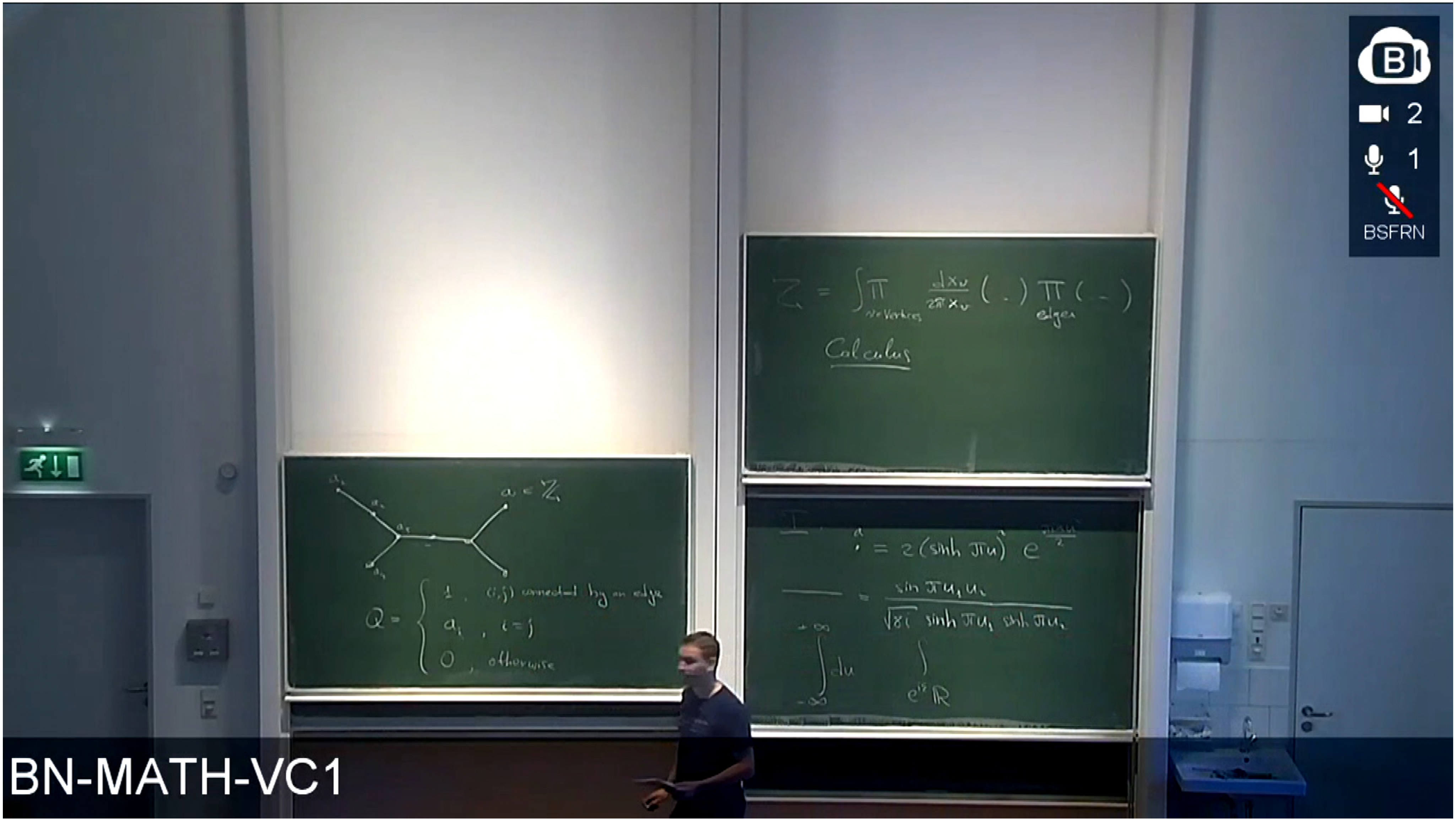
Title: MPIM/PI teleseminar on categorified knot invariants - Zed-hat

Date: May 17, 2018 09:00 AM

URL: http://pirsa.org/18050052

Abstract: <p>In this talk, intended for a broader audience, I promise to use techniques only at the level of university calculus. While staying at this level, our goal will be to learn conceptual lessons for categorification of quantum group invariants of knots and 3-manifolds, also known as the Witten-Reshetikhin-Turaev (or WRT) invariants. In particular, we will introduce new q-series invariants of 3-manifolds that have integer powers and integer coefficients and, if time permits, discuss their various constructions and properties. The talk is based on several papers with D. Pei and/or P. Putrov, C. Vafa, as well as ongoing work with C. Manolescu.</p>



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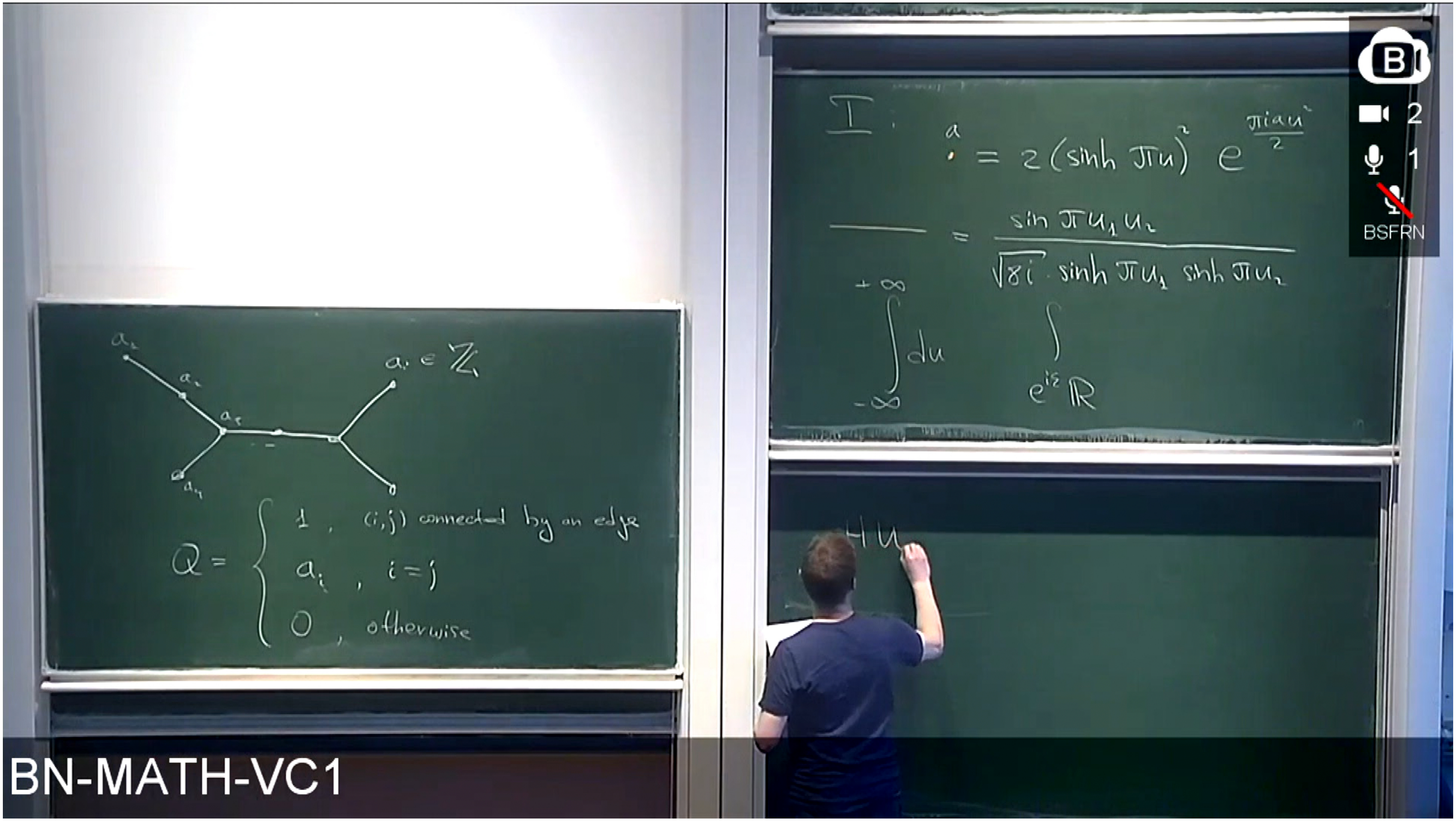
BSFRN
Calculus
$a \in \mathbb{Z}$
connected by an edge
otherwise
BN-MATH-VC1



