Title: BMS current algebras in three and four dimensions

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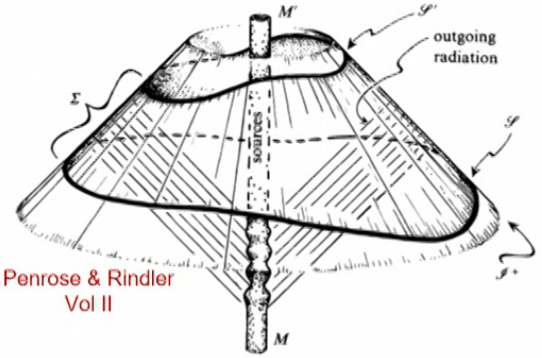
Abstract: <p>The Bondi mass loss formula has been central in the context of early research on gravitational waves. We show how it can be understood as a particular case of BMS current algebra and discuss the associated central extension. We then move down to three dimensions where a more complete picture emerges.</p>

Quantum Gravity seminar
Perimeter Institute
May 24, 2018

BMS current algebra and central extension



Penrose & Rindler
Vol II

Glenn Barnich

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Overview

BMS symmetry

Would-be conserved BMS current algebra

The field dependent central charge

3d gravity as group theory

In collaboration with
C. Troessaert, H. Gonzalez, B. Oblak

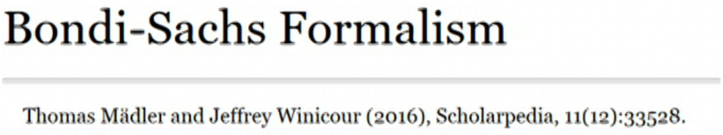
Introduction

Bondi mass loss due to gravitational radiation :

non-linear GR effect that was important to settle the controversy on the existence of gravitational waves

D. Kennefick

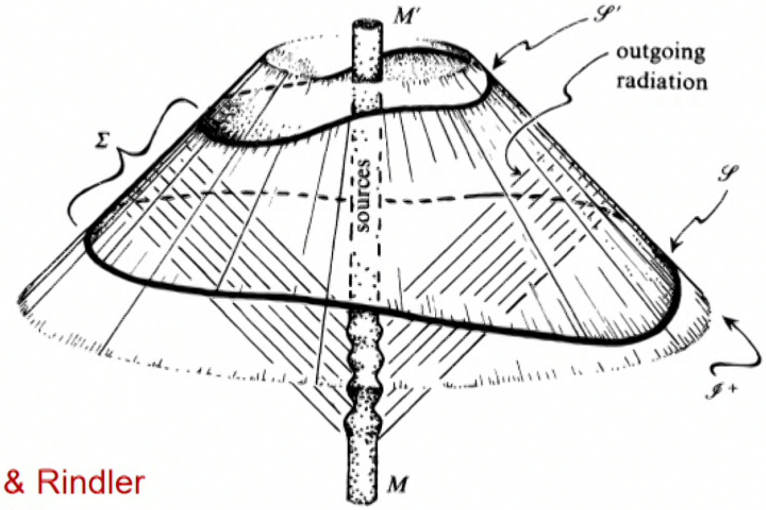
King's College and the story of how gravitational waves became real



Bondi-Sachs Formalism

Thomas Mädler and Jeffrey Winicour (2016), Scholarpedia, 11(12):33528.

The set-up



u

retarded time

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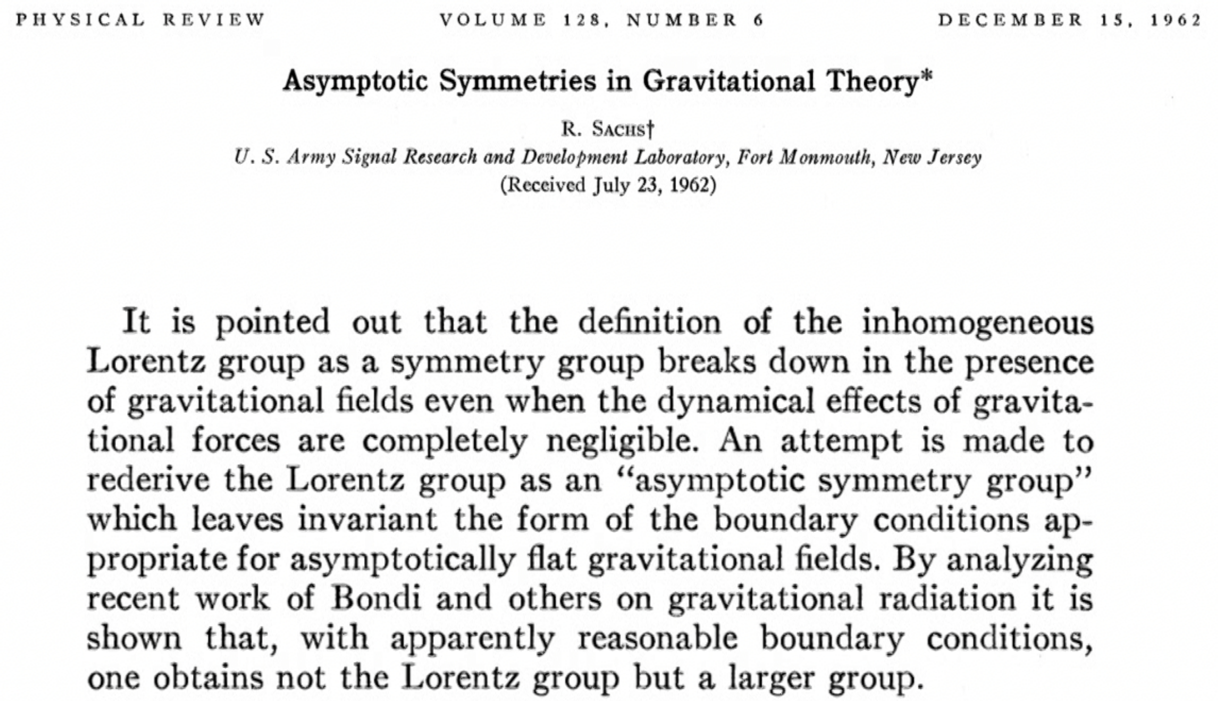
complex coordinates on celestial sphere

