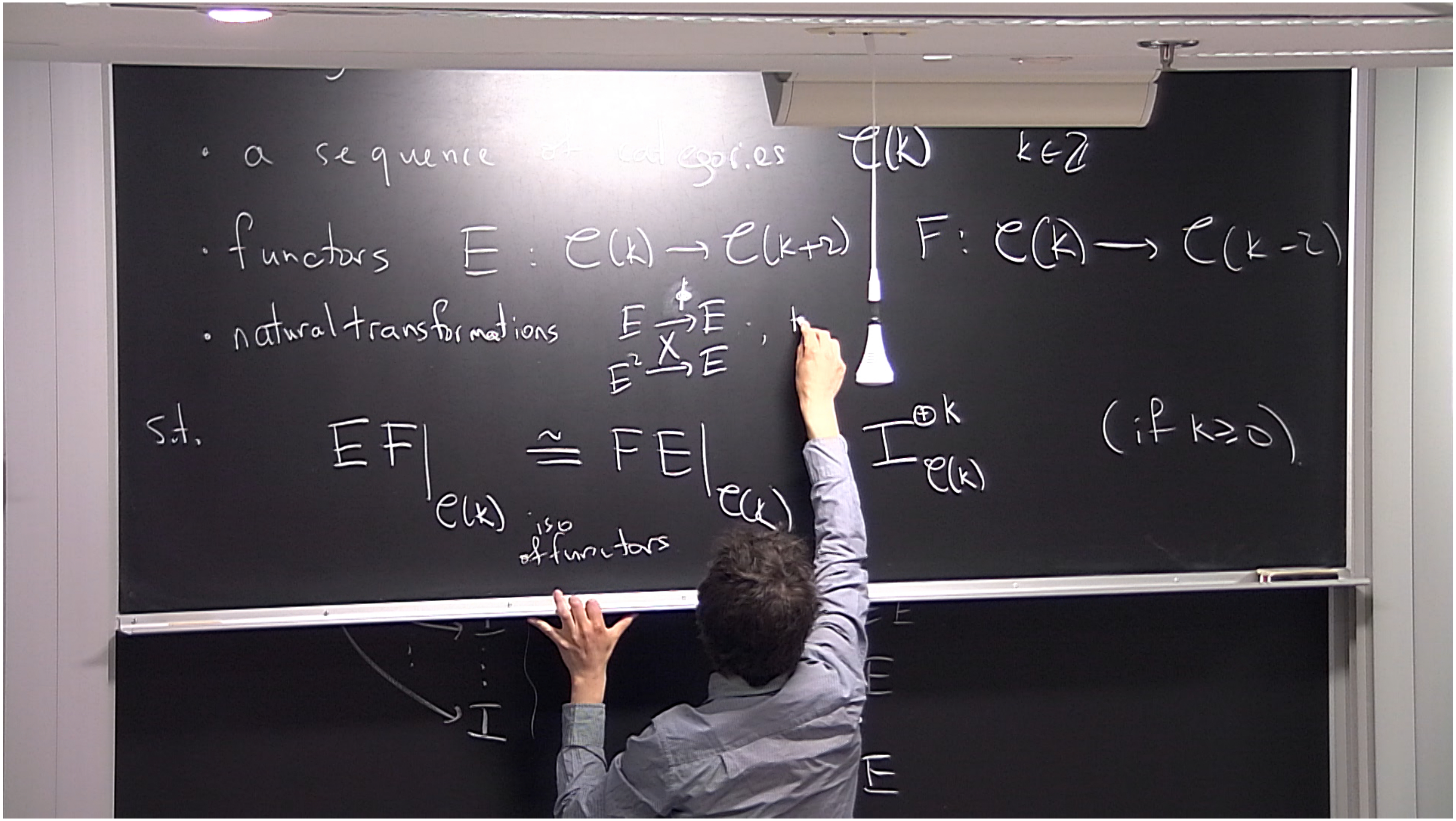
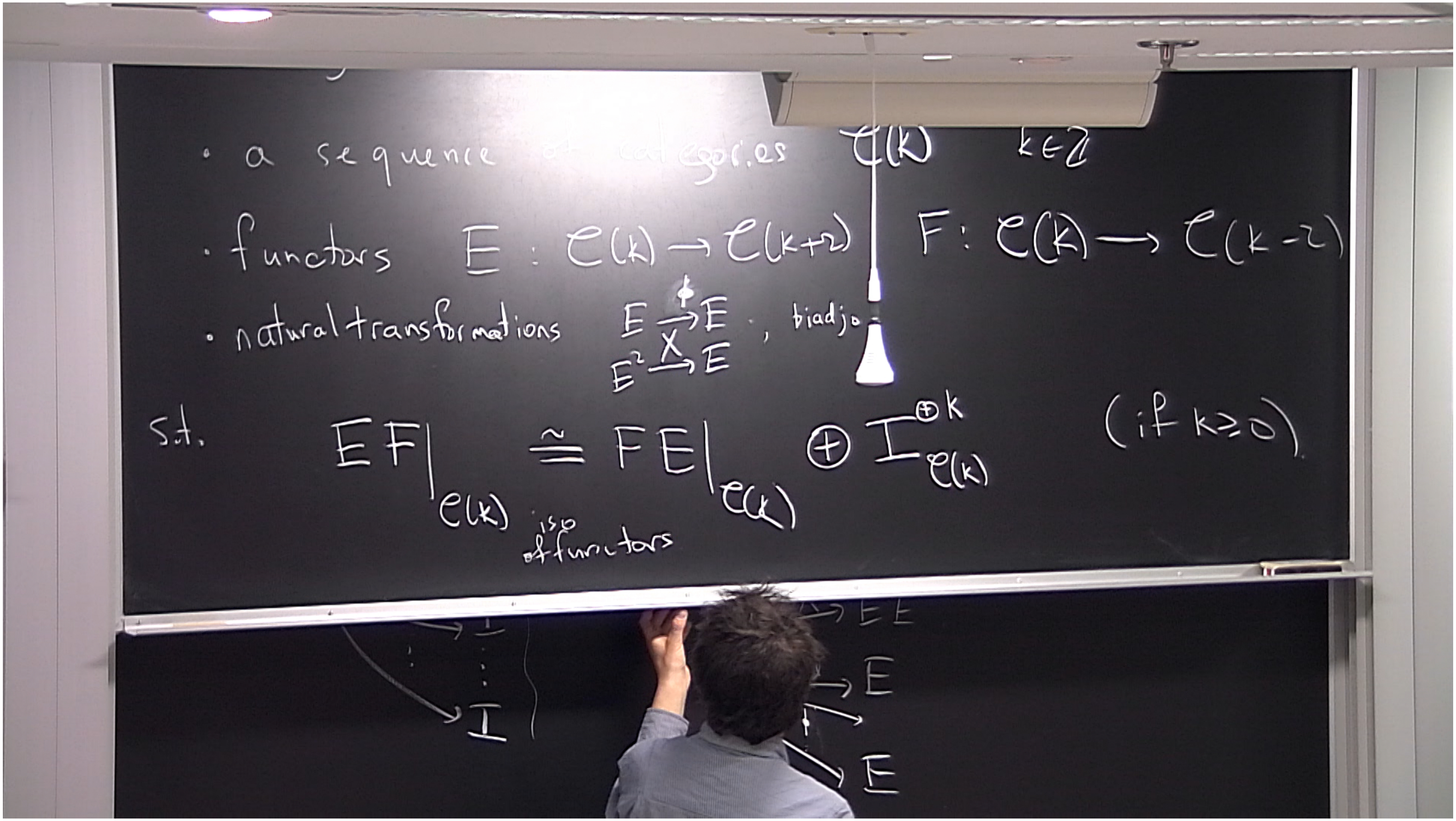
Title: MPIM/PI teleseminar on categorified knot invariants - 4

Date: May 18, 2018 11:30 AM

URL: http://pirsa.org/18050055

Abstract:


• a sequence of categories $\mathcal{C}(k)$ $k \in \mathbb{Z}$
• functors $E : \mathcal{C}(k) \to \mathcal{C}(k+2)$ $F : \mathcal{C}(k) \longrightarrow \mathcal{C}(k-2)$
• natural transformations $E \xrightarrow{X} E$, $E^2 \xrightarrow{T} E$
s.t. $EF|_{\mathcal{C}(k)} \cong FE|_{\mathcal{C}(k)}$ $I^{\oplus k}_{\mathcal{C}(k)}$ (if $k \geq 0$)
iso of functors


• a sequence of categories C(k) k∈Z
• functors E : C(k) → C(k+2) F : C(k) → C(k-2)
• natural transformations E → E , biadjo
X
E² → E
s.t. EF|C(k) ≅ FE|C(k) ⊕ I^⊕k C(k) (if k≥0)
iso of functors

