Title: Free Probability Theory for Floquet and Quantum Many-Body Systems

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Abstract: <p>Suppose the eigenvalue distributions of two matrices M_1 and M_2 are known. What is the eigenvalue distribution of the sum M_1+M_2? This problem has a rich pure mathematics history dating back to H. Weyl (1912) with many applications in various fields. Free probability theory (FPT) answers this question under certain conditions, which often involves some degree of randomness (disorder). We will describe FPT and show examples of its powers for the qualitative understanding (often approximations) of physical quantities such as density of states, and gapped vs. gapless phases of some Floquet systems. These physical quantities are often hard to compute exactly. Nevertheless, using FPT and other ideas from random matrix theory excellent approximations can be obtained. Besides the applications presented, we believe the techniques will find new applications in new contexts.</p>

Free Probability Theory for Floquet and QMBS

Ramis Movassagh[1]

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PI, Waterloo, Canada May 23, 2018

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Problem

We have the eigenvalues of Hermitian matrices A and B.
Find the eigenvalues of

$$A \quad + \quad B$$

In this talk matrices are $m \times m$

Motivation

Many problems in application are of such

1. Schrödinger Operator: $-\nabla^2 + V(x)$
   [quantum chemistry, material science]
2. Lattice problems
   $H = \sum_{<k,j>} H_{k,j} \overset{1D}{=} \sum_{k\ odd} H_{k,k+1} + \sum_{k\ even} H_{k,k+1}$
   [Condensed matter and quantum info science]
3. Anderson model: $L + \text{diag}(\varepsilon_1, \dots, \varepsilon_m)$
4. Noisy systems $M + E$, where E is a random matrix
   (communication, engineering)

etc. etc.

Previous works

Posing the problem : 1912 H. Weyl , Math. Ann. 71 (1912),
Given the eigenvalues of two *m x m* Hermitian matrices, how does one determine all the possible set of the eigenvalues of the sum?

Problem: What are the eigenvalues of $A + B$?

Problem: Suppose we have the eigenvalues of A and B. Find the eigenvalue distribution of:

$$A + B = Q_A^{-1} \Lambda_A Q_A + Q_B^{-1} \Lambda_B Q_B$$

Previous works

Posing the problem : 1912 H. Weyl , Math. Ann. 71 (1912),
Given the eigenvalues of two *m x m* Hermitian matrices, how does one determine all the possible set of the eigenvalues of the sum?

A. Horn's conjecture : A. Horn, Pacific J. Math. 12 (1962)
Conjectured a (over-complete) set of recursive inequalities for Eigenvalues of sums of Hermitian matrices

A. Horn's theorem : A. A. Klyachko, Selecta Math. (N.S.) 4 no. 3, 419–445 (1998)
Klyachko proved it.

A. Knudsen and T. Tao: Soc. 48, no. 2, 175–186. (2001)
More practical proof based on Honey-combs and Schubert Calculus, Notices Amer. Math.

Despite the great success, there are not many results that enable drawing a picture ! The bounds in particular are *not* good for very sparse matrices (e.g., those that model local interactions on lattices) .

Problem: What are the eigenvalues of $A + B$?

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Problem: What are the eigenvalues of M ?

Problem: Suppose we have the eigenvalues of A and B. Find the eigenvalue distribution of:

$$\begin{aligned} A+B &= Q_A^{-1}\Lambda_A Q_A + Q_B^{-1}\Lambda_B Q_B \\ M &\equiv \Lambda_A + Q_s^{-1}\Lambda_B Q_s, \end{aligned}$$

$\Lambda_A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$
$\Lambda_B = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_m)$
$Q_s \equiv Q_B Q_A^{-1}$ is the main source of difficulty.

Impossible to answer without a knowledge of eigenvectors.

Goal

Assuming (strongest needed) :
Λ_A or Λ_B are independent and random. Q_s is random and permutation invariant [But not necessarily Haar].

Draw : On a computer the eigenvalue distribution (density) of:

$$M = \Lambda_A + Q_s^{-1} \Lambda_B Q_s,$$

Problem: What are the eigenvalues of $A+B$?