$$\zeta = e^{i\phi} \cot\frac{\theta}{2}$$

$$d\theta^2 + \sin^2\theta d\phi^2 = 2P_S^{-2} d\zeta d\bar{\zeta}$$

$$P_S(\zeta,\bar{\zeta}) = \frac{1}{\sqrt{2}}(1+\zeta\bar{\zeta})$$

BMS symmetry The paper



PHYSICAL REVIEW VOLUME 128, NUMBER 6 DECEMBER 15, 1962

Asymptotic Symmetries in Gravitational Theory*

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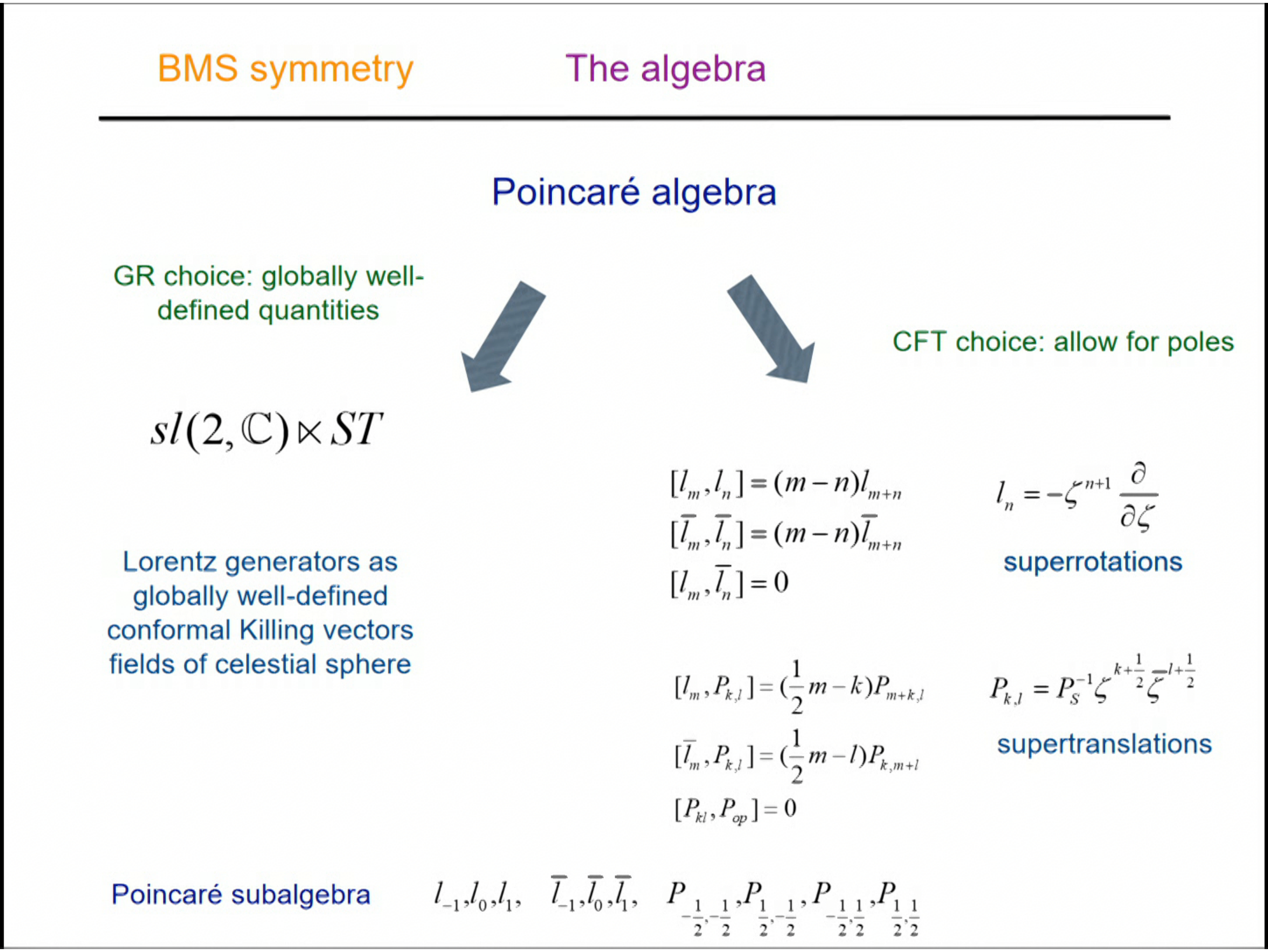
(Received July 23, 1962)

It is pointed out that the definition of the inhomogeneous Lorentz group as a symmetry group breaks down in the presence of gravitational fields even when the dynamical effects of gravitational forces are completely negligible. An attempt is made to rederive the Lorentz group as an "asymptotic symmetry group" which leaves invariant the form of the boundary conditions appropriate for asymptotically flat gravitational fields. By analyzing recent work of Bondi and others on gravitational radiation it is shown that, with apparently reasonable boundary conditions, one obtains not the Lorentz group but a larger group.



BMS symmetry The algebra

Poincaré algebra

GR choice: globally well-defined quantities

CFT choice: allow for poles

$$sl(2,\mathbb{C}) \ltimes ST$$

Lorentz generators as globally well-defined conformal Killing vectors fields of celestial sphere

$$[l_m, l_n] = (m-n) l_{m+n}$$
$$[\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n}$$
$$[l_m, \bar{l}_n] = 0$$

$$l_n = -\zeta^{n+1} \frac{\partial}{\partial \zeta}$$

superrotations

$$[l_m, P_{k,l}] = (\frac{1}{2} m - k) P_{m+k,l}$$
$$[\bar{l}_m, P_{k,l}] = (\frac{1}{2} m - l) P_{k,m+l}$$
$$[P_{kl}, P_{op}] = 0$$

$$P_{k,l} = P_S^{-1} \zeta^{k+\frac{1}{2}} \bar{\zeta}^{l+\frac{1}{2}}$$

supertranslations

Poincaré subalgebra $l_{-1}, l_0, l_1, \quad \bar{l}_{-1}, \bar{l}_0, \bar{l}_1, \quad P_{-\frac{1}{2},-\frac{1}{2}}, P_{\frac{1}{2},-\frac{1}{2}}, P_{-\frac{1}{2},\frac{1}{2}}, P_{\frac{1}{2},\frac{1}{2}}$

BMS current algebra

Currents

$$J^u_\xi = -\frac{1}{8\pi G P_S^2}\Big[\Big(f(\Psi_2^0 + \underbrace{\sigma^0\dot{\bar{\sigma}}^0}_{\text{mass aspect}}) + \mathcal{Y}(\Psi_1^0 + \underbrace{\sigma^0\eth\bar{\sigma}^0 + \frac{1}{2}\eth(\sigma^0\bar{\sigma}^0)}_{\text{angular momentum aspect}})\Big) + \text{c.c.}\Big]$$

$$\xi \begin{cases} \mathcal{Y} = P_S^{-1}\bar{Y}, \quad \bar{\mathcal{Y}} = \bar{P}_S^{-1}Y \\ f = P_S^{-1}T + \frac{u}{2}\psi \end{cases}$$

conformal Killing vectors

$T(\zeta,\bar{\zeta})$ supertranslation generators

$\psi = \eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}}$

Ψ_i^0 Weyl tensor

$\sigma^0, \dot{\sigma}^0$ asymptotic part of shear & news

information on TT polarizations of gravity waves

BMS current algebra   Action on fields

$$-\delta_{\xi}\sigma^{0} = [f\partial_{u} + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\bar{\eth}\overline{\mathcal{Y}}]\sigma^{0} \underline{-\eth^{2} f}$$

