Title: Why initial system-environment correlations do not imply the failure of complete positivity: a causal perspective

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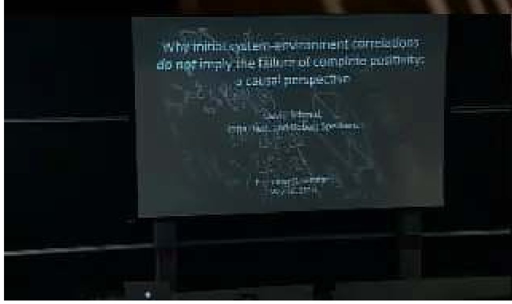
Abstract: <p>When a system interacts with an environment with which it is initially uncorrelated, its evolution is described by a completely positive map. The common wisdom in the field of quantum information theory, however, is that when the system is initially correlated with the environment, the map describing its evolution may fail to be completely positive. This has motivated many researchers to try and characterize this putatively more general sort of dynamics, and even the textbook of Nielsen and Chuang suggests that "It is an interesting problem for further research to study quantum information processing beyond the quantum operations formalism." This talk will demonstrate that this common wisdom is mistaken. The error can be traced to the standard argument for how the evolution map ought to be defined in such circumstances. One can show that anomolous dynamics would arise even in completely classical examples if one were to follow the prescription of the standard argument. The framework of classical causal models specifies how dynamics ought to be defined in such circumstances, and</p>

<p>the quantum analogue of this framework provides the correct definition of the quantum evolution map, which is found to be always completely positive.</p>

<p>Joint work with David Schmid</p>

Why initial system-environment correlations ***do not*** imply the failure of complete positivity: a causal perspective

David Schmid,
Katja Ried, and Robert Spekkens



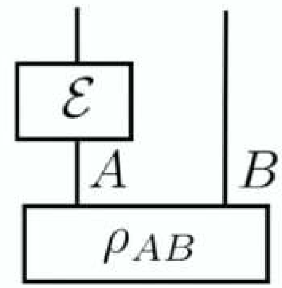
Perimeter QI seminar
May 16, 2018

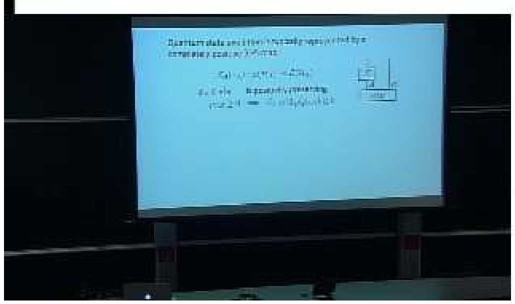
Quantum state evolution is typically represented by a completely positive (CP) map

$$\mathcal{E}_A(\,\cdot_A) : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$$

$\mathcal{E}_A \otimes \mathrm{id}_B$ is positivity preserving

$$\rho_{AB} \geq 0 \implies \mathcal{E}_A \otimes \mathrm{id}_B(\rho_{AB}) \geq 0$$





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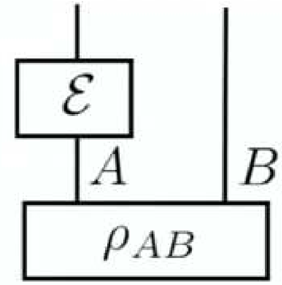
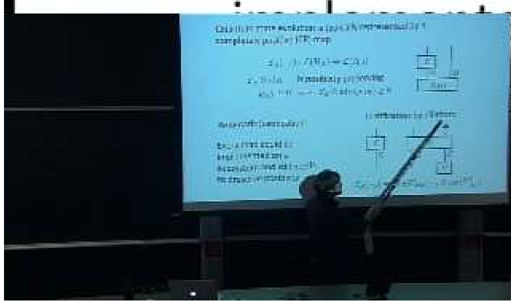
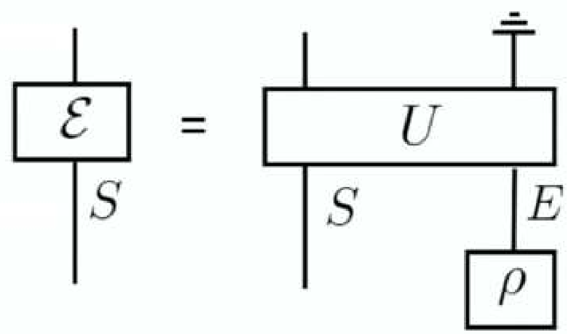


Axiomatic justification:

Every map could be
...ed on a
... and still needs
... positivity

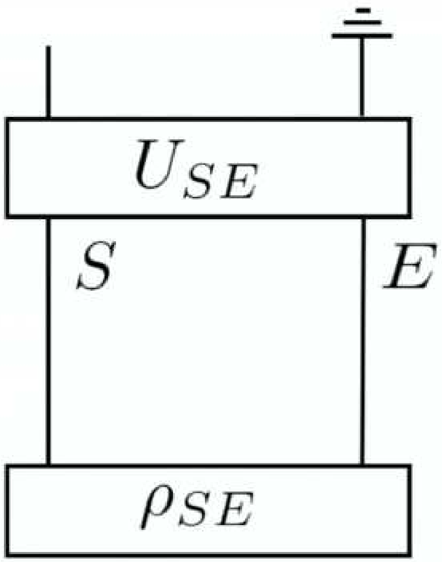


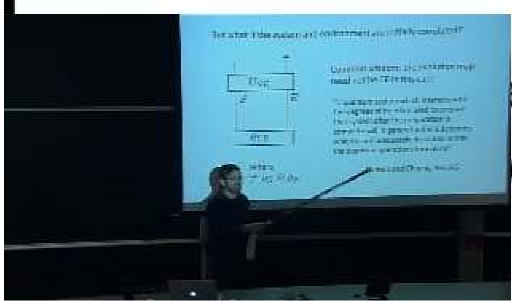
Justification by dilation:



$$\mathcal{E}_S(\,\cdot_S) = \mathrm{Tr}_E[U_{SE}(\,\cdot_S \otimes \rho_E)U_{SE}^\dagger]$$

But what if the system and environment are initially correlated?





ere

$\rho_S \otimes \rho_E$

Common wisdom: the evolution map need not be CP in this case

"a quantum system which interacts with the degrees of freedom used to prepare that system after the preparation is complete will in general suffer a dynamics which is not adequately described within the quantum operations formalism"

---Nielsen and Chuang, Sec. 8.5

Articles endorsing the common wisdom:

[2] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, "Non-markovian quantum processes: Complete framework and efficient characterization," PRA 97, 012127, 2018.

[3] M. Ringbauer, C. J. Wood, K. Modi, A. Gilchrist, A. G. White, and A. Fedrizzi, "Characterizing quantum dynamics with initial system-environment correlations," PRL 114, 090402, 2015.

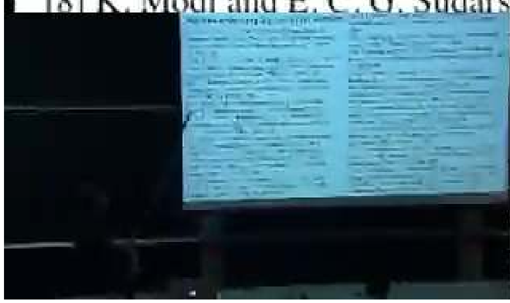
[4] C. A. Rodríguez-Rosario, K. Modi, and A. Aspuru-Guzik, "Linear assignment maps for correlated system-environment states," PRA 81, 012313, 2010.

[5] K. Modi, "Operational approach to open dynamics and quantifying initial correlations," Scientific Reports 2, 2012.

[6] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, "Operational markov condition for quantum processes," PRL, 120, 2018.

[7] K. Modi, "Preparation of states in open quantum mechanics," Open Systems and Information Dynamics, 18, 253, 2011.

[8] K. Modi and E. C. G. Sudarshan, "Role of preparation in



PRAA, 81, 052119, 2010.

nics need not be completely

lies:," PRL, 75, 3021, 1995.

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3020, 1995.

[12] T. F. Jordan, A. Shaji, and E. C. G. Sudarshan, "Dynamics of initially entangled open quantum systems," PRA 70, 052110, 2004.

[13] A. Shaji and E. Sudarshan, "Who's afraid of not completely positive maps?," Physics Letters A 341, 48, 2005.

[14] A. Shabani and D. A. Lidar, "Vanishing quantum discord is necessary and sufficient for completely positive maps", PRL 102, 100402, 2009.

[15] A. Brodutch, A. Datta, K. Modi, A. Rivas, and C. A. Rodríguez-Rosario, "Vanishing quantum discord is not necessary for completely positive maps," PRA 87, 042301, 2013.

[16] H. Hayashi, G. Kimura, and Y. Ota, "Kraus representation in the presence of initial correlations," PRA 67, 062109, 2003.

[17] C. A. Rodríguez-Rosario, K. Modi, A. meng Kuah, A. Shaji, and E. C. G. Sudarshan, "Completely positive maps and classical correlations," Journal of Physics A 41, 205301, 2008.

[18] H. A. Carteret, D. R. Terno, and K. Zyczkowski, "Dynamics beyond completely positive maps: Some properties and applications," PRA 77, 042113,, 2008.

[19] P. Štelmachovic and V. Bužek, "Dynamics of open quantum systems initially entangled with environment: Beyond the Kraus representation," PRA, 64, 062106, 2001.

[20] K. M. F. Romero, P. Talkner, and P. Hänggi, "Is the dynamics of open quantum systems always linear?," PRA, 69, 052109, 2004.

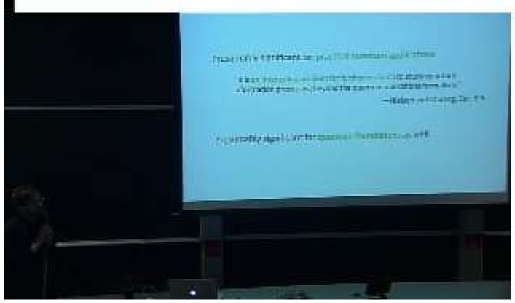
[21] T. F. Jordan, A. Shaji, and E. C. G. Sudarshan, "Maps for Lorentz transformations of spin," PRA 73, 032104, 2006.

Presumably significant for practical quantum applications

"It is an interesting problem for further research to study quantum information processing beyond the quantum operations formalism."

---Nielsen and Chuang, Sec. 8.5

Presumably significant for quantum foundations as well



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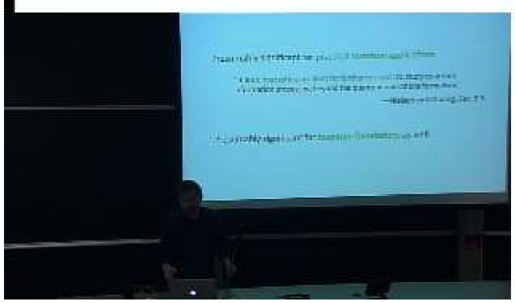


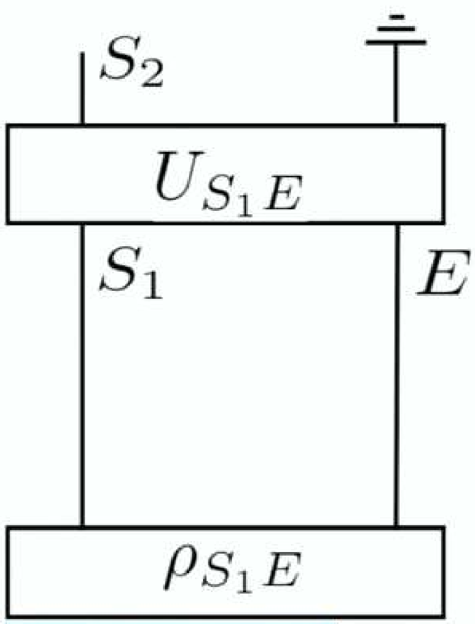
This talk:

The common wisdom is **mistaken**

Not an interesting problem for further research

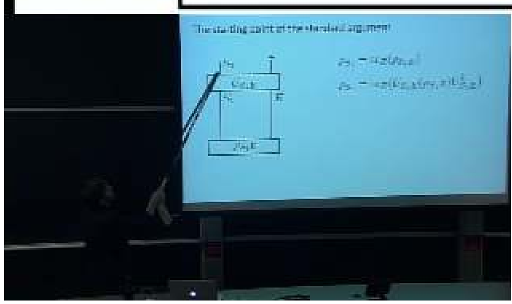
No foundational significance

The starting point of the standard argument



$$\rho_{S_1} = \mathrm{tr}_E(\rho_{S_1E})$$

$$\rho_{S_2} = \mathrm{tr}_E\big(U_{S_1E}(\rho_{S_1E})U^\dagger_{S_1E}\big)$$



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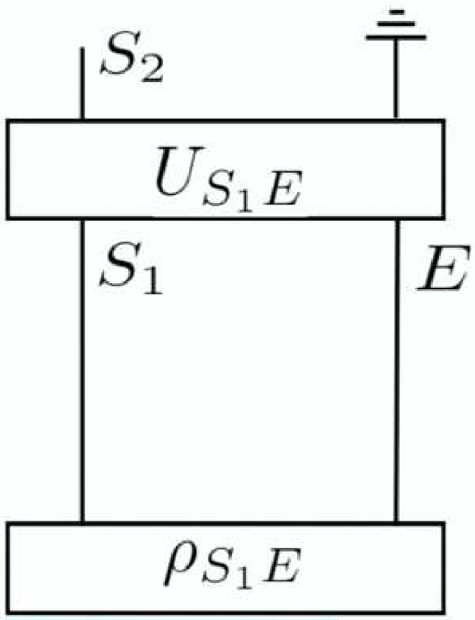
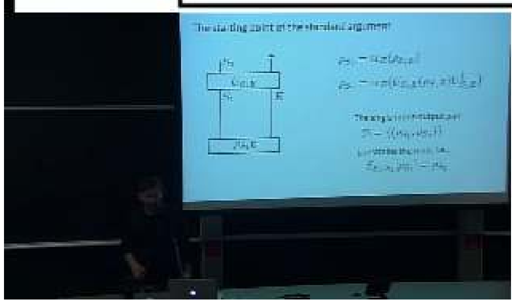