$$EF\big|_{e(k)} \cong FE\big|_{e(k)} \oplus I_{e(k)}$$

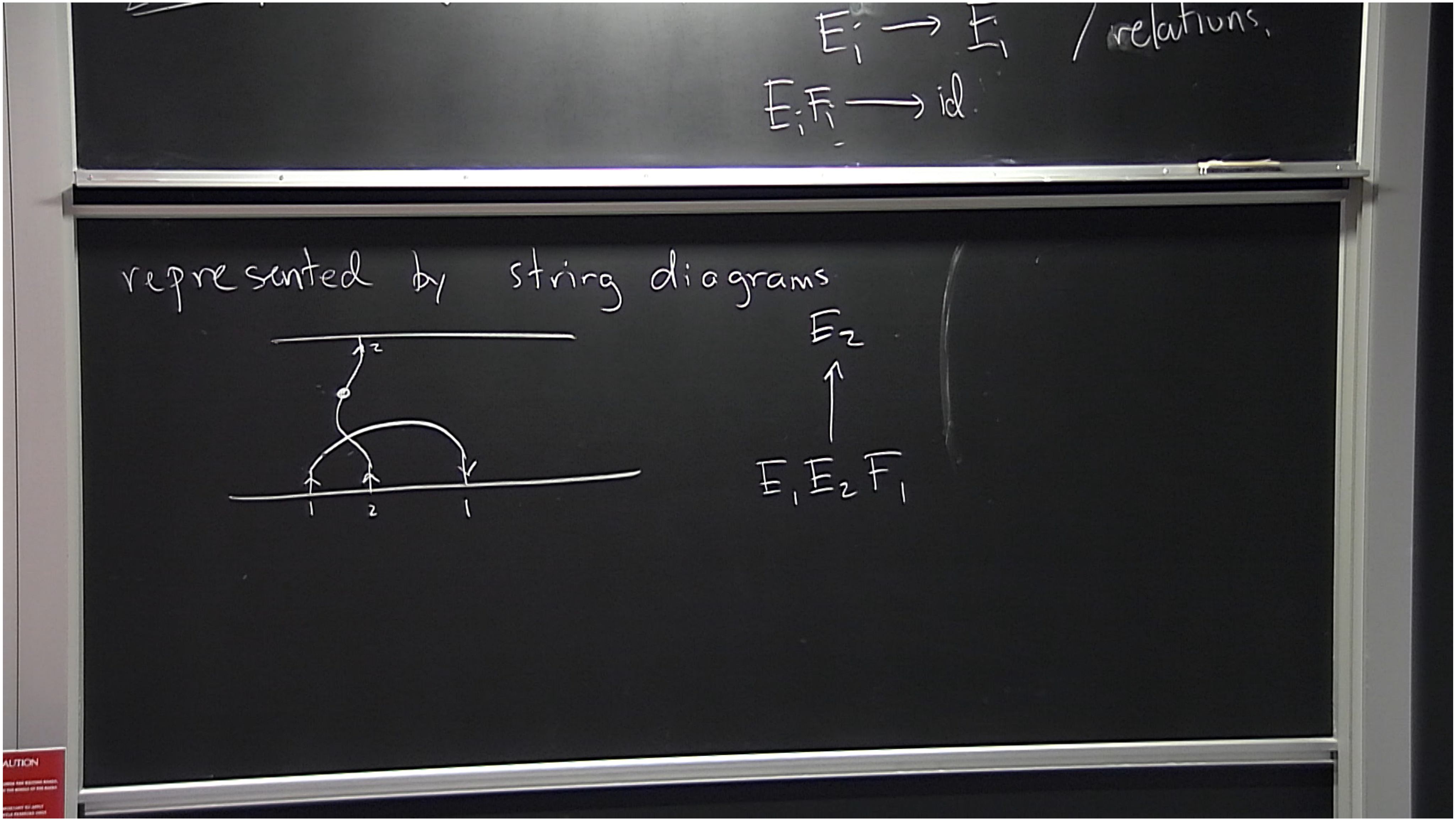
iso of functors

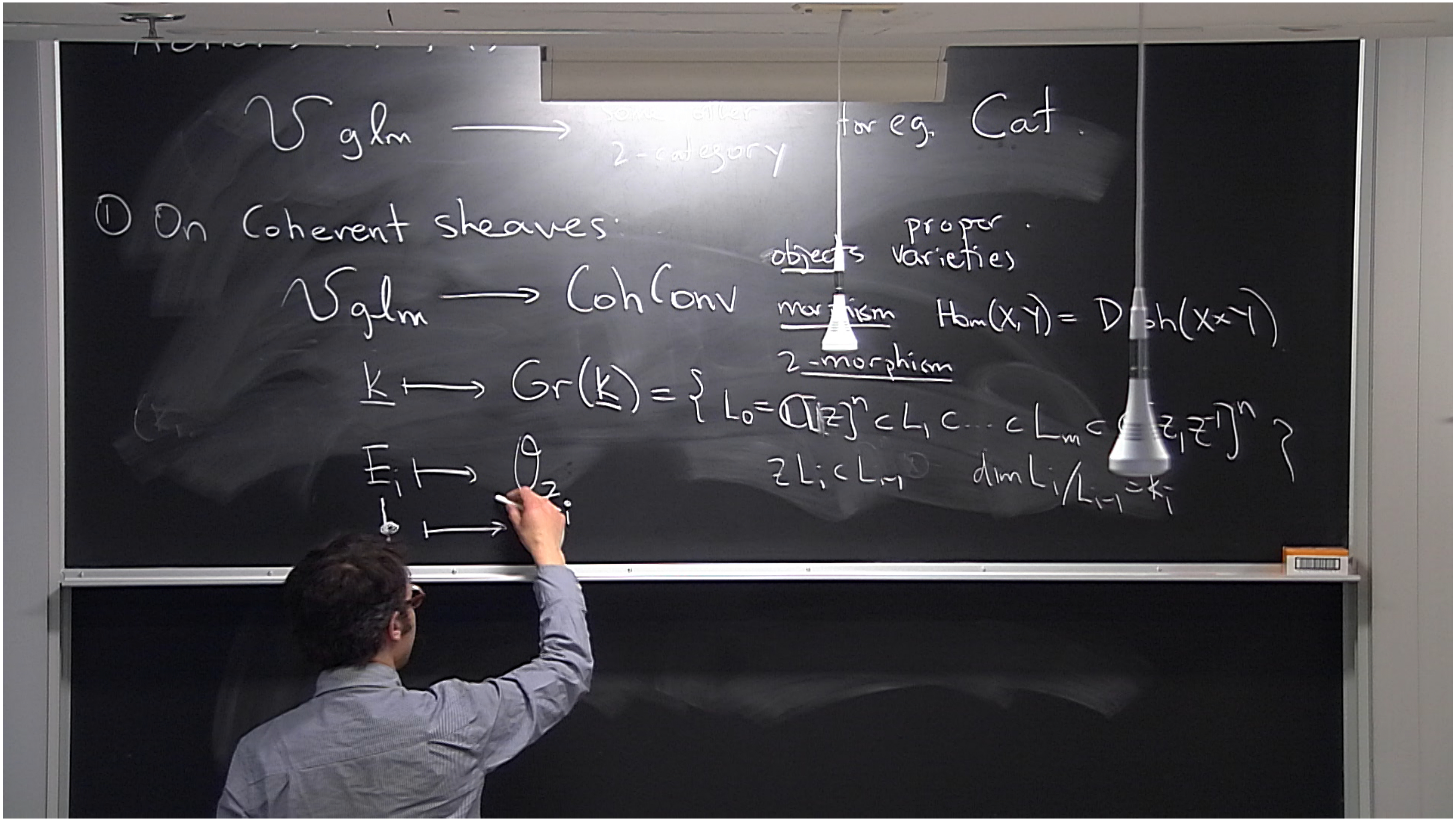
$$\begin{tikzcd} EF \arrow[r] \arrow[dr] \arrow[ddr] & FE \\ & I \\ & \vdots \\ & I \end{tikzcd}$$

E, F are biadjoint

$$EE \xrightarrow{X} EE$$

$$\begin{tikzcd} E \arrow[r, "id"] \arrow[r, bend right] \arrow[dr] & E \\ & E \end{tikzcd}$$


$E_i \longrightarrow E_i$ / relations.
$E_i F_i \longrightarrow \mathrm{id}$
represented by string diagrams
E_2
$E_1 E_2 F_1$
2
1
2
1


V glm → some other 2-category for eg. Cat.
① On Coherent sheaves:
V glm → CohConv
objects proper varieties
morphism Hom(X,Y) = D Coh(X×Y)
2-morphism
k ↦ Gr(k) = { L0 = C[[z]]^n ⊂ L1 ⊂ ... ⊂ Lm ⊂ C[z,z^-1]^n }
zLi ⊂ Li-1 dim Li/Li-1 = ki
Ei ↦ OZi

① On Coherent shea... ...per ...eties

$\mathcal{U}\mathfrak{gl}_m \longrightarrow \mathrm{CohConv}$

morphism $\mathrm{Hom}(X,Y) = D\,\mathrm{Coh}(X \times Y)$

2-morphism

$\underline{k} \longmapsto Gr(\underline{k}) = \{ L_0 = \mathbb{C}[z]^n \subset L_1 \subset \cdots \subset L_m \subset \mathbb{C}[z,z^{-1}]^n \}$

$zL_i \subset L_{i-1}$ $\quad \dim L_i/L_{i-1} = k_i$

$E_i \longmapsto \mathcal{O}_{Z_i}$

$\phi \longmapsto$ obstruction to deforming $\mathcal{O}_{Z_i}$.

②

CKL: $E_i \longmapsto \mathcal{O}_{Z_i}$ $\quad Z_i$ $\quad L_i \supset L_{i-1}$ $\quad \dim L_i/L_{i-1} = k_i$

$\phi \longmapsto$ obstruction to deforming $\mathcal{O}_{Z_i}$.

(2) $\mathcal{V}\mathfrak{gl}_m \longrightarrow \{$ rings, bimodules $\}$

Khovanov–Lauda

$\underline{k} \longmapsto H^\circ(Fl(k_1, \dots, k_m; \mathbb{C}^{k_1+\dots+k_m}))$

$E_i \longmapsto$ bimodule coming from a correspondence.

(3) category $\mathcal{O}$

Bernstein–Frenkel–Khovanov

Stroppel

Sussan, ...

$\mathcal{V}\mathfrak{gl}_m \longrightarrow Cat$

$\underline{k} \longrightarrow$ a singular block of category $\mathcal{O}$ for $\mathfrak{sl}_N$ $\quad N = k_1 + \dots + k_m$.

$\mathcal{O}_{\underbrace{1,\dots,1}_{k_1},\underbrace{2,\dots,2}_{k_2},\dots,\underbrace{m,\dots,m}_{k_m} - \rho}$

CKL: $E_i \longmapsto \mathcal{O}_{Z_i}$ $Z \quad L_i \supset L_{i-1} \quad \dim L_i / L_{i-1} = k_i$

$\phi \longmapsto$ obstruction to deforming $\mathcal{O}_{Z_i}$.

(2) $U\mathfrak{gl}_m \longrightarrow \left\{ \begin{array}{l} \text{rings} \\ \text{bimodules} \end{array} \right.$

Khovanov-Lauda

$\underline{k} \longmapsto H^*(F\ell(k_1, \dots, k_m; \mathbb{C}^{k_1 + \dots + k_m}))$

$E_i \longmapsto$ bimodule coming from a correspondence.

(3) category $\mathcal{O}$

Bernstein-Frenkel-Khovanov

Stroppel

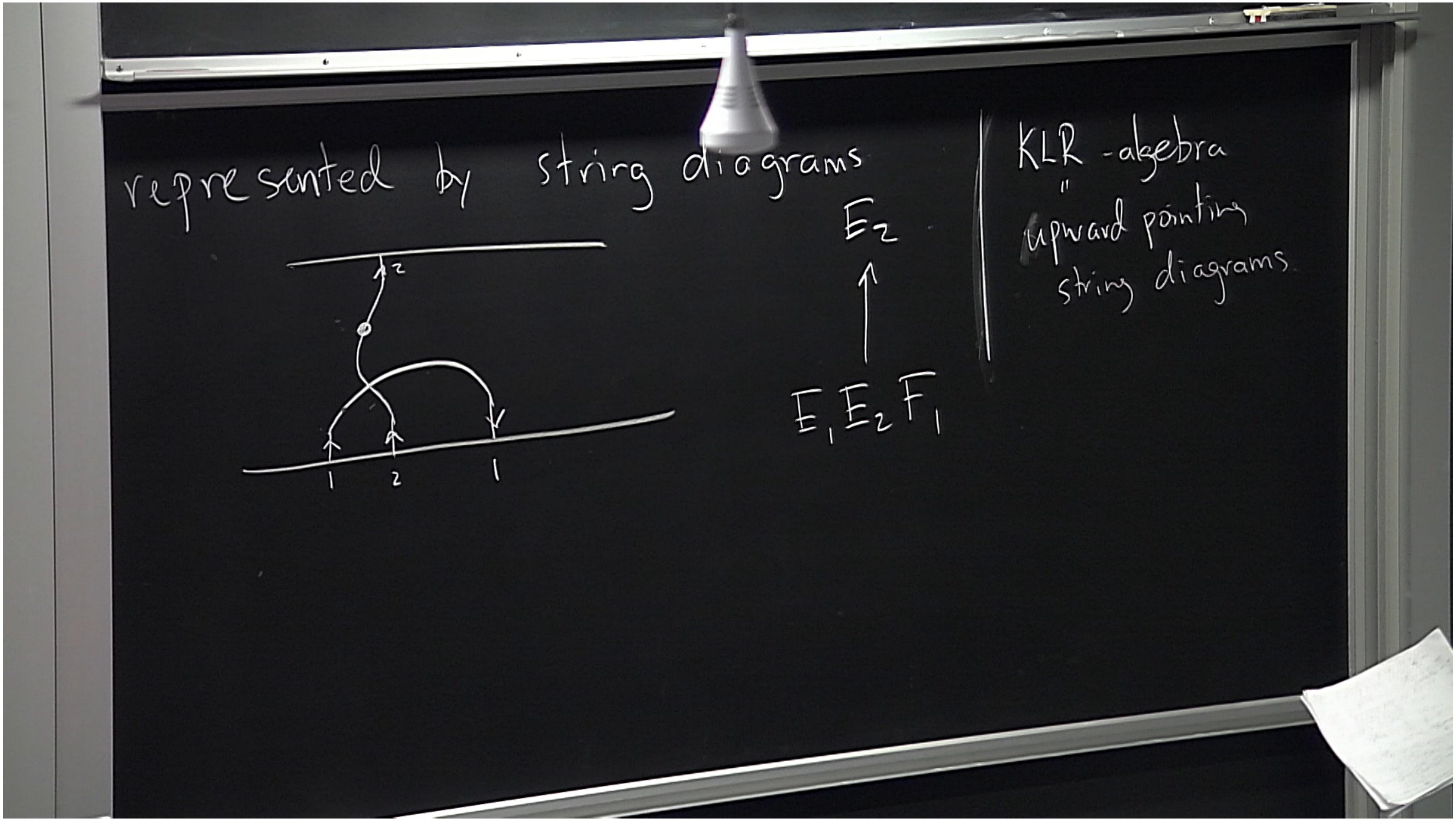
Sussan, ...