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The β-Orthogonal Group

$M_c = \Lambda_A + \Pi^{-1}\Lambda_B \Pi$	$M = \Lambda_A + Q_s^{-1}\Lambda_B Q_s$	$M_f = \Lambda_A + Q^{-1}\Lambda_B Q$
$d\nu_c = d\nu_A * d\nu_B$	$d\nu = ?$	$d\nu_F = d\nu_A \boxplus_F d\nu_B$
***Classical*: "Commuting"**		***Free*: "Haar" Eigenvectors**

The β-Orthogonal Group

$M_c = \Lambda_A + \Pi^{-1}\Lambda_B \Pi$ $\qquad$ $M = \Lambda_A + Q_s^{-1}\Lambda_B Q_s$ $\qquad$ $M_f = \Lambda_A + Q^{-1}\Lambda_B Q$

$d\nu_c = d\nu_A * d\nu_B$ $\qquad$ $d\nu$ $\approx$ $d\nu_F = d\nu_A \boxplus_F d\nu_B$

***Classical*: "Commuting"** $\qquad$ ***Free*: "Haar" Eigenvectors**

Conventional Probability Theory

Commuting random variables

$$ab = ba$$

Classical independence implies

$$\langle \delta a \delta b \rangle = 0,$$

where, $\delta a = a - \langle a \rangle$

Consider: $a + b$

Densities are NOT additive:

- **Log-characteristics** are additive !

Free Probability Theory (FPT)

Generalizes to non-Commuting random variables

$$AB \neq BA$$

D. Voiculescu

Free independence implies *

$$\varphi\left(\delta A \delta B\right) = 0,$$

where, $\delta A = A - \varphi(A)$

$$\varphi\left(A\right) = \frac{1}{\dim(A)} \mathrm{E}Tr[A]$$

** I am over-simplifying the free condition.*

Consider: $A + B$

Densities are NOT additive:

- **R-Transforms** are additive !

Free probability technology

1) Input to the theory are the Cauchy transforms of the densities: $G_A(z)$, $G_B(z)$.

$$G(z) = \frac{1}{2\pi i}\int_R dx \frac{f(x)}{z-x};$$

2) R-Transforms: $R_A(z)$, $R_B(z)$.

$$R(G(z)) = z - \frac{1}{G(z)}$$

3) R-Transform is additive (key property!). Good practice to let $w \equiv G(z)$

$$R_{A+B}(w) = R_A(w) + R_B(w).$$

4) Having the R-Transform of the sum, we undo the steps to obtain the density.

$$z = G^{-1}(w) = R(w) + \frac{1}{w} \quad \Rightarrow \quad w = G(z) = \int_R \frac{f_{A+B}(x)}{G^{-1}(w) - x} dx$$

5) Invert using Plemelj-Sokhotsky formula to obtain the of the sum:

$$f_{A+B}(x) = \frac{1}{\pi}\lim_{\epsilon+}\left(\mathrm{Im}\left(G(z)\right)\right)$$

Reference: RM-, A. Edelman, arXiv (2017)

Applications

- Anderson model (in any dimension)
- Driven Floquet systems with disorder
- Spin chains (more complicated, requires more than just FPT)

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Anderson's Model

PRL **109,** 036403 (2012) PHYSICAL REVIEW LETTERS week ending 20 JULY 2012

Error Analysis of Free Probability Approximations to the Density of States of Disordered Systems

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Ramis Movassagh[**] and Alan Edelman[††]
Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139, USA
(Received 29 February 2012; published 17 July 2012)

Theoretical studies of localization, anomalous diffusion and ergodicity breaking require solving the electronic structure of disordered systems. We use free probability to approximate the ensemble-averaged density of states without exact diagonalization. We present an error analysis that quantifies the accuracy using a generalized moment expansion, allowing us to distinguish between different approximations. We identify an approximation that is accurate to the eighth moment across all noise strengths, and contrast this with perturbation theory and isotropic entanglement theory.

