Title: 8d gauge anomalies and the topological Green-Schwarz mechanism

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Abstract: <p>String theory provides us with 8d supersymmetric gauge theories with gauge algebras su(N), so(2N), sp(N), e_6, e_7 and e_8, but no construction for so(2N+1), f_4 and g_2 is known. If string theory is universal in 8 dimensions, this pattern requires explanation. I will show that the theories for f_4 and so(2N+1) have a global gauge anomaly in flat space, while g_2 does not have it. Surprisingly, we also find that the sp(N) theories, arising from example from O7^+ planes in string theory, have a subtler gauge anomaly. This subtler anomaly, in contrast to the one in flat space, could in principle be canceled by a topological analogue of the Green-Schwarz mechanism. I will discuss one simple example of such a generalized anomaly cancellation mechanism in three dimensions, and then explain why the generalized Green-Schwarz term required in 8 dimensions to make the O7^+ consistent is necessarily a more subtle generalization of a Chern-Simons coupling</p>

8d gauge anomalies

and the

topological Green-Schwarz mechanism

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with H. Hayashi, K. Ohmori, Y. Tachikawa and K. Yonekura.

String theory and Quantum Field Theory

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This is too wide a question at this point, so I will ask instead whether all supersymmetric QFTs admit an embedding in String Theory. (One could impose further conditions, such as having a consistent coupling to gravity in the UV, but I will start here.)

Known results for supersymmetric universality

Once we assume supersymmetry we can make progress:

- In more than 11 dimensions every susy Lagrangian theory includes excitations of helicity larger than two (except, possibly, for F-theory viewed as a theory of signature $(10, 2)$ [Vafa '96]).

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- In 10d we have IIA and IIB sugra in the $\mathcal{N}=2$ sector. Pure $\mathcal{N}=1$ gauge theory is anomalous. If we couple it to supergravity we can cancel the anomalies using the Green-Schwarz mechanism [Green, Schwarz '84]. This is possible for the gauge algebras

$$\mathfrak{g} \in \{\mathfrak{e}_8 \oplus \mathfrak{e}_8, \mathfrak{so}(32), \mathfrak{e}_8 \oplus \mathfrak{u}(1)^{248}, \mathfrak{u}(1)^{496}\}\,. \tag{1}$$

The first two possibilities are realized by string theory, while the second two are more subtly inconsistent [Adams, DeWolfe, Taylor '10]. So string theory is also universal in 10d.

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(In 6d local anomalies are powerful again. Here we have many choices, both in the field theory and the string theory sides. The status of 6d universality is not clear yet, but important progress is being made regularly. [Kumar, Taylor '09], [...])

String compactifications down to 9d

There are four known components of the $\mathcal{N} = 1$ moduli space one can construct this way (for a detailed analysis see [Aharony, Komargodski, Patir '07])

- **Rank 2 (a):**
  - M-theory on the Klein bottle.
- **Rank 2 (b):**
  - IIA with $O8^+$ and $O8^-$.
- **Rank 10:**
  - M-theory on Möbius band.
  - CHL string. [Chaudhury, Hockney, Lykken '95]
- **Rank 18:**
  - M-theory on the cylinder.
  - Heterotic on S^1.
  - IIA with two $O8^-$ planes and 16 D8s.

String compactifications down to 8d

We obtain three possible $\mathcal{N}=1$ 8d theories by putting the previous $\mathcal{N}=1$ theories on an S^1. The resulting theories are neatly described in IIB language (on $T^2/(\mathcal{I}\Omega(-1)^{F_L})$):

- **Rank 4:** IIB with two $O7^-$ and two $O7^+$.
- **Rank 12:** IIB with three $O7^-$, one $O7^+$ and 8 D7s.
- **Rank 20:** IIB with four $O7^-$ and 16 D7s.

All these cases can also be described in F-theory, possibly with frozen singularities. (For a detailed discussion of the moduli spaces and dual pictures, see [de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi '01] and [Taylor '11].)

Non-abelian enhancements

$\mathcal{N} = 1$ theories in 8d have a complex scalar in the vector multiplet. Giving a generic vev to these scalars costs no energy, and breaks the gauge algebra to $\mathfrak{u}(1)^{\mathrm{rk}}$. The set of all vacua accessed in this way is the *Coulomb branch*.

At certain points in the Coulomb branch there can be non-abelian enhancements. The enhancements in the known backgrounds are to $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{sp}(N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

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I will aim to explain why the other algebras

$$\mathfrak{so}(2N+1) \quad ; \quad \mathfrak{f}_4 \quad \text{and} \quad \mathfrak{g}_2$$

do not appear.

Summary of results

We find that 8d $\mathcal{N} = 1$ theories with algebra $\mathfrak{f}_4$ and $\mathfrak{so}(2N+1)$ for $N \geq 3$ do not exist quantum mechanically, due to an anomaly.

We find no anomaly for $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ and $\mathfrak{g}_2$.

We find no ordinary global anomaly for $\mathfrak{sp}(N)$ (associated to π_8), but there is an anomaly of a more subtle kind.

- The $d = 8$ $\mathcal{N} = 1$ $\mathfrak{sp}(N)$ theories *are* inconsistent.
- But perhaps this inconsistency can be cured by coupling to a TQFT (the *topological Green-Schwarz mechanism*). We conjecture that this is what happens on the worldvolume of an O7$^+$.
- The needed TQFT is necessarily somewhat involved (I will explain why), and we have not been able to construct it.

A motivating puzzle: $\mathfrak{so}(2N+1)$ in 8d?

In the IIB picture, in perturbation theory, we also have the possibility of putting "half" a D7 on top of the $O7^-$ plane. This would lead to a $\mathfrak{so}(2N+1)$ gauge algebra in 8d.

This seems problematic non-perturbatively:

- There is no monodromy associated to $\mathfrak{so}(2N+1)$ in the Kodaira classification.
- There is no natural "frozen" flux in the F-theory realization that could lead to this. [Hyakutake, Imamura, Sugimoto '00] [de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi '01] [Bergman, Gimon, Sugimoto '01] [Tachikawa '15]
- A D3 probe has a global $SU(2)$ anomaly. [Hyakutake, Imamura, Sugimoto '00] [Witten '82].