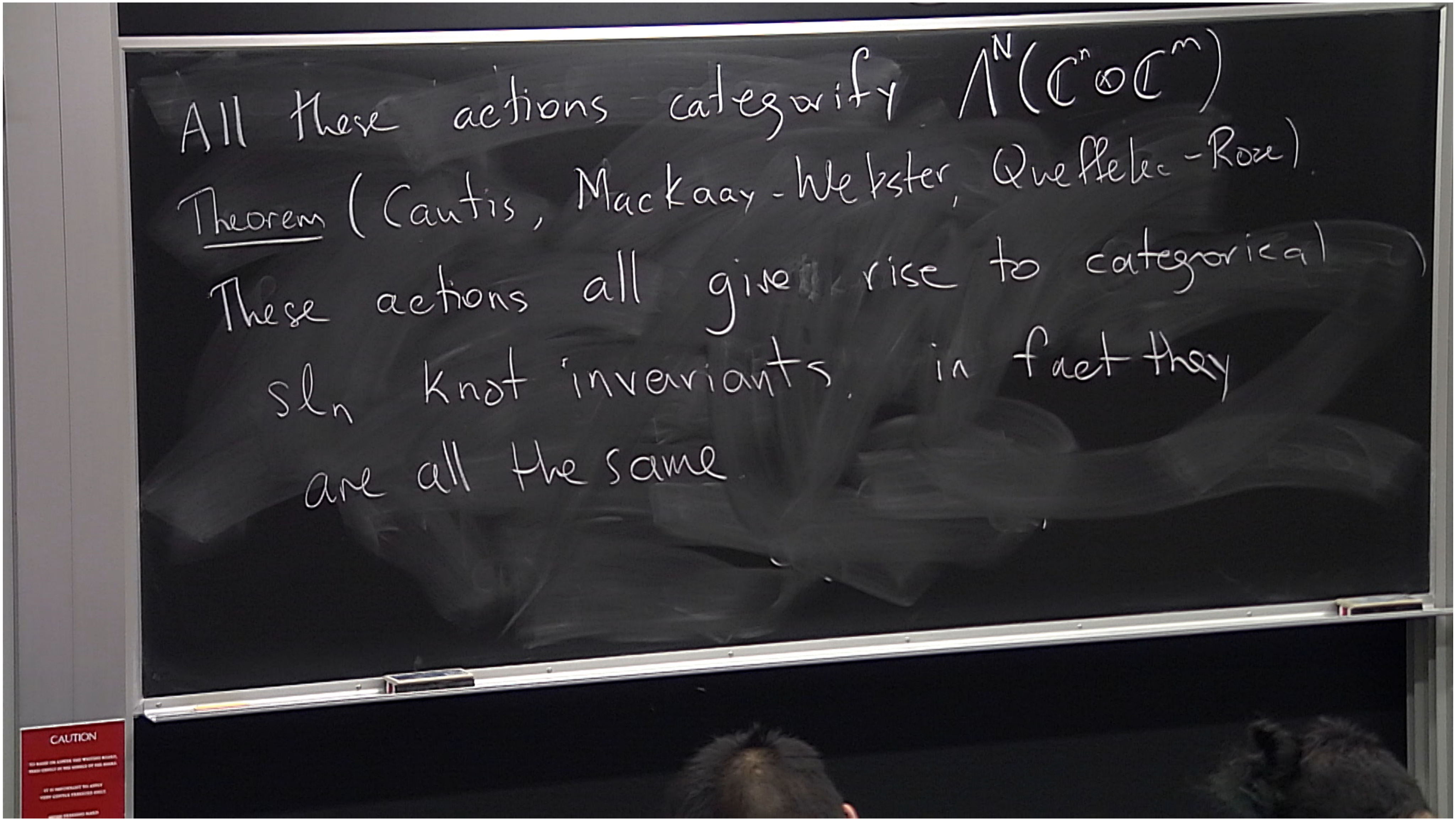
$U\mathfrak{gl}_m \longrightarrow Cat$

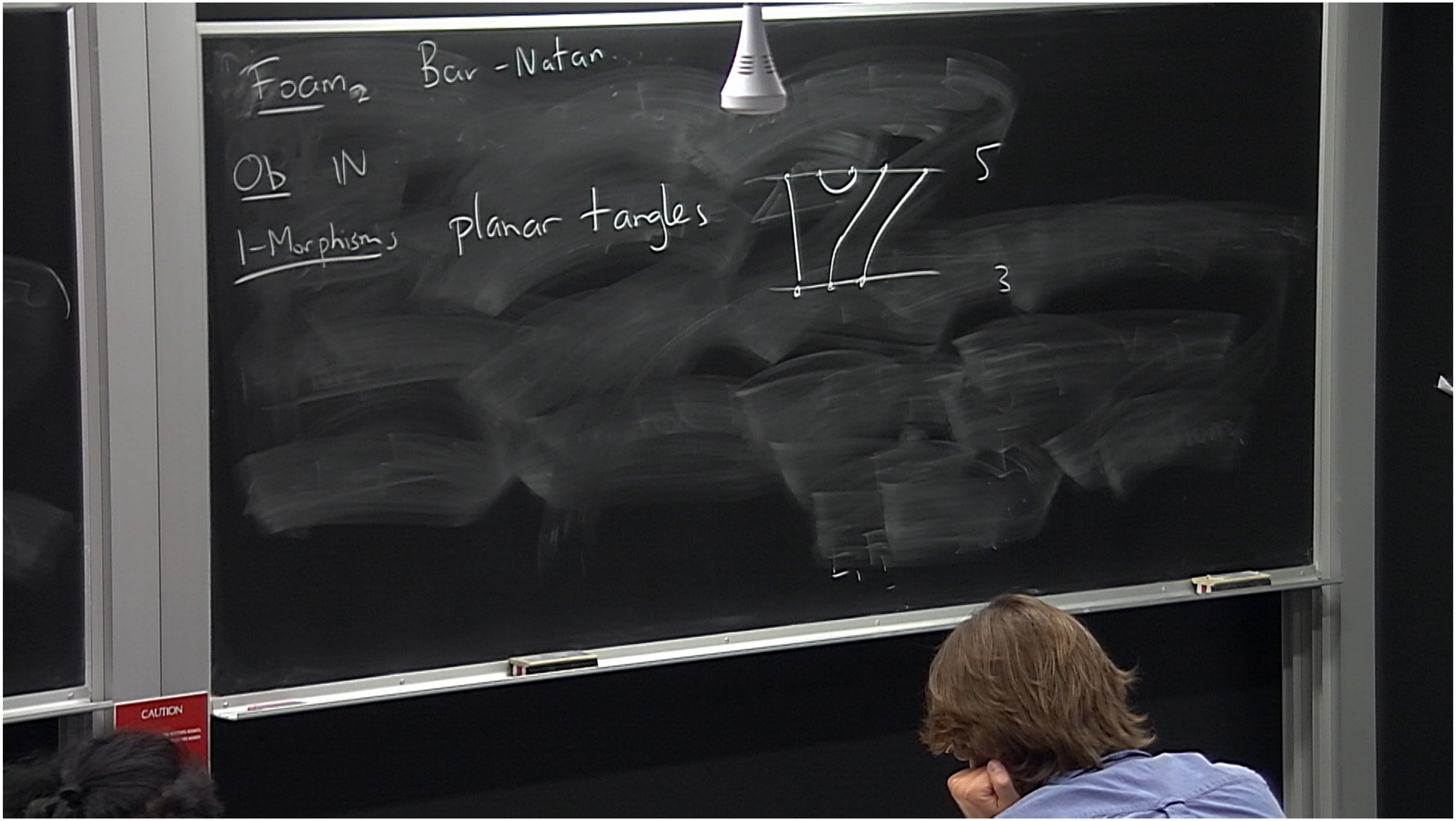
$\underline{k} \longmapsto$ a singular block of category $\mathcal{O}$ for $\mathfrak{sl}_N$ $\quad N = k_1 + \dots + k_m$.

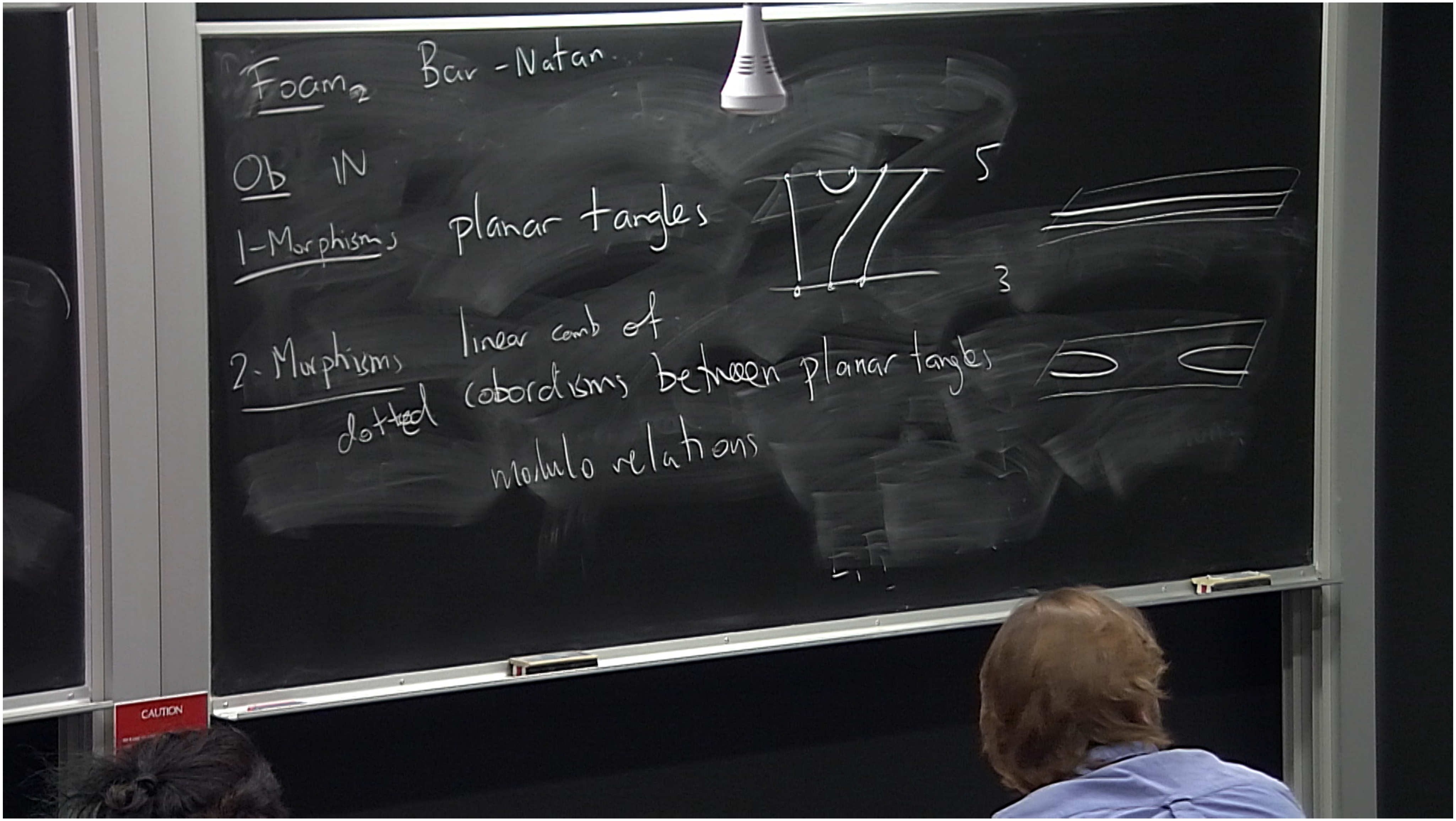
$E_i \longmapsto$ translation functor.

$\mathcal{O}_{\underbrace{1,\dots,1}_{k_1},\underbrace{2,\dots,2}_{k_2},\dots,\underbrace{m,\dots,m}_{k_m} - \rho}$

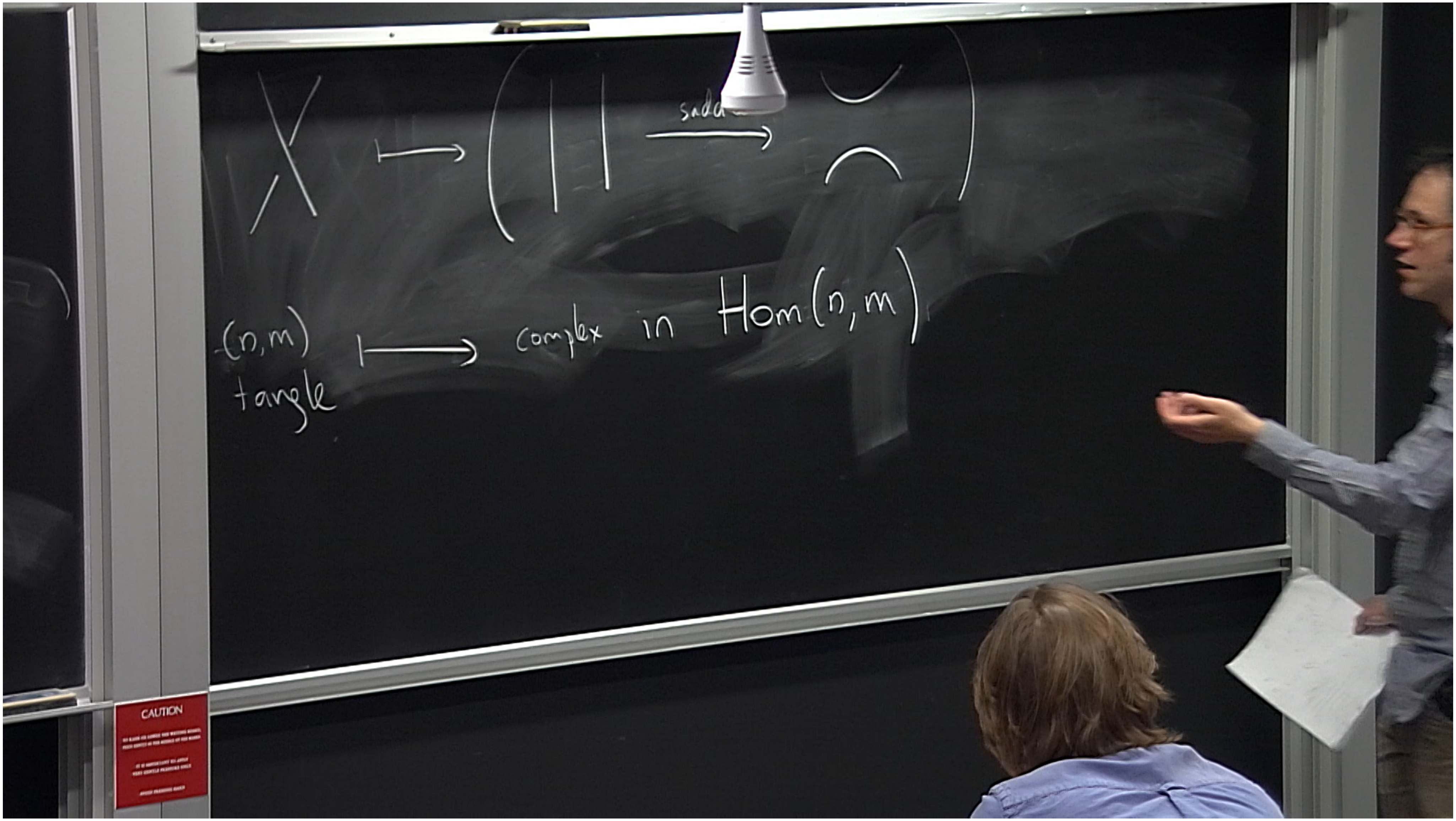

represented by string diagrams
2
1 2 1
E_2
$E_1 E_2 F_1$
KLR-algebra
"
upward pointing
string diagrams

