in integrated conformal anomaly
  - boundary terms in the entanglement entropy

$\rightarrow$ Calculations at weak/strong coupling match for $N \gg 1$

Open questions and outlook

- **Interpretation** of the 'tension' T (or m) of the brane in the BCFT side?
  - (a) The minimal surface prescription reproduces weak coupling results for BCFT_4 with 1/2 susy $\Rightarrow$ boundary charges do not receive quantum corrections. Understand this better.
  - (b) For restricted Takayanagi's prescription ($T \neq 0$), there is T-dependence in the holographic anomaly and HEE:
  - $\rightarrow$ Does T corresponds to **different BC** in the BCFT side?
- What is the **"right" prescription** for AdS/BCFT? (if there is one)
- Study **non-orthogonal intersection** between Σ and $\partial\mathcal{M}$:
  In $d = 3$, the log term is expected to capture the angle dependence and thus should carry interesting info about the BCFT_3.



- **Numerical** calculations on the lattice (work in progress...)

Thank you!