The first two arguments are potentially a limitation of model building tools. The last one is more serious, in that it can signal an inconsistency, but one needs to be careful.

Review of anomalies

Consider a (Lagrangian) theory $\mathcal{T}$ with some global symmetry G. We can introduce a background connection A_G for G, and compute the path integral

$$Z(A_G) = \int [D\psi] e^{-S(A_G,\psi)} \tag{2}$$

where ψ are some fundamental fields. (Only the fermionic fields, and the connection they couple to, matter for my discussion.)

Denote by $\mathcal{M}$ the space of all A_G. We have an anomaly whenever $Z(A_G)$ is not well defined as a function on the manifold $\mathcal{M}/G$:

- Non-invariance under small loops (curvature) in $\mathcal{M}/G$: *local anomaly*.
- Non-invariance under parallel transport for non-trivial loops in $\mathcal{M}/G$: *global anomalies*.

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This means that for Lagrangian theories anomalies are at most phases: for any field ψ in a representation R, we can include an extra field $\tilde{\psi}$ in a rep $\overline{R}$ (and with an action which is the conjugate of that for ψ), and then the full matter content can be made massive. So

$$Z(A_G) = Z_\psi(A_G) Z_{\tilde{\psi}}(A_G) = Z_\psi(A_G)\overline{Z_\psi(A_G)} = \left|Z_\psi(A_G)\right|^2 . \quad (3)$$

Since the $\psi + \tilde{\psi}$ theory is gappable, we have that $\left|Z_\psi(A_G)\right|$ is a well defined function on $\mathcal{M}/G$.

Review of anomalies

In general, $Z(A_G)$ is a section of some bundle over $\mathcal{M}/G$. If the bundle is non-trivial the theory is still consistent; we say that we have a 't Hooft anomaly. For example, the $SU(4)_R$ symmetry of $\mathcal{N}=4$ $SU(N)$ SYM has such an anomaly in 4d $(\mathrm{Tr}(F_R^3) \neq 0)$, but the theory is fine, and the symmetry is unbroken.

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What an anomaly means is that the symmetry G cannot be gauged, since gauging involves integration of $Z(A_G)$ over orbits of G in $\mathcal{M}$.

We will consider the case in which there are no local anomalies. How do we detect a possible global anomaly?

The "traditional" global anomaly

Consider a symmetry transformation $g\colon \mathbb{R}^d \to G$. We impose that $g \to 1$ at infinity, so it could be a gauge transformation. The resulting set of transformations are topologically classified by maps $S^d \to G$ up to continuous deformations, i.e. by $\pi_d(G)$.

Now, for any choice of $[g] \in \pi_d(G)$, pick a representative g and consider the family of (not pure gauge) connections

$$A_G(g;t) = f(t)g^{-1}dg \tag{4}$$

for some smooth $f(t)$ such that $f(-\infty) = 0$ and $f(+\infty) = 1$. This defines a loop in the space of connections (modulo gauge transformations). So there is a global anomaly if

$$\frac{Z(A_G(g;+\infty))}{Z(A_G(g;-\infty))} = \frac{Z(0^g)}{Z(0)} = e^{i\mathcal{A}} \neq 1\,. \tag{5}$$

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The "traditional" global anomaly

We will be interested in the case in which the fermions are *real*. This means that the mass coupling

$$m\psi\psi = 0 \tag{6}$$

does not break G, but it identically vanishes. (More generally, m is a symmetric bilinear form, leading to a singlet under G.) But we can add an extra copy of the fermions, and introduce a mass coupling

$$m\psi_1\psi_2 \neq 0 \tag{7}$$

This implies that $Z(A_G)^2$ is well defined, so the anomaly is $\mathbb{Z}_2$-valued (i.e. $e^{i\mathcal{A}} = \pm 1$ at most).

The "traditional" global anomaly

Consider an example: 4d Weyl fermion ψ_1 in the fundamental of $SU(2)$. This is a real fermion (the mass term is allowed, but it identically vanishes), since the fundamental of $SU(2)$ is pseudoreal, and the Weyl spinor of $\mathrm{Spin}(4) = SU(2) \times SU(2)$ is pseudoreal.

Famously [Witten '82], this system has a global anomaly:

$$Z(0) = -Z(0^g) \tag{8}$$

for $[g]$ the non-trivial generator of $\pi_4(SU(2)) = \mathbb{Z}_2$.

The Elitzur and Nair approach

In his original argument, Witten relates the anomaly to the mod 2 index in the five dimensional mapping torus for (S^4, g), by viewing $A_G(t)$ as a connection on the mapping torus. This is hard to use for computations. An easier argument to use in practice is due to Elitzur and Nair. [Witten '83] [Elitzur, Nair '84] [Lundell, Tosa '88]

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The basic idea is embedding $SU(2)$ into $SU(3)$, and deform (using the fact that $\pi_4(SU(3)) = 0$) $A_G(t)$ into an interpolating pure gauge connection in $SU(3)$

$$B_G(t) = 0^{f_t} = f_t^{-1} df_t \tag{9}$$

for f_t some homotopy in $SU(3)$ between 1 and $g \in SU(2)$.

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To go to $SU(3)$ we need to add a $SU(2)$ singlet, since under $SU(3) \supset SU(2)$ we have $\mathbf{3} \to \mathbf{2} \oplus \mathbf{1}$. Note that

$$Z_{\mathbf{3}} = Z(A_G) Z_{\mathbf{1}} \tag{10}$$

has the same $SU(2)$ anomaly as our original partition function.

The Elitzur and Nair approach

We now view the anomaly

$$Z_{\mathbf{3}}(0^g) = e^{i\mathcal{A}} Z_{\mathbf{3}}(0) \tag{11}$$

as coming from the local anomaly we obtain by a series of $SU(3)$ transformations. By the usual descent arguments ($\delta\omega = d\mathscr{A}$, with $d\omega = I_{d+2}$), we have

$$\mathcal{A} = \int_{S^d} \mathscr{A} = \pi \int_{B_{d+1}} \omega(0^g) - \omega(0) = \pi \int_{B_{d+1}} \omega(0^g) \tag{12}$$

with B_{d+1} the $d+1$ ball with boundary S^d (for us, $d = 4$), and $\omega(A)$ is the Chern-Simons density for a connection A.