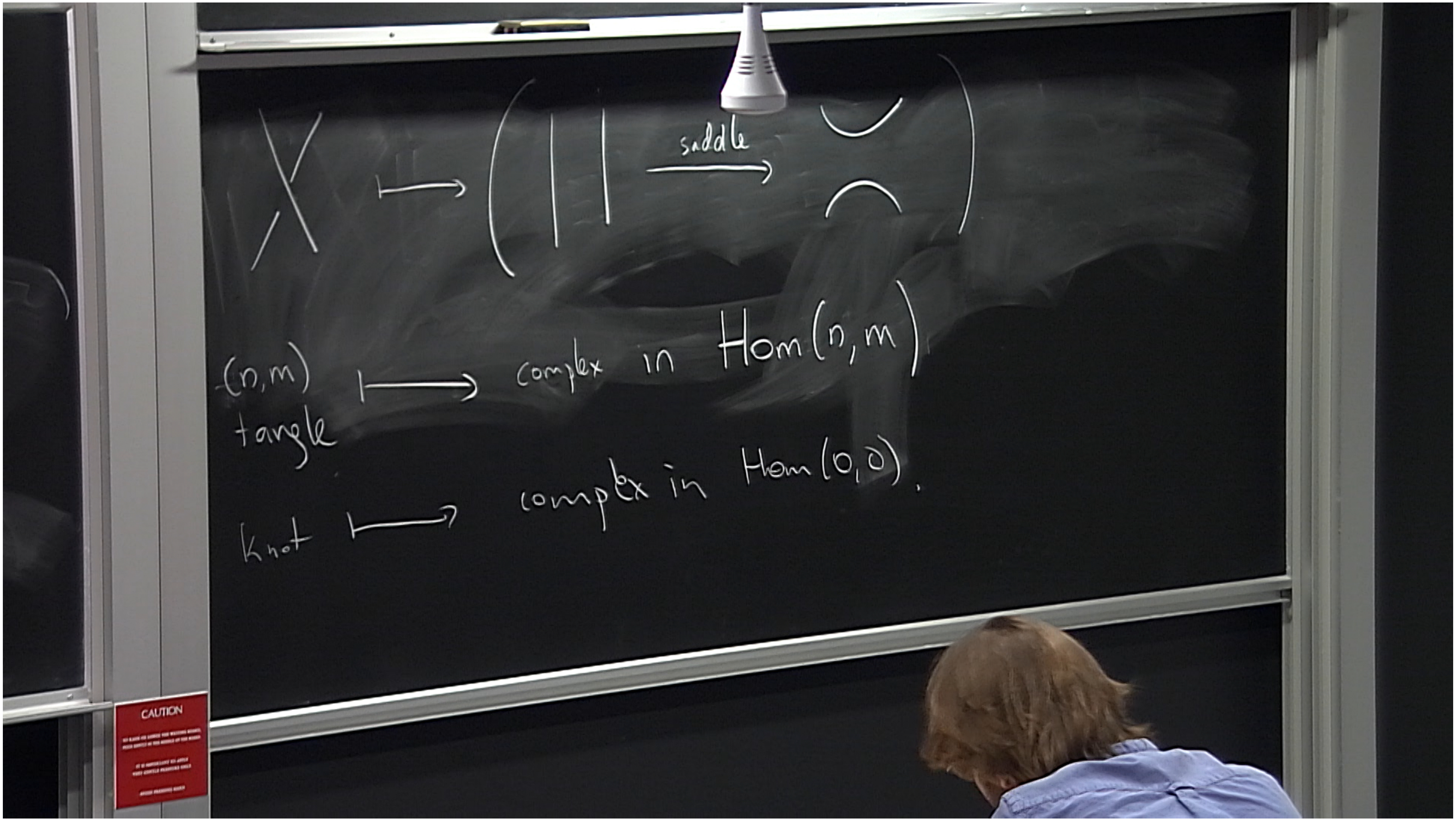

All these actions categorify $\Lambda^N(\mathbb{C}^n \otimes \mathbb{C}^m)$
Theorem (Cautis, Mackaay-Webster, Queffelec-Rose)
These actions all give rise to categorical
sl_n knot invariants, in fact they
are all the same
CAUTION


Foam2 Bar-Natan
Ob IN
1-Morphisms planar tangles
5
3
CAUTION



Foam2 Bar-Natan
Ob N
1-Morphisms planar tangles
5
3
2-Morphisms linear comb of
dotted cobordisms between planar tangles
modulo relations
CAUTION


sadd
(n,m)
tangle
complex in Hom(n,m)
CAUTION


saddle
(n,m) tangle
complex in Hom(n,m)
knot
complex in Hom(0,0).
CAUTION

(2) $U\mathfrak{gl}_m \longrightarrow \left\{ \begin{array}{l} \text{rings} \\ \text{bimodules} \end{array} \right.$

Khovanov-Lauda

$\underline{k} \longmapsto H^\bullet(Fl(k_1,\dots,k_m;\mathbb{C}^{k_1+\dots+k_m}))$

$E_i \longmapsto$ bimodule coming from a correspondence.

(3) category $\mathcal{O}$

Bernstein–Frenkel–Khovanov

Stroppel

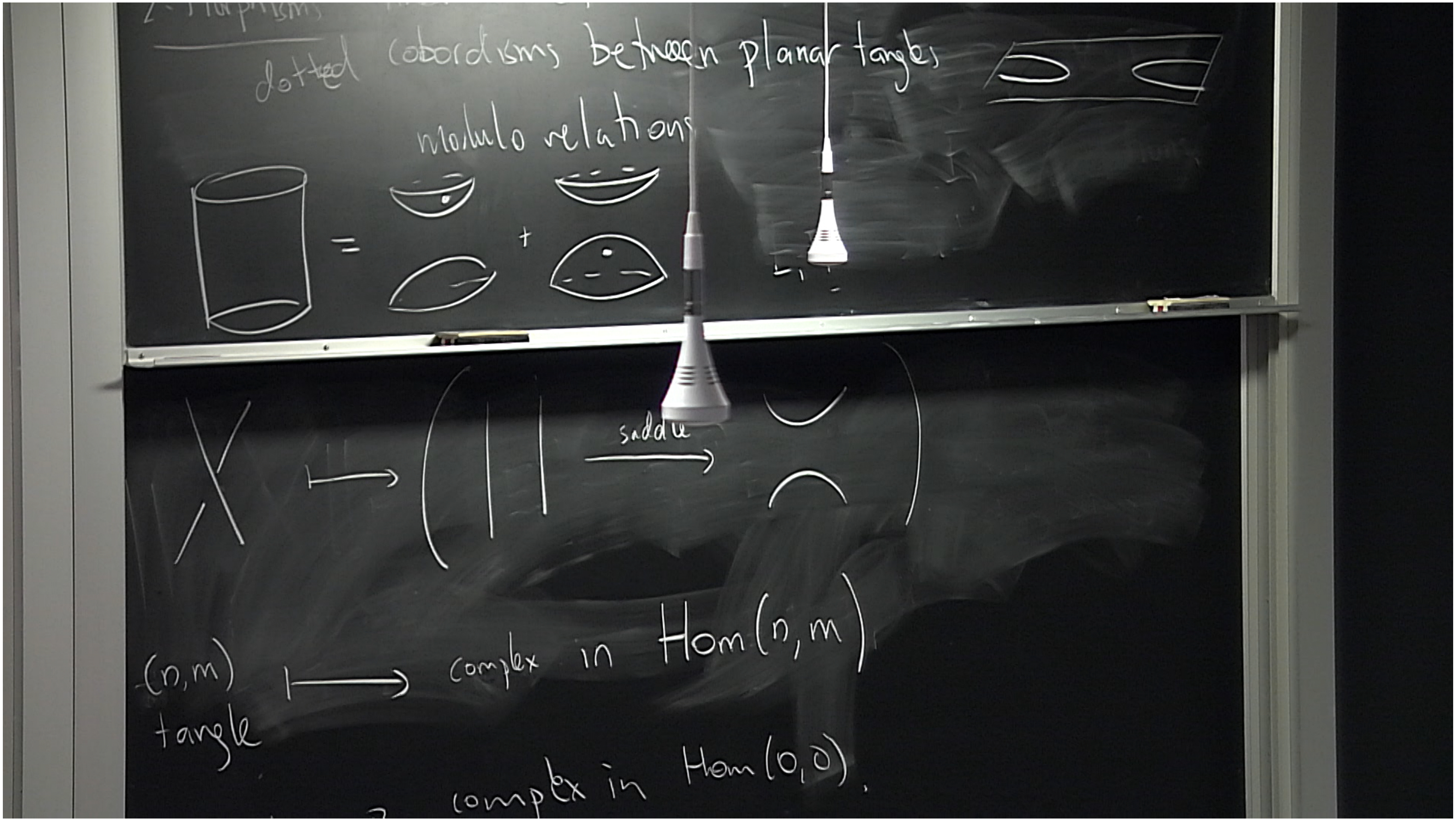
Sussan, ...

$U\mathfrak{gl}_m \longrightarrow Cat$

$\underline{k} \longrightarrow$ a singular block of category $\mathcal{O}$ for $\mathfrak{sl}_N$ $\quad N=k_1+\dots+k_m$.