$a_i \in \mathbb{Z}$
$Q = \begin{cases} 1, & (i,j) \text{ connected by an edge} \\ a_i, & i=j \\ 0, & \text{otherwise} \end{cases}$
I.
$= 2(\sinh \pi u)^2 e^{\frac{\pi i a u^2}{2}}$
$= \frac{\sin \pi u_1 u_2}{\sqrt{8i}\, \sinh \pi u_1 \sinh \pi u_2}$
$\int_{-\infty}^{+\infty} du$
$\int_{e^{i\varepsilon}\mathbb{R}}$
BN-MATH-VC1
BSFRN

$$Q = \begin{cases} 1 \,, & (i,j) \text{ connected by an edge} \\ a_i \,, & i=j \\ 0 \,, & \text{otherwise} \end{cases}$$

II.

$$\oint_{|x_i|=1} \prod_{i \in \text{Vertices}} \frac{dx_i}{2\pi i \, x_i} \cdot \frac{1}{1-\prod_j x_j^{-Q_{ij}}} \times (1-x_i)^{\deg(i)-2}$$

I.

$$\overset{a}{\bullet} = 2(\sinh \pi u)^2 \, e^{\frac{\pi i a u^2}{2}}$$

$$\text{———} = \frac{\sin \pi u_1 u_2}{\sqrt{8i} \cdot \sinh \pi u_1 \, \sinh \pi u_2}$$

$$\int_{-\infty}^{+\infty} du \qquad \int_{e^{i\varepsilon}\mathbb{R}}$$

HW:

$$Z\left( \overset{4}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \right)$$

$$Z\left( \overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{5}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \right)$$

$$Z\left( \overset{3}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{5}{\bullet} \right)$$

$$Q = \begin{cases} 1\,, & (i,j) \text{ connected by an edge} \\ a_i\,, & i=j \\ 0\,, & \text{otherwise} \end{cases}$$

I. $\overset{a}{\bullet} = 2(\sinh \pi u)^2\, e^{\frac{\pi i a u^2}{2}}$

$\underline{\qquad} = \dfrac{\sin \pi u_1 u_2}{\sqrt{8i}\cdot \sinh \pi u_1 \, \sinh \pi u_2}$

$\displaystyle\int_{-\infty}^{+\infty} du \qquad \int_{e^{i\varepsilon}\mathbb{R}}$

II. $\displaystyle\oint_{|x_i|=1} \prod_{i\in \text{Vertices}} \frac{dx_i}{2\pi i\, x_i} \cdot \frac{1}{1-\prod_j x_j^{-Q_{ij}}} \times (1-x_i)^{\deg(i)-2}\, x_i^{h_i}$

$h \in \operatorname{coker}(Q)$

HW: $Z(\overset{4}{\bullet}\!\!-\!\!-\!\!\overset{2}{\bullet})$

$Z(\overset{1}{\bullet}\!\!-\!\!-\!\!\overset{5}{\bullet}\!\!-\!\!-\!\!\overset{2}{\bullet})$

$Z(\overset{3}{\bullet}\!\!-\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!-\!\!\overset{5}{\bullet})$

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$$\oint_{|x_i|=1} \prod_{i\in\text{Vertices}} \frac{dx_i}{2\pi i\, x_i} \cdot \frac{1}{1-\prod_j x_j^{-Q_{ij}}}$$

$$\times \left(1-x_i\right)^{\deg(i)-2} \; x_i^{h_i}$$

$$h \in \operatorname{coker}(Q)$$

$\underline{\mathrm{I}}$ :

$$a = 2(\sinh \pi u)^2 \; e^{\frac{\pi i a u^2}{2}}$$

$$\underline{\qquad} = \frac{\sin \pi u_1 u_2}{\sqrt{8i}\cdot \sinh \pi u_1 \sinh \pi u_2}$$