$$\rho_{S_1} = \mathrm{tr}_E(\rho_{S_1E})$$

$$\rho_{S_2} = \mathrm{tr}_E\left(U_{S_1E}(\rho_{S_1E})U^\dagger_{S_1E}\right)$$

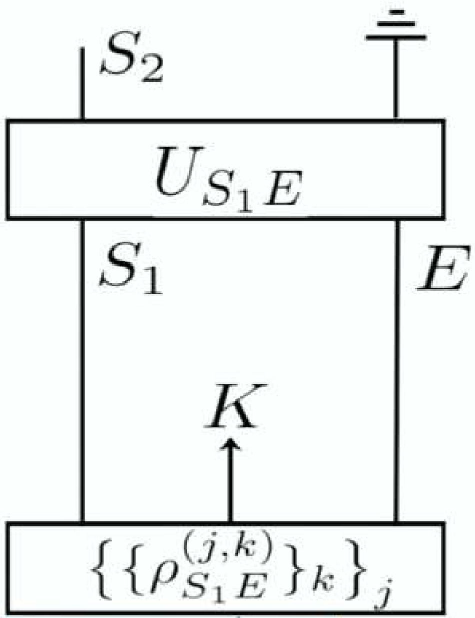
The single input-output pair

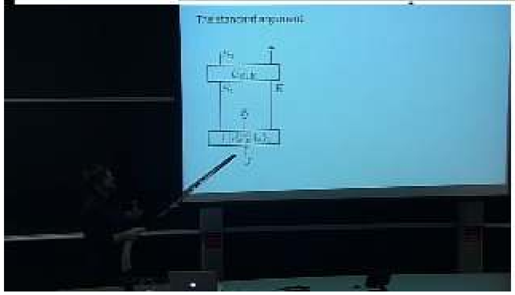
$$\mathfrak{R} = \{(\rho_{S_1}, \rho_{S_2})\}$$

constrains the map, i.e.,

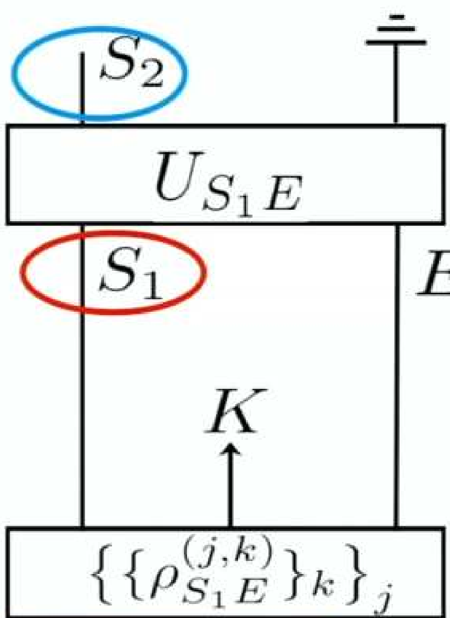
$$\mathcal{E}_{S_2|S_1}[\rho_{S_1}] = \rho_{S_2}$$

The standard argument





The standard argument



$$J = j, K = k : \ \rho^{(jk)}_{S_1 E}$$

$$\rho^{(jk)}_{S_1} = \mathrm{tr}_E(\rho^{(jk)}_{S_1 E})$$

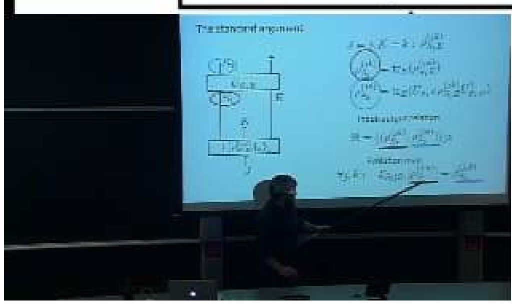
$$\rho^{(jk)}_{S_2} = \mathrm{tr}_E(U_{S_1 E}\rho^{(jk)}_{S_1 E}U^{\dagger}_{S_1 E})$$

Input-output relation

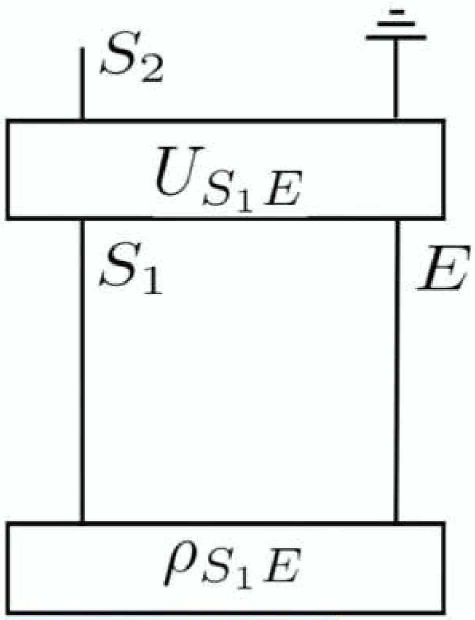
$$\mathfrak{R} = \{(\rho^{(jk)}_{S_1}, \rho^{(jk)}_{S_2})\}_{jk}$$

Evolution map

$$\forall j, k : \quad \mathcal{E}_{S_2|S_1}[\rho^{(j,k)}_{S_1}] = \rho^{(j,k)}_{S_2}$$



The starting point of the standard argument



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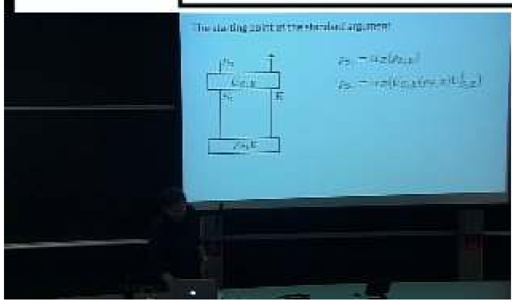

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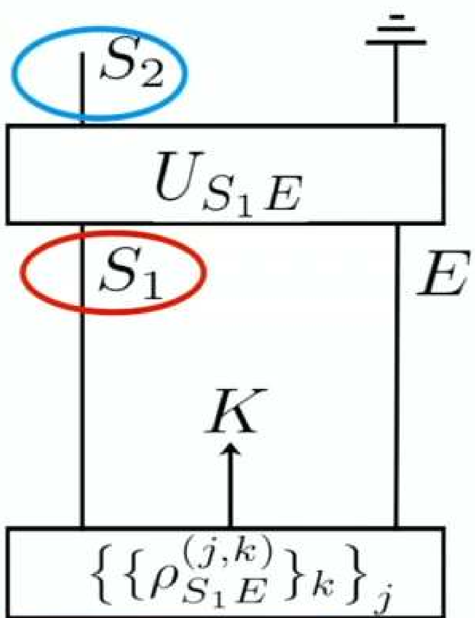

$$J = j, K = k : \ \rho_{S_1E}^{(jk)}$$

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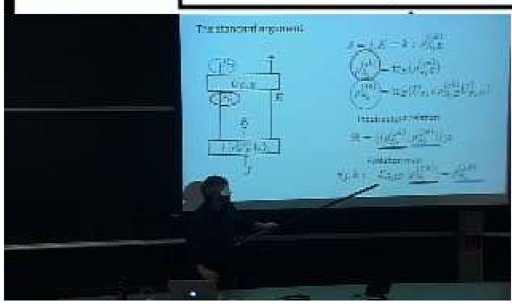
Input-output relation

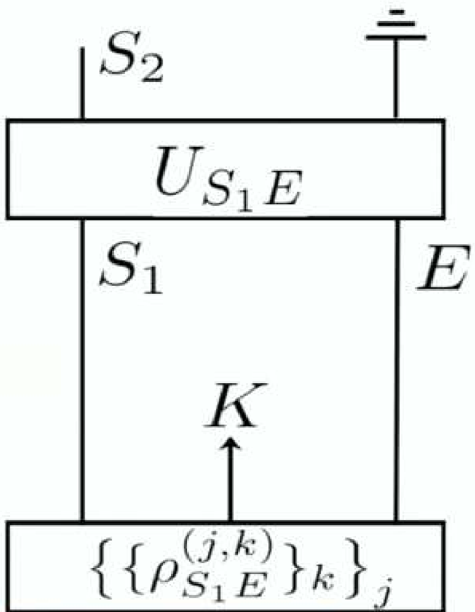
$$\mathfrak{R} = \{(\rho_{S_1}^{(jk)}, \rho_{S_2}^{(jk)})\}_{jk}$$

Evolution map

$$\forall j, k : \quad \mathcal{E}_{S_2|S_1}[\rho_{S_1}^{(j,k)}] = \rho_{S_2}^{(j,k)}$$



The standard argument

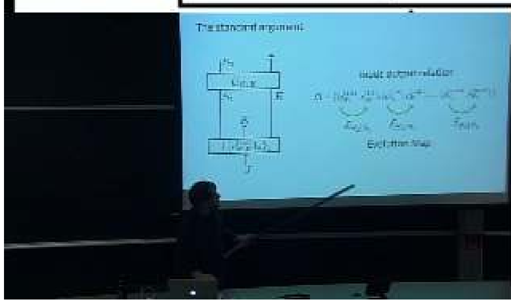


Input-output relation

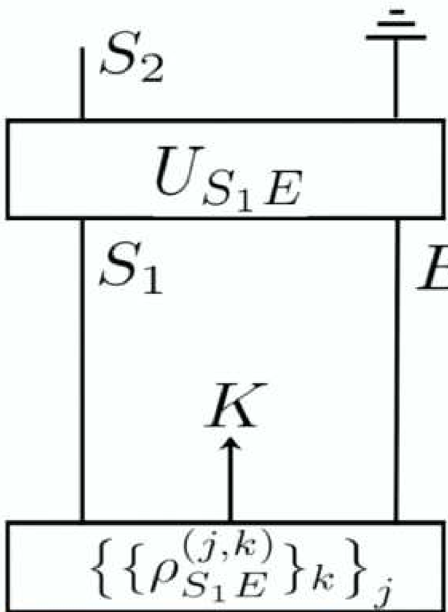
$$\mathfrak{R} = \{(\rho_{S_1}^{(1,1)}, \rho_{S_2}^{(1,1)}), (\rho_{S_1}^{(1,2)}, \rho_{S_2}^{(1,2)}), \dots (\rho_{S_1}^{(n,m)}, \rho_{S_2}^{(n,m)})\}$$

$\mathcal{E}_{S_2|S_1}$ $\mathcal{E}_{S_2|S_1}$ $\mathcal{E}_{S_2|S_1}$

Evolution Map



The standard argument

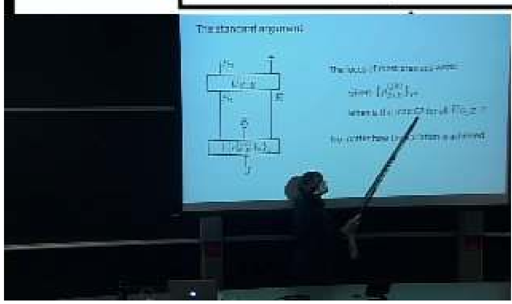


The focus of most previous work:

Given $\{\rho^{(jk)}_{S_1E}\}_{jk}$

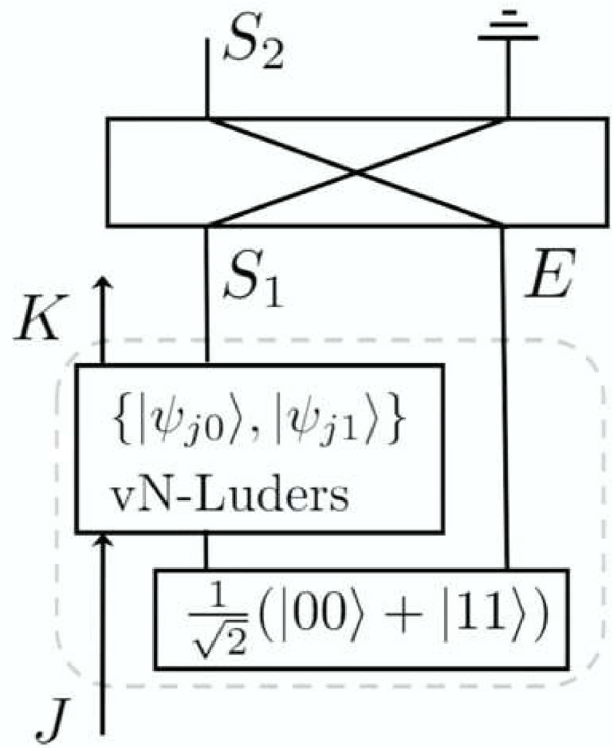
when is the map CP for all U_{S_1E} ?

No matter how the variation is achieved



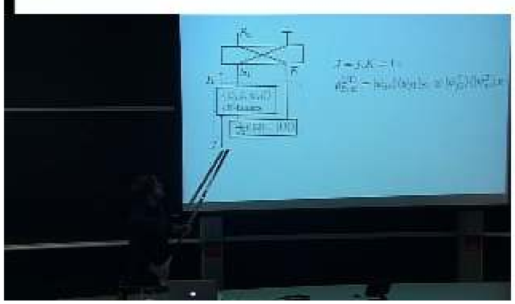
Problematic consequences of the standard argument

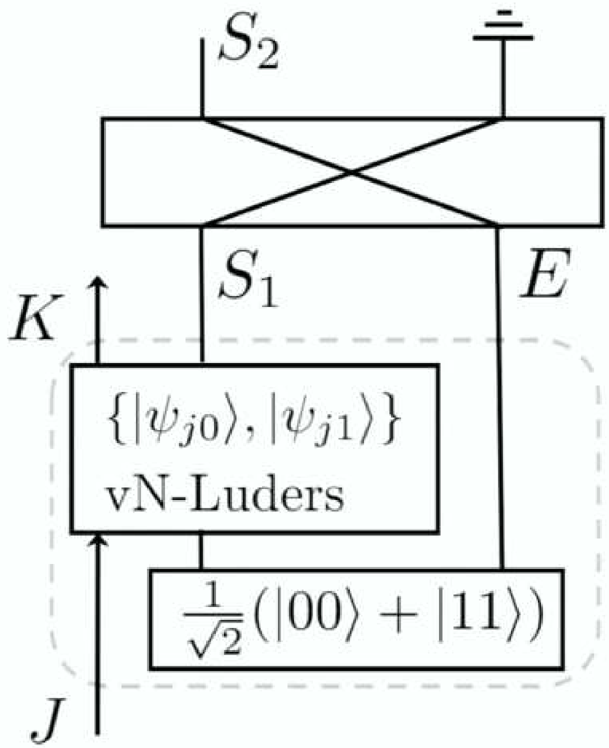
Example 1:
Failure of complete positivity



$J = j, K = 1:$

$$\rho^{(j1)}_{S_1 E} = |\psi_{j1}\rangle\langle\psi_{j1}|_{S_1} \otimes |\psi^T_{j1}\rangle\langle|\psi^T_{j1}|_E$$





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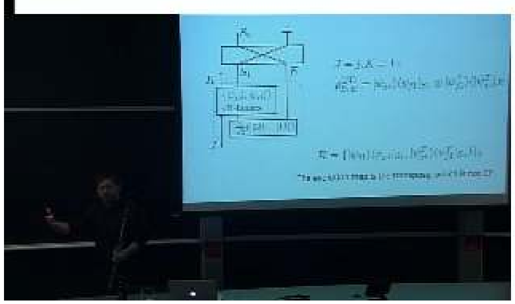
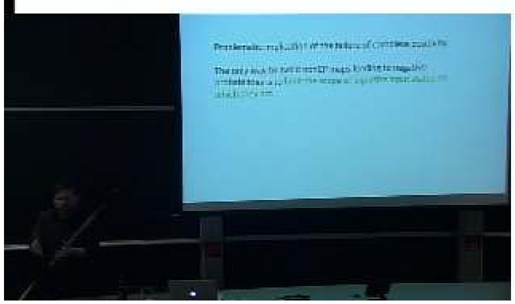


$$\mathcal{R} = \{|\psi_{j1}\rangle\langle\psi_{j1}|_{S_1}, |\psi^T_{j1}\rangle\langle\psi^T_{j1}|_{S_2})\}_j$$

The evolution map is the transpose, which is not CP

Problematic implication of the failure of complete positivity

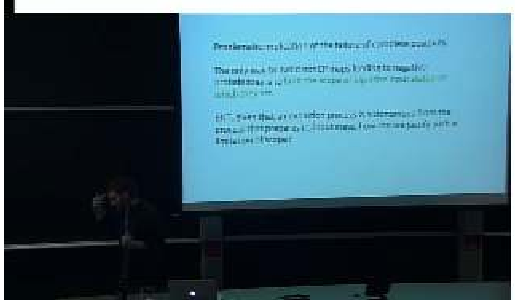
The only way to avoid nonCP maps leading to negative probabilities is to limit the scope of bipartite input states on which they act.