The Elitzur and Nair approach

Now we are in $SU(3)$, so we can deform the homotopically non-trivial (in $\pi_4(SU(2))$) gauge transformation on the boundary to nothing, via $SU(3)$ gauge transformations.

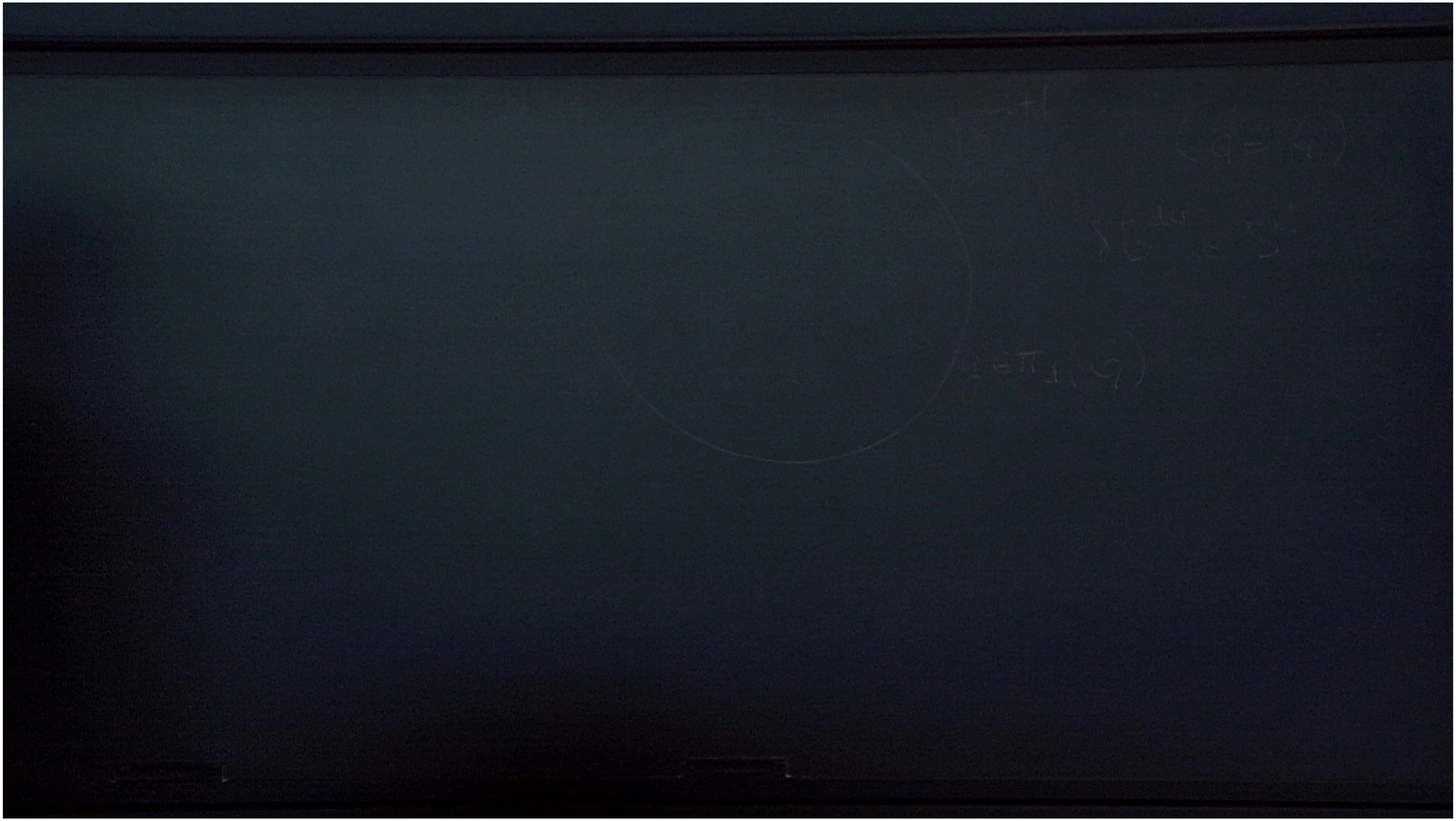
The Chern-Simons density is invariant under transformations without local anomalies (such as those in the $SU(2)$ subgroup). So the integral really depends on the $SU(3)/SU(2)$ structure only.