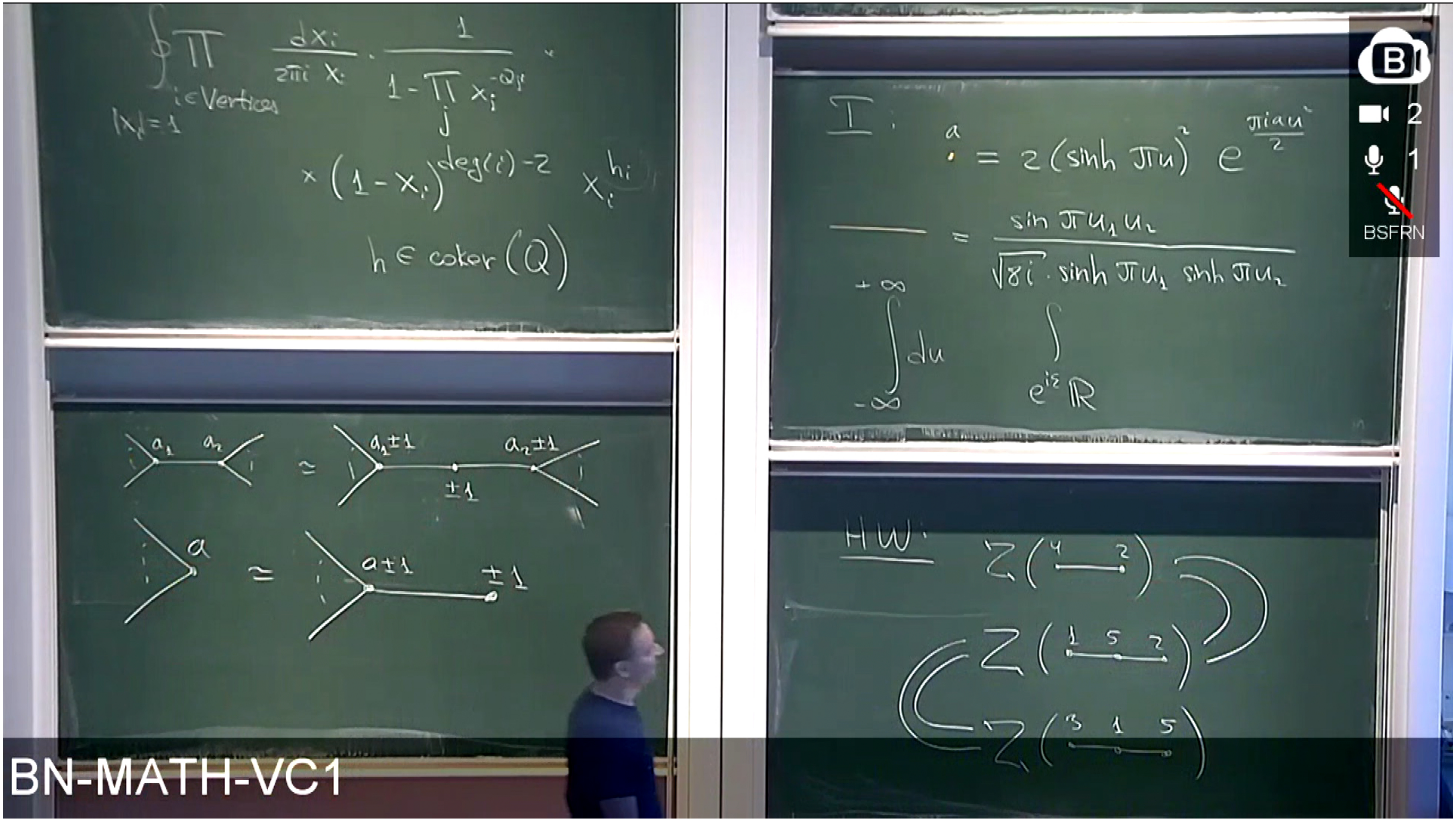
$$\int_{-\infty}^{+\infty} du \qquad \int_{e^{i\varepsilon}\mathbb{R}}$$

$\underline{\text{HW}}$:

$Z(4 \text{ — } 2)$

$Z(1 \text{ — } 5 \text{ — } 2)$

$Z(3 \text{ — } 1 \text{ — } 5)$


$\oint_{|x_i|=1} \prod_{i \in \text{Vertices}} \frac{dx_i}{2\pi i\, x_i} \cdot \frac{1}{1-\prod_j x_j^{-Q_{ij}}} \times (1-x_i)^{\deg(i)-2}\, x_i^{h_i}$
$h \in \text{coker}(Q)$
$a_1$ $a_2$ $\simeq$ $a_1 \pm 1$ $\pm 1$ $a_2 \pm 1$
$a$ $\simeq$ $a \pm 1$ $\pm 1$
I:
$= 2(\sinh \pi u)^2\, e^{\frac{\pi i a u^2}{2}}$
$= \frac{\sin \pi u_1 u_2}{\sqrt{8i}\cdot \sinh \pi u_1 \sinh \pi u_2}$
$\int_{-\infty}^{+\infty} du$
$\int_{e^{i\varepsilon}\mathbb{R}}$
HW:
$Z(4 \; 2)$
$Z(1 \; 5 \; 2)$
$Z(3 \; 1 \; 5)$
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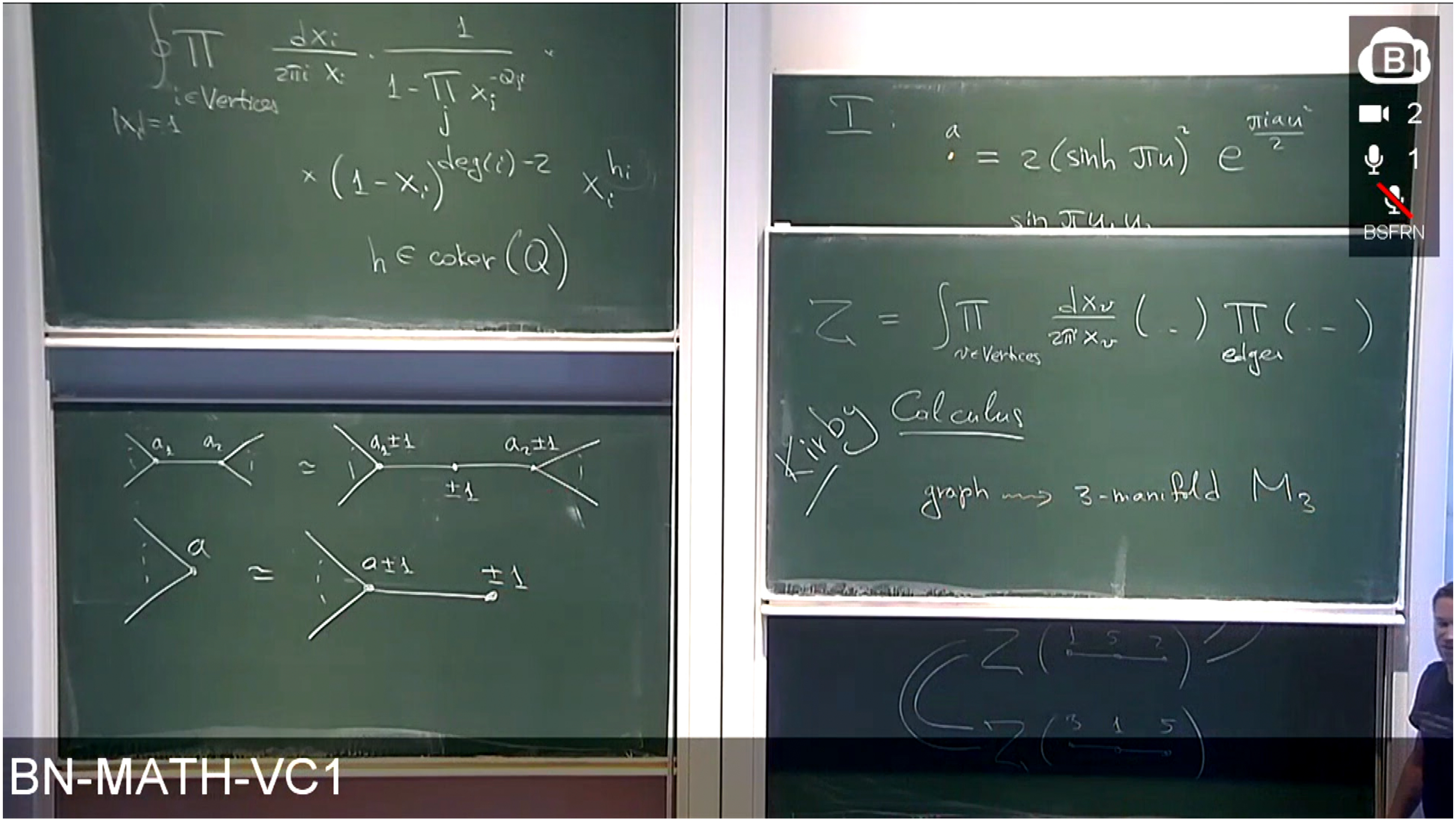
∮ ∏ dxi/2πi xi · 1/(1 − ∏j xj^(−Qij))
|xi| = 1, i ∈ Vertices
× (1 − xi)^(deg(i) − 2) xi^hi
h ∈ coker(Q)
a1 a2
a1±1 a2±1
±1
a
a±1
±1
I : a = 2(sinh πu)² e^(πiau²/2)
Z = ∫ ∏ dxv/2πi xv ( . ) ∏ ( . . )
v ∈ Vertices
edges
Kirby Calculus
graph ⇝ 3-manifold M3
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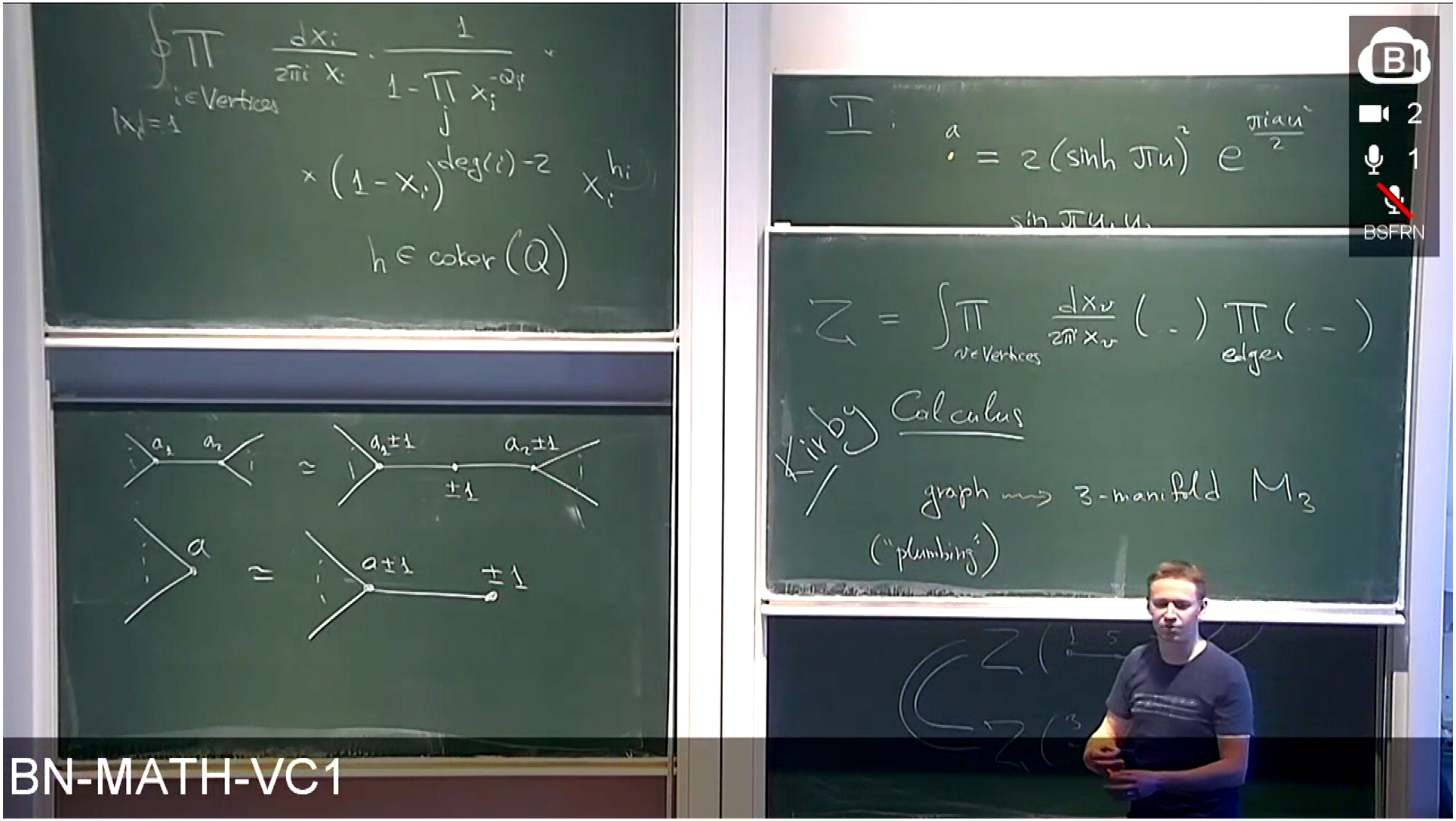