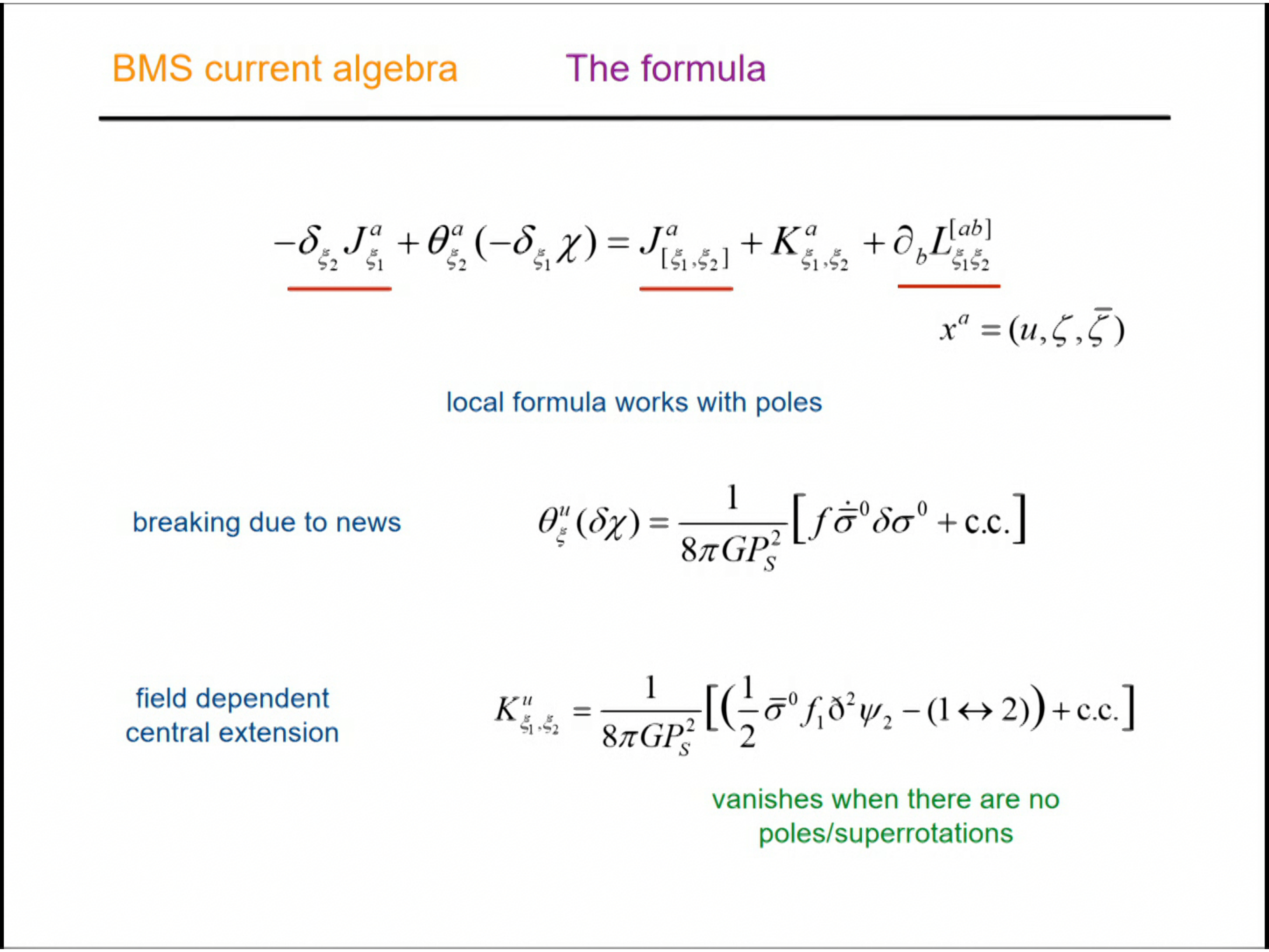
$$-\delta_{\xi}\dot{\sigma}^{0} = [f\partial_{u} + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y}]\dot{\sigma}^{0} \underline{-\eth^{2}\psi}$$

$$-\delta_{\xi}\Psi^{0}{}_{2} = [f\partial_{u} + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\overline{\mathcal{Y}}]\Psi^{0}_{2} + 2\eth f\Psi^{0}_{3}$$

$$-\delta_{\xi}\Psi^{0}{}_{1} = [f\partial_{u} + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y} + \bar{\eth}\overline{\mathcal{Y}}]\Psi^{0}_{1} + 3\eth f\Psi^{0}_{2}$$

transformations of fields involve inhomogeneous terms

Minkowski vacuum breaks BMS invariance



BMS current algebra The formula

$$\underline{-\delta_{\xi_2} J^a_{\xi_1}} + \theta^a_{\xi_2}(-\delta_{\xi_1}\chi) = \underline{J^a_{[\xi_1,\xi_2]}} + K^a_{\xi_1,\xi_2} + \underline{\partial_b L^{[ab]}_{\xi_1\xi_2}}$$

$$x^a = (u, \zeta, \bar{\zeta})$$

local formula works with poles

breaking due to news

$$\theta^u_\xi(\delta\chi) = \frac{1}{8\pi G P_S^2}\left[f\dot{\bar{\sigma}}^0 \delta\sigma^0 + \text{c.c.}\right]$$

field dependent
central extension

$$K^u_{\xi_1,\xi_2} = \frac{1}{8\pi G P_S^2}\left[\left(\frac{1}{2}\bar{\sigma}^0 f_1 \eth^2 \psi_2 - (1 \leftrightarrow 2)\right) + \text{c.c.}\right]$$

vanishes when there are no
poles/superrotations

BMS current algebra Non-conservation

current non-conservation for $\xi_1 = \xi, \xi_2 = \partial_u$

$$\partial_u \mathcal{J}^u_\xi + \eth \mathcal{J}_\xi + \bar{\eth} \overline{\mathcal{J}_\xi} \approx \Theta^u_{\partial_u}(\delta_\xi \chi) + \mathcal{K}^u_{\xi,\partial_u}$$

charges when there
are no poles

$$Q_\xi = \oint_{S^2} d^2\Omega \, \mathcal{J}^u_\xi$$

$$\frac{d}{du} Q_\xi = \frac{1}{8\pi G} \oint_{S^2} d^2\Omega \, [\dot{\bar{\sigma}}^0 \delta_\xi \sigma^0 + \mathrm{c.c.}]$$

Bondi mass loss formula for $\xi = \partial_u$

$$\frac{d}{du} Q_{\partial_u} = -\frac{1}{8\pi G} \oint_{S^2} d^2\Omega \, [\underbrace{\dot{\bar{\sigma}}^0 \dot{\sigma}^0 + \mathrm{c.c.}}_{\geq 0}]$$

BMS current algebra   The formula

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vanishes when there are no
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Central extension WZ consistency condition

in QFT the Adler-Bardeen anomaly satisfies
the Wess-Zumino consistency condition

$$\gamma A^a_\mu = D_\mu C^a \quad \gamma C^a = -\frac{1}{2} C^b C^c f^a_{bc}$$

$$\left\{ \begin{array}{l} \mathrm{Tr}\, F^3 = d_H \omega^{0,5} \\ \gamma\omega^{0,5} + d_H \omega^{1,4} = 0 \\ \boxed{\gamma\omega^{1,4} + d_H \omega^{2,3} = 0} \\ \quad\vdots \\ \gamma\omega^{4,1} + d_H \mathrm{Tr}\, C^5 = 0 \\ \gamma \mathrm{Tr}\, C^5 = 0 \end{array} \right.$$

fermionic generators $Y \to \eta(\zeta) \qquad T \to C(\zeta, \bar{\zeta})$

$$\omega^{2,2} = d\zeta d\bar{\zeta} K'' - dud\bar{\zeta} K + dud\zeta \bar{K}$$

$$\left\{ \begin{array}{l} \boxed{\gamma\omega^{2,2} + d_H \omega^{3,1} = 0} \\ \gamma\omega^{3,1} + d_H \omega^{4,0} = 0 \\ \gamma\omega^{4,0} = 0 \end{array} \right.$$