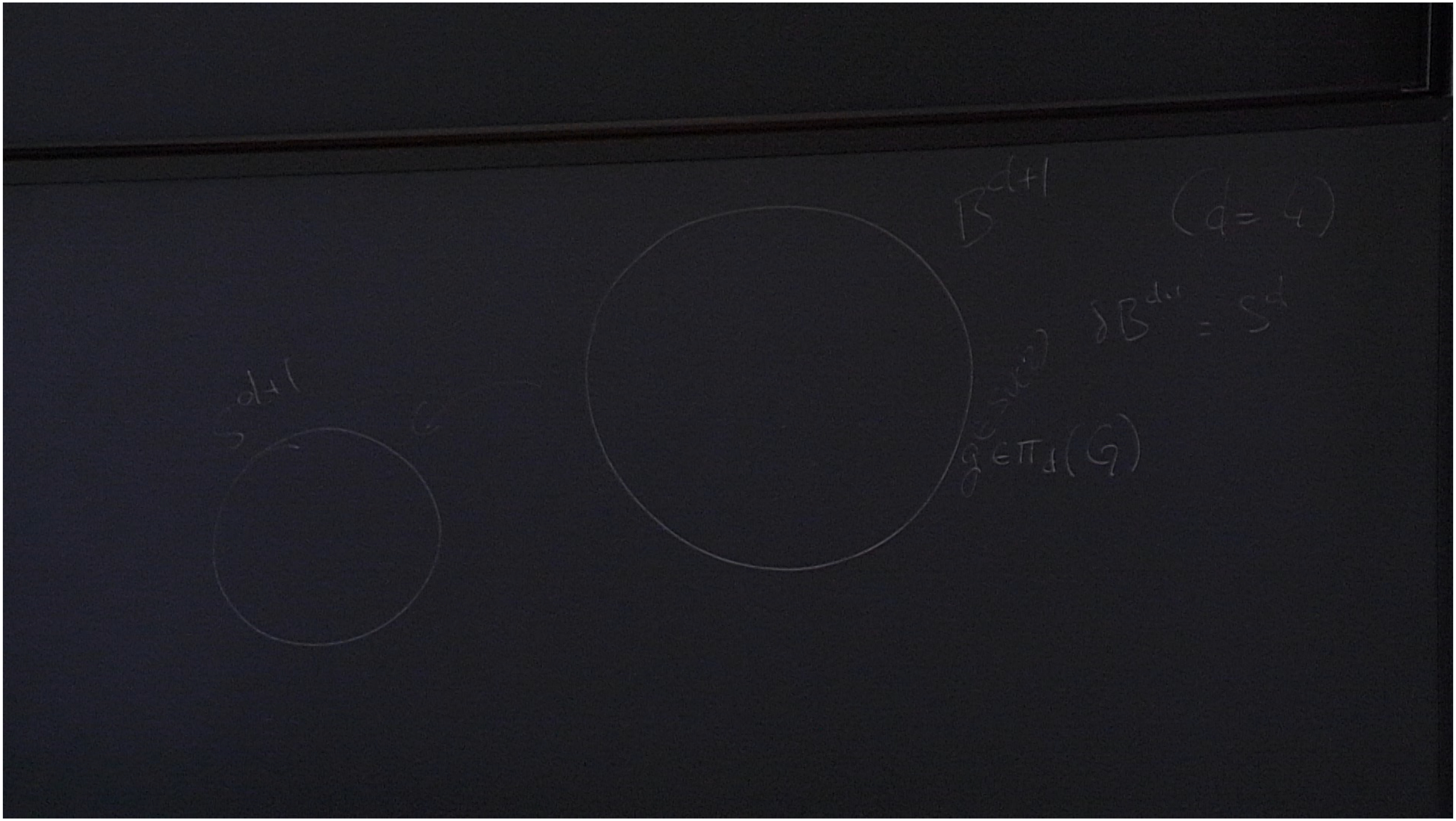
In particular, we can contract the boundary to a point in order to compute the anomaly

$$\mathcal{A} = \int_{B_{d+1}} \omega(0^g) \to \mathcal{A} = \int_{S^{d+1}} \omega(0^g)\,. \tag{13}$$

So we can view the above anomaly as a homomorphism from appropriate homotopy classes to $\mathbb{R}$

$$\mathcal{A}\colon \pi_{d+1}(SU(3)/SU(2)) \to \mathbb{R}\,. \tag{14}$$




B^{d+1}
$(d = 4)$
$\partial B^{d+1} = S^d$
$g \in \pi_d(G)$
S^{d+1}

The Elitzur and Nair approach

Consider the short exact sequence

$$0 \to SU(2) \to SU(3) \to SU(3)/SU(2) \to 0 \qquad (15)$$

which induces (since $\pi_4(SU(3)) = 0$)

$$\ldots \to \underbrace{\pi_5(SU(3))}_{\mathbb{Z}} \xrightarrow{\alpha} \underbrace{\pi_5\left(\frac{SU(3)}{SU(2)}\right)}_{\mathbb{Z}} \xrightarrow{\beta} \underbrace{\pi_4(SU(2))}_{\mathbb{Z}_2} \to 0\,. \qquad (16)$$

By exactness, α is multiplication by 2, and β is reduction modulo 2.

The homomorphism in the $SU(3)$ case is well understood. We have that $\int_{S^{d+1}} \omega(0^g) = \int_{S^{d+1}} \mathrm{tr}\big((g^{-1}dg)^{d+1}\big)$. So $\mathcal{A}(f) = 2\pi$ with f the generator of $\pi_5(SU(3))$.

Which implies $\mathcal{A}(g) = \pi$, so there is an anomaly in this case.

The Elitzur and Nair approach

All this was a fairly roundabout argument, but it generalizes easily when we know enough about homotopy groups of Lie groups.

We have $\pi_8(G) \neq 0$ for

$$G \in \{SU(2), SU(3), SU(4), SO(7) \ldots SO(10), SO(N), G_2, F_4\} .$$

So only these groups can have global anomalies of the kind we are computing.

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Example: $\mathcal{N} = 1$ $\mathfrak{f}_4$ in 8d

For instance, we can embed F_4 into E_6. This embedding is useful since a lot of known about the homotopy groups of E_6/F_4. [Conlon '66] (This, is the symmetric space EIV, in the classification by Cartan.) In particular, we have the exact sequence

$$0 \to \underbrace{\pi_9(E_6)}_{\mathbb{Z}} \xrightarrow{\alpha} \underbrace{\pi_9\left(\frac{E_6}{F_4}\right)}_{\mathbb{Z}} \xrightarrow{\beta} \underbrace{\pi_8(F_4)}_{\mathbb{Z}_2} \to 0\,. \tag{17}$$

Since $\mathbf{27} \to \mathbf{26} \oplus \mathbf{1}$ for $E_6 \supset F_4$, we find that the fundamental $(\mathbf{26})$ of F_4 has a $\mathbb{Z}_2$ discrete anomaly, by the same arguments as before.

The adjoint $\mathbf{78}$ of E_6 is free of local anomalies, and decomposes as $\mathbf{26} \oplus \mathbf{52}$, so the adjoint of F_4 also has a $\mathbb{Z}_2$ discrete anomaly!

Results

The other cases are also tractable. For the cases with $\pi_8(G) \neq 0$ we find:

$$G \in \{SU(2), SU(3), SU(4), SO(2N+1), SO(2N+2), G_2, F_4\}$$

with $N \geq 3$. Red means that we have proven the theory inconsistent, blue that we found no inconsistency from this particular check. (But other checks may reveal inconsistencies!)

Beyond the traditional anomaly

In principle, it is not enough to show that the theory is anomaly-free in $\mathbb{R}^8 \approx S^8$. We would like to know whether the theory makes sense on any manifold, with any gauge bundle.

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An important loophole: we could potentially couple to a TQFT that forbids the problematic gauge bundles, as in [Seiberg '10]. Or more generally, we could couple to a TQFT with the opposite anomaly.