DOI: 10.1103/PhysRevLett.109.036403 PACS numbers: 71.23.An

Anderson's model

$$
\begin{aligned}
H &= J\sum_{i=1}^{n}\{|i\rangle\langle i+1| + |i+1\rangle\langle i|\} \\
&= \begin{bmatrix} & J & & & J \\ J & & J & & \\ & J & & \ddots & \\ & & \ddots & & J \\ J & & & J & \end{bmatrix} \\
&\equiv A
\end{aligned}
$$

Anderson's model

$$
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H &= J\sum_{i=1}^{n}\{|i\rangle\langle i+1| + |i+1\rangle\langle i|\} + \sum_{i=1}^{n}\varepsilon_i |i\rangle\langle i| \\
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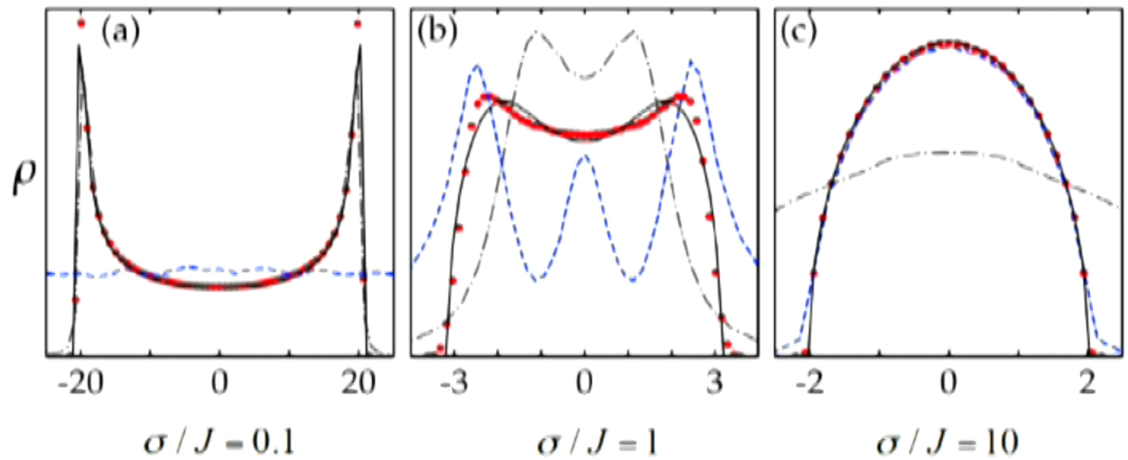
&\equiv \quad A \quad + \quad B
\end{aligned}
$$

Anderson's model

$$\mathcal{H} = \begin{pmatrix} G & J & & & \\ J & G & J & & \\ & J & G & J & \\ & & J & G & J \\ & & & J & G \end{pmatrix} = \begin{pmatrix} G & & & & \\ & G & & & \\ & & G & & \\ & & & G & \\ & & & & G \end{pmatrix} + \begin{pmatrix} & J & & & \\ J & & J & & \\ & J & & J & \\ & & J & & J \\ & & & J & \end{pmatrix}$$

Anderson's model: prl 2012



Disorder: low moderate high

- Exact diagonalization
- Free probability theory
- Perturbation theory: Reference Hopping
- Perturbation theory: Reference disorder

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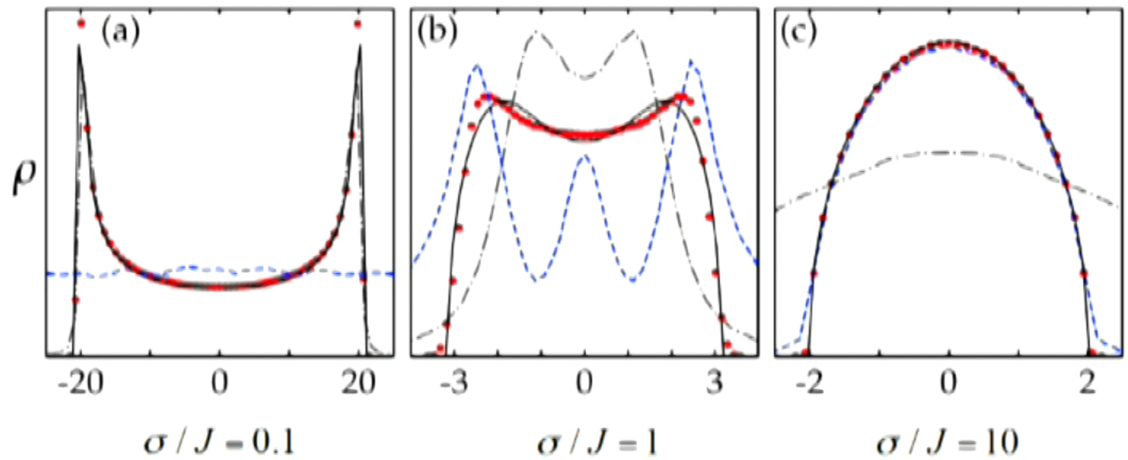