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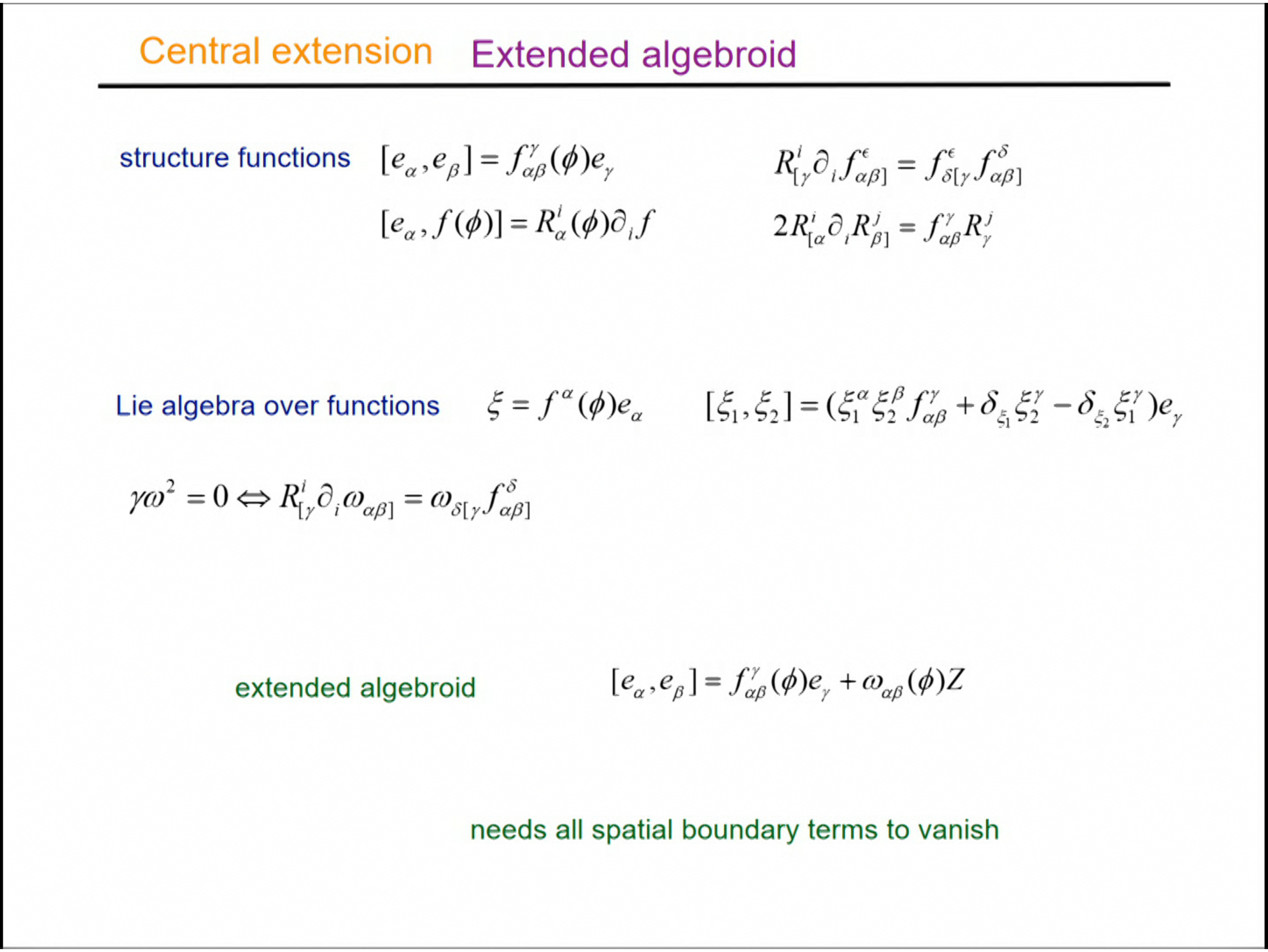

Central extension Extended algebroid

structure functions $[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta}(\phi) e_\gamma$ $\quad R^i_{[\gamma}\partial_i f^\epsilon_{\alpha\beta]} = f^\epsilon_{\delta[\gamma} f^\delta_{\alpha\beta]}$

$[e_\alpha, f(\phi)] = R^i_\alpha(\phi)\partial_i f$ $\quad 2R^i_{[\alpha}\partial_i R^j_{\beta]} = f^\gamma_{\alpha\beta} R^j_\gamma$

Lie algebra over functions $\xi = f^\alpha(\phi) e_\alpha$ $\quad [\xi_1, \xi_2] = (\xi_1^\alpha \xi_2^\beta f^\gamma_{\alpha\beta} + \delta_{\xi_1}\xi_2^\gamma - \delta_{\xi_2}\xi_1^\gamma) e_\gamma$

$$\gamma\omega^2 = 0 \Leftrightarrow R^i_{[\gamma}\partial_i \omega_{\alpha\beta]} = \omega_{\delta[\gamma} f^\delta_{\alpha\beta]}$$

extended algebroid $[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta}(\phi) e_\gamma + \omega_{\alpha\beta}(\phi) Z$

needs all spatial boundary terms to vanish

Central extension WZ consistency condition

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fermionic generators $Y \rightarrow \eta(\zeta) \qquad T \rightarrow C(\zeta, \bar{\zeta})$

$$\omega^{2,2} = d\zeta d\bar{\zeta} K^{u} - dud\bar{\zeta} K + dud\zeta \bar{K}$$

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Central extension Extended algebroid

structure functions $[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta}(\phi) e_\gamma$ $R^i_{[\gamma} \partial_i f^\epsilon_{\alpha\beta]} = f^\epsilon_{\delta[\gamma} f^\delta_{\alpha\beta]}$

$[e_\alpha, f(\phi)] = R^i_\alpha(\phi) \partial_i f$ $2R^i_{[\alpha} \partial_i R^j_{\beta]} = f^\gamma_{\alpha\beta} R^j_\gamma$

Lie algebra over functions $\xi = f^\alpha(\phi) e_\alpha$ $[\xi_1, \xi_2] = (\xi_1^\alpha \xi_2^\beta f^\gamma_{\alpha\beta} + \delta_{\xi_1} \xi_2^\gamma - \delta_{\xi_2} \xi_1^\gamma) e_\gamma$

$$\gamma\varpi^2 = 0 \Leftrightarrow R^i_{[\gamma} \partial_i \omega_{\alpha\beta]} = \omega_{\delta[\gamma} f^\delta_{\alpha\beta]}$$

extended algebroid $[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta}(\phi) e_\gamma + \omega_{\alpha\beta}(\phi) Z$

needs all spatial boundary terms to vanish

Central extension   Getting numbers

$$P_R = 1 \qquad K_{\xi^g_1,\xi^g_2} = \int d\zeta \int d\bar{\zeta} \Big[\big(\sigma^0 f_1 \partial^3 Y_2 - (1 \leftrightarrow 2) \big) + \text{c.c.} \Big]$$

from the conformal dimensions :
$$\partial_u^n \sigma(u,\zeta,\bar{\zeta}) = \sum_{k,l} (\partial_u^n \sigma)_{k,l}(u)\, \zeta^{-k-\frac{n-1}{2}} \bar{\zeta}^{-l-\frac{n+3}{2}}$$

admit Laurent series (delta function singularities), integrals as residues

$$\left\{ \begin{aligned}
K_{l_m,l_n} &= \frac{1}{2} u(m+1)(n+1) \sigma^0{}_{m+n-\frac{1}{2},-\frac{1}{2}} [n(n-1) - m(m-1)] \\
K_{l_m,\bar{l}_n} &= -\frac{1}{2} u(m+1)(n+1) [\sigma^0{}_{m-\frac{1}{2},n-\frac{1}{2}} m(m-1) - \bar{\sigma}^0{}_{m-\frac{1}{2},n-\frac{1}{2}} n(n-1)] \\
K_{l_m,P_{k,l}} &= \sigma^0{}_{m+k,l}\, m(m^2-1) \\
K_{P_{k,l},P_{o,p}} &= 0
\end{aligned} \right.$$

Central extension   Cardyology at null infinity ?