∮ ∏ dxᵢ/2πi xᵢ · 1/(1 − ∏ⱼ xⱼ^−Qᵢⱼ)
|xᵢ|=1, i∈Vertices
× (1 − xᵢ)^deg(i)−2 xᵢ^hᵢ
h ∈ coker(Q)
a₁ a₂ ≃ a₁±1 ±1 a₂±1
a ≃ a±1 ±1
I: a = 2(sinh πu)² e^(πiau²/2)
Z = ∫ ∏ dx_v/2πi x_v ( . ) ∏ ( . . )
v∈Vertices edges
Kirby Calculus
graph ⇝ 3-manifold M₃
("plumbing")
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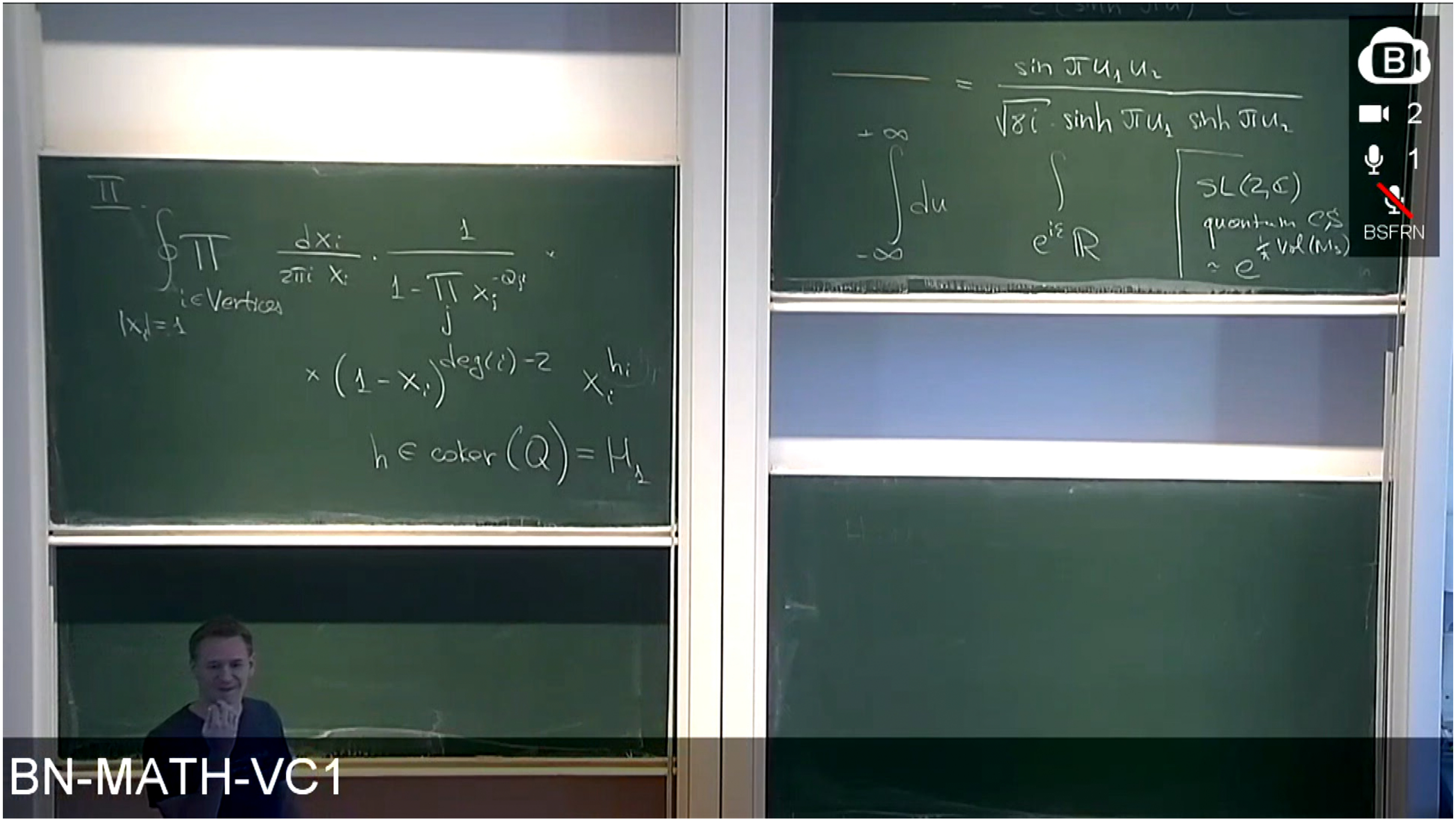
II.
$\oint_{|x_i|=1} \prod_{i \in \text{Vertices}} \frac{dx_i}{2\pi i\, x_i} \cdot \frac{1}{1 - \prod_j x_j^{-Q_{ij}}} \times$
$\times (1 - x_i)^{\deg(i) - 2} \; x_i^{h_i}$
$h \in \operatorname{coker}(Q) = H_1$
$= \frac{\sin \pi u_1 u_2}{\sqrt{8i} \cdot \sinh \pi u_1 \, \sinh \pi u_2}$
$\int_{-\infty}^{+\infty} du$
$\int_{e^{i\varepsilon}\mathbb{R}}$
$SL(2,\mathbb{C})$
quantum CS
$\sim e^{\frac{1}{\pi} Vol(M_3)}$
BN-MATH-VC1
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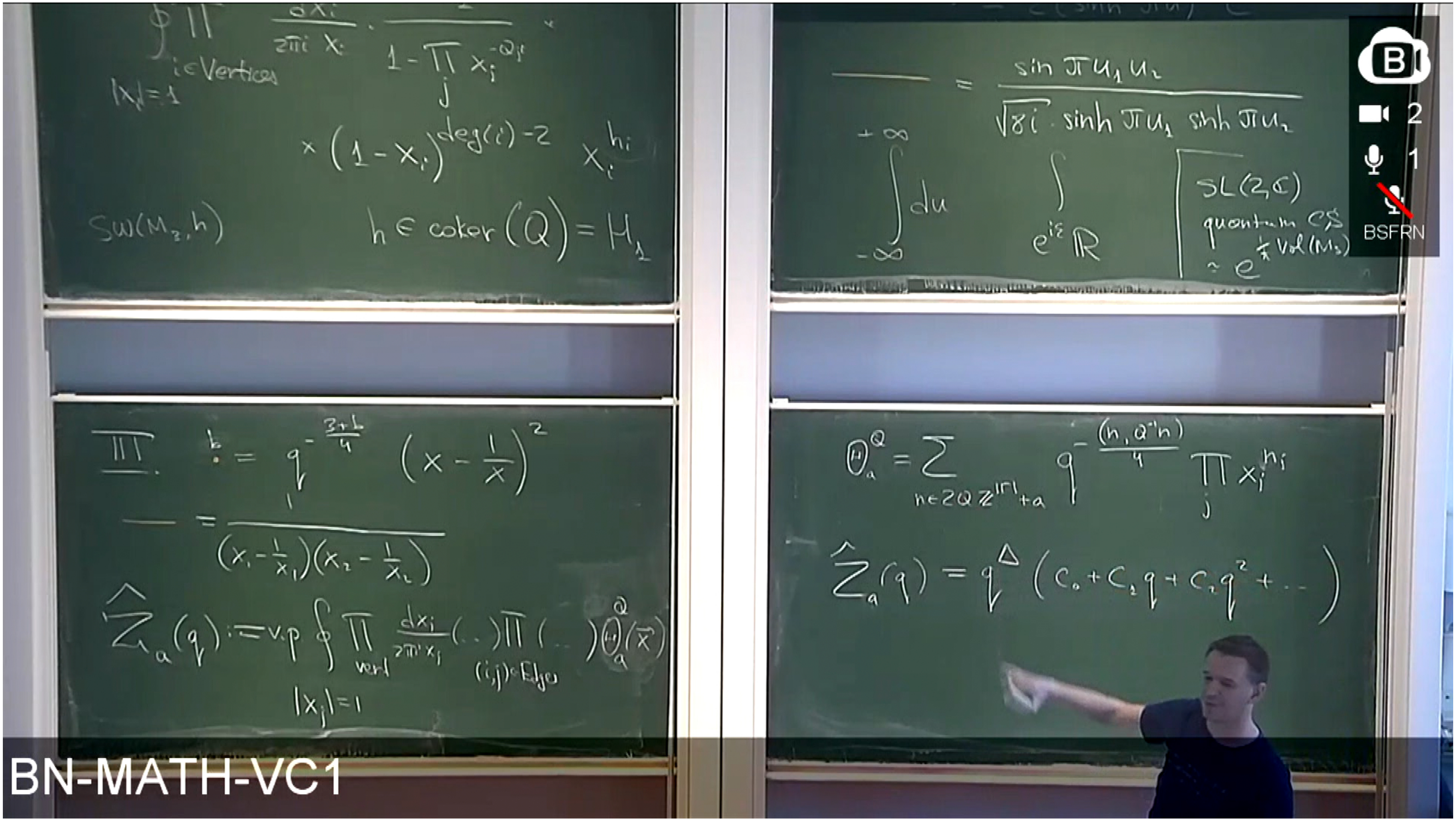