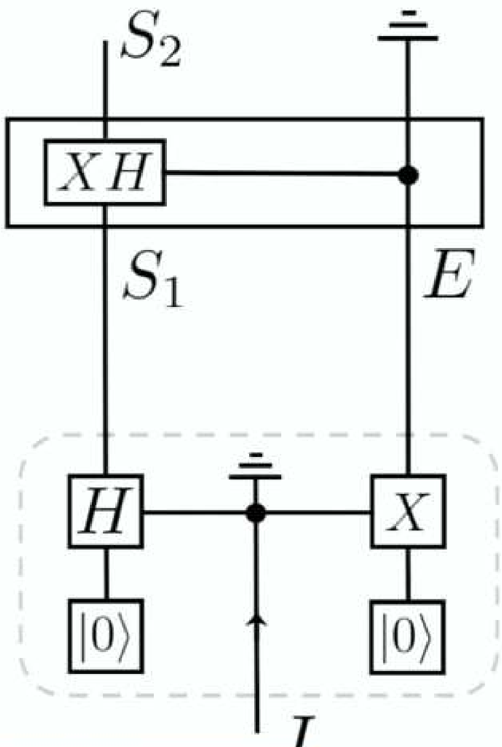


Problematic implication of the failure of complete positivity

The only way to avoid nonCP maps leading to negative probabilities is to limit the scope of bipartite input states on which they act.

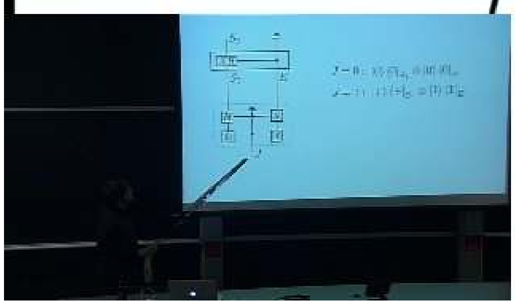
BUT, given that an evolution process is autonomous from the process that prepares its input state, how can we justify such a limitation of scope?

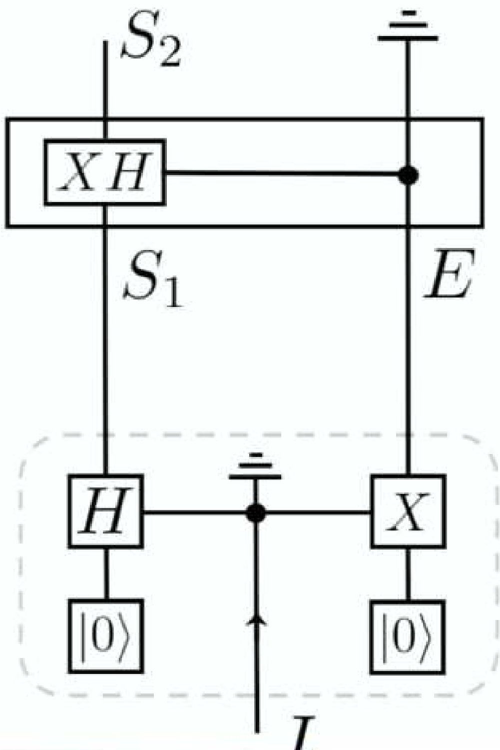




$$J = 0: \ |0\rangle \langle 0|_{S_1} \otimes |0\rangle \langle 0|_E$$

$$J = 1: \ |+\rangle \langle +|_{S_1} \otimes |1\rangle \langle 1|_E$$





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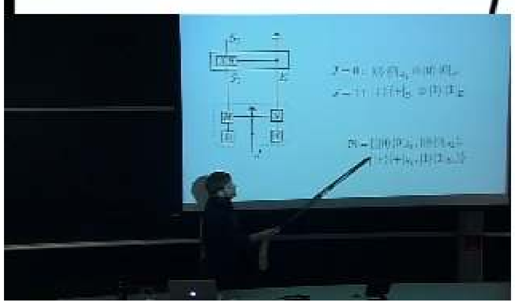
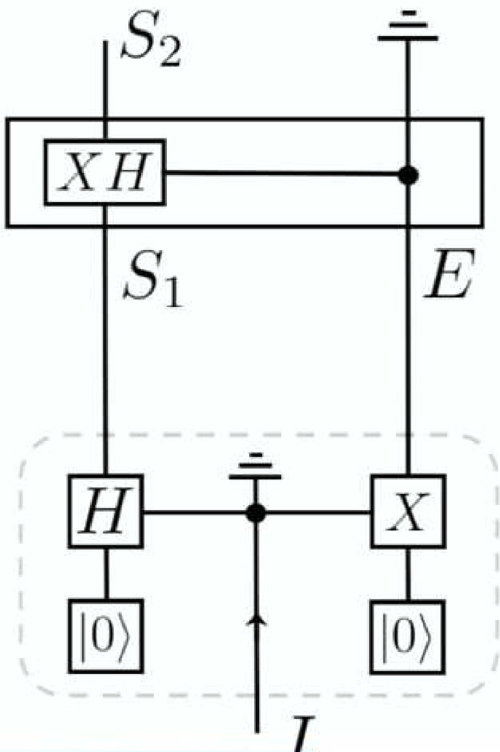
$$\mathfrak{R} = \{ (|0\rangle\langle 0|_{S_1}, |0\rangle\langle 0|_{S_2}), \\ (|+\rangle\langle +|_{S_1}, |1\rangle\langle 1|_{S_2}) \}$$





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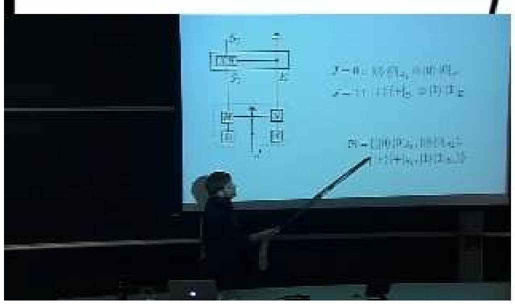
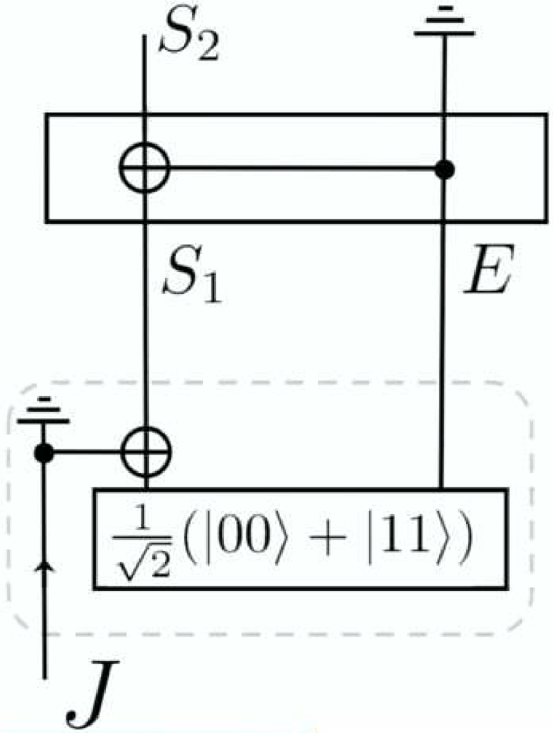


$$\mathfrak{R} = \{(|0\rangle\langle 0|_{S_1}, |0\rangle\langle 0|_{S_2}), \\ (|+\rangle\langle +|_{S_1}, |1\rangle\langle 1|_{S_2})\}$$

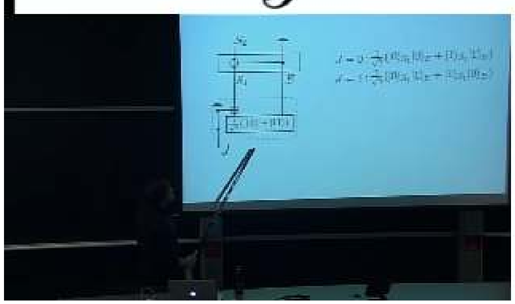
Problematic implications of the failure of linearity:

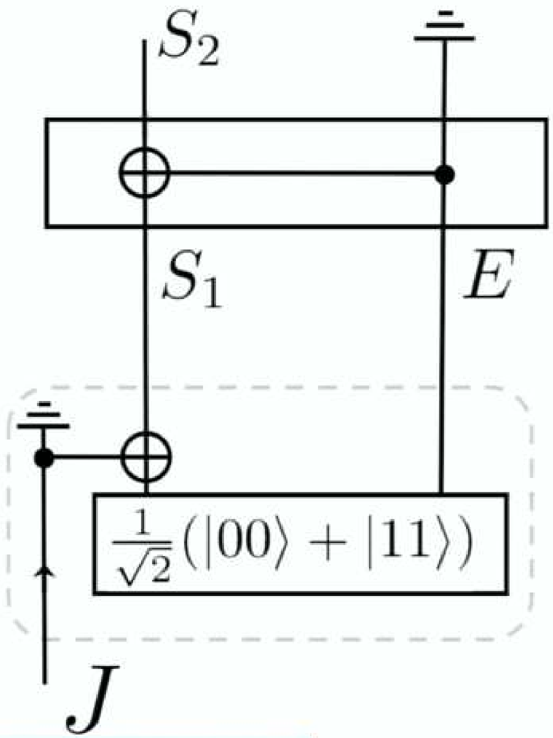
Evolution is a kind of data processing and should not be able to increase the information content of a system!



$$J = 0 : \frac{1}{\sqrt{2}}(|0\rangle_{S_1}|0\rangle_E + |1\rangle_{S_1}|1\rangle_E)$$

$$J = 1 : \frac{1}{\sqrt{2}}(|0\rangle_{S_1}|1\rangle_E + |1\rangle_{S_1}|0\rangle_E)$$





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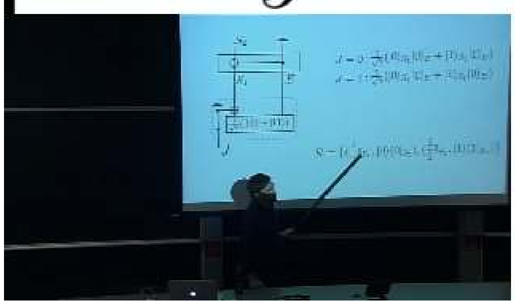
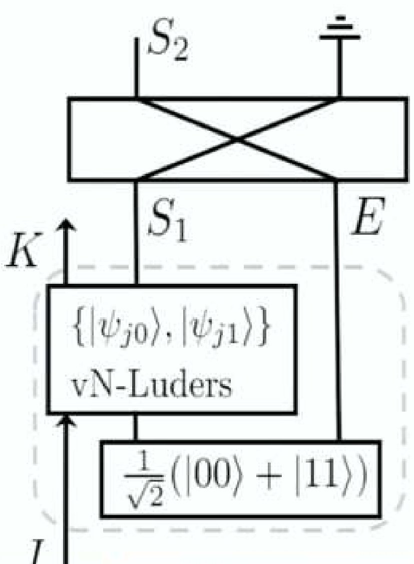
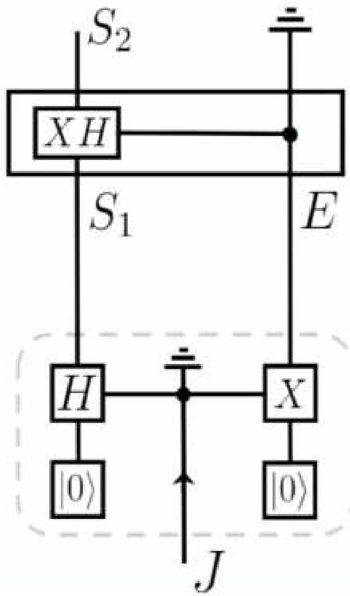
$$J = 1 : \frac{1}{\sqrt{2}}(|0\rangle_{S_1}|1\rangle_E + |1\rangle_{S_1}|0\rangle_E)$$

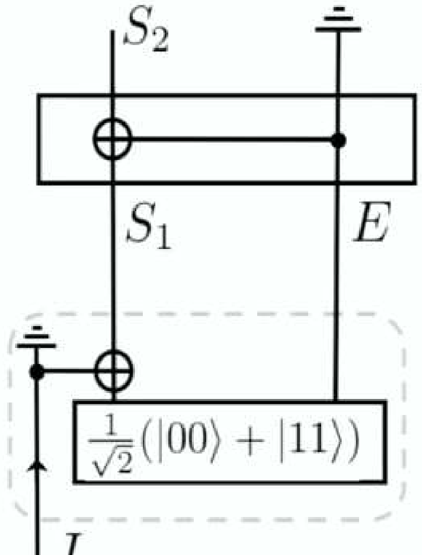
$$\mathcal{R} = \{(\frac{1}{2}\mathbb{I}_{S_1}, |0\rangle\langle 0|_{S_2}), (\frac{1}{2}\mathbb{I}_{S_1}, |1\rangle\langle 1|_{S_2})\}$$