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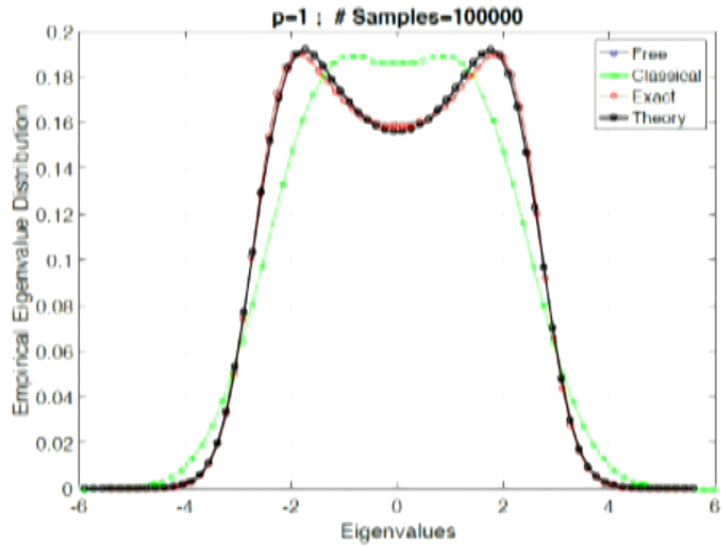
Anderson's model: It should really be free



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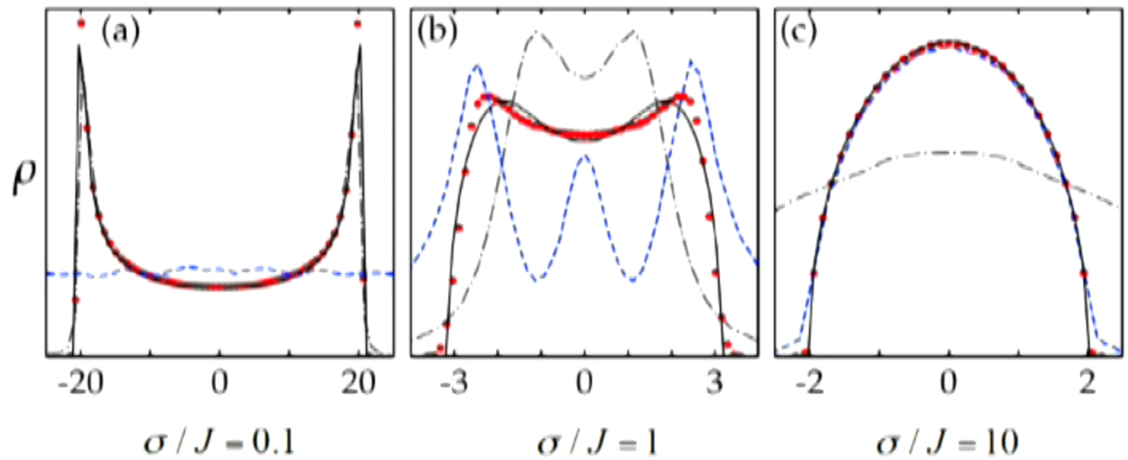