For example, we could have a theory with gauge group G/C, with an anomaly coming from bundles with non-trivial Stiefel-Whitney class. We can fix this by coupling to a TQFT that effectively removes the problematic bundles [Kapustin, Seiberg '14].

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- We give an explicit 3d example of such fixing by coupling to a TQFT in the paper ($SU(N)/\mathbb{Z}_N$ with an adjoint Majorana).
- We have not been able to construct the right TQFT in the problematic 8d cases. Probably our own mathematical limitations.
- This mechanism cannot cancel the "traditional" (flat space) anomaly.

Anomalies on $S^4 \times \mathbb{R}^{d-4}$ with an instanton

We consider a spacetime of the form $S^4 \times \mathbb{R}^{d-4}$. Choose a decomposition of $G \supset (SU(2) \times H)/\mathcal{C}$. Now put a single instanton of the $SU(2)$ factor on the S^4.

Assume that the original theory had a fermion in a representation

$$R_G \to \bigoplus_{n \geq 1} (\mathbf{n}_{SU(2)} \otimes R_H^n) \,. \tag{18}$$

A fermion in the representation $\mathbf{n}_{SU(2)}$ gives rise to $N_n = \frac{1}{6}(n^3 - n)$ zero modes, so there is an effective theory in $\mathbb{R}^{d-4}$, with gauge group H, and fermions in the representation

$$r_H = \bigoplus_{n \geq 1} N_n R_H^n \,. \tag{19}$$

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$$R_G \to \bigoplus_{n \geq 1} (\mathbf{n}_{SU(2)} \otimes R_H^n) \,. \tag{18}$$

A fermion in the representation $\mathbf{n}_{SU(2)}$ gives rise to $N_n = \frac{1}{6}(n^3 - n)$ zero modes, so there is an effective theory in $\mathbb{R}^{d-4}$, with gauge group H, and fermions in the representation

$$r_H = \bigoplus_{n \geq 1} N_n R_H^n \,. \tag{19}$$

The original theory is anomalous if this effective theory in $\mathbb{R}^{d-4}$ is.

Example: $\mathfrak{so}(2N+1)$ with $N > 2$

Up to discrete factors, any global form of this algebra has a subgroup

$$SU(2) \times SU(2)' \times SO(2N-3)\,. \tag{20}$$

We choose $H = SU(2)' \times SO(2N-3)$. The adjoint of $\mathfrak{so}(2N+1)$ decomposes as

$$\begin{aligned} &\mathbf{2}_{SU(2)} \otimes \mathbf{2}_{SU(2)'} \otimes (\mathbf{2N-3})_{SO(2N-3)} \\ &\oplus \mathrm{Adj}(SU(2) \times SU(2)' \times SO(2N-3)) \end{aligned} \tag{21}$$

The resulting representation in four dimensions of H is

$$r_H = \mathbf{2}_{SU(2)'} \otimes (\mathbf{2N-3})_{SO(2N-3)} + (H \text{ singlets}). \tag{22}$$

which manifestly has a global anomaly.

A new anomaly for $\mathfrak{sp}(2N)$

Analysing the other cases is straightforward, and for the most part reproduces the results of the analysis in $\mathbb{R}^8$.

There is one exception. Consider the decomposition $USp(2N) \supset USp(2) \times USp(2N-2)$. The adjoint decomposes as

$$\text{Adj} \to (\mathbf{2} \otimes (\mathbf{2N-2})) \oplus (\text{Adj} \otimes \mathbf{1}) \oplus (\mathbf{1} \oplus \text{Adj}) .$$

so the effective $USp(2N-2)$ theory in $\mathbb{R}^4$ has fermions in the representation

$$r_H = \square \oplus (\text{singlets}) . \tag{23}$$

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So there is a Witten anomaly!

The Dai-Freed viewpoint on anomalies

Consider the case that your space-time X is the boundary of some manifold Y, over which all the relevant structures on X extend. We define the path integral of a fermion ψ on X as [Dai, Freed '04]

$$Z_\psi = |Z_\psi| e^{-2\pi i\, \eta(\mathcal{D}_Y)} \tag{24}$$

with

$$\eta(\mathcal{D}_Y) = \frac{\dim \ker \mathcal{D}_Y + \sum_{\lambda \neq 0} \operatorname{sign}(\lambda)}{2}\,. \tag{25}$$

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More generally, we would like to consider something like

$$Z_\psi = |Z_\psi| Z_{\mathcal{A}} \tag{26}$$

with $\mathcal{A}$ some TQFT in $d+1$ dimensions, the "anomaly theory". (I am restricting to *invertible* TQFTs, where $|Z_{\mathcal{A}}| = 1$.)