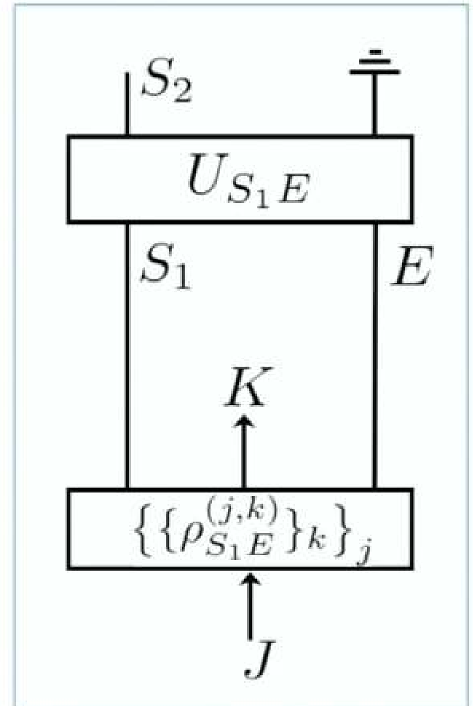




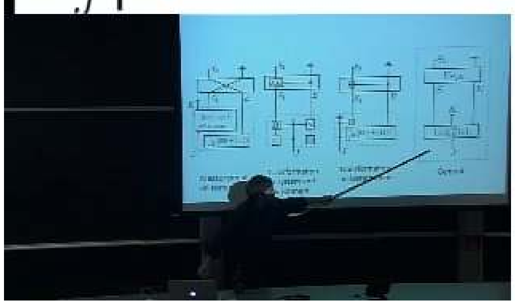
Transformation on system and environment



Transformation on system alone

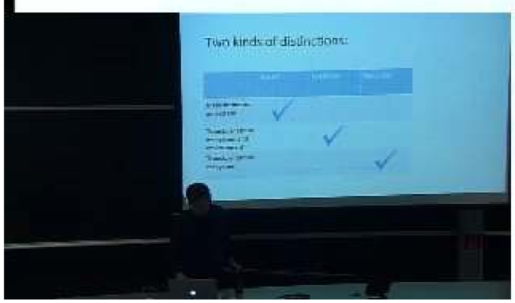


General



Two kinds of distinctions:

	Not CP	Not linear	Not a map
Measurements on system	✓		
Transformations on system and environment		✓	
			✓



Pathologies of the standard argument are generic

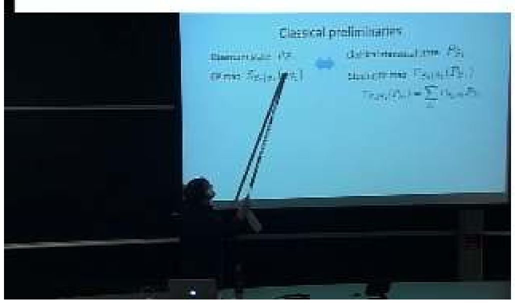
	Not CP	Not linear	Not a map
Measurements on system	✓	✓	✓
Transformations on system and environment	✓	✓	✓
	✓	✓	✓

Classical preliminaries

Quantum state ρ_{S_1} ⟺ Classical statistical state P_{S_1}

CP map $\mathcal{E}_{S_2|S_1}(\rho_{S_1})$ ⟺ Stochastic map $\Gamma_{S_2|S_1}(P_{S_1})$

$$\Gamma_{S_2|S_1}(P_{S_1}) = \sum_{S_1} P_{S_2|S_1} P_{S_1}$$



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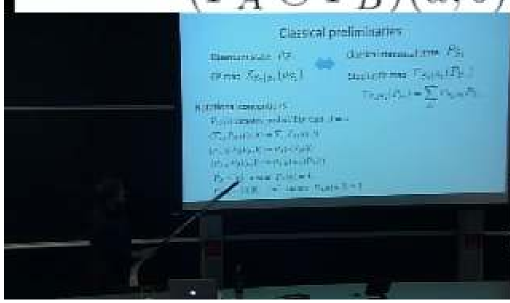
Notational conventions:

$P_A(a)$ denotes probability that $A = a$

$(\sum_A P_{AB})(a,b) := \sum_a P_{AB}(a,b)$

$(P_A \otimes P_B)(a,b) := P_A(a)P_B(b)$



$:= P_{A|B}(a,b)P_B(b)$

ns $P_A(a) = 1,$

$\equiv [ab]$ means $P_{A,B}(a,b) = 1$

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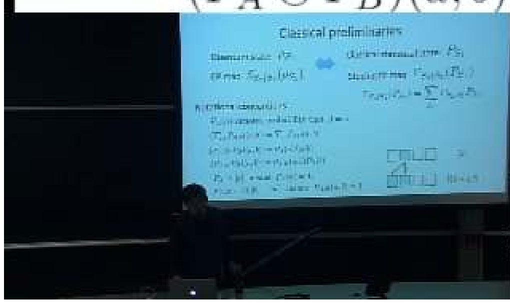

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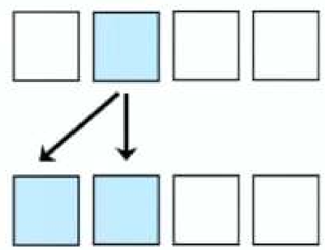
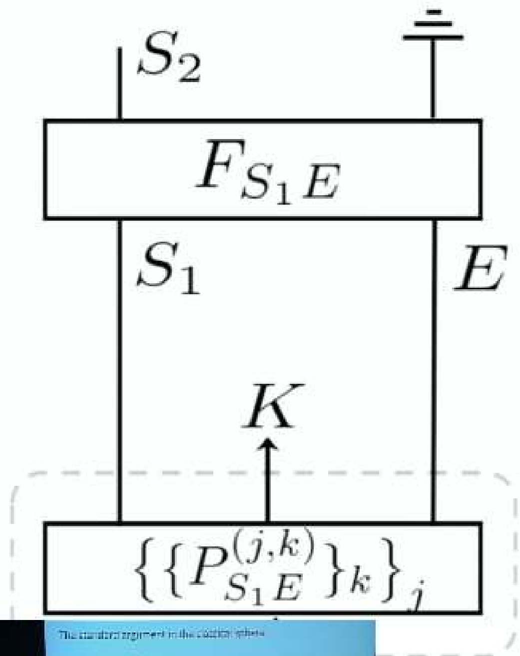
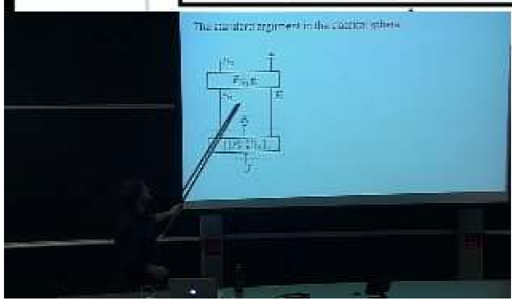


$[2]$

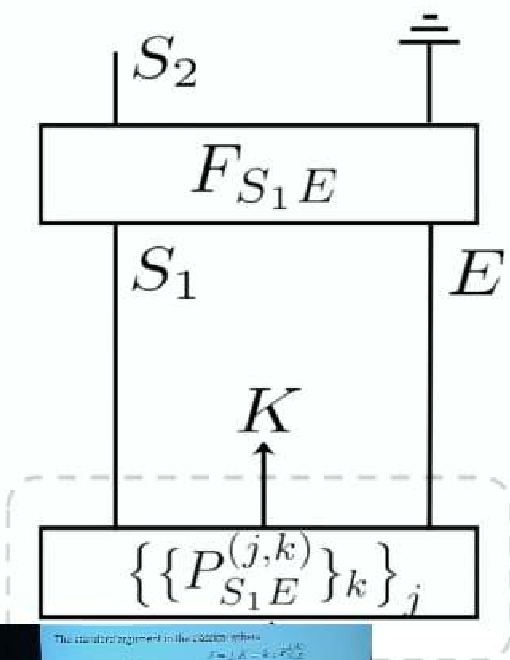
$\frac{1}{2}[1] + \frac{1}{2}[2]$

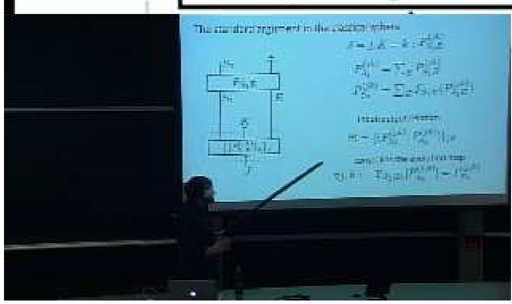
The standard argument in the classical sphere





The standard argument in the classical sphere





$$J = j, K = k : \; P^{(jk)}_{S_1E}$$

$$P^{(jk)}_{S_1} = \sum_E P^{(jk)}_{S_1E}$$

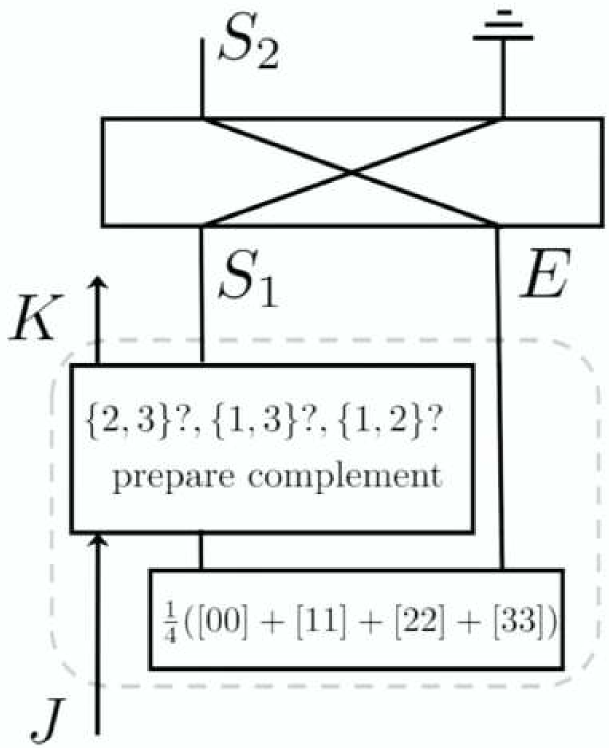
$$P^{(jk)}_{S_2} = \sum_E F_{S_1E}(P^{(jk)}_{S_1E})$$

Input-output relation

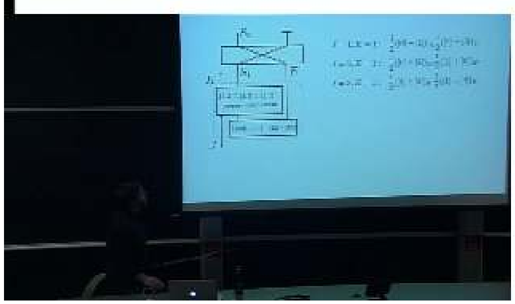
$$\mathfrak{R} = \{(P^{(jk)}_{S_1}, P^{(jk)}_{S_2})\}_{jk}$$

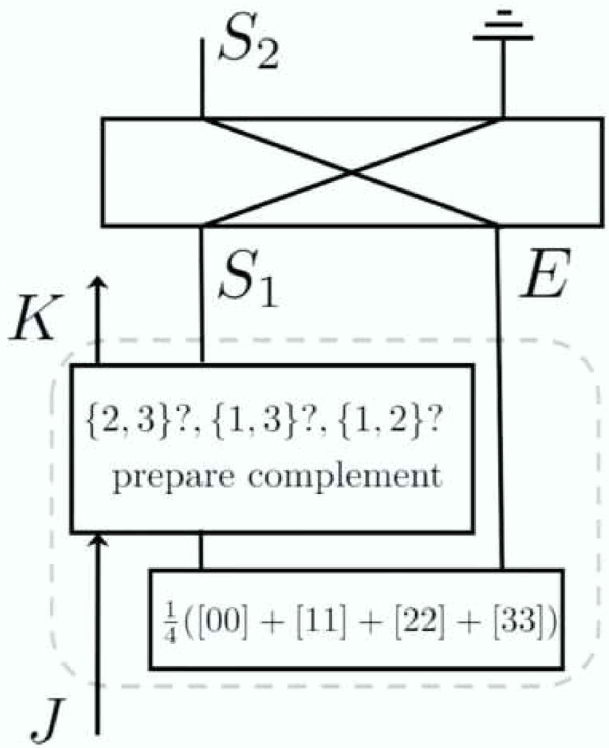
constrains the evolution map

$$\forall j, k : \quad \Gamma_{S_2|S_1}[P^{(j,k)}_{S_1}] = P^{(j,k)}_{S_2}$$



$$J=1, K=1: \quad \frac{1}{2}([0]+[1])_{S_1}\frac{1}{2}([2]+[3])_E$$
$$J=2, K=1: \quad \frac{1}{2}([0]+[2])_{S_1}\frac{1}{2}([1]+[3])_E$$
$$J=3, K=1: \quad \frac{1}{2}([0]+[3])_{S_1}\frac{1}{2}([1]+[2])_E$$



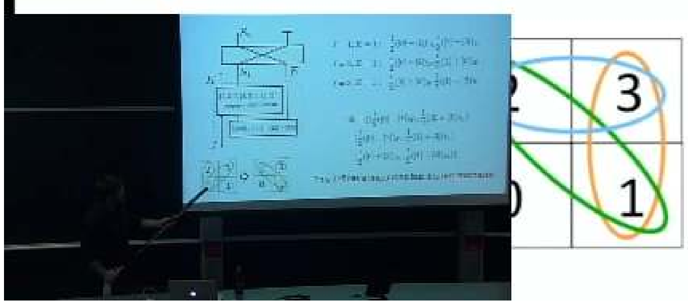


$$J=1, K=1: \quad \frac{1}{2}([0]+[1])_{S_1}\frac{1}{2}([2]+[3])_E$$

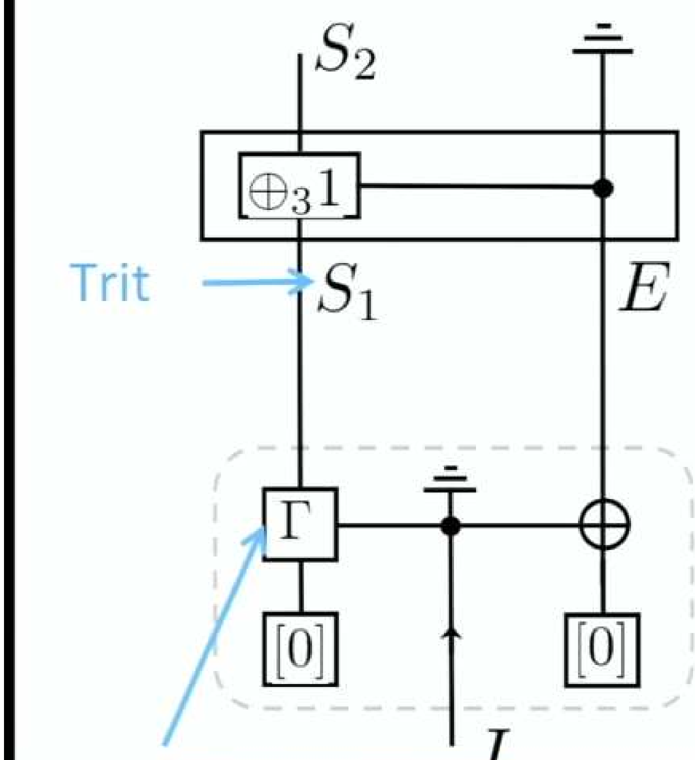
$$J=2, K=1: \quad \frac{1}{2}([0]+[2])_{S_1}\frac{1}{2}([1]+[3])_E$$

$$J=3, K=1: \quad \frac{1}{2}([0]+[3])_{S_1}\frac{1}{2}([1]+[2])_E$$

$$\mathfrak{R} = \{(\frac{1}{2}([0]+[1])_{S_1}, \frac{1}{2}([2]+[3])_{S_2})$$
$$(\frac{1}{2}([0]+[2])_{S_1}, \frac{1}{2}([1]+[3])_{S_2})$$
$$(\frac{1}{2}([0]+[3])_{S_1}, \frac{1}{2}([1]+[2])_{S_2})\}$$