But $\sigma^0 = 0$ for Kerr black hole

transform Scri to a cylinder times a line by a finite superrotation $\zeta = e^{\frac{2\pi}{L}\omega}$

$$\partial_{u'}^n \sigma'^0 (u', \omega, \bar{\omega}) = (\frac{2\pi}{L})^{n+1} [(\partial_u^n \sigma^0)_{k,l}(u) e^{-\frac{2\pi}{L}k\omega} e^{-\frac{2\pi}{L}l\bar{\omega}}] + (\frac{2\pi}{L})^2 \frac{1}{4} \underbrace{(\delta_n^0 u' + \delta_n^1)}_{\text{finite shift !}}$$

thermal circle $iu \sim iu + \beta$

Work in progress ....

BMS current algebra Action on fields

$$-\delta_\xi \sigma^0 = [f\partial_u + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\bar{\eth}\overline{\mathcal{Y}}]\sigma^0 \underline{-\eth^2 f}$$

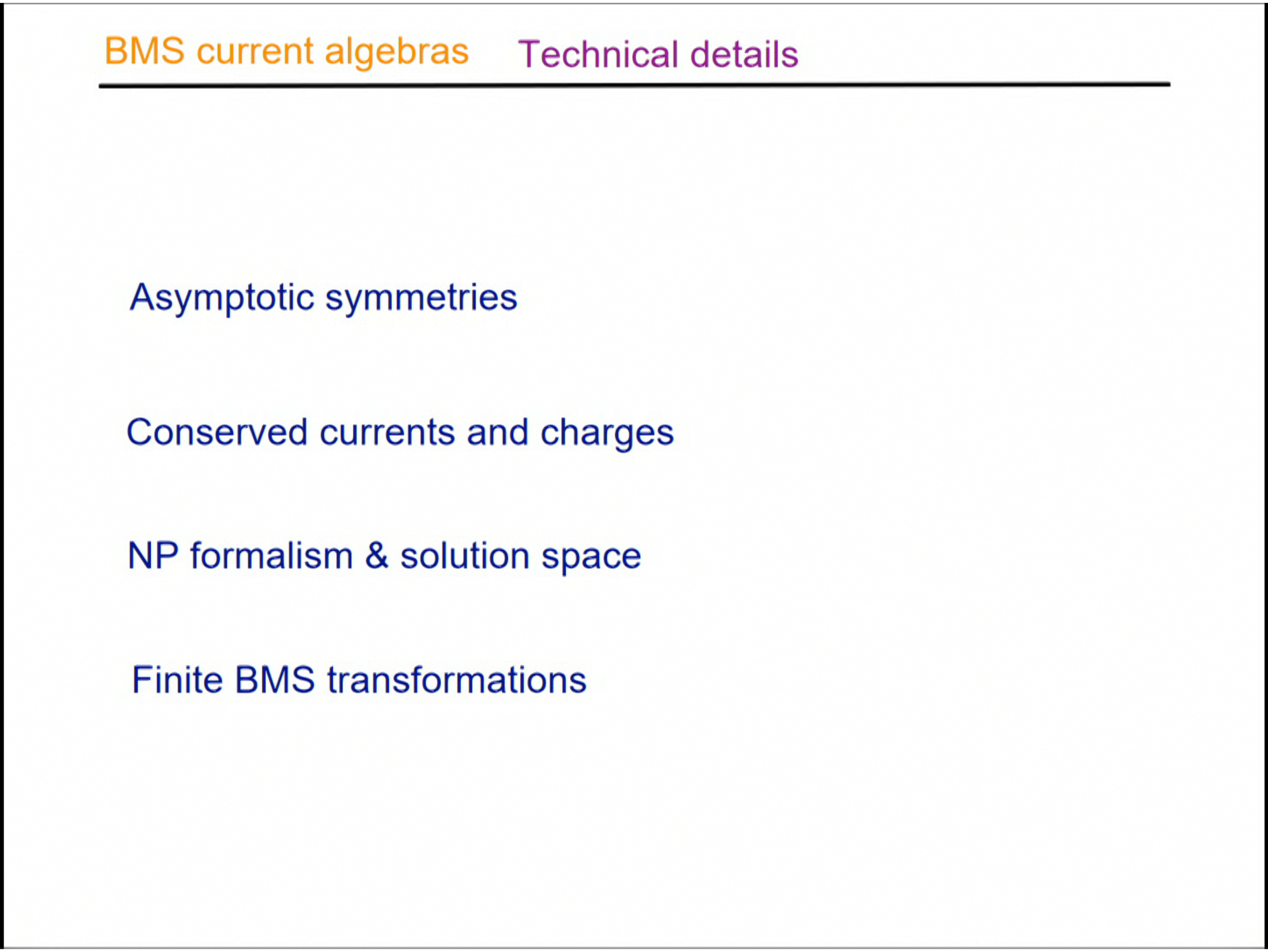
$$-\delta_\xi \dot{\sigma}^0 = [f\partial_u + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y}]\dot{\sigma}^0 \underline{-\eth^2 \psi}$$

$$-\delta_\xi \Psi^0{}_2 = [f\partial_u + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\overline{\mathcal{Y}}]\Psi_2^0 + 2\eth f\Psi_3^0$$

$$-\delta_\xi \Psi^0{}_1 = [f\partial_u + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y} + \bar{\eth}\overline{\mathcal{Y}}]\Psi_1^0 + 3\eth f\Psi_2^0$$

transformations of fields involve inhomogeneous terms

Minkowski vacuum breaks BMS invariance



BMS current algebras   Technical details

Asymptotic symmetries

Conserved currents and charges

NP formalism & solution space

Finite BMS transformations

Finite transformations BMS and Weyl group

"integrate" BMS Lie algebra → group
finite transformations of solution space

Residual gauge symmetries : find the local Lorentz transformations + diffeomorphisms that leave NPU solution space invariant
How do they act on solution space ?

$$(\zeta'(\zeta), \bar{\zeta}'(\bar{\zeta}), \beta(\zeta,\bar{\zeta}), E(u,\zeta,\bar{\zeta}) = E_R + iE_I)$$

finite superrotations, supertranslations, complex Weyl rescalings

β, E_R determine $u' = u'(u,\zeta,\bar{\zeta})$

$$\beta(\zeta,\bar{\zeta}) = \int_{\hat{u}}^{0} dv (P\bar{P})^{\frac{1}{2}}$$

$$u'(u,\zeta,\bar{\zeta}) = \int_{\hat{u}}^{u} dv e^{E_R}$$

Weyl invariant time coordinate

$$\tilde{u}(u,\zeta,\bar{\zeta}) = \int_0^u dv (P\bar{P})^{\frac{1}{2}}(v,\zeta,\bar{\zeta})$$

$$\tilde{u}'(u',\zeta',\bar{\zeta}') = J^{-\frac{1}{2}}\left[\tilde{u}(u,\zeta,\bar{\zeta}) + \beta(\zeta,\bar{\zeta})\right]$$

$$P'(u',\zeta',\bar{\zeta}') = P(u,\zeta,\bar{\zeta}) e^{-\bar{E}} \frac{\partial\bar{\zeta}'}{\partial\bar{\zeta}}$$

$$J = \frac{\partial\zeta}{\partial\zeta'}\frac{\partial\bar{\zeta}}{\partial\bar{\zeta}'}$$