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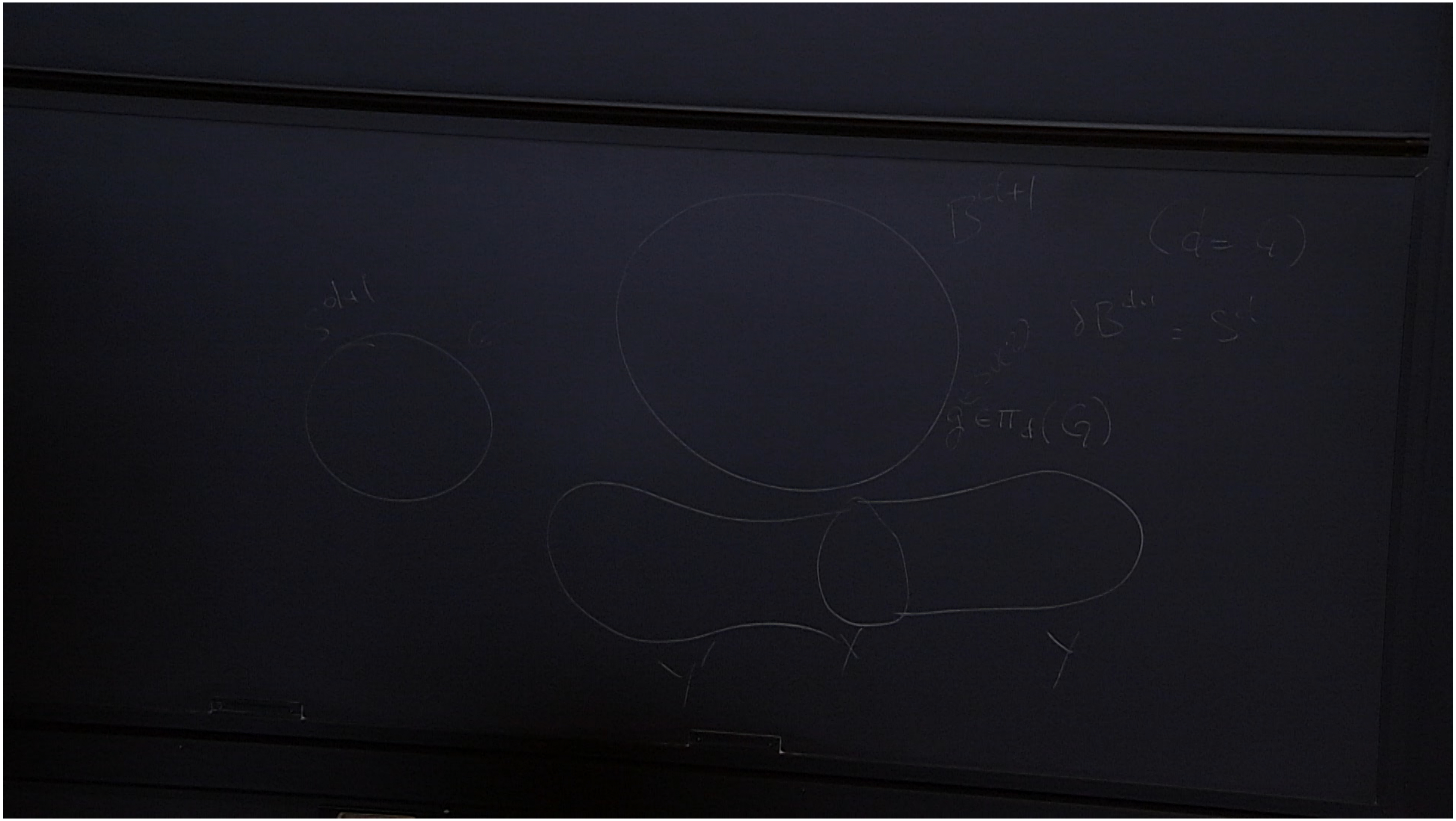
The Dai-Freed viewpoint on anomalies

Anomalies, in this language, come from situations in which the phase of the partition function depends on the choice of Y:

$$e^{-2\pi i\,\eta(\mathcal{D}_Y)} \neq e^{-2\pi i\,\eta(\mathcal{D}_Z)} \tag{27}$$

even if $\partial Y = \partial Z = X$.


B^{d+1}
$(d = 4)$
S^{d+1}
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$g \in \pi_d(G)$
Y'
X
Y

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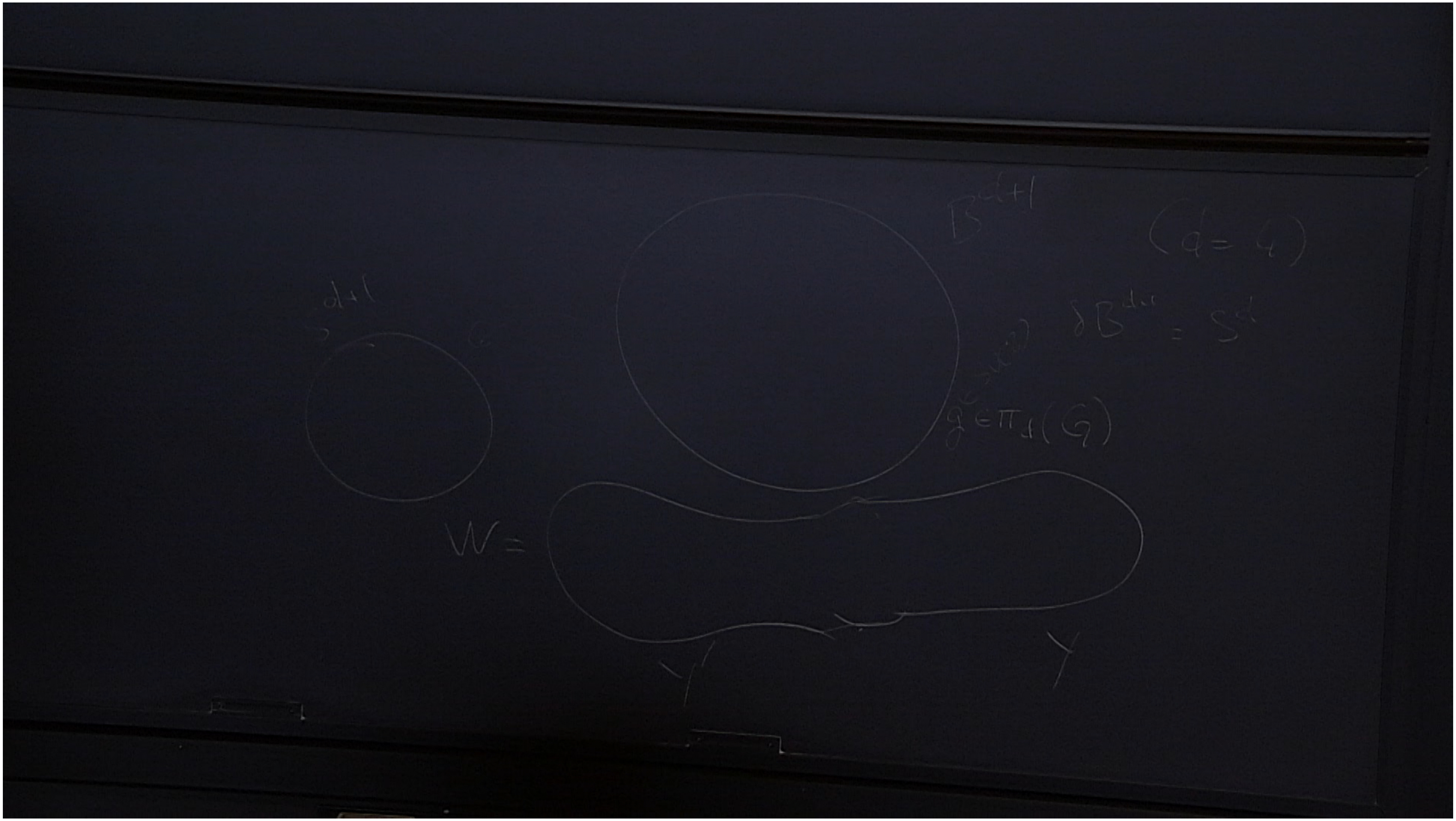
even if $\partial Y = \partial Z = X$.