This defines a linear map but it is not stochastic

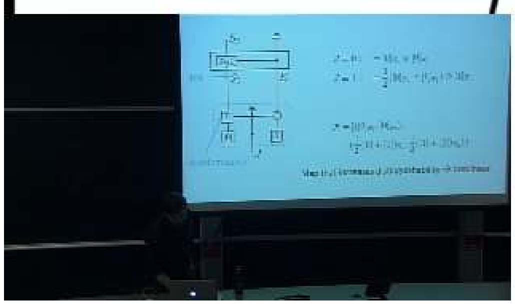


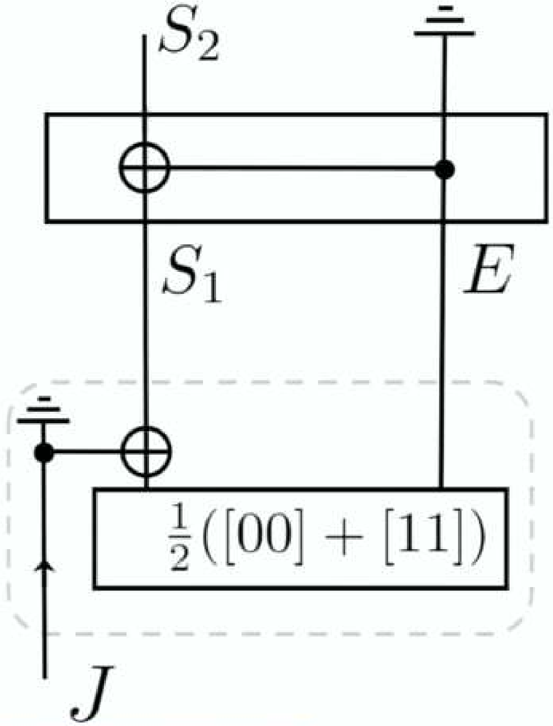
$$J = 0: \quad = [0]_{S_1} \otimes [0]_E$$

$$J = 1: \quad = \frac{1}{2}([0]_{S_1} + [1]_{S_1}) \otimes [1]_E,$$

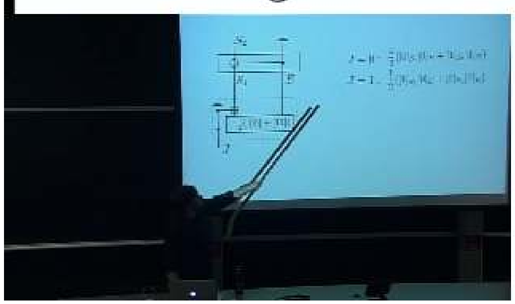
$$\mathfrak{R} = \{([0]_{S_1}, [0]_{S_2}), \\ (\frac{1}{2}([0] + [1])_{S_1}, \frac{1}{2}([1] + [2])_{S_2})\}$$

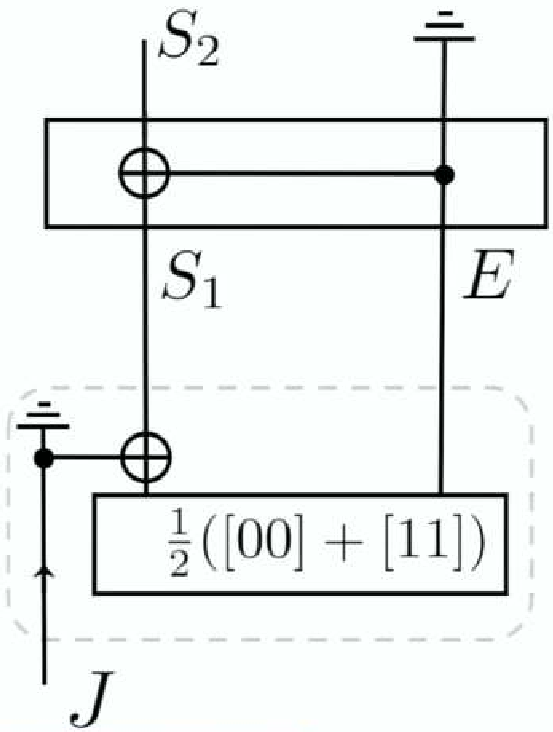
Map that increases distinguishability → nonlinear





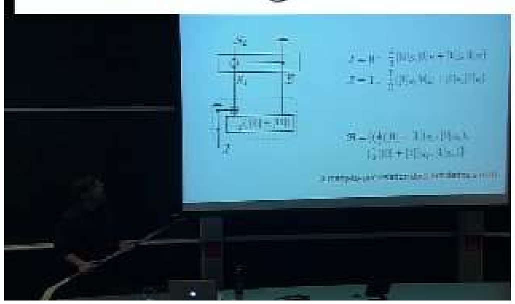
$$J = 0: \quad \frac{1}{2}\big([0]_{S_1}[0]_E + [1]_{S_1}[1]_E\big)$$
$$J = 1: \quad \frac{1}{2}\big([1]_{S_1}[0]_E + [0]_{S_1}[1]_E\big)$$





$$J=0: \ \frac{1}{2}\left([0]_{S_1}[0]_E+[1]_{S_1}[1]_E\right)$$
$$J=1: \ \frac{1}{2}\left([1]_{S_1}[0]_E+[0]_{S_1}[1]_E\right)$$

$$\mathfrak{R}=\{(\tfrac{1}{2}([0]+[1])_{S_1},[0]_{S_2}),\\ (\tfrac{1}{2}([0]+[1])_{S_1},[1]_{S_2})\}$$

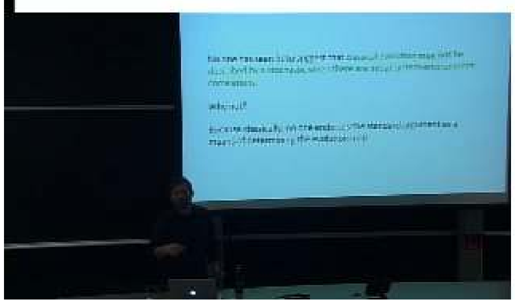


A many-to-one relation does not define a map

No one has seen fit to suggest that classical evolution **may not** be described by a stochastic when there are initial system-environment correlations

Why not?

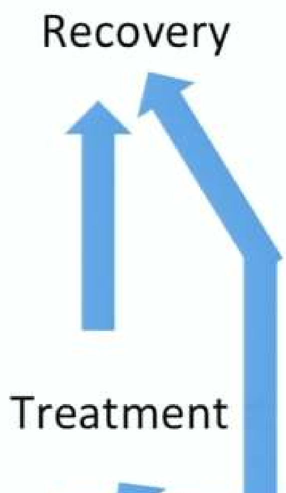
Because classically, no one endorses the standard argument as a means of determining the evolution map



The framework of classical causal models and the do-calculus

See: J. Pearl, *Causality*

Causal model

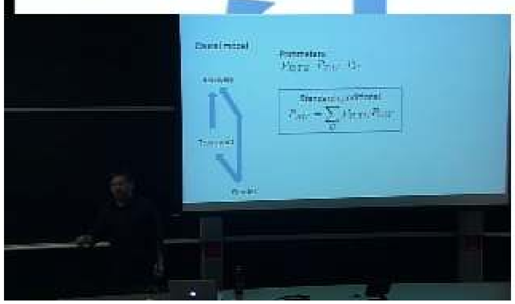


Parameters:

$$P_{R|TG} \quad P_{T|G} \quad P_G$$

Standard conditional

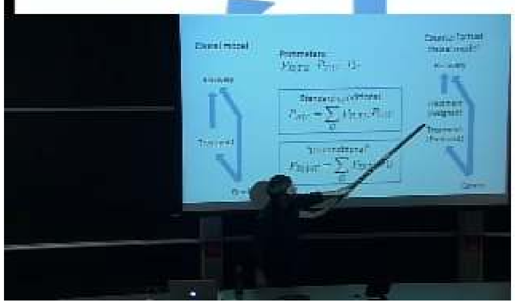
$$P_{R|T} = \sum_G P_{R|TG} P_{G|T}$$



Causal model

Recovery

Treatment



Parameters:

$P_{R|TG} \quad P_{T|G} \quad P_G$

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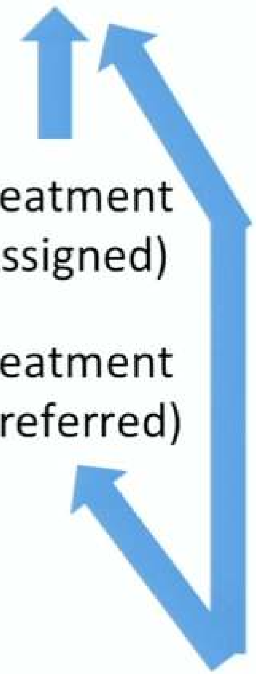
$$P_{R|T} = \sum_G P_{R|TG} P_{G|T}$$

"Do conditional"

$$P_{R|\mathrm{do}T} = \sum_G P_{R|TG} P_G$$

Counterfactual causal model

Recovery

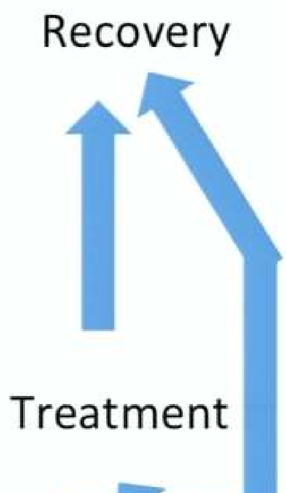


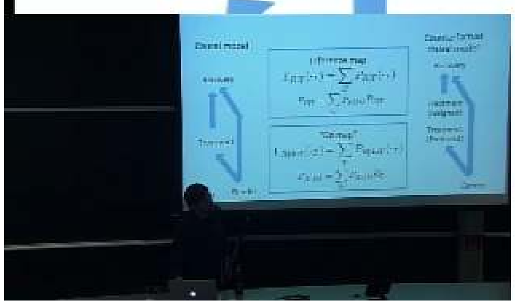
Treatment (Assigned)

Treatment (Preferred)

Gender

Causal model





Inference map

$$\Gamma_{R|T}(\cdot_T) = \sum_T P_{R|T}(\cdot_T)$$

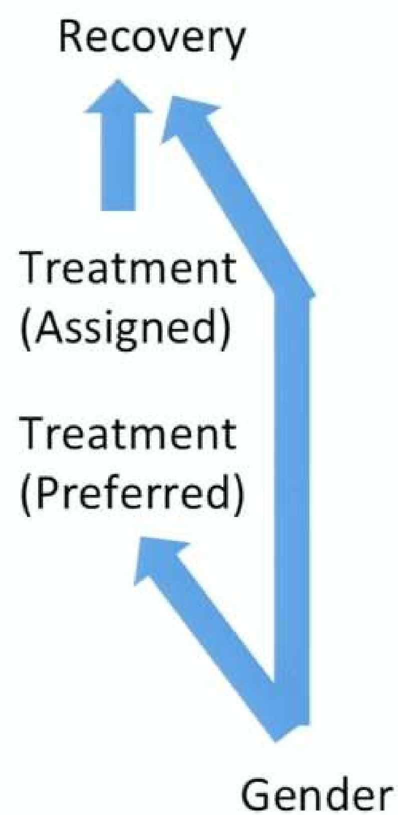
$$P_{R|T} = \sum_G P_{R|TG} P_{G|T}$$

"Do map"

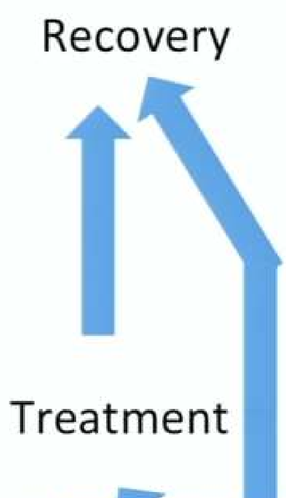
$$\Gamma_{R|\mathrm{do}T}(\cdot_T) = \sum_T P_{R|\mathrm{do}T}(\cdot_T)$$

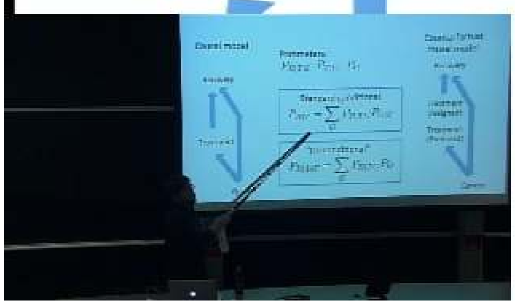
$$P_{R|\mathrm{do}T} = \sum_G P_{R|TG} P_G$$