$\mathcal{O}_{\underbrace{1,\dots,1}_{k_1},\underbrace{2,\dots,2}_{k_2},\dots,\underbrace{m,\dots,m}_{k_m}-\rho}$

$E_i \longmapsto$ translation functor.


dotted cobordisms between planar tangles
modulo relations
=
+
saddle
(n,m)
tangle
complex in Hom(n,m)
complex in Hom(0,0).

local relations:

Khovanov defined Foam_3

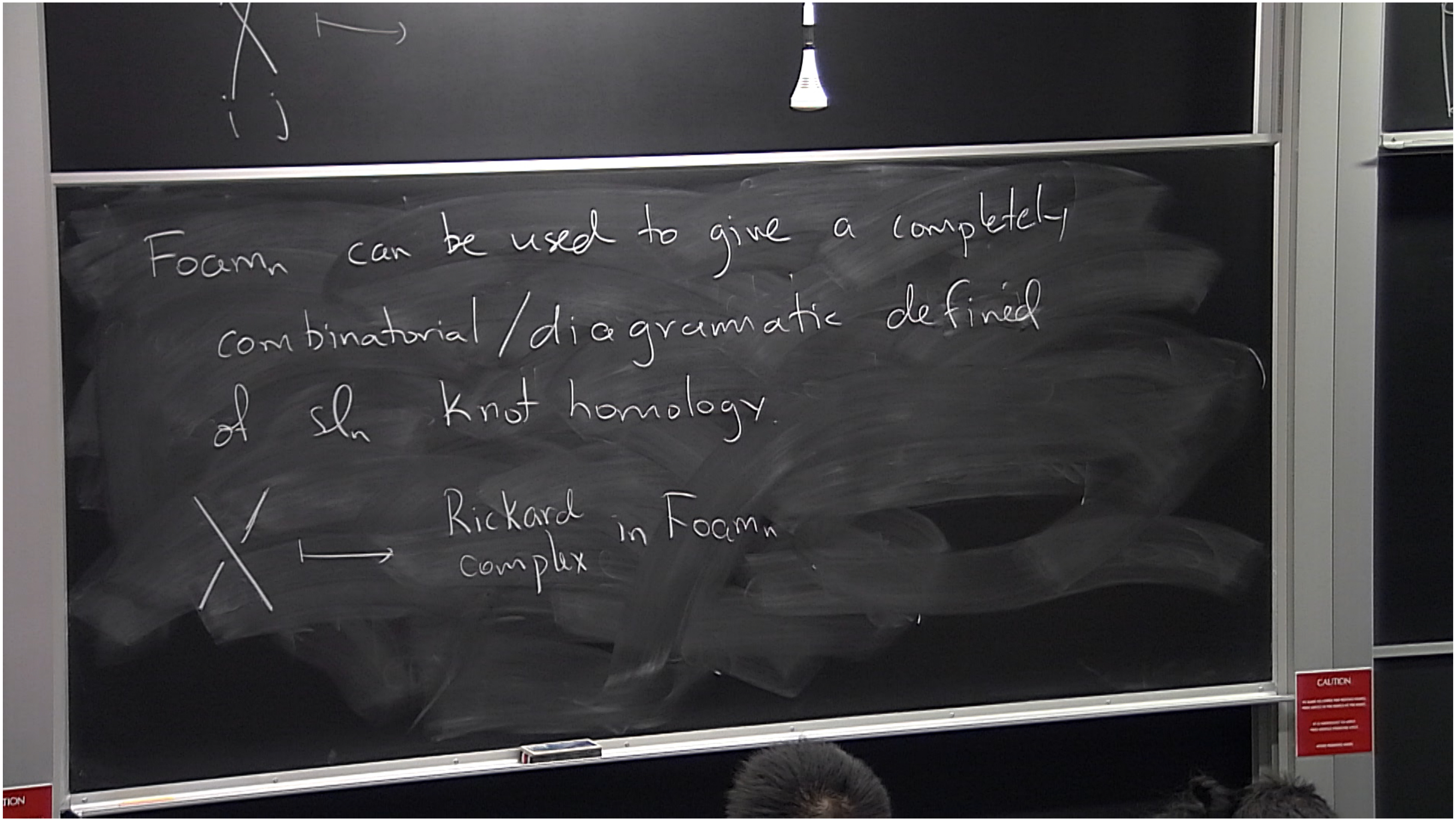
used it to define sl_3 knot homology.

<u>Theorem</u> (Queffelec - Rose)

There is a def. of Foam_n and a 2-functor

$$\mathcal{U}\mathfrak{gl}_m \longrightarrow \mathrm{Foam}_n$$

Foam_n can be used to give a completely combinatorial/diagramatic defined of $\mathfrak{sl}_n$ knot homology.


i j
Foam$_n$ can be used to give a completely
combinatorial/diagramatic defined
of sl_n knot homology.
Rickard complex in Foam$_n$
CAUTION

Khovanov defined Foam_3

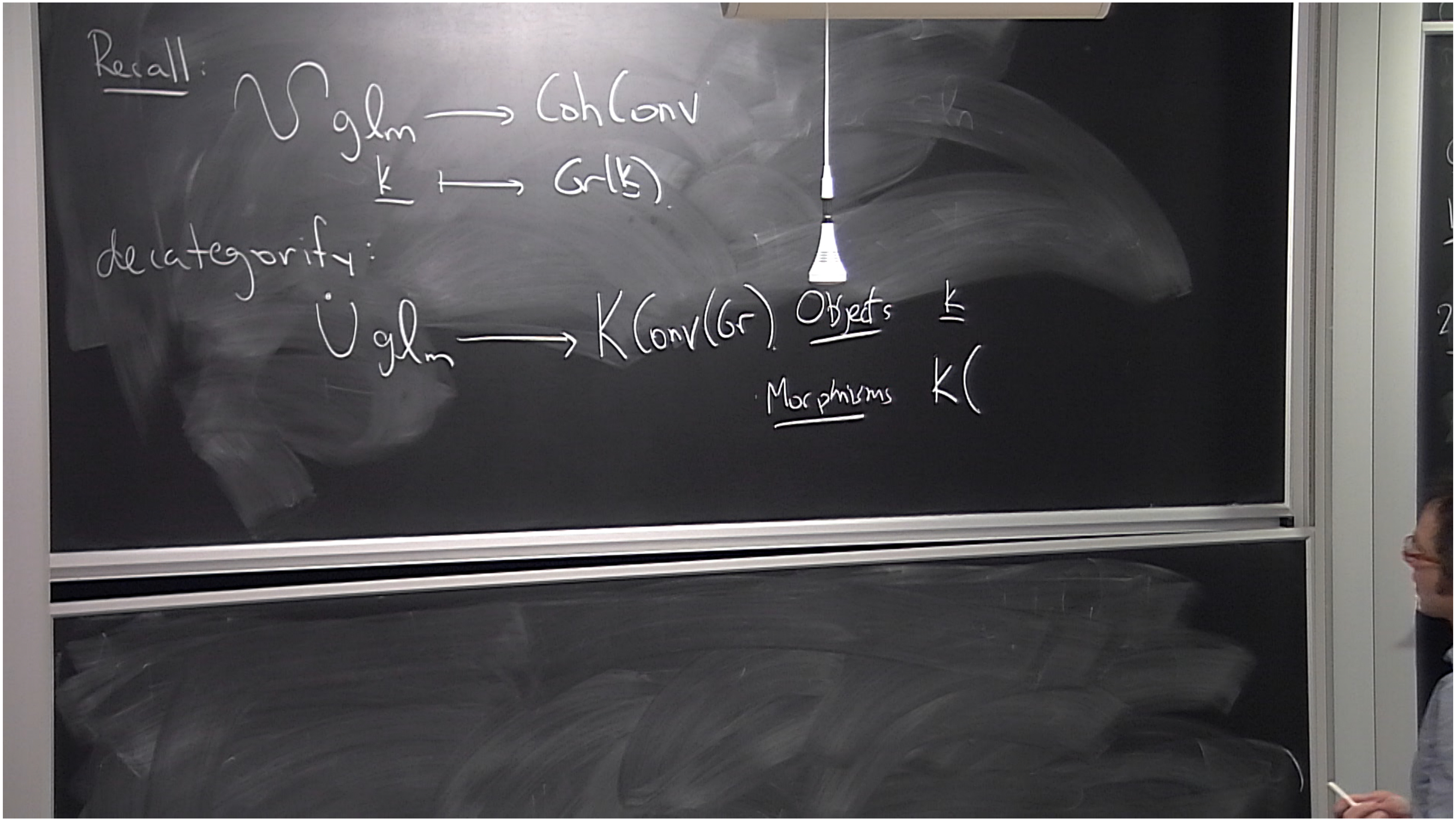
used it to define $\mathfrak{sl}_3$ knot homology.

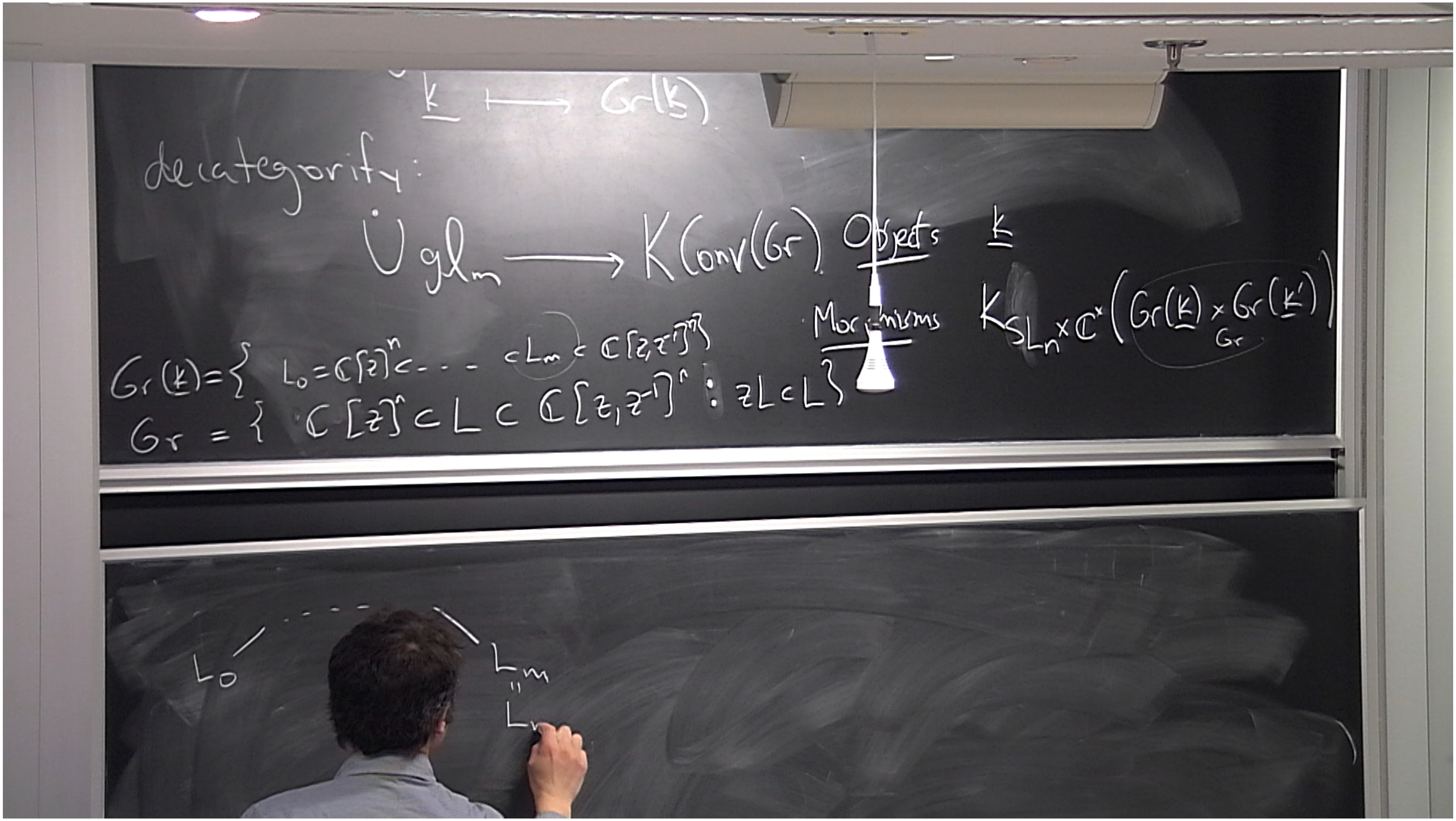
<u>Theorem</u> (Queffelec - Rose)