Gluing Y and $\overline{Z}$ over X to form the closed manifold W, we find that the partition function is well defined as a function of the fields on X only if $e^{-2\pi i\,\eta(\mathcal{D})} = 1$ on every such W.


B^{d+1}
(d=4)
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The $SU(2)$ Witten anomaly revisited

As an example of this proposal, let us rephrase Witten's 4d $SU(2)$ anomaly [Witten '82] in this language. He argues that the change in the path integral of a Weyl fermion on the fundamental of $SU(2)$ under a gauge transformation g that cannot be deformed to the identity $(0 \neq [g] \in \pi_4(SU(2)))$ is given by the mod 2 index

$$Z(A^g) = (-1)^{\dim \ker \mathcal{D}_Y} Z(A) \tag{28}$$

where Y is the *mapping torus*, a non-trivial gauge bundle over $S^1 \times S^4$. There is an anomaly if $\dim \ker \mathcal{D}_Y \in 2\mathbb{Z} + 1$.

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$$\mathcal{D}_Y = i\gamma^\mu D_\mu \tag{29}$$

is formally antisymmetric and real (forgetting about the "i"). So formally it can be block-diagonalized to

$$\gamma^\mu D_\mu \to \underbrace{(0) \oplus \cdots \oplus (0)}_{\ker \mathcal{D}_Y} \oplus \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix} \oplus \cdots \tag{30}$$

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So we have that

$$\eta(\mathcal{D}_Y) = \frac{\dim\ker\mathcal{D}_Y + \sum_{\lambda\neq 0}\operatorname{sign}(\lambda)}{2} = \frac{1}{2}\dim\ker\mathcal{D}_Y \tag{31}$$

and thus $e^{-2\pi i\,\eta(\mathcal{D}_Y)} = (-1)^{\dim\ker\mathcal{D}_Y}$.

The parity anomaly in three dimensions revisited

Consider a Dirac fermion Ψ in three dimensions, in the fundamental of $SU(N)$. It is well known [Redlich '84] that it is not possible to quantize this theory preserving both parity invariance and invariance under $SU(N)$.

In the language above, this is encoded in an anomaly TQFT given by

$$S_{\mathsf{TQFT}} = \frac{1}{2\pi i}\log \mathcal{A}(Y) = \frac{1}{2}\int_Y \mathrm{tr}_{\Box}(F^2) = \frac{1}{2}\int_Y c_2(F)\,, \qquad (32)$$

the instanton number, and we can define the partition function to be

$$Z_\Psi = |Z_\Psi|\mathcal{A}(Y) = |Z_\Psi|(-1)^{\int_Y c_2(F)}\,. \qquad (33)$$

(If $\partial Y = X$, this reduces to the familiar statement that the parity anomaly is encoded in a Chern-Simons term $\frac{1}{2}\int_X \omega$, since $d\omega = \mathrm{tr}_{\Box}(F^2)$.)

The anomaly for $SU(N)/\mathbb{Z}_N$ with an adjoint

Now consider the case in which Ψ is *Majorana*, and transforms in the adjoint of $\mathfrak{su}(N)$. We choose the global form of the gauge group to be $SU(N)/\mathbb{Z}_N$. The anomaly theory is, as before,

$$\frac{1}{2\pi i}\log \mathcal{A}(Y) = \frac{1}{2}\cdot\frac{1}{2}\int_Y \mathrm{Tr}_{\mathsf{Adj}}(F^2) = \frac{N}{2}\int_Y c_2(F)\,. \qquad (34)$$

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This is a non-trivial theory (for any N), since $SU(N)/\mathbb{Z}_N$ configurations can have instanton charge $1/N$. For this to happen, it must be the case that they cannot lift to $SU(N)$ instantons. This obstruction to lifting is measured by a generalized Stiefel-Whitney class $w_2 \in H^2(Y,\mathbb{Z}_N)$, with $n_{\mathsf{inst}} \equiv w_2^2/2N \bmod 1$. The anomaly theory can then be written as

$$\mathcal{A}(Y) = \frac{1}{4}\langle w_2 \cup w_2, Y\rangle \approx \frac{1}{4}\int_Y w_2\wedge w_2\,. \qquad (35)$$

Anomalies of a gapped boundary phase

Perhaps surprisingly, we *can* cancel this anomaly by coupling to a TQFT, of a form formally analogous to the Green-Schwarz action. (Inspired by [Kapustin '14].) In general, choose

$$S = \int_M \left(kb'da' + b'X + (-1)^{d-p+1}Ya'\right) \quad (36)$$

where $X \in H^p(\mathcal{M}, \mathbb{Z}_k)$ and $Y \in H^{d-p+1}(\mathcal{M}, \mathbb{Z}_k)$ and b' and a' are $(d-p)$-form and $(p-1)$-form fields. Now introduce

$$f \equiv da' + \frac{1}{k}X \quad ; \quad g \equiv db' + \frac{1}{k}Y\,. \quad (37)$$

Then the expression above, if $\partial N = M$, can be rewritten as

$$\mathcal{A}(N) = k\int_N fg\,. \quad (38)$$

This anomaly, when put on a compact manifold L, becomes

$$\mathcal{A}(L) = \frac{1}{k}\int_L XY \quad (39)$$

so we cancel the anomaly choosing $d = 3$, $p = 2$ and $X = Y = w_2$.

Comments on the 8d generalization

Unfortunately the generalization to 8d is not straightforward. Naively:

$$X \sim \mathrm{Tr}(F^2), \quad Y \sim [\text{mod 2 index in 5d}] \tag{40}$$

with q_1 the Pontryagin class of the instanton. But Y is not something that can admit an expression in terms of characteristic classes, as is well known.