Counterfactual causal model



Causal model





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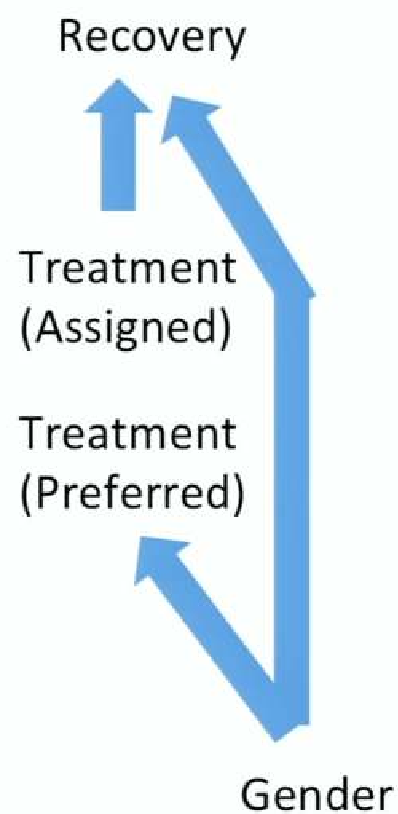
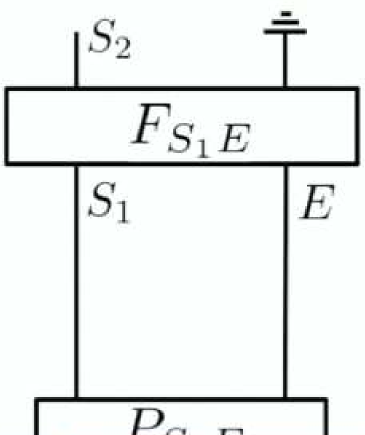
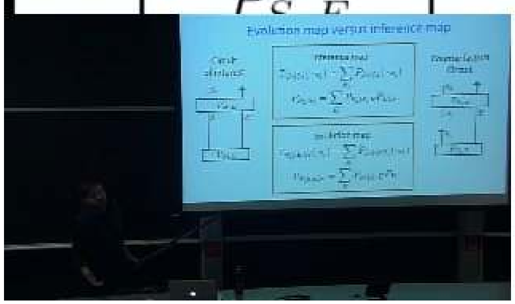


Evolution map versus inference map

Circuit of interest





Inference map

$$\Gamma_{S_2|S_1}(\,\cdot S_1) = \sum_{S_1} P_{S_2|S_1}(\,\cdot S_1)$$

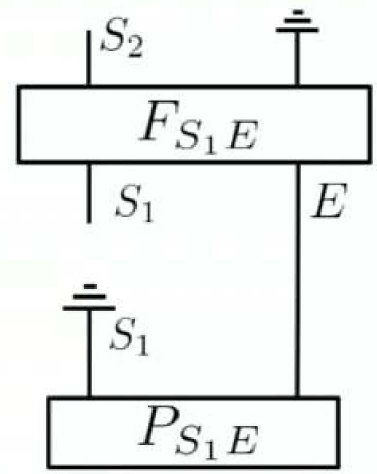
$$P_{S_2|S_1} = \sum_E P_{S_2|S_1E} P_{E|S_1}$$

Evolution map

$$\Gamma_{S_2|\mathrm{do}S_1}(\cdot S_1) = \sum_{S_1} P_{S_2|\mathrm{do}S_1}(\cdot S_1)$$

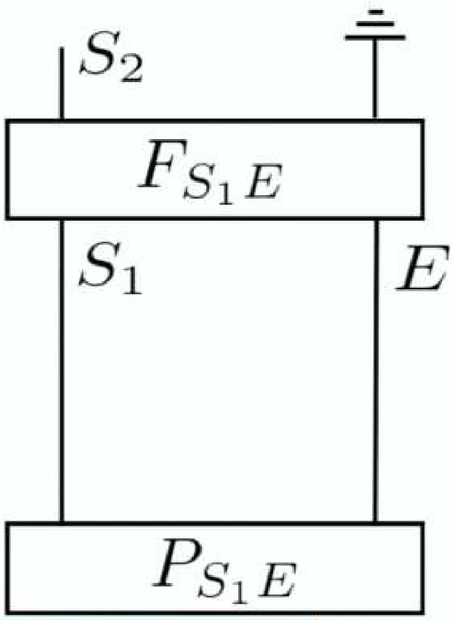
$$P_{S_2|\mathrm{do}S_1} = \sum_E P_{S_2|S_1E} P_E$$

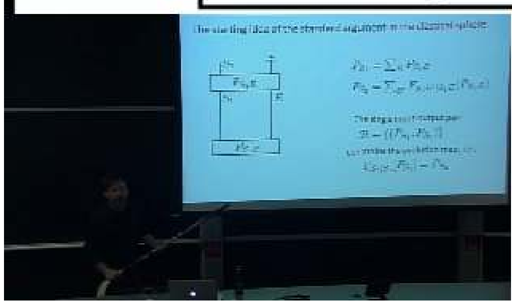
Counterfactual Circuit



The mistaken assumptions of the standard argument

The starting idea of the standard argument in the classical sphere





$$P_{S_1} = \sum_E P_{S_1 E}$$

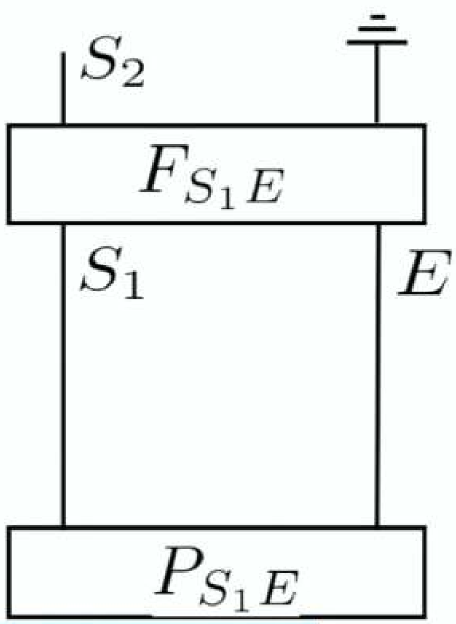
$$P_{S_2} = \sum_{E'} F_{S_2 E' | S_1 E}(P_{S_1 E})$$

The single input-output pair

$$\mathfrak{R} = \{(P_{S_1}, P_{S_2})\}$$

constrains the evolution map, i.e.,

$$\Gamma_{S_2 | S_1}[P_{S_1}] = P_{S_2}$$



Classical probability theory dictates that

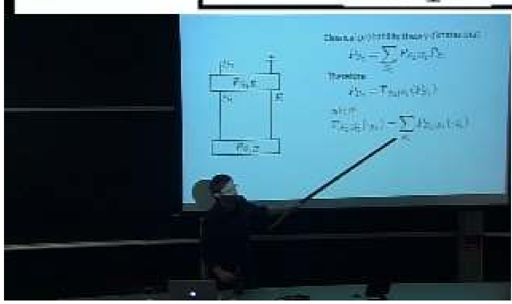
$$P_{S_2} = \sum_{S_1} P_{S_2|S_1} P_{S_1}$$

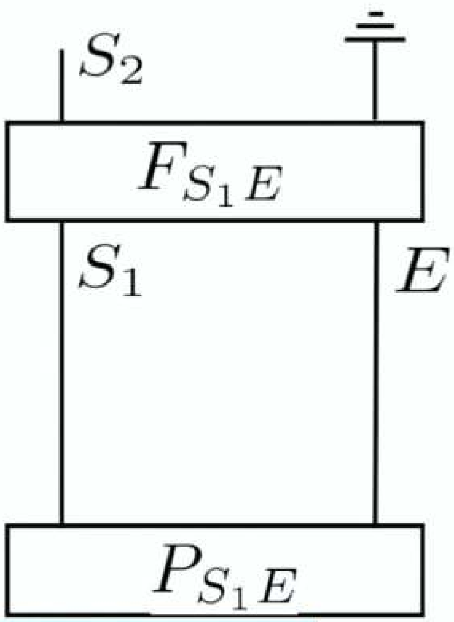
Therefore

$$P_{S_2} = \Gamma_{S_2|S_1}(P_{S_1})$$

where

$$\Gamma_{S_2|S_2}(\cdot_{S_1}) = \sum_{S_1} P_{S_2|S_1}(\cdot_{S_1})$$





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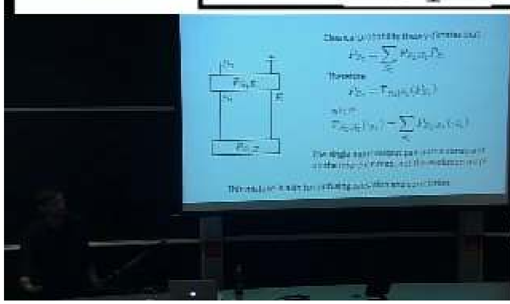
where

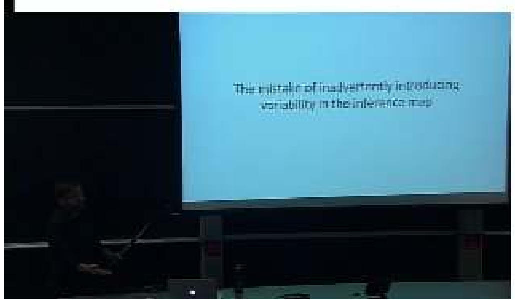
$$\Gamma_{S_2|S_2}(\cdot_{S_1}) = \sum_{S_1} P_{S_2|S_1}(\cdot_{S_1})$$

The single input-output pair puts a constraint on the inference map, not the evolution map!



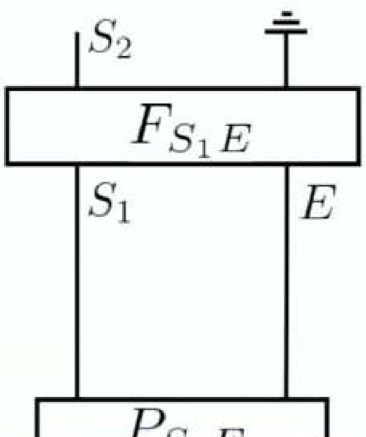
ake is akin to confusing causation and correlation

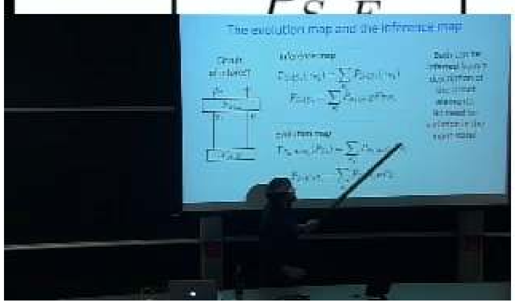
The mistake of inadvertently introducing variability in the inference map



The evolution map and the inference map

Circuit of interest





Inference map

$$\Gamma_{S_2|S_1}(\,\cdot\, s_1) = \sum_{S_1} P_{S_2|S_1}(\,\cdot\, s_1)$$

$$P_{S_2|S_1} = \sum_{E} P_{S_2|S_1 E} P_{E|S_1}$$

Evolution map

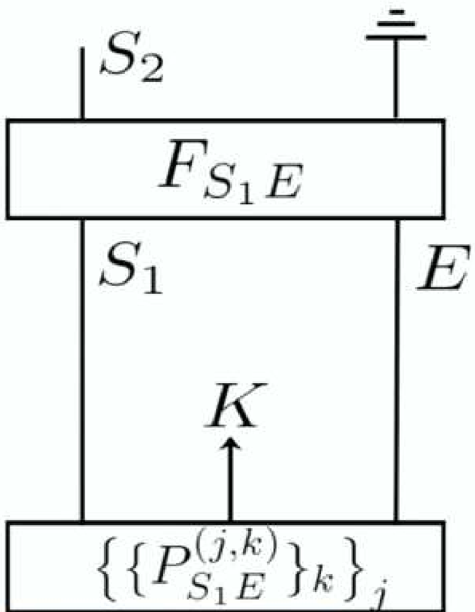
$$\Gamma_{S_2|\mathrm{do}S_1}(P_{S_1}) = \sum_{S_1} P_{S_2|\mathrm{do}S_1} P_{S_1}$$

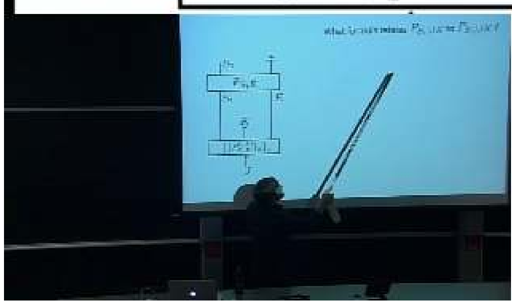
$$P_{S_2|\mathrm{do}S_1} = \sum_{E} P_{S_2|S_1 E} P_{E}$$

Both can be inferred from a description of the circuit elements.

No need for variation in the input state!

What formula relates $P_{S_1|JK}$ to $P_{S_2|JK}$?





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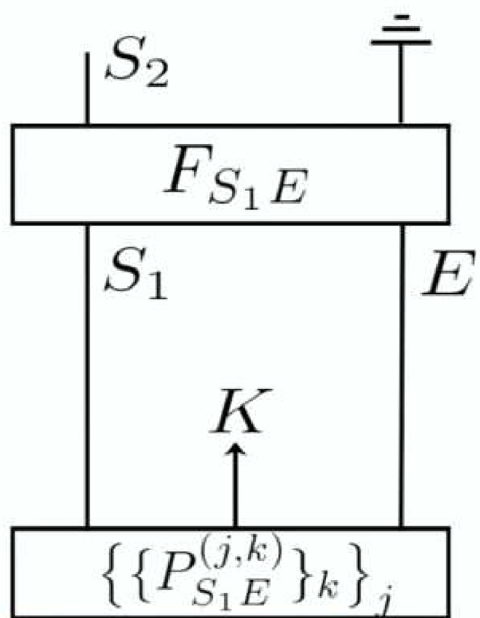
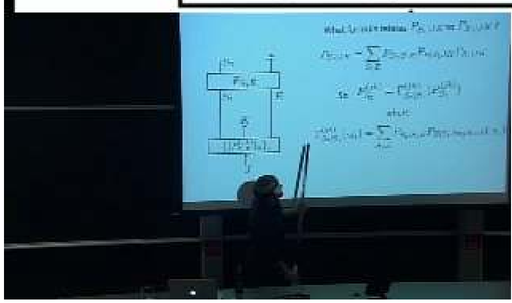


$$P_{S_2|JK} = \sum_{S_1 E} P_{S_2|S_1 E} P_{E|S_1 JK} P_{S_1|JK}$$

So $$P^{(jk)}_{S_2} = \Gamma^{(jk)}_{S_2|S_1}(P^{(jk)}_{S_1})$$

where

$$\Gamma^{(jk)}_{S_2|S_1}(\cdot_{S_1}) = \sum_{S_1 E} P_{S_2|S_1 E} P_{E|S_1 J=jK=k}(\cdot_{S_1})$$



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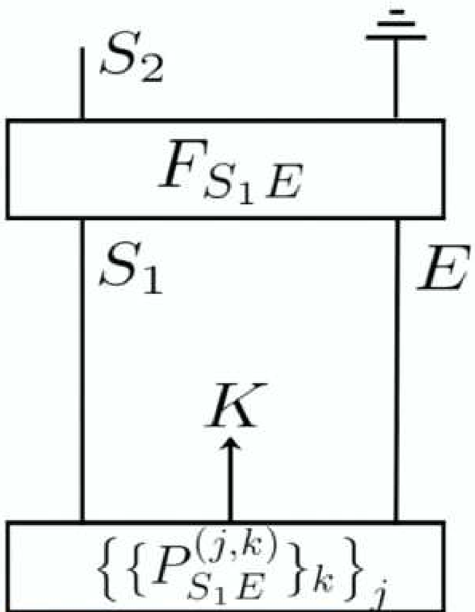

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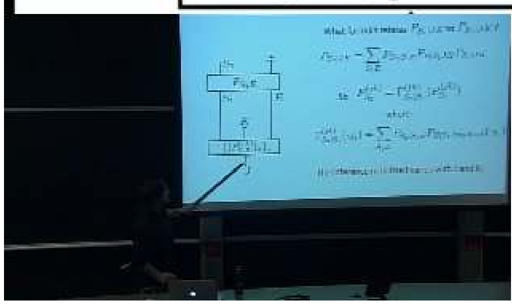
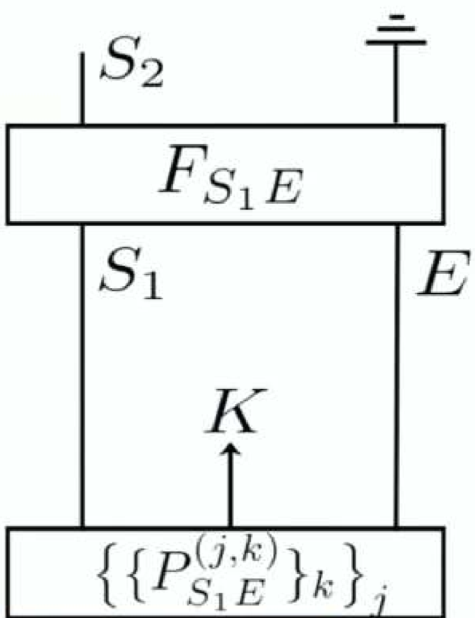


The inference map itself varies with J and K!