i ∈ Vertices
|x_j| = 1
× (1 − x_i)^{deg(i) − 2} x_i^{h_i}
SW(M_3, h)
h ∈ coker(Q) = H_1
= sin π u_1 u_2 / (√(8i) · sinh π u_1 sinh π u_2)
du
e^{iε} ℝ
SL(2,ℂ)
quantum CS
≈ e^{(1/π) Vol(M_3)}
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2
1
:= q^{−(3+b)/4}
(x − 1/x)^2
(x_1 − 1/x_1)(x_2 − 1/x_2)
Ẑ_a(q) := v.p ∮ ∏_{vert} dx_j/(2πi x_j) ( ) ∏_{(i,j) ∈ Edges} ( ) Θ_a^Q(x)
|x_j| = 1
Θ_a^Q = Σ_{n ∈ 2Q ℤ^{|V|} + a} q^{−(n, Q^{−1} n)/4} ∏_j x_j^{n_j}
Ẑ_a(q) = q^Δ (c_0 + c_1 q + c_2 q^2 + ...)
BN-MATH-VC1

$$\oint_{|x_j|=1} \prod_{i \in \text{Vertices}} \frac{dx_i}{2\pi i\, x_i} \cdot \frac{1}{1-\prod_j x_j^{-Q_{ij}}} \times (1-x_i)^{\deg(i)-2}\, x_i^{h_i}$$

$SW(M_3, h)$ $\qquad h \in \operatorname{coker}(Q) = H_1$

$$\text{—} = \frac{\sin \pi u_1 u_2}{\sqrt{8i}\,\sinh \pi u_1 \,\sinh \pi u_2}$$

$$\int_{-\infty}^{+\infty} du \qquad \int_{e^{i\varepsilon}\mathbb{R}}$$

$SL(2,\mathbb{C})$ quantum CS $\sim e^{\frac{1}{\hbar}\mathrm{Vol}(M_3)}$

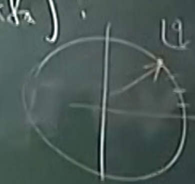
III. $\quad \overset{b}{\bullet} := q^{-\frac{3+b}{4}} \left(x - \frac{1}{x}\right)^2$

$$\text{—} = \frac{1}{\left(x_1 - \frac{1}{x_1}\right)\left(x_2 - \frac{1}{x_2}\right)}$$

$$\hat{Z}_a(q) := \text{v.p.} \oint_{|x_j|=1} \prod_{\text{vert}} \frac{dx_i}{2\pi i\, x_i} (\ldots) \prod_{(i,j)\in \text{Edges}} (\ldots)\, \Theta_a^Q(\vec{x})$$

$$\Theta_a^Q = \sum_{n \in 2Q\mathbb{Z}^{|\Gamma|}+a} q^{-\frac{(n, Q^{-1}n)}{4}} \prod_j x_j^{n_j}$$

Thm [w/ D. Pei, P. Putrov, C. Vafa].




- convergent in $|q|<1$
- $\hat{Z}_a(q) \in q^{\Delta}\mathbb{Z}[[q]]$
- $\lim_{q \to \exp\left(\frac{2\pi i}{k}\right)} \hat{Z}_a(q) = WRT(M_3, k)$

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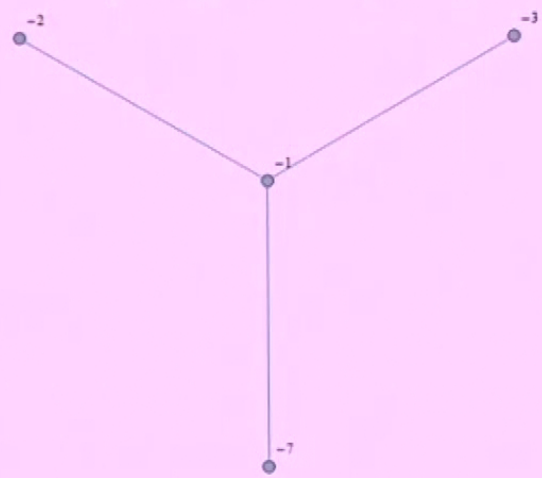
Plumbed0.nb

```
Framings = -{2, 1, 3, 7};
L = Length[Framings];
Plumbing = Graph[{1 → 2, 2 → 3, 2 → 4}, DirectedEdges → False,
  VertexWeight → Framings,
  VertexLabels → Table[i → Framings[[i]], {i, L}],
  GraphLayout → "SpringElectricalEmbedding"]
```




```
VL = VertexList[Plumbing]
{1, 2, 3, 4}
```

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```
VL = VertexList[Plumbing]
{1, 2, 3, 4}