NB: simple formulas when $\partial_u P = 0 = \partial_{u'} P'$ standard BMS group when P is fixed

Finite transformations   Schwarzian derivatives

transformation of the Weyl invariant quantities

$$\sigma'^0_R = (\frac{\partial\zeta}{\partial\zeta'})^{-\frac{1}{2}}(\frac{\partial\bar\zeta}{\partial\bar\zeta'})^{\frac{3}{2}}\Big[\sigma^0_R + \bar\partial^2\beta + \underline{\frac{1}{2}\{\bar\zeta',\bar\zeta\}(\tilde u+\beta)}\Big],$$

$$\dot{\bar\sigma}'^0_R = (\frac{\partial\zeta}{\partial\zeta'})^2[\dot{\bar\sigma}^0_R + \underline{\frac{1}{2}\{\zeta',\zeta\}}\Big],$$

$$\Psi'^0_{4R} = (\frac{\partial\zeta}{\partial\zeta'})^{\frac{5}{2}}(\frac{\partial\bar\zeta}{\partial\bar\zeta'})^{\frac{1}{2}}\Psi^0_{4R},$$

$$\Psi'^0_{3R} = (\frac{\partial\zeta}{\partial\zeta'})^2\frac{\partial\bar\zeta}{\partial\bar\zeta'}\Big[\Psi^0_{3R} - Y\Psi^0_{4R}\Big], \quad Y = \bar\partial\beta + \frac{1}{2}\bar\partial\ln\frac{\partial\bar\zeta'}{\partial\bar\zeta}(\tilde u+\beta),$$

$$\Psi'^0_{2R} = (\frac{\partial\zeta}{\partial\zeta'})^{\frac{3}{2}}(\frac{\partial\bar\zeta}{\partial\bar\zeta'})^{\frac{3}{2}}\Big[\Psi^0_{2R} - 2Y\Psi^0_{3R} + Y^2\Psi^0_{4R}\Big],$$

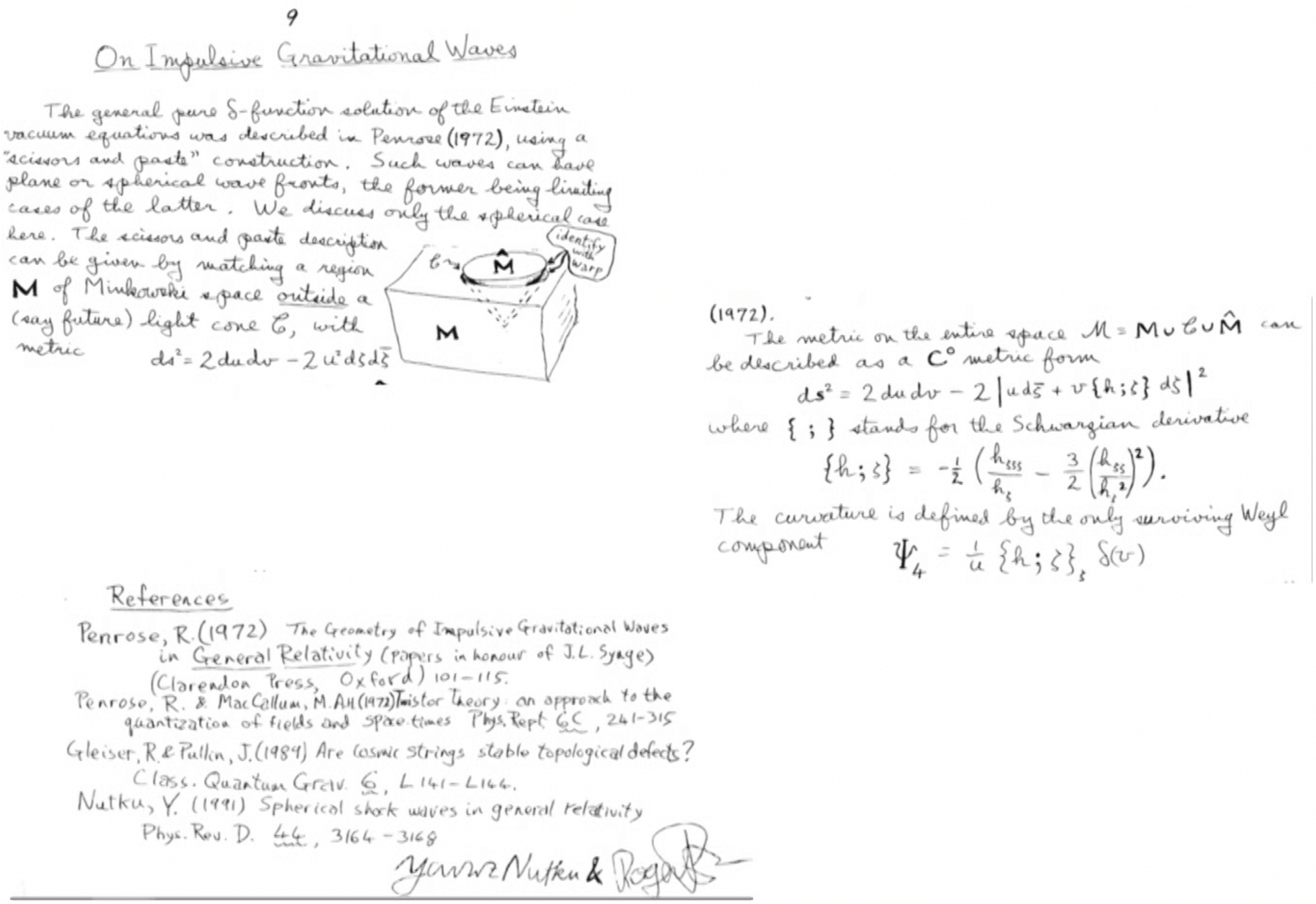
$$\Psi'^0_{1R} = \frac{\partial\zeta}{\partial\zeta'}(\frac{\partial\bar\zeta}{\partial\bar\zeta'})^2\Big[\Psi^0_{1R} - 3Y\Psi^0_{2R} + 3Y^2\Psi^0_{3R} - Y^3\Psi^0_{4R}\Big],$$

$$\Psi'^0_{0R} = (\frac{\partial\zeta}{\partial\zeta'})^{\frac{1}{2}}(\frac{\partial\bar\zeta}{\partial\bar\zeta'})^{\frac{5}{2}}\Big[\Psi^0_{0R} - 4Y\Psi^0_{1R} + 6Y^2\Psi^0_{2R} - 4Y^3\Psi^0_{3R} + Y^4\Psi^0_{4R}\Big],$$

$$(-4\pi G)M'_R = (\frac{\partial\zeta}{\partial\zeta'}\frac{\partial\bar\zeta}{\partial\bar\zeta'})^{\frac{3}{2}}\Big[(-4\pi G)M_R + \bar\partial^2\partial^2\beta + \frac{1}{2}\{\bar\zeta',\bar\zeta\}(\bar\sigma^0_R + \partial^2\beta) +$$

$$+ \frac{1}{2}\{\zeta',\zeta\}(\sigma^0_R + \bar\partial^2\beta) + \frac{1}{4}\{\bar\zeta',\bar\zeta\}\{\zeta',\zeta\}(\tilde u+\beta)\Big]$$

Finite transformations Schwarzian derivatives



9

On Impulsive Gravitational Waves

The general pure δ-function solution of the Einstein vacuum equations was described in Penrose (1972), using a "scissors and paste" construction. Such waves can have plane or spherical wave fronts, the former being limiting cases of the latter. We discuss only the spherical case here. The scissors and paste description can be given by matching a region **M** of Minkowski space outside a (say future) light cone $\mathscr{C}$, with metric

$$ds^2 = 2\,du\,dv - 2u^2\,d\zeta\,d\bar\zeta$$

(1972).