We have:

$$\mathfrak{R} = \{(P_{S_1}^{(1,1)}, P_{S_2}^{(1,1}), (P_{S_1}^{(1,2)}, P_{S_2}^{(1,2)}), \ldots (P_{S_1}^{(n,m)}, P_{S_2}^{(n,m)})\}$$

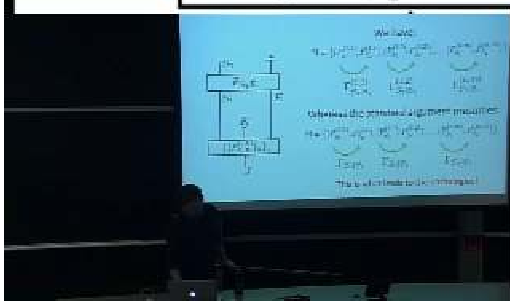
$$\Gamma^{(1,1)}_{S_2|S_1} \qquad \Gamma^{(1,2)}_{S_2|S_1} \qquad \Gamma^{(n,m)}_{S_2|S_1}$$

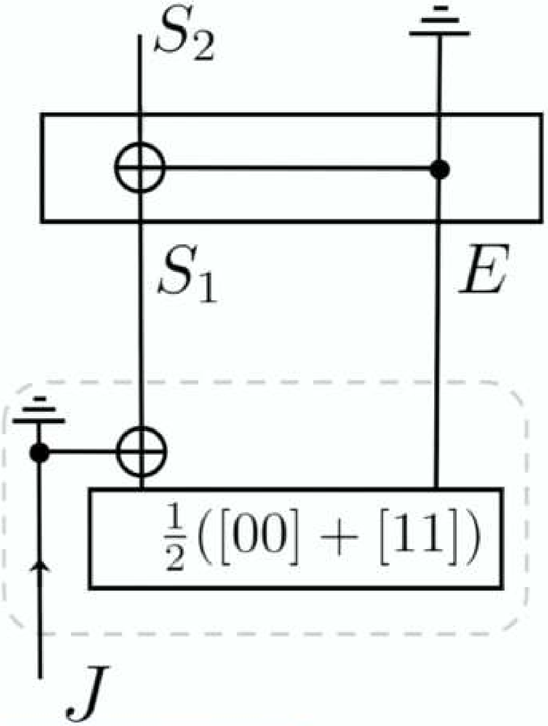
Whereas the standard argument presumes:

$$\mathfrak{R} = \{(P_{S_1}^{(1,1)}, P_{S_2}^{(1,1}), (P_{S_1}^{(1,2)}, P_{S_2}^{(1,2)}), \ldots (P_{S_1}^{(n,m)}, P_{S_2}^{(n,m)})\}$$

$$\Gamma_{S_2|S_1} \qquad \Gamma_{S_2|S_1} \qquad \Gamma_{S_2|S_1}$$

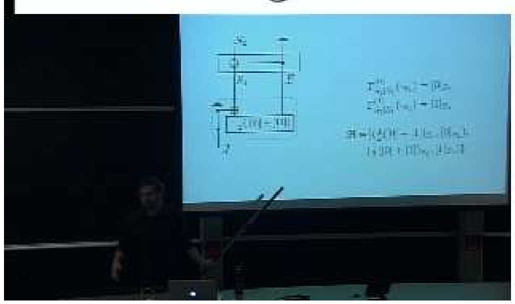
This is what leads to the pathologies!

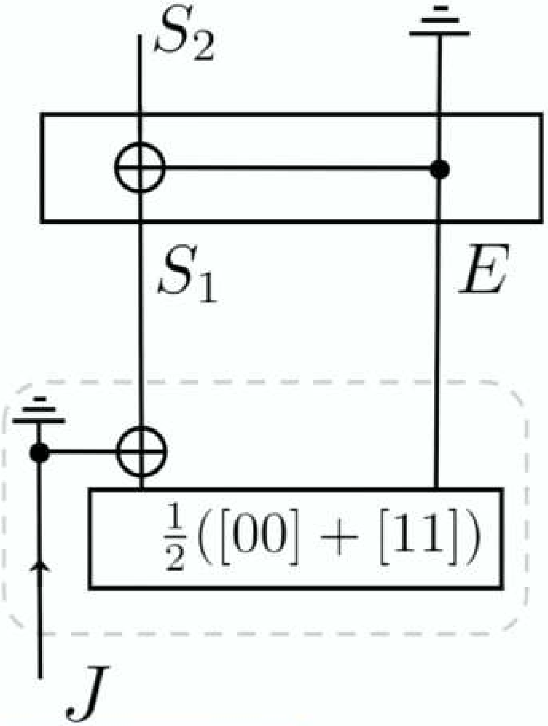


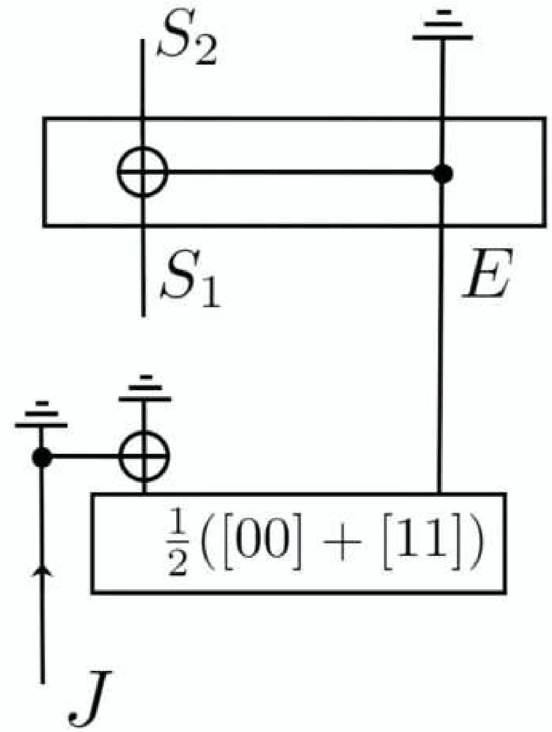


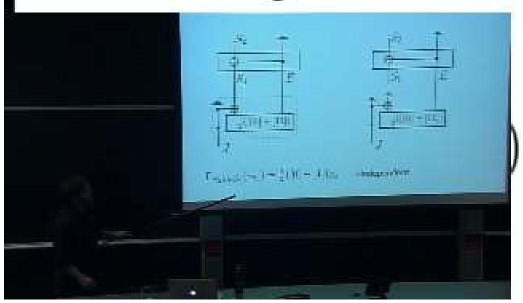
$$\Gamma^{(0)}_{S_2|S_1}(\cdot_{S_1}) = [0]_{S_2}$$
$$\Gamma^{(1)}_{S_2|S_1}(\cdot_{S_1}) = [1]_{S_2}$$

$$\mathfrak{R} = \{(\tfrac{1}{2}([0] + [1])_{S_1}, [0]_{S_2}), \\ (\tfrac{1}{2}([0] + [1])_{S_1}, [1]_{S_2})\}$$

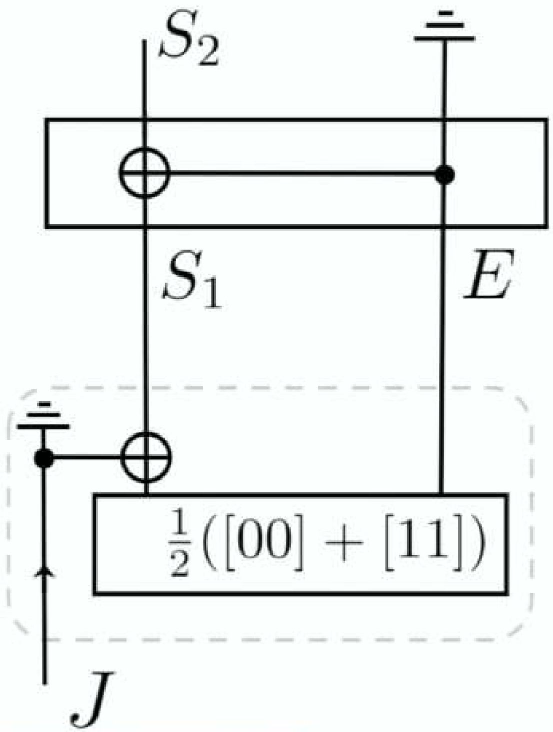








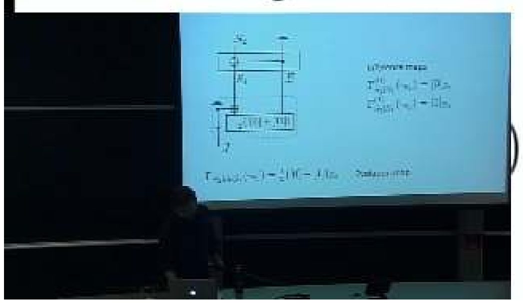
$) = \frac{1}{2}([0] + [1])_{S_2}$ J-independent



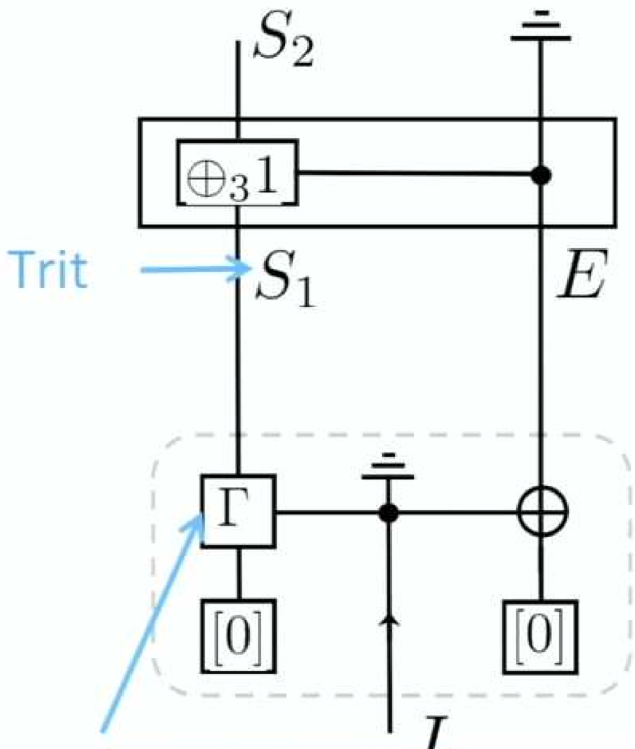
Inference maps

$$\Gamma^{(0)}_{S_2|S_1}(\cdot_{S_1}) = [0]_{S_2}$$

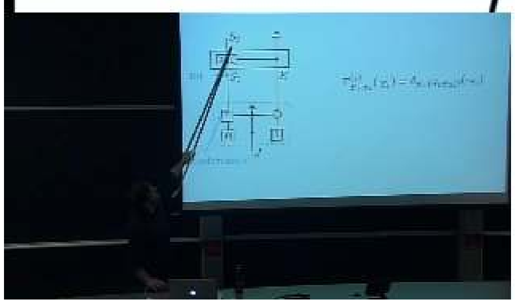
$$\Gamma^{(1)}_{S_2|S_1}(\cdot_{S_1}) = [1]_{S_2}$$

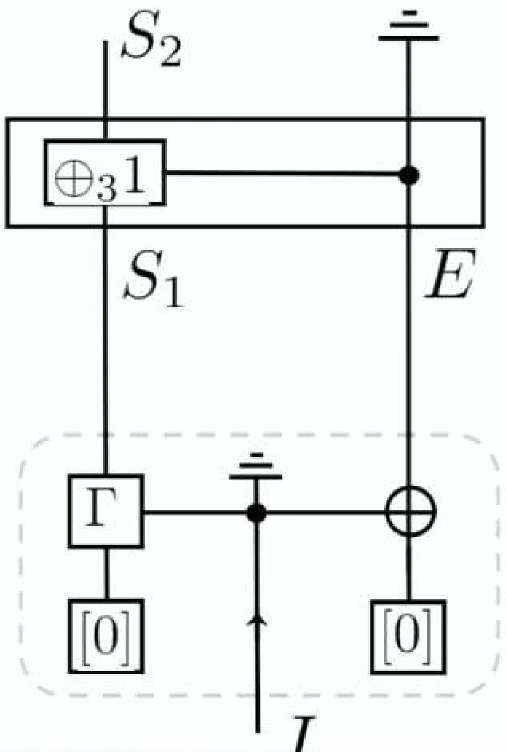


$$) = \frac{1}{2}([0]+[1])_{S_2}$$ Evolution map



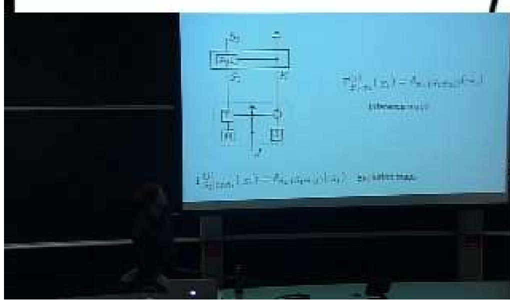
$$\Gamma^{(j)}_{S_2|S_1}(\cdot_{S_1}) = \delta_{S_2,(S_1 \oplus_3 j)}(\cdot_{S_1})$$





$$\Gamma^{(j)}_{S_2|S_1}(\cdot_{S_1}) = \delta_{S_2,(S_1 \oplus_3 j)}(\cdot_{S_1})$$

Inference maps



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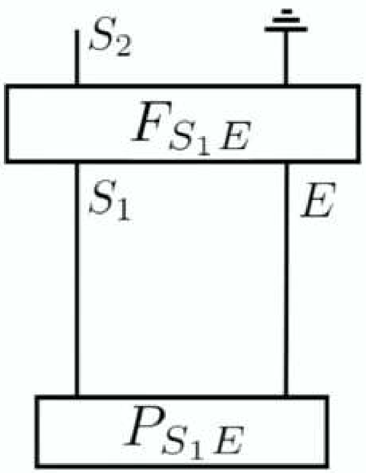
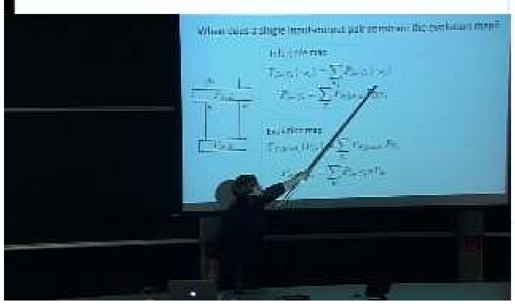
When does a single input-output pair constrain the evolution map?