M = AdjacencyMatrix[Plumbing];
Do[M[[i, i]] = Framings[[VL[[i]]]], {i, L}];
M // MatrixForm
```

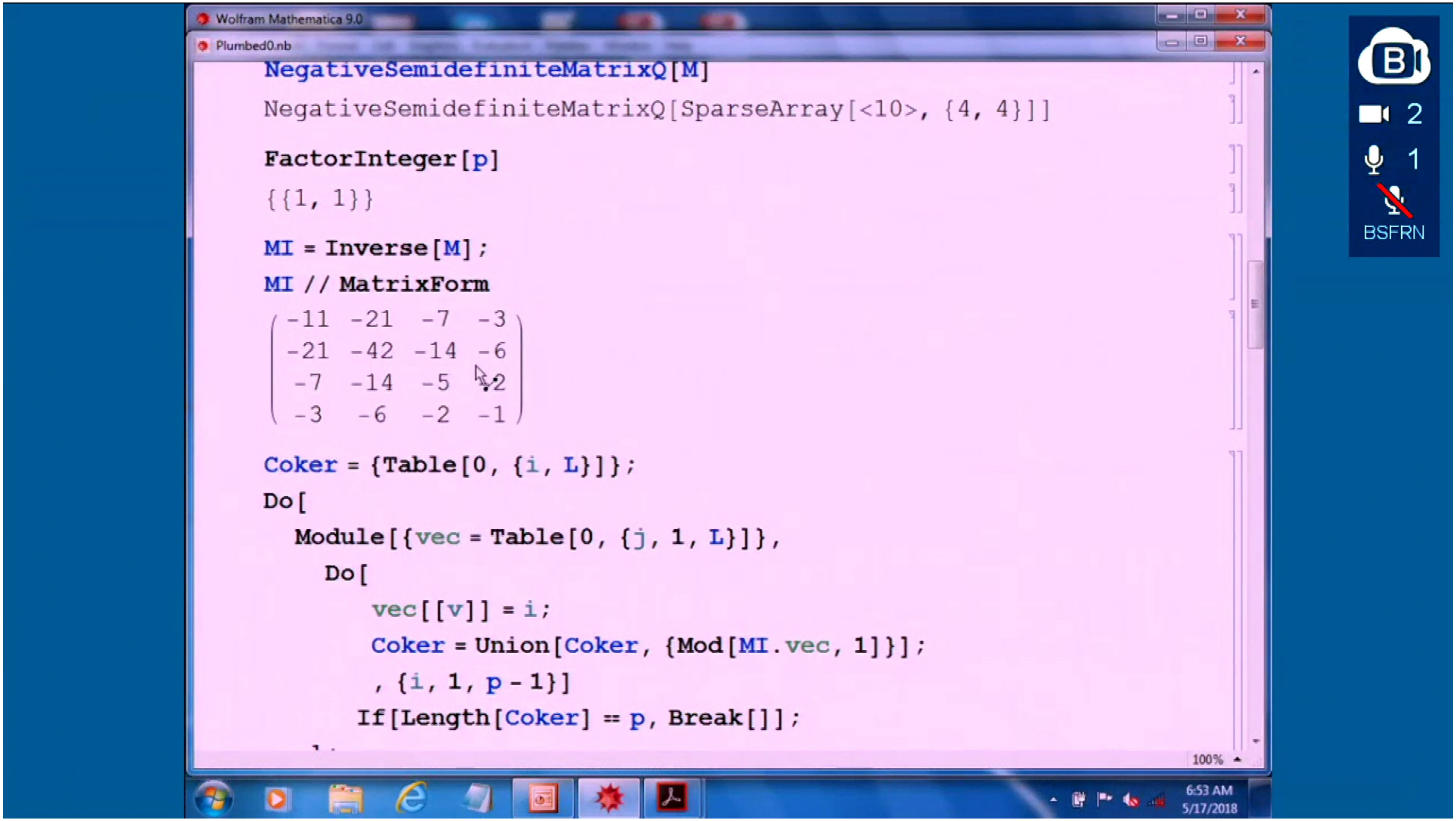
$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -7 \end{pmatrix}$$

```
p = Abs[Det[M]]
1

Delta = VertexDegree[Plumbing] - 2
{-1, 1, -1, -1}

Eigenvalues[M] // N
{-7.17418, -3.47914, -2.32949, -0.0171987}

NegativeSemidefiniteMatrixQ[M]
NegativeSemidefiniteMatrixQ[SparseArray[<10>, {4, 4}]]
```


Wolfram Mathematica 9.0
Plumbed0.nb
NegativeSemidefiniteMatrixQ[M]
NegativeSemidefiniteMatrixQ[SparseArray[<10>, {4, 4}]]
FactorInteger[p]
{{1, 1}}
MI = Inverse[M];
MI // MatrixForm
-11 -21 -7 -3
-21 -42 -14 -6
-7 -14 -5 -2
-3 -6 -2 -1
Coker = {Table[0, {i, L}]};
Do[
Module[{vec = Table[0, {j, 1, L}]},
Do[
vec[[v]] = i;
Coker = Union[Coker, {Mod[MI.vec, 1]}];
, {i, 1, p - 1}]
If[Length[Coker] == p, Break[]];
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```
        , {i, 1, p - 1}]
      If[Length[Coker] == p, Break[]];
    ];
   , {v, L}];
Coker;
Length[Coker]

1

TwCoker = Mod[2 * Coker, 1];

Cok0 = Function[vec,
   Mod[MI.vec, 1]
  ];

cD = Cok0[Delta]
Position[TwCoker, cD]

{0, 0, 0, 0}

{{1}}

Cok1 = Function[vec,
   Cok0[(vec - Delta) / 2]];
```

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```
Cok1 = Function[vec,
   Cok0[(vec - Delta) / 2]];

Deg = Function[{mon, var},
  D[mon, var] / mon * var // Simplify]
```

$$\mathrm{Function}\left[\{\mathrm{mon},\ \mathrm{var}\},\ \mathrm{Simplify}\left[\frac{\partial_{\mathrm{var}}\mathrm{mon}\ \mathrm{var}}{\mathrm{mon}}\right]\right]$$

```
Degv = Function[mon,
   Table[Deg[mon, x_i], {i, 1, L}]
  ];

qMon = Function[{mon, cokel},
   Module[{dv = Degv[mon]},
    If[Cok1[dv] == cokel, q^(-dv.(MI.dv) / 4) * (mon /. {x_ → 1}), 0]
   ]
  ];

qLap = Function[{expr, cokel},
  q^(-3 * L / 4 - Sum[M[[i, i]], {i, L}] / 4) *
   (qMon[#, cokel] & /@ expr)]
```

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```
  (qMon[#, cokel] & /@ expr)]

Function[{expr, cokel}, q^(-3L/4 - 1/4 Σ_1^L M[[i, i]]) (qMon[#1, cokel] &) /@ expr]

ordx = 20;

Fx0 = 1;
Do[
  Module[{vfac = 1 / (x_v - 1 / x_v) ^Delta[[v]]},
   Fx0 *=
     (1 * Normal[Series[vfac, {x_v, 0, ordx}]] +
        1 * Normal[Series[vfac, {x_v, Infinity, ordx}]]) / 2;
  ]
  , {v, L}];
Fx = Fx0 // Expand;


tLap =
  Table[{Series[qLap[Fx, cel], {q, 0, 300}] // Normal // Simplify,
     cel, Mod[-cel - MI.Delta, 1]}, {cel, Coker}] // Simplify;
tLap // TableForm
```

Wolfram Mathematica 9.0

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```
    cel, Mod[-cel - MI.Delta, 1]}, {cel, Coker}] // Simplify;
tLap // TableForm
```

$$\sqrt{q}\ \left(1 - q - q^{5} + q^{10} - q^{11} + q^{18} + q^{30} - q^{41} + q^{43} - q^{56} - q^{76} + q^{93} - q^{96} + q^{115} + q^{1}\right.$$

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Wolfram Mathematica 9.0 - [Plumbed0.nb *]

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Plumbed0.nb *

$$\begin{pmatrix} -21 & -42 & -14 & -6 & -6 \\ -7 & -14 & -5 & -2 & -2 \\ -3 & -6 & -2 & -1 & -1 \\ -3 & -6 & -2 & -1 & -2 \end{pmatrix}$$