The metric on the entire space $\mathcal{M} = \mathbf{M} \cup \mathscr{C} \cup \hat{\mathbf{M}}$ can be described as a $\mathbf{C}^0$ metric form

$$ds^2 = 2\,du\,dv - 2\left|u\,d\bar\zeta + v\{h;\zeta\}\,d\zeta\right|^2$$

where $\{\,;\,\}$ stands for the Schwarzian derivative

$$\{h;\zeta\} = -\frac{1}{2}\left(\frac{h_{\zeta\zeta\zeta}}{h_\zeta} - \frac{3}{2}\left(\frac{h_{\zeta\zeta}}{h_\zeta{}^2}\right)^2\right).$$

The curvature is defined by the only surviving Weyl component

$$\Psi_4 = \frac{1}{u}\{h;\zeta\}_\zeta\,\delta(v)$$

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Yavuz Nutku & Roger

3d AdS & flat   Symmetries and solutions

residual symmetries $l^{-1} \neq 0 \qquad \xi = Y^{+}(x^{+})\partial_{+} + Y^{-}(x^{-})\partial_{-} \qquad x^{\pm} = \frac{u}{l} \pm \phi$

$l^{-1} = 0 \qquad \xi = Y(\phi)\partial_{\phi} + (T + uY')\partial_{u}$

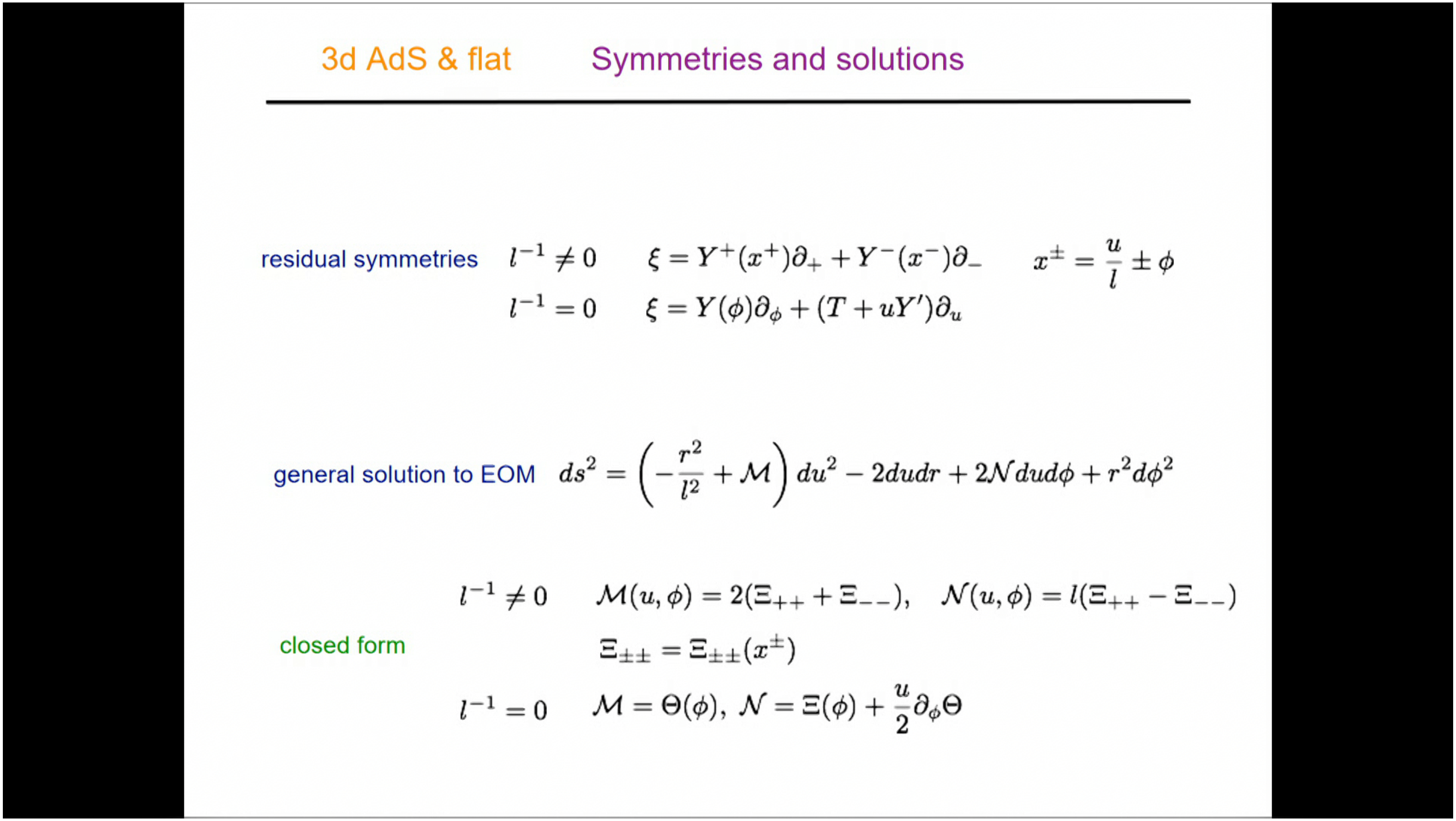
general solution to EOM $ds^2 = \left(-\frac{r^2}{l^2} + \mathcal{M}\right) du^2 - 2dudr + 2\mathcal{N} dud\phi + r^2 d\phi^2$

closed form

$l^{-1} \neq 0 \qquad \mathcal{M}(u,\phi) = 2(\Xi_{++} + \Xi_{--}), \quad \mathcal{N}(u,\phi) = l(\Xi_{++} - \Xi_{--})$

$\Xi_{\pm\pm} = \Xi_{\pm\pm}(x^{\pm})$

$l^{-1} = 0 \qquad \mathcal{M} = \Theta(\phi), \ \mathcal{N} = \Xi(\phi) + \frac{u}{2}\partial_{\phi}\Theta$



3d AdS & flat   Symmetries and solutions

residual symmetries $l^{-1} \neq 0 \qquad \xi = Y^{+}(x^{+})\partial_{+} + Y^{-}(x^{-})\partial_{-} \qquad x^{\pm} = \frac{u}{l} \pm \phi$