Inference map

$$\Gamma_{S_2|S_1}(\,\cdot s_1) = \sum_{S_1} P_{S_2|S_1}(\,\cdot s_1)$$

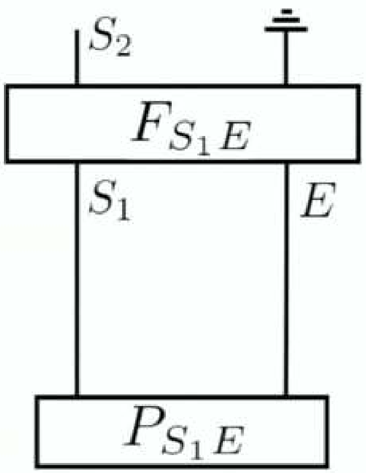
$$P_{S_2|S_1} = \sum_{E} P_{S_2|S_1 E} P_{E|S_1}$$

Evolution map

$$\Gamma_{S_2|\mathrm{do}S_1}(P_{S_1}) = \sum_{S_1} P_{S_2|\mathrm{do}S_1} P_{S_1}$$

$$P_{S_2|\mathrm{do}S_1} = \sum_{E} P_{S_2|S_1 E} P_E$$

When does a single input-output pair constrain the evolution map?



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Evolution map

$$\Gamma_{S_2|\mathrm{do}S_1}(P_{S_1}) = \sum_{S_1} P_{S_2|\mathrm{do}S_1} P_{S_1}$$

$$P_{S_2|\mathrm{do}S_1} = \sum_{E} P_{S_2|S_1 E} P_E$$

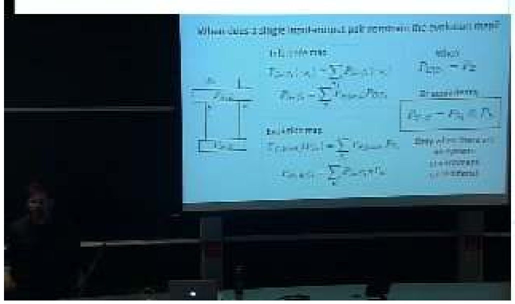
When

$$P_{E|S_1} = P_E$$

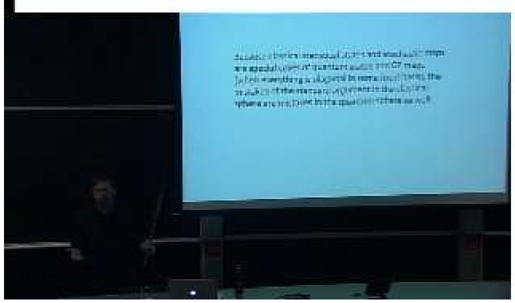
Or equivalently

$$P_{S_1 E} = P_{S_1} \otimes P_E$$

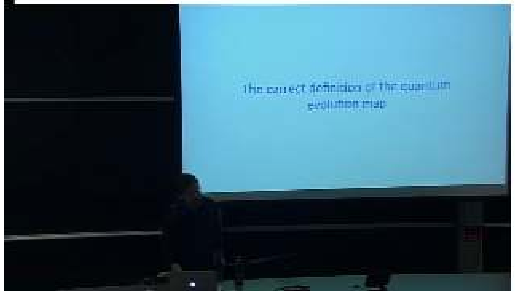
Only when there are **no** system-environment correlations!

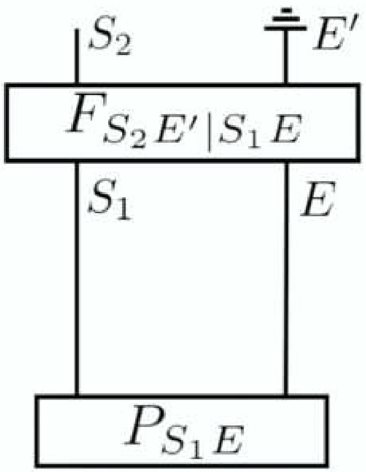


Because classical statistical states and stochastic maps are **special cases** of quantum states and CP maps (when everything is diagonal in some fixed basis), the mistakes of the standard argument in the classical sphere are mistakes in the quantum sphere as well.



The correct definition of the quantum evolution map

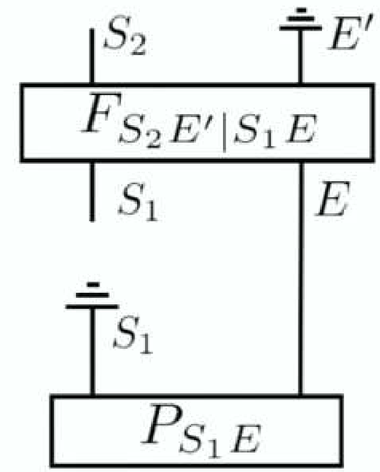


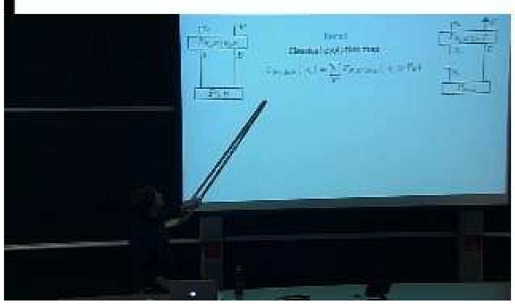


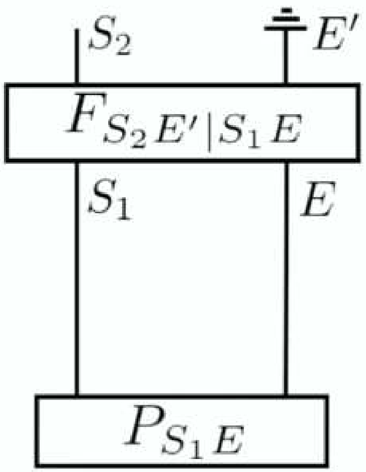
Recall

Classical evolution map

$$\Gamma_{S_2|\mathrm{do}S_1}(\,\cdot_{S_1}) = \sum_{E'} F_{S_2E'|S_1E}(\,\cdot_{S_1} \otimes P_E)$$







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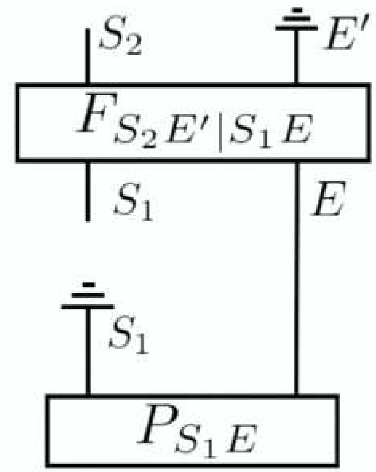
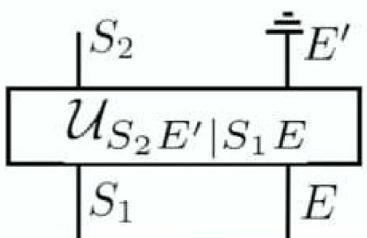
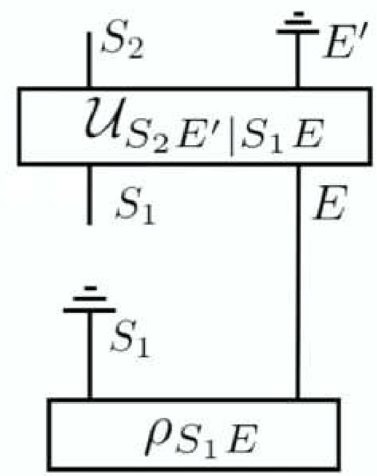
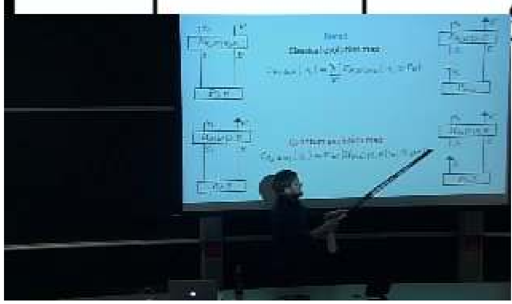




Quantum evolution map

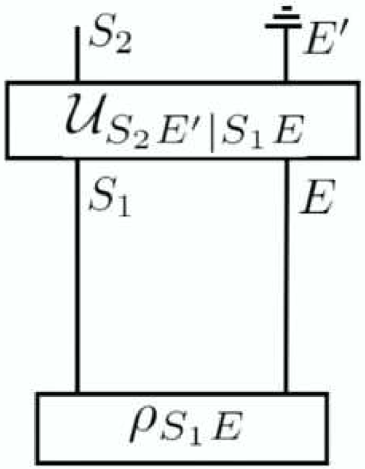
$$\mathcal{E}_{S_2|\mathrm{do}S_1}(\cdot_{S_1}) = \mathrm{tr}_{E'}\left(\mathcal{U}_{S_2E'|S_1E}(\cdot_{S_1} \otimes \rho_E)\right)$$





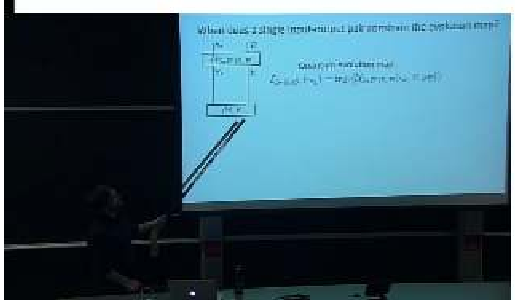
Can one define a quantum inference map?
(i.e., for general quantum circuits,
not just those in the classical subtheory)

When does a single input-output pair constrain the evolution map?



Quantum evolution map

$$\mathcal{E}_{S_2|\mathrm{do}S_1}(\cdot_{S_1}) = \mathrm{tr}_{E'}\left(\mathcal{U}_{S_2E'|S_1E}(\cdot_{S_1} \otimes \rho_E)\right)$$



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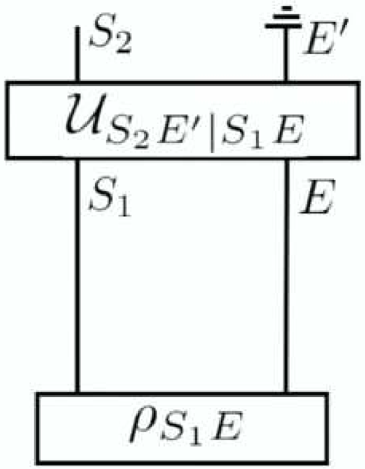
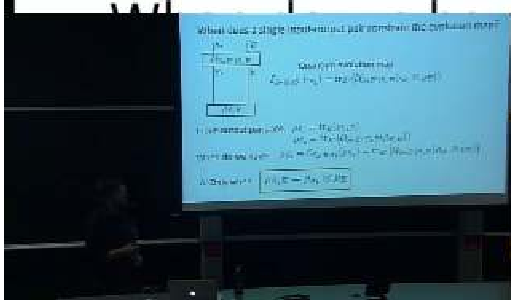


Quantum evolution map

$$\mathcal{E}_{S_2|\mathrm{do}S_1}(\cdot_{S_1}) = \mathrm{tr}_{E'}\left(\mathcal{U}_{S_2E'|S_1E}(\cdot_{S_1} \otimes \rho_E)\right)$$

Input-output pairs are: $\rho_{S_1} = \mathrm{tr}_E(\rho_{S_1E})$

$$\rho_{S_2} = \mathrm{tr}_{E'}\left(\mathcal{U}_{S_2E'|S_1E}(\rho_{S_1E})\right)$$



e $\rho_{S_2} = \mathcal{E}_{S_2|\mathrm{do}S_1}(\rho_{S_1}) = \mathrm{tr}_{E'}\left(\mathcal{U}_{S_2E'|S_1E}(\rho_{S_1} \otimes \rho_E)\right)$

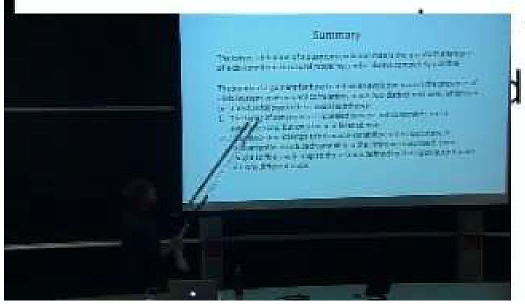
$$\rho_{S_1E} = \rho_{S_1} \otimes \rho_E$$

Summary

The correct definition of a quantum evolution map is the quantum analogue of a do-conditional in causal modeling, and is always completely positive

The standard argument for how to define an evolution map in the presence of initial system-environment correlations made two distinct mistakes, which can be stated as follows in the classical subtheory:

1. The types of constraints it appealed to were not constraints on an evolution map, but only on an inference map
2. In a misguided attempt to introduce variability in the input state, it inadvertently introduced variability in the inference map itself, then … fit a single map to the relation defined by the input-output pairs … different maps



Looking forward

Is there a sensible notion of an inference map in quantum theory?
What are the quantum analogues of other concepts and techniques from classical causal models?
How to experimentally determine the quantum evolution map given limited experimental control?