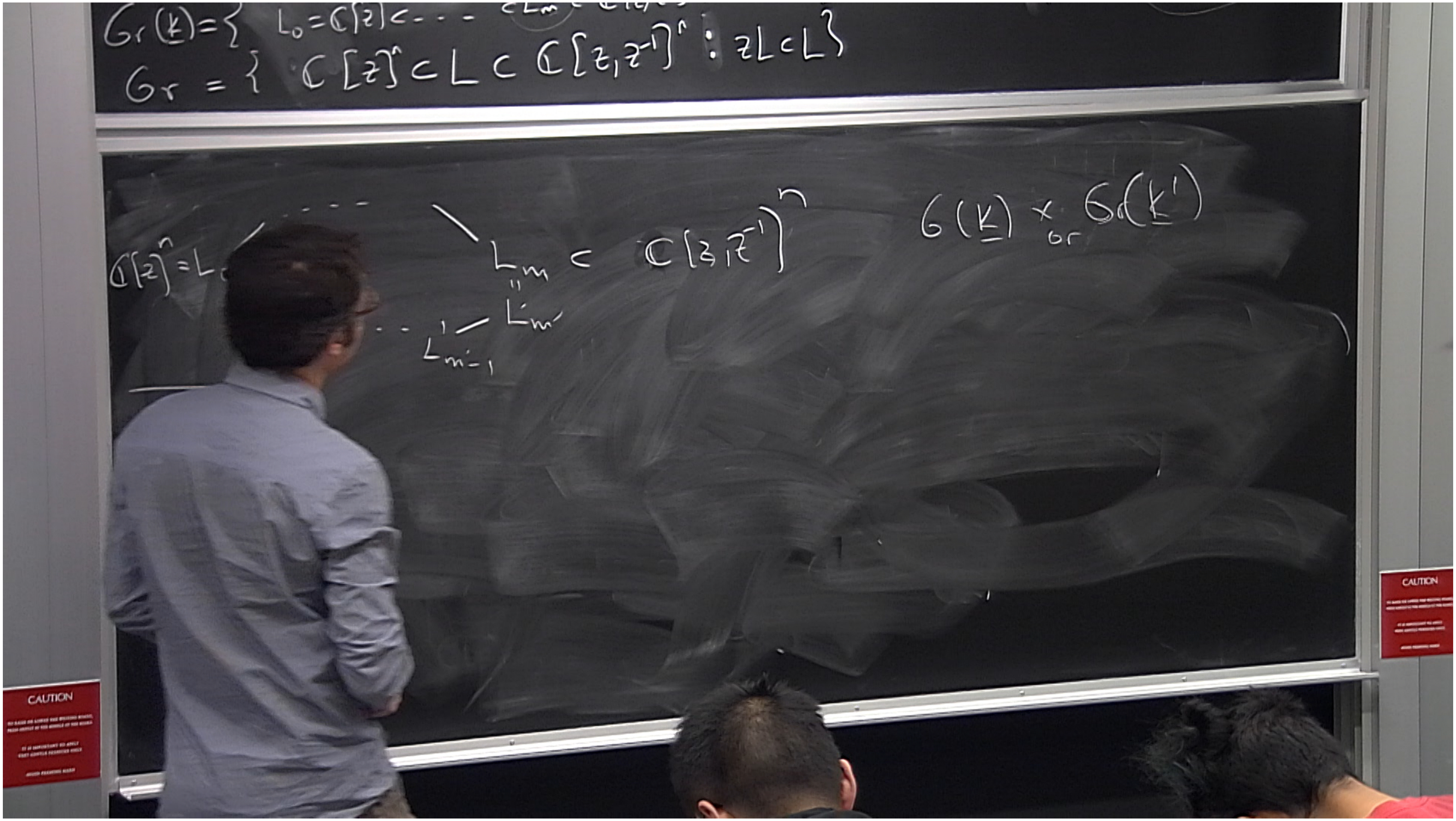
There is a def. of Foam_n and a 2-functor

$$\dot{\mathcal{U}}\mathfrak{gl}_m \longrightarrow \mathrm{Foam}_n$$

$$\dot{U}\mathfrak{gl}_m \longrightarrow \mathrm{Sp}_n.$$


Recall:
$\widetilde{\mathfrak{gl}_m} \longrightarrow$ CohConv
$\underline{k} \longmapsto Gr(\underline{k})$
decategorify:
$\dot{U}\mathfrak{gl}_m \longrightarrow K(Conv(Gr))$
Objects $\underline{k}$
Morphisms K(


k ⟼ Gr(k)
decategorify:
U̇gl_m ⟶ K(Conv(Gr)) Objects k
Morphisms K_{SL_n × C^×}(Gr(k) ×_Gr Gr(k'))
Gr(k) = { L_0 = C[z]^n ⊂ . . . ⊂ L_m ⊂ C[z,z^-1]^n }
Gr = { C[z]^n ⊂ L ⊂ C[z,z^-1]^n : zL ⊂ L }
L_0
L_m
L_n


Gr = { C[z]^n ⊂ L ⊂ C[z,z^-1]^n : zL ⊂ L }
L_m ⊂ C[z,z^-1]^n
G(k) ×_Gr Gr(k')

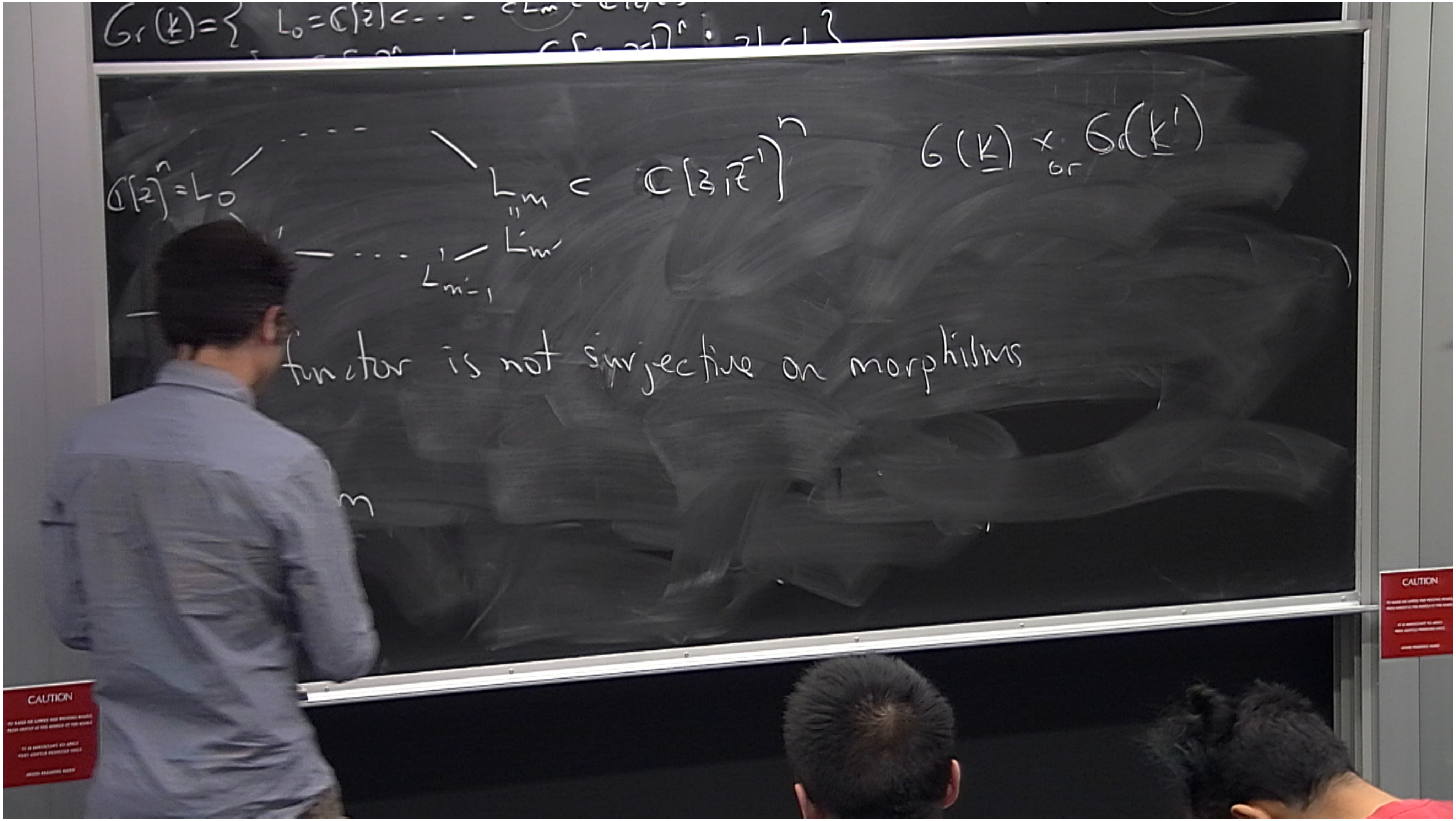
$\mathrm{Gr}(\underline{k}) = \{ \; L_0 = \mathbb{C}[z] \subset \cdots \subset L_m \subset \ldots$

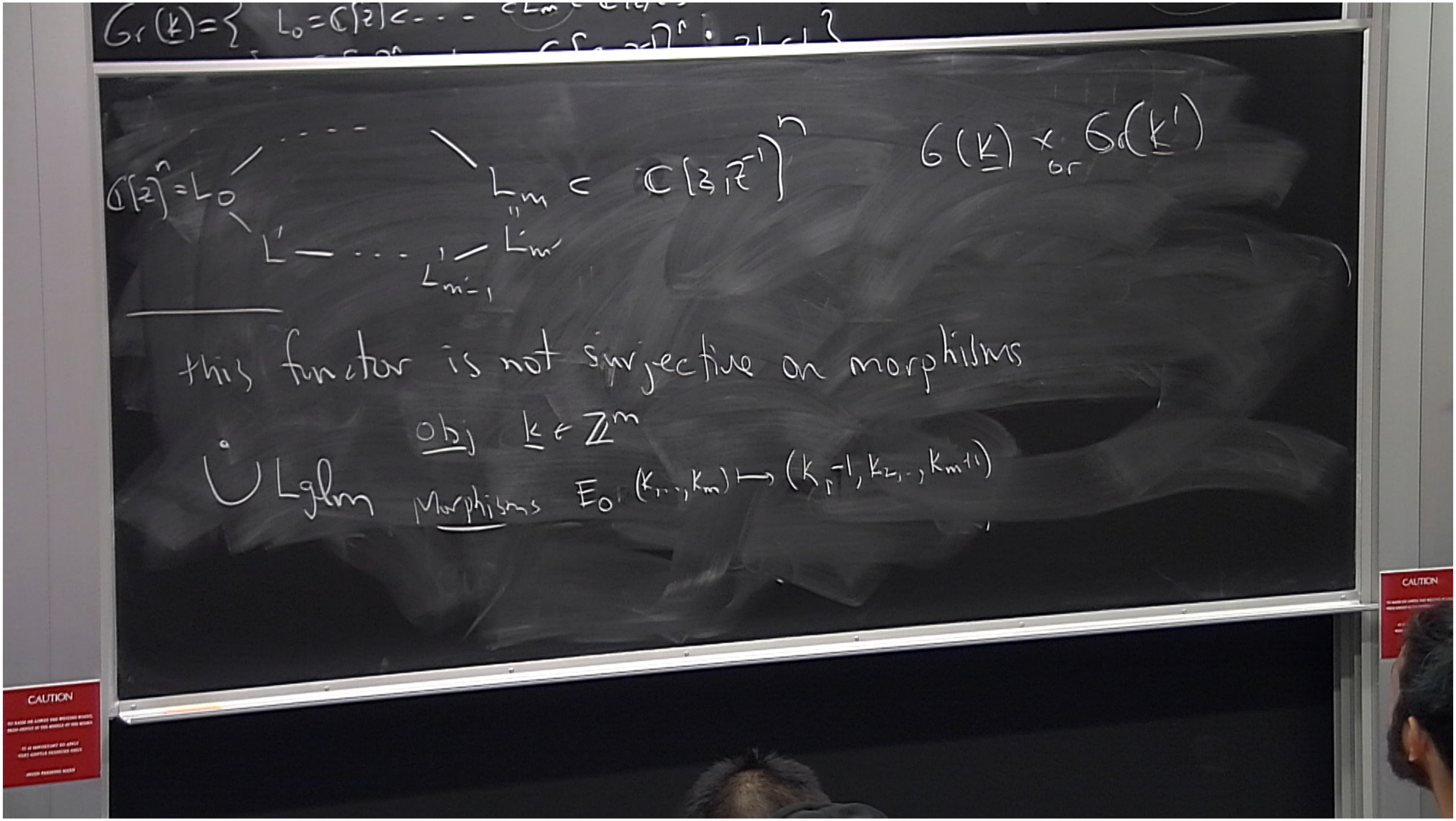
$\mathrm{Gr} = \{ \; \mathbb{C}[z]^n \subset L \subset \mathbb{C}[z, z^{-1}]^n : zL \subset L \}$