```
In[15]:= Coker = {Table[0, {i, L}]};
Do[
  Module[{vec = Table[0, {j, 1, L}]},
    Do[
      vec[[v]] = i;
      Coker = Union[Coker, {Mod[MI.vec, 1]}];
      , {i, 1, p - 1}]
     If[Length[Coker] == p, Break[]];
   ];
  , {v, L}];
Coker;
Length[Coker]

Out[15]= 1
```

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*"The (partly conjectural) theory of motives is an attempt to find a universal way to linearize algebraic varieties, i.e. motives are supposed to provide a cohomology theory that embodies all these particular cohomologies."*

*- Wikipedia*

Alexander Grothendieck

BSFRN

$$\mathrm{TQFT}_d : \quad \mathrm{Bord}_d \longrightarrow d\text{-category}$$

# Fiber Integration

$\mathrm{TQFT}_d : \quad \mathrm{Bord}_d \longrightarrow d\text{-category}$

$\mathrm{TQFT}_{d\text{-}n}(\ \ldots\ ) \quad := \quad \mathrm{TQFT}_d(\ \ldots \times M_n)$

labeled by $M_n$

# EFT-valued Topological Invariants

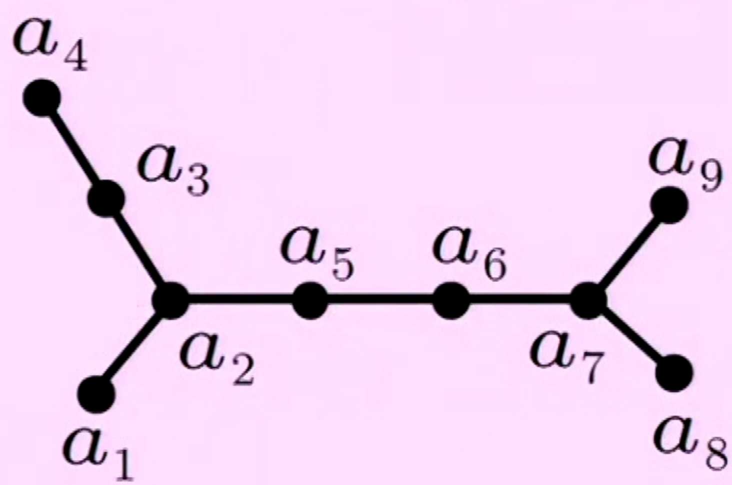
$$\mathrm{EFT}_{d-n}(\ \dots\ ) \ := \ \mathrm{EFT}_{d}(\ \dots \times M_n)$$

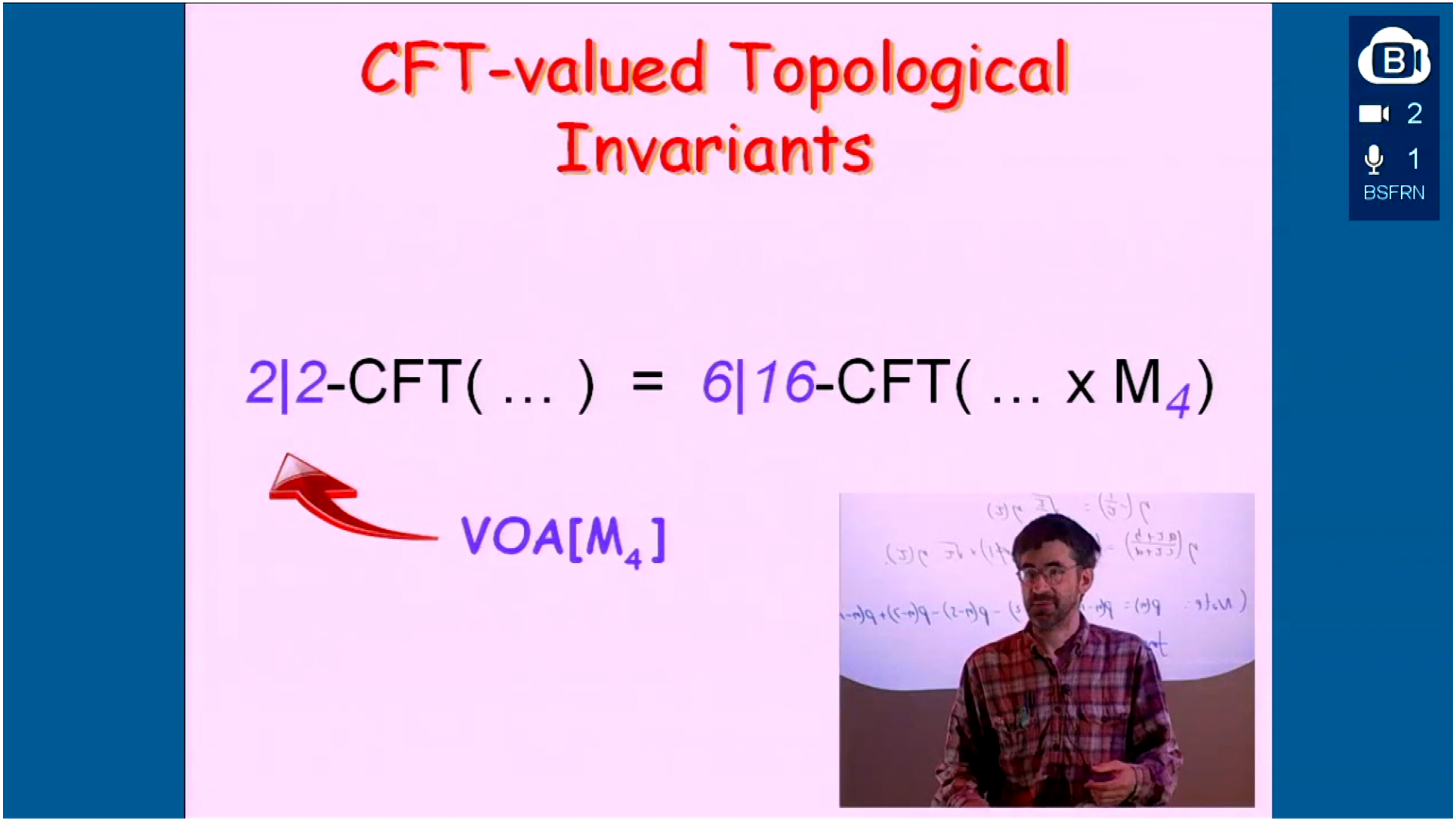
$$\mathrm{TQFT}_{d-n}(\ \dots\ ) \ := \ \mathrm{TQFT}_{d}(\ \dots \times M_n)$$

labeled by $M_n$

# CFT-valued Topological Invariants

$$3|4\text{-CFT}(\ \dots\ ) \;=\; 6|16\text{-CFT}(\ \dots \times M_3)$$




CFT-valued Topological
Invariants
2|2-CFT( … ) = 6|16-CFT( … x M$_4$)
VOA[M$_4$]
2
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