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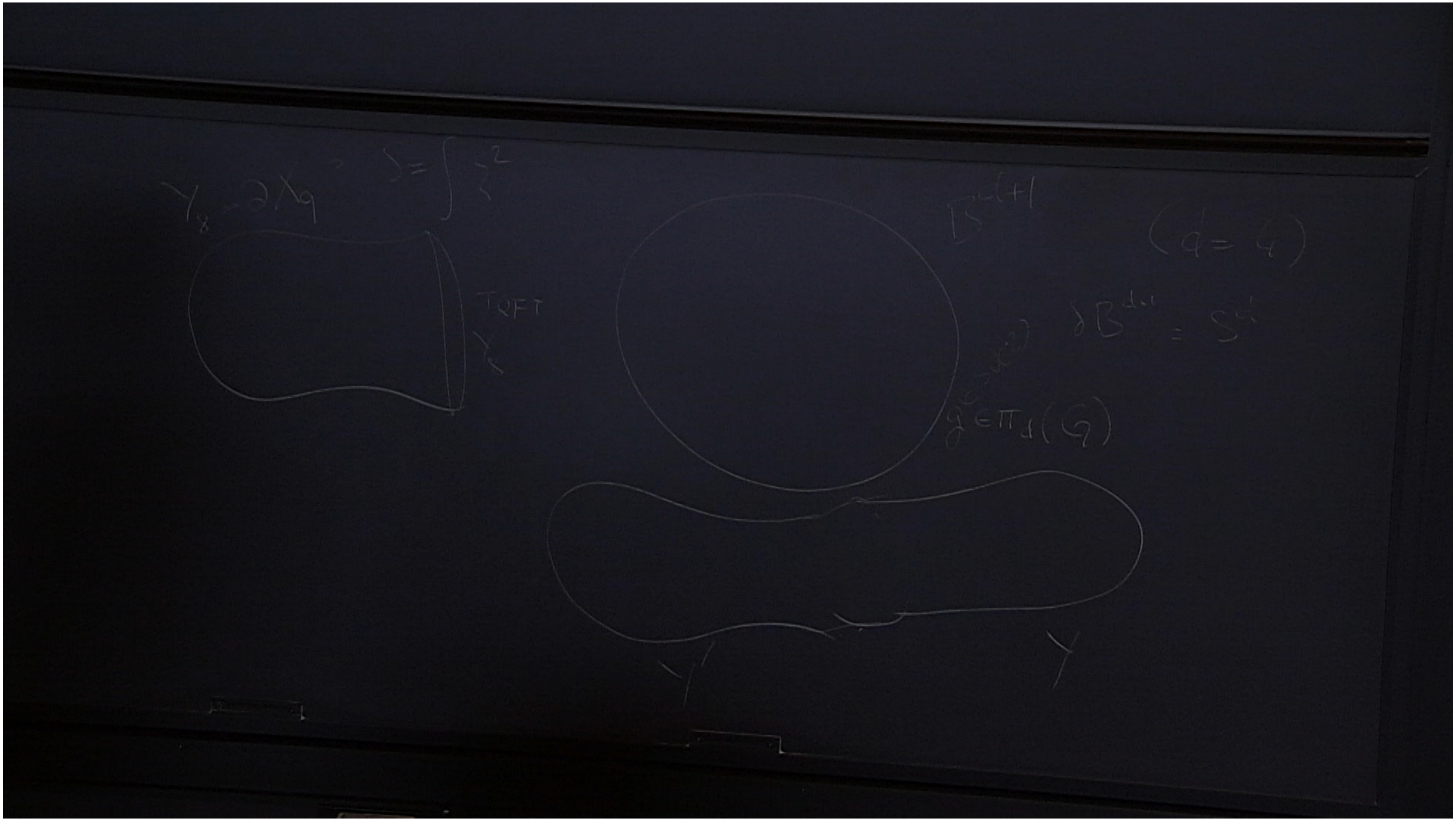
with q_1 the Pontryagin class of the instanton. But Y is not something that can admit an expression in terms of characteristic classes, as is well known. For example, an instanton on S^5 can have a single zero mode [Witten '82], but there are no characteristic classes in S^5 one can integrate to detect this.

The right notion seems to be K-theory instead. It should involve some KSp-theory class $\xi \in \mathrm{KSP}^0(X) = \mathrm{KO}^4(X)$, and the natural anomaly theory seems to be $\int \xi^2$. (More details about our guess in the paper, but the problem remains open in any case.)


TQFT

No way to fix "local" global anomalies

The traditional global anomaly of Witten is, in some sense, local: it deals with gauge transformations which can be restricted to arbitrarily small neighbourhoods of arbitrary points.

This suggests that we cannot remove the anomaly via a coupling to a TQFT: it would have to be a topological theory that

- Reduces effectively to a polynomial on the local massless fields being varied (A), due to locality.
- Has an effective action invariant under gauge transformations connected to the identity.
- While the same action picks up a sign under gauge transformations not connected to the identity.

In even dimensions there is no such action, so it seems like those anomalies associated with $\pi_d(G) \cong [S^d \to G]$ cannot be fixed. While those associated to more complicated $X \to G$ maps might.

Conclusions

We reduced (a little bit) the gap between what you seemed to be able to do in $d = 8$ $\mathcal{N} = 1$ QFT (everything!) to what you seem to be able to do in string theory (very little!).

Local anomalies are absent, but global anomalies are powerful enough to say some things.

We find that:

- $\mathfrak{so}(2N+1)$ and $\mathfrak{f}_4$ are anomalous, as you might have suspected.
- $\mathcal{N} = 1$ $\mathfrak{g}_2$ seems to be omalous in 8d, to the extent that we checked. But no known construction!
- An unexpected anomaly for $\mathfrak{sp}(2N)$, i.e. the $O7^+$! Potentially fixable, but we have not worked out the details.

Future directions

Patterns unexplained:

- The very specific choices of rank: $\mathrm{rk}(G) \in \{4, 12, 20\}$.
- Correlation of rank with available algebras.

Roads not taken:

- One should impose the necessity of being able to couple consistently to gravity, including on non-orientable spacetimes.
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And this was all about anomalies, what about the swampland?
[Vafa '05]