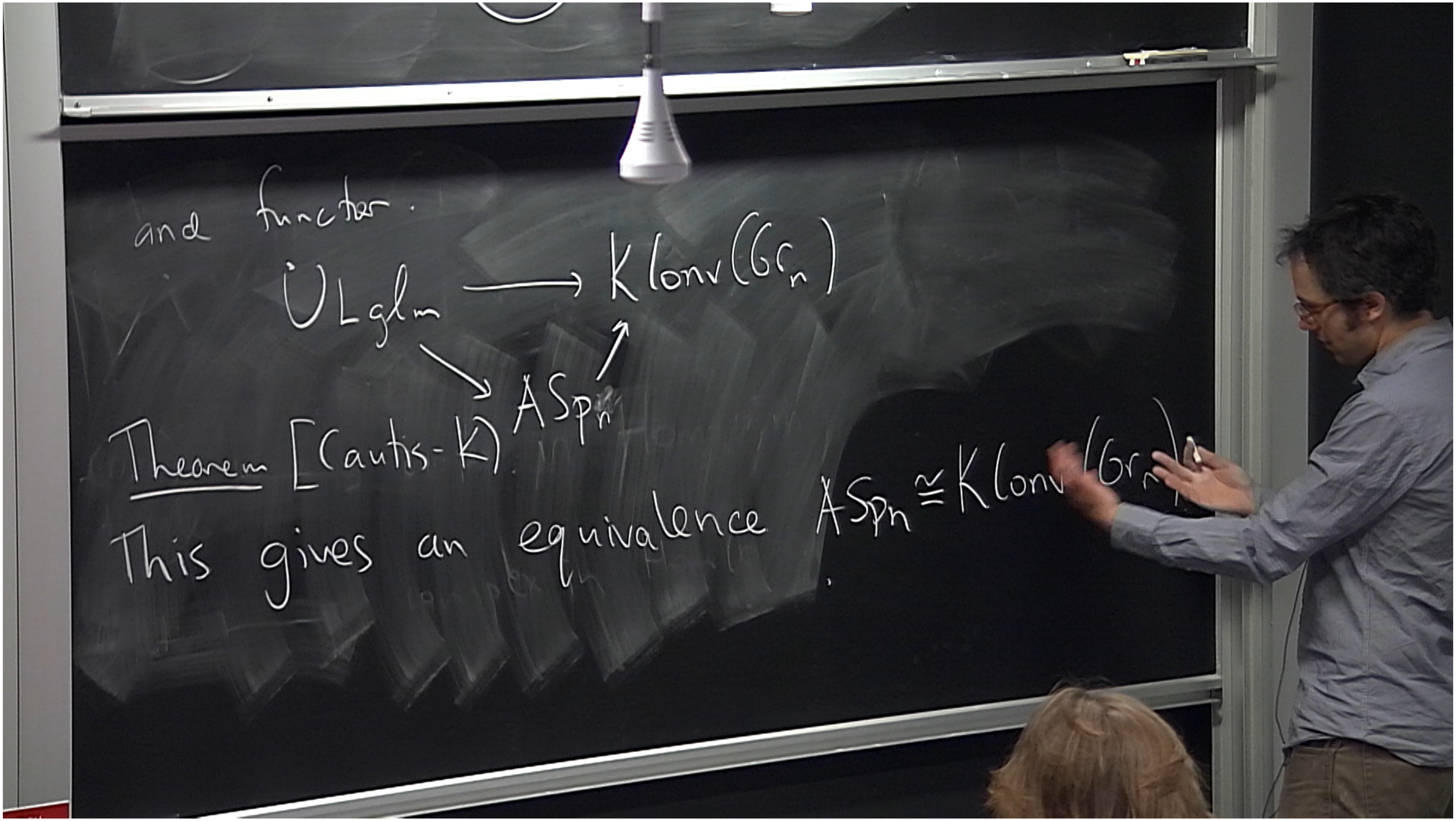
$$\begin{tikzcd}[column sep=small]
 & \cdots \arrow[r, dash] & L_m \arrow[d, equal] & \subset \mathbb{C}[z, z^{-1}]^n \\
\mathbb{C}[z]^n = L_0 \arrow[ur, dash] \arrow[r, dash] & L' \cdots L'_{m-1} \arrow[r, dash] & L'_m &
\end{tikzcd}$$

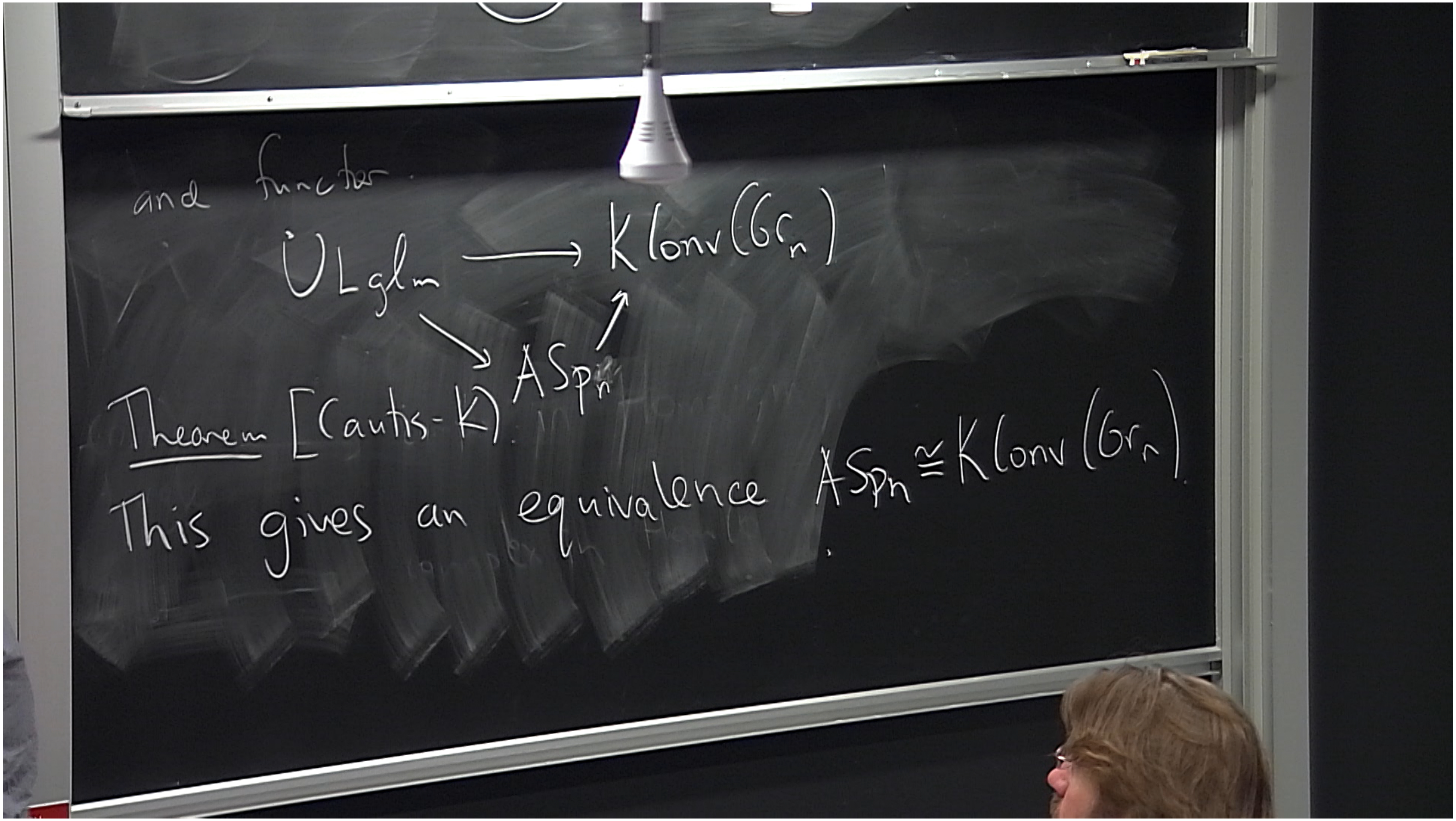
$G(\underline{k}) \times_{\mathrm{Gr}} \mathrm{Gr}(\underline{k}')$

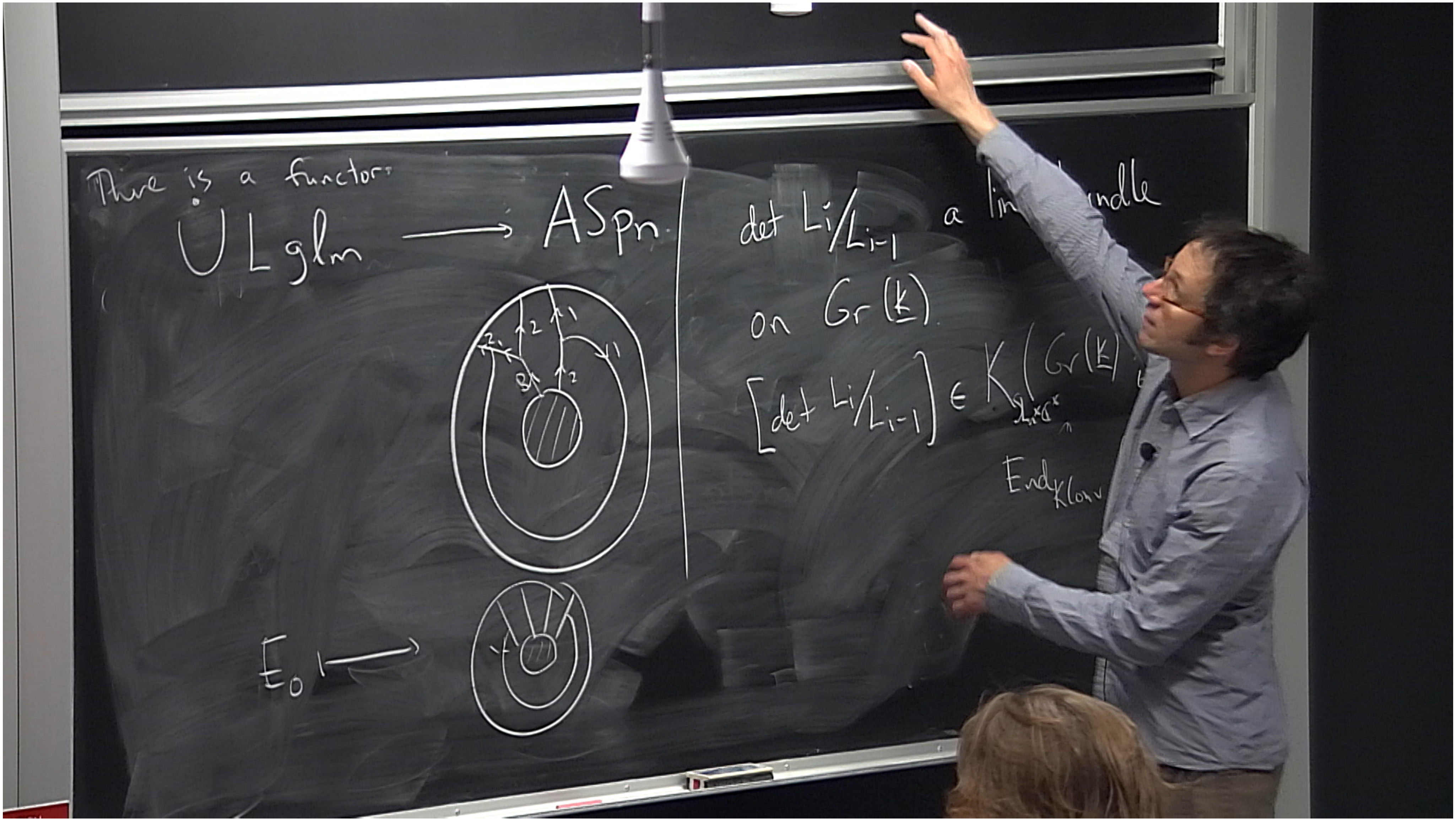
this functor is not surjective on morphisms

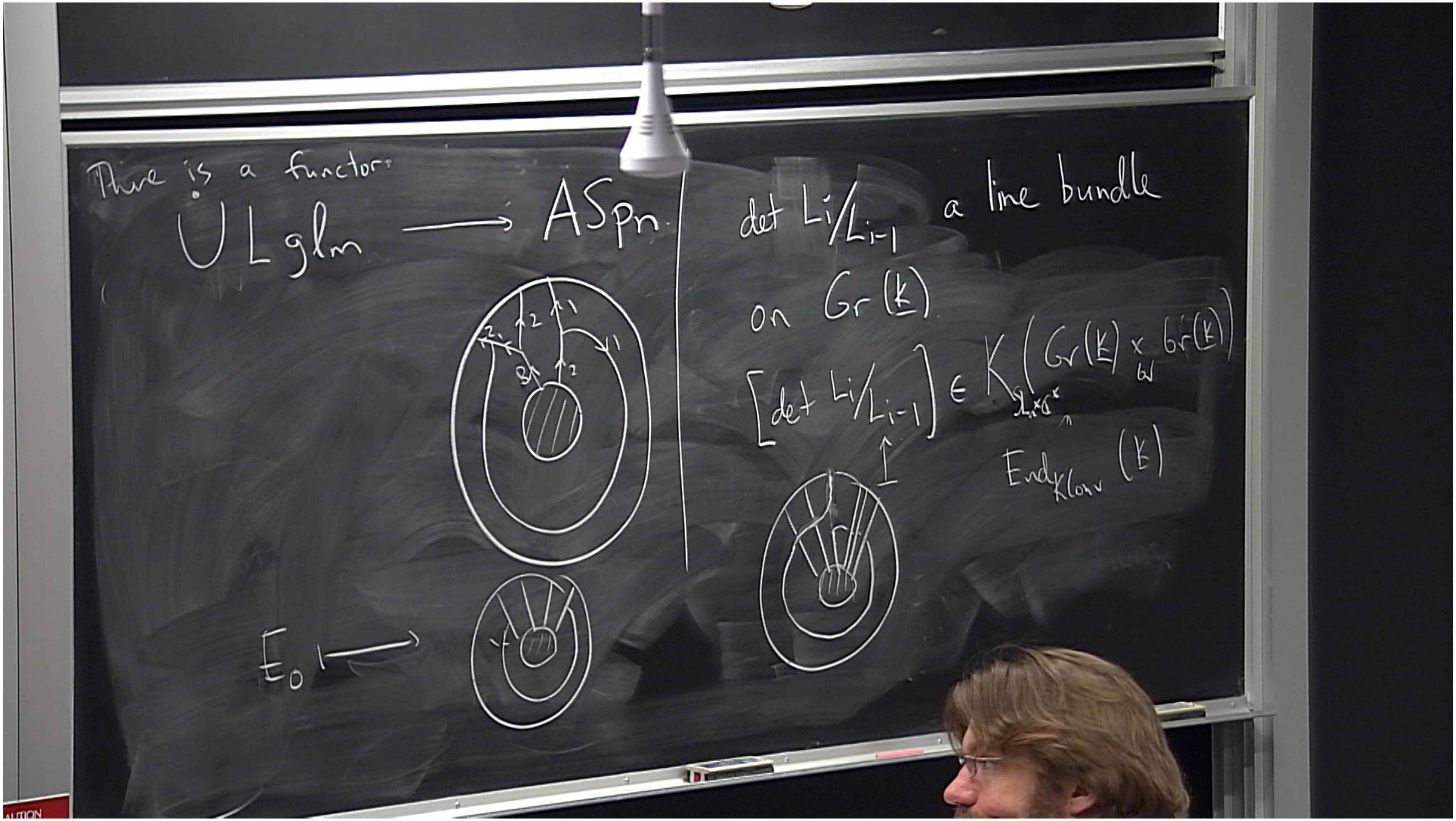

Gr(k) = {
C[z]^n = L_0
L_m ⊂ C[z, z^-1]^n
L'_m'
L'_m'-1
G(k) ×_Gr Gr(k')
functor is not surjective on morphisms
CAUTION
CAUTION