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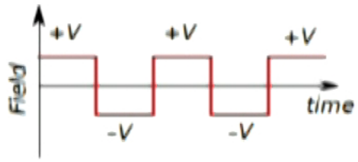
Floquet Driven systems

Oles Shtanko

$$H(t) = H_0 + V(t) \qquad V(t+\tau) = V(t)$$

H_0 : **Clean systems e.g., Kitaev chain, or BHZ model**

$V(t)$: **Periodic driving**



Joint work with Oles Shtanko: arXiv [Cond-Mat] 1803.08519

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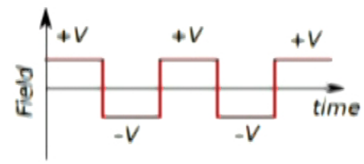


By Floquet-Block Theorem: Total time evolution at $t = n\tau$ is: $U_n = (U_F)^n$

Floquet Hamiltonian : $U_F \equiv \exp(-i\tau H_F) \equiv T\exp(-i\int_0^\tau dt' H(t'))$

Joint work with Oles Shtanko: arXiv [Cond-Mat] 1803.08519

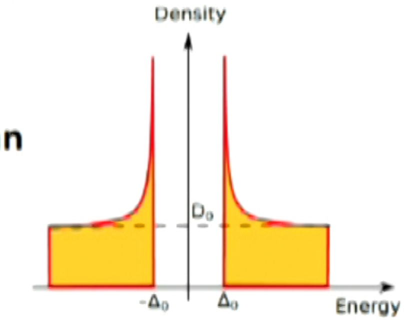
Disordered Floquet Driven systems

$$H(t) = H_0 + V(t) + \delta V \qquad V(t+\tau) = V(t)$$

δV : Disorder (can even be a diagonal random matrix)

Induces disorder in the Floquet Hamiltonian

$$H'_F = H_F + \delta V_F \qquad \delta V_F = \sum_{\ell \ge 1} \delta V_\ell \tau^{\ell-1}$$



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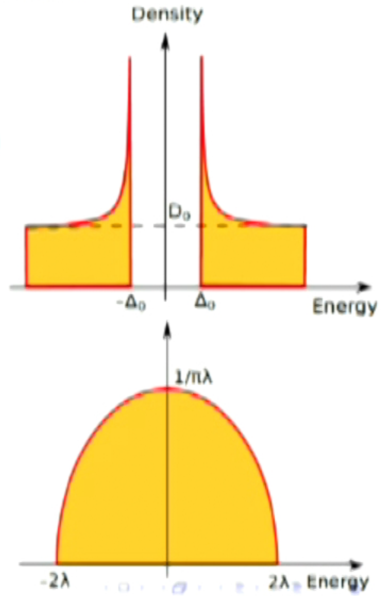
Idea: Make the effective model

$$H_F^{eff} = H_F + M \qquad M :$$



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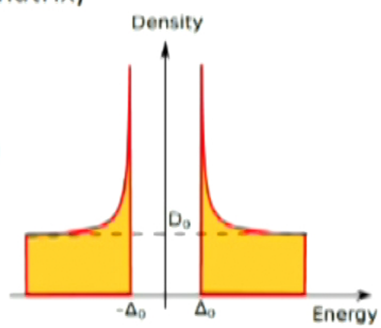

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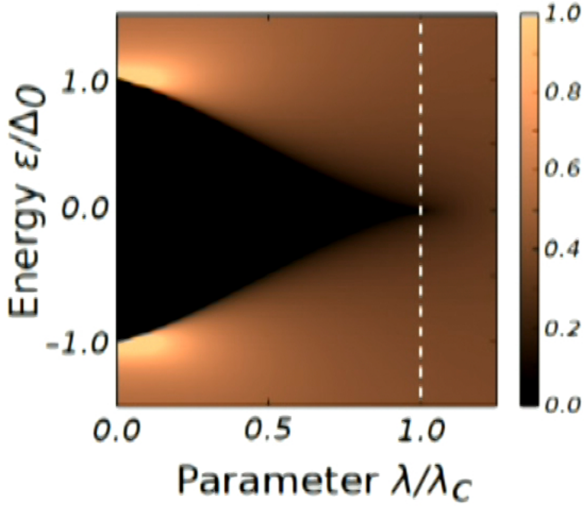
Universal Features: Analytically computed