$l^{-1} = 0 \qquad \xi = Y(\phi)\partial_{\phi} + (T + uY')\partial_{u}$

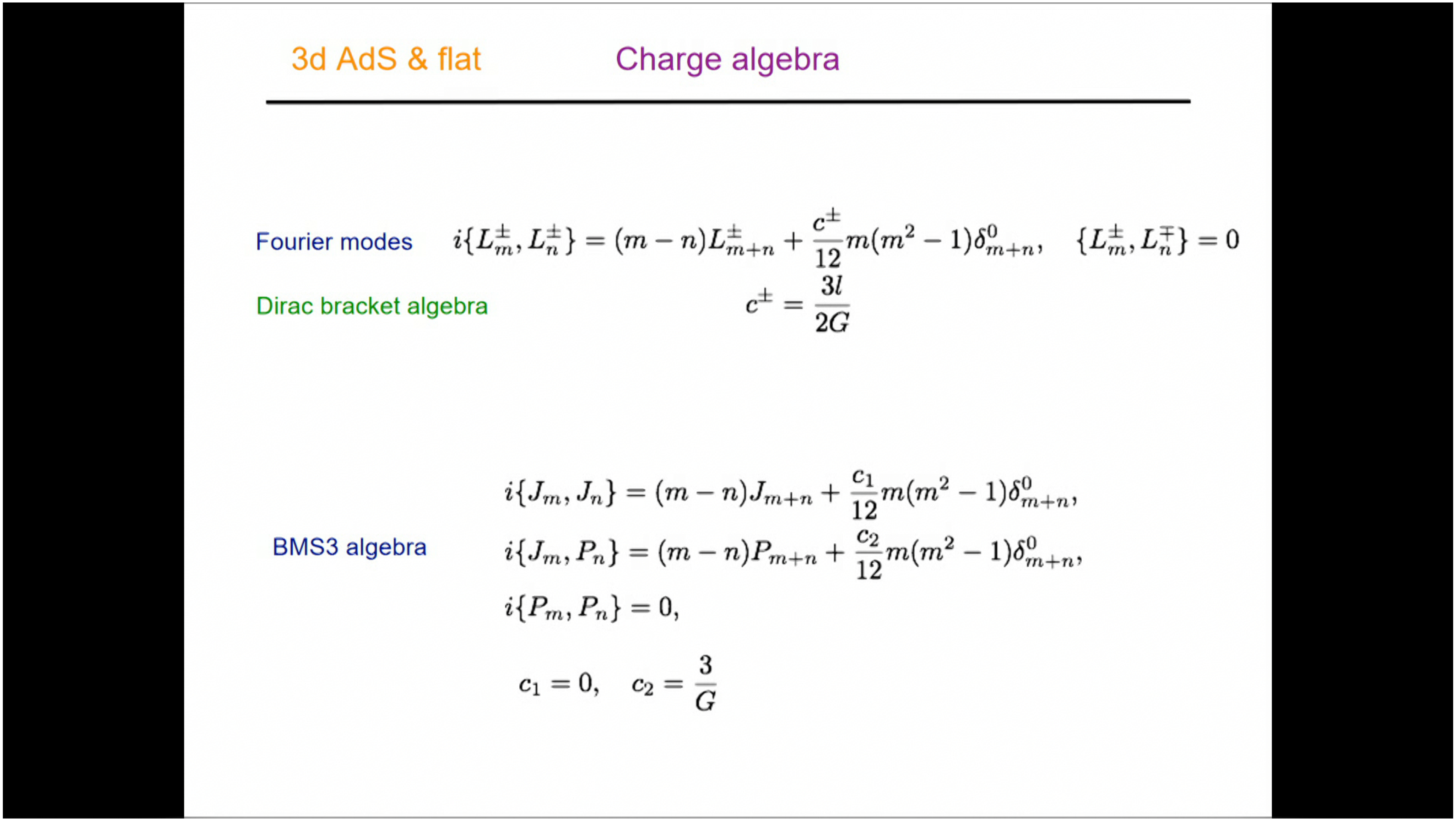
general solution to EOM $ds^2 = \left(-\frac{r^2}{l^2} + \mathcal{M}\right) du^2 - 2dudr + 2\mathcal{N} dud\phi + r^2 d\phi^2$

closed form

$$l^{-1} \neq 0 \qquad \mathcal{M}(u,\phi) = 2(\Xi_{++} + \Xi_{--}), \quad \mathcal{N}(u,\phi) = l(\Xi_{++} - \Xi_{--})$$

$$\Xi_{\pm\pm} = \Xi_{\pm\pm}(x^{\pm})$$

$$l^{-1} = 0 \qquad \mathcal{M} = \Theta(\phi), \ \mathcal{N} = \Xi(\phi) + \frac{u}{2}\partial_{\phi}\Theta$$



3d AdS & flat — Charge algebra

Fourier modes $$i\{L^\pm_m, L^\pm_n\} = (m-n)L^\pm_{m+n} + \frac{c^\pm}{12}m(m^2-1)\delta^0_{m+n}, \quad \{L^\pm_m, L^\mp_n\} = 0$$

Dirac bracket algebra $$c^\pm = \frac{3l}{2G}$$

BMS3 algebra

$$i\{J_m, J_n\} = (m-n)J_{m+n} + \frac{c_1}{12}m(m^2-1)\delta^0_{m+n},$$

$$i\{J_m, P_n\} = (m-n)P_{m+n} + \frac{c_2}{12}m(m^2-1)\delta^0_{m+n},$$

$$i\{P_m, P_n\} = 0,$$

$$c_1 = 0, \quad c_2 = \frac{3}{G}$$

AdS3 & 3d flat Zero mode solutions

zero mode solutions in both cases

BMS form

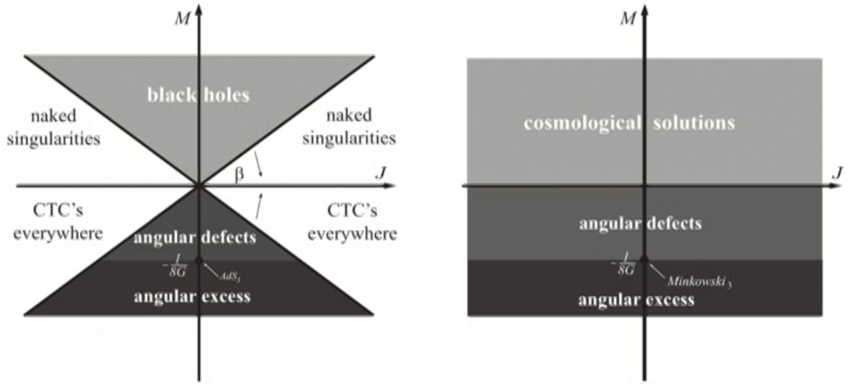
$$ds^2 = (-\frac{r^2}{l^2} + 8GM)du^2 - 2dudr + 8GJdud\phi + r^2 d\phi^2$$

ADM form

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\varphi + N^\varphi dt)^2,$$

$$N^2 = \frac{r^2}{l^2} - 8MG + \frac{16G^2J^2}{r^2}, \quad N^\varphi = \frac{4GJ}{r^2}$$

$$\Xi_{\pm\pm} = 2G(M \pm \frac{J}{l})$$



(a)

(b)

repeat derivation of entropy of cosmological solutions from Cardy type formula

Solution space   Coadjoint representation

action of symmetry group on solution space   coadjoint action

$$\mathrm{Ad}^{*}_{(f,\alpha)^{-1}}(j, ic_1; p, ic_2) = \left(\tilde{j}d\phi^2, ic_1; \tilde{p}d\phi^2, ic_2\right)$$

$$\tilde{p} = (f')^2 p \circ f - \frac{c_2}{24\pi} S[f], \quad \longrightarrow$$

energy-momentum tensor of CFT2

$$\tilde{j} = (f')^2 \left[j + \underbrace{\alpha p' + 2\alpha' p}_{\alpha \times p} - \frac{c_2}{24\pi}\alpha''' \right] \circ f - \frac{c_1}{24\pi} S[f]$$

change in orbital part
due to supertranslation

$$p \sim \textcircled{H} \; , \quad j \sim \Xi$$