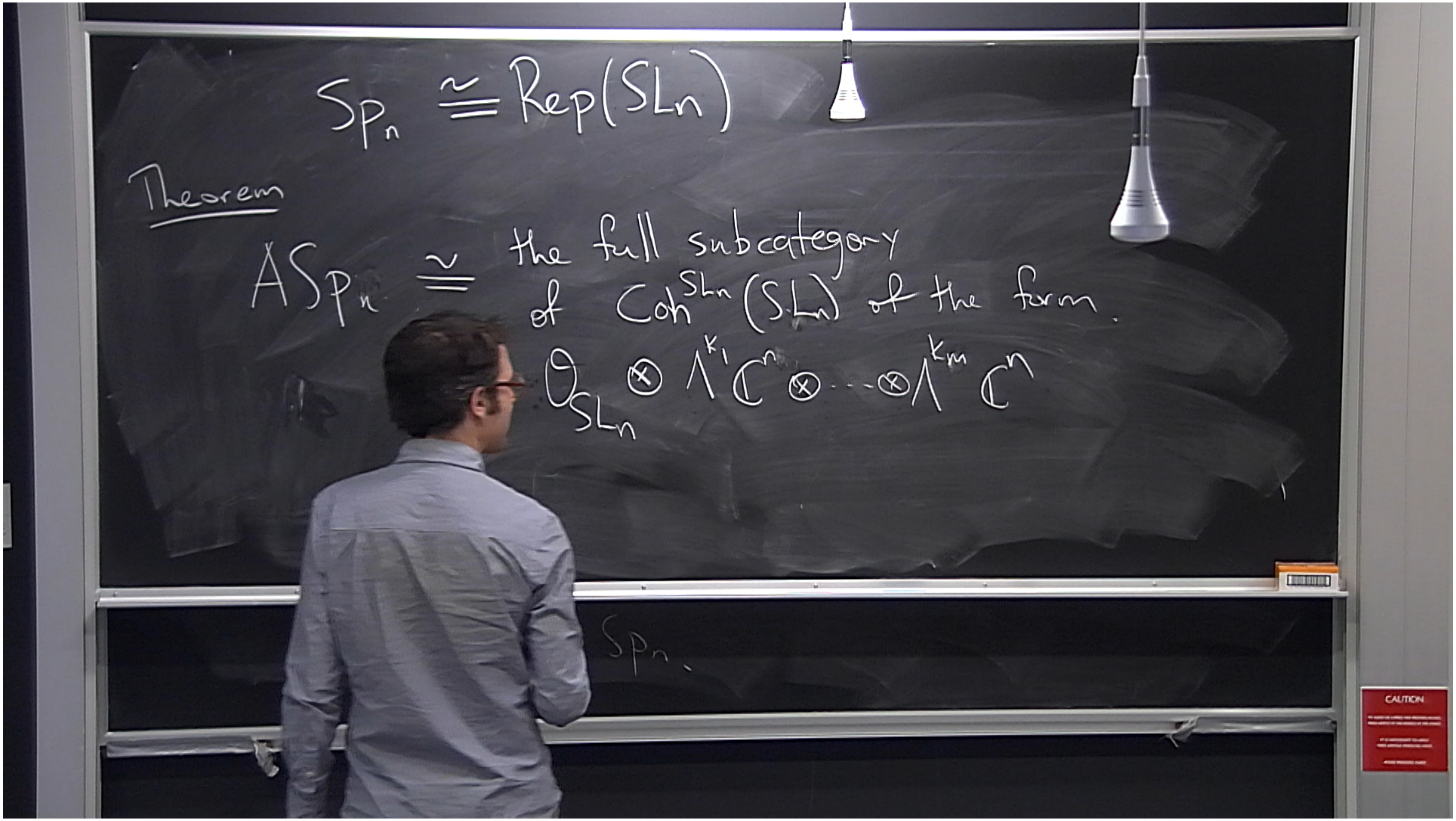

$Gr(\underline{k}) = \{ L_0 = \mathbb{C}[z] \subset \cdots$
$\mathbb{C}[z]^n = L_0 \cdots L_m \subset \mathbb{C}[z, z^{-1}]^n$
$L' \cdots L'_{m-1}$ L'_m
$G(\underline{k}) \times_{Gr} Gr(\underline{k}')$
this functor is not surjective on morphisms
Obj $\underline{k} \in \mathbb{Z}^m$
$\dot{\bigcup}$ Lglm
Morphisms E_0: $(k_1, .., k_m) \mapsto (k_1 - 1, k_2, .., k_m + 1)$

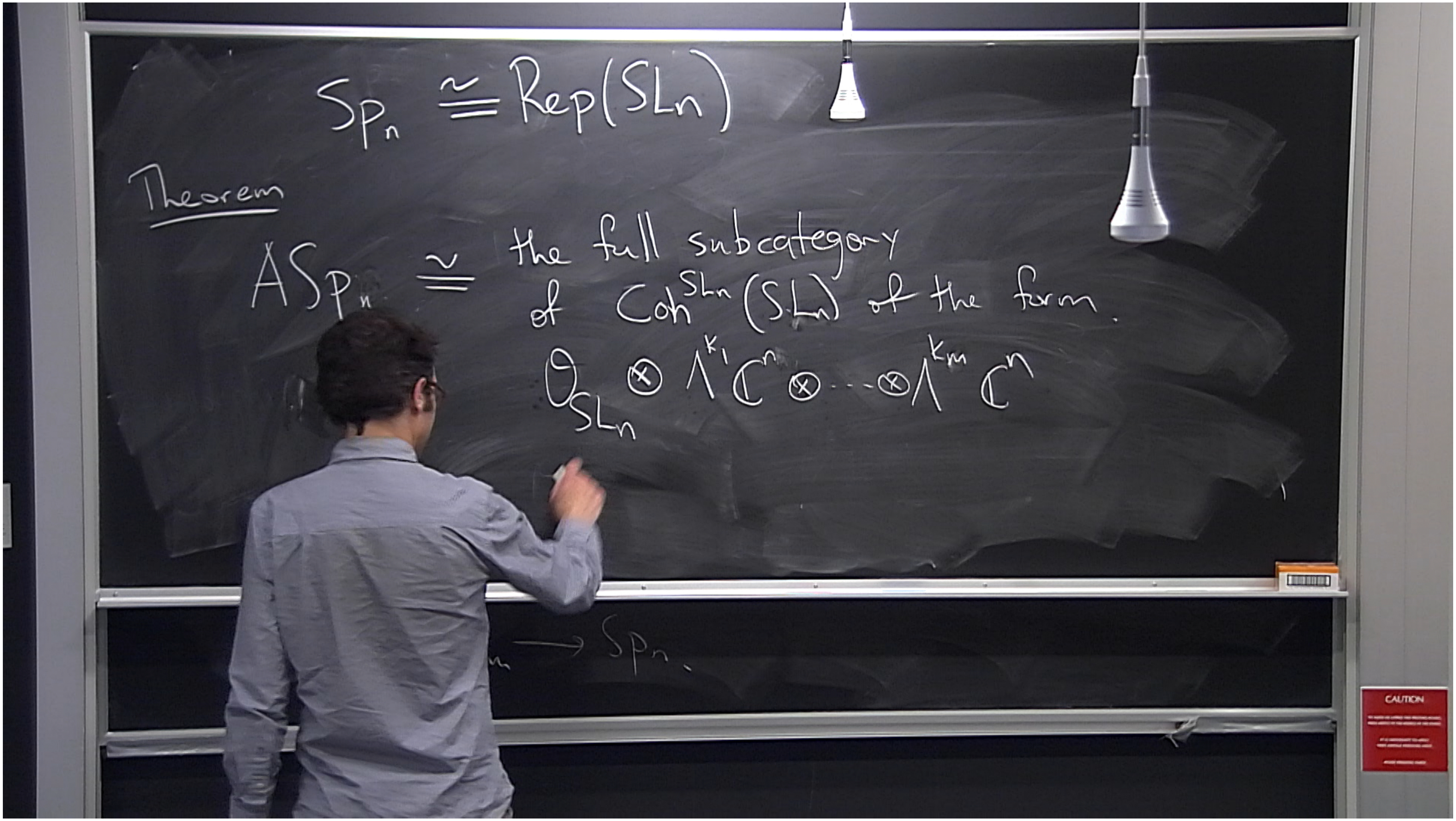

and functor.
U̇Lgl_m → KConv(Gr_n)
ASp_n
Theorem [Cautis-K]
This gives an equivalence ASp_n ≅ KConv(Gr_n).


and functor
$\dot{U}L\mathfrak{gl}_m \longrightarrow K\mathrm{Conv}(Gr_n)$
$\dot{U}L\mathfrak{gl}_m \to ASp_n \to K\mathrm{Conv}(Gr_n)$
Theorem [Cautis-K]
This gives an equivalence $ASp_n \cong K\mathrm{Conv}(Gr_n)$.


There is a functor
U L glm ⟶ ASpn
det Li/Li-1
on Gr(k)
[det Li/Li-1] ∈ K(Gr(k))
E0 ⟼


There is a functor:
ULgl_m → ASp_n
det Li/Li-1 a line bundle
on Gr(k)
[det Li/Li-1] ∈ K(Gr(k) × Gr(k))
End_Kconv (k)
E_0 ↦


$Sp_n \cong Rep(SL_n)$
Theorem
$ASp_n \cong$ the full subcategory
of $Coh^{SL_n}(SL_n)$ of the form.
$\mathcal{O}_{SL_n} \otimes \Lambda^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n$
Sp_n.
CAUTION


$Sp_n \cong Rep(SL_n)$
Theorem
$ASp_n \cong$ the full subcategory
of $Coh^{SL_n}(SL_n)$ of the form.
$\mathcal{O}_{SL_n} \otimes \Lambda^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n$
$\longrightarrow Sp_n.$

$$Sp_n \cong Rep(SL_n)$$

Theorem

$ASp_n \cong$ the full subcategory of $Coh^{SL_n}(SL_n)$ of the form.

$$\mathcal{O}_{SL_n} \otimes \Lambda^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n$$

$U_q(\mathfrak{sl}_n)$-equiv $\mathcal{O}_q(SL_n)$-modules.

$$\mathcal{O}_q(SL_n) \otimes \Lambda_q^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n .$$

$$U\mathfrak{gl}_m \longrightarrow Sp_n .$$

$$Sp_n \cong Rep(SL_n)$$

Corollary

$$\mathcal{O}_q\left(\frac{SL_n}{SL_n}\right)\text{-mod} \cong K(\mathrm{Conv}(Gr_{SL_n}))$$

Theorem

$ASp_n \cong$ the full subcategory of $Coh^{SL_n}(SL_n)$ of the form

$$\mathcal{O}_{SL_n} \otimes \Lambda^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n$$

$U_q(sl_n)$-equiv $\mathcal{O}_q(SL_n)$-modules.

$$\mathcal{O}_q(SL_n) \otimes \Lambda_q^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n .$$