The Gap as a function of disorder:

$$\Delta(\lambda) = \Delta_0 \left[1 - \left(\pi \rho_0 \frac{\lambda^2}{\Delta_0} \right)^{2/3} \right]^{3/2}$$

$$\lambda = \sqrt{\varphi(\delta V_F^2)}$$



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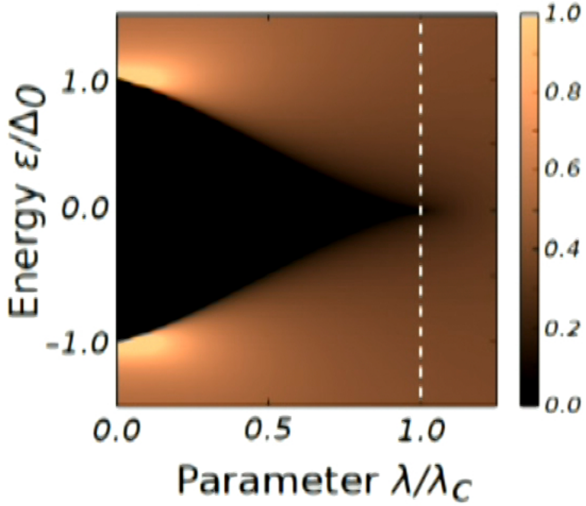

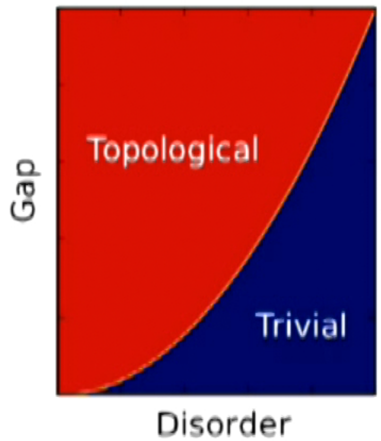

The critical disorder strength:

$$\lambda_c = \sqrt{\Delta_0 / \pi\rho_0}$$

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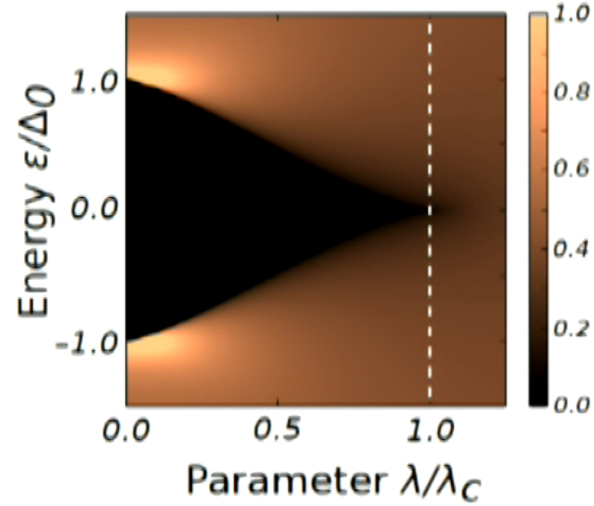

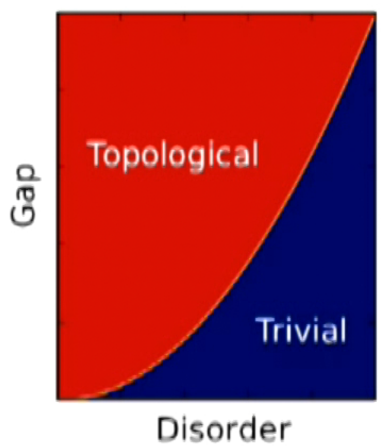

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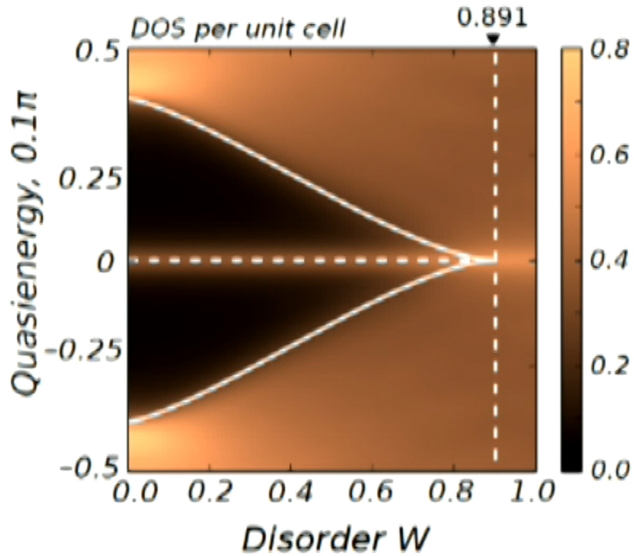
For narrow bandgap: $\lambda_c >> \Delta_0$

Critical Exponents $\Delta \sim (\lambda_c - \lambda)^{\nu z} \rightarrow \nu z = 3/2$

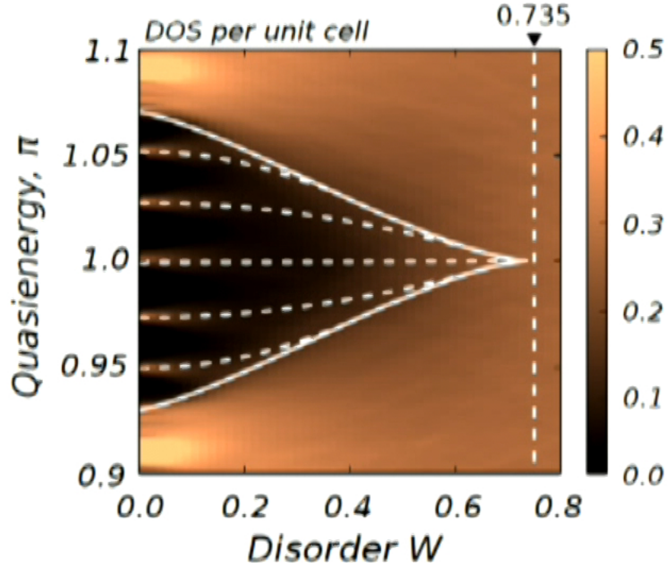
Kitaev chain



$$H(t)=\sum_{\mathbf{r}}[\sum_{\mathbf{a}}(\Gamma_a|\mathbf{r}\rangle\langle\mathbf{r}+\mathbf{a}|+h.c.)+M_{\mathbf{r}}|\mathbf{r}\rangle\langle\mathbf{r}|+f\sigma_z\Theta(t)|\mathbf{r}\rangle\langle\mathbf{r}|]$$

$$\Gamma_x=iD\sigma_y+J\sigma_z,\ M_{\mathbf{r}}=(\mu+h_{\mathbf{r}})\sigma_z,\quad \Theta(t)\equiv\text{sgn}(\sin(2\pi t/\tau))$$

Bernevig-Hughes-Zhang (BHZ)Hamiltonian



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$$\Gamma_{x,y}=-i\frac{A}{2}\sigma_{x,y}+B\sigma_z,\ M_{\mathbf{r}}=h_{\mathbf{r}}+(\mu-4B)\sigma_z,$$

Conclusions: Free Probability Theory

- Approximates (often very accurately) the spectrum of a wide range of models
- Non-Perturbative. Not many such techniques.
- The technique has been distilled out for applications

References:

1. O. Shtanko, RM-, arXiv (2018)
2. RM-, A. Edelman, arXiv (2017)
3. RM- with (lot of MIT Chemists), PRL (2011)
4. RM-, A. Edelman, PRL (2010)

What if matrices are close to commuting?

The β-Orthogonal Group

0 | $o(\beta m^2)$ | $\Theta(\beta m^2)$

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p

$p = 0$ $\quad$ $p = 1$

$$d\nu \approx p\, d\nu_F + (1-p)\, d\nu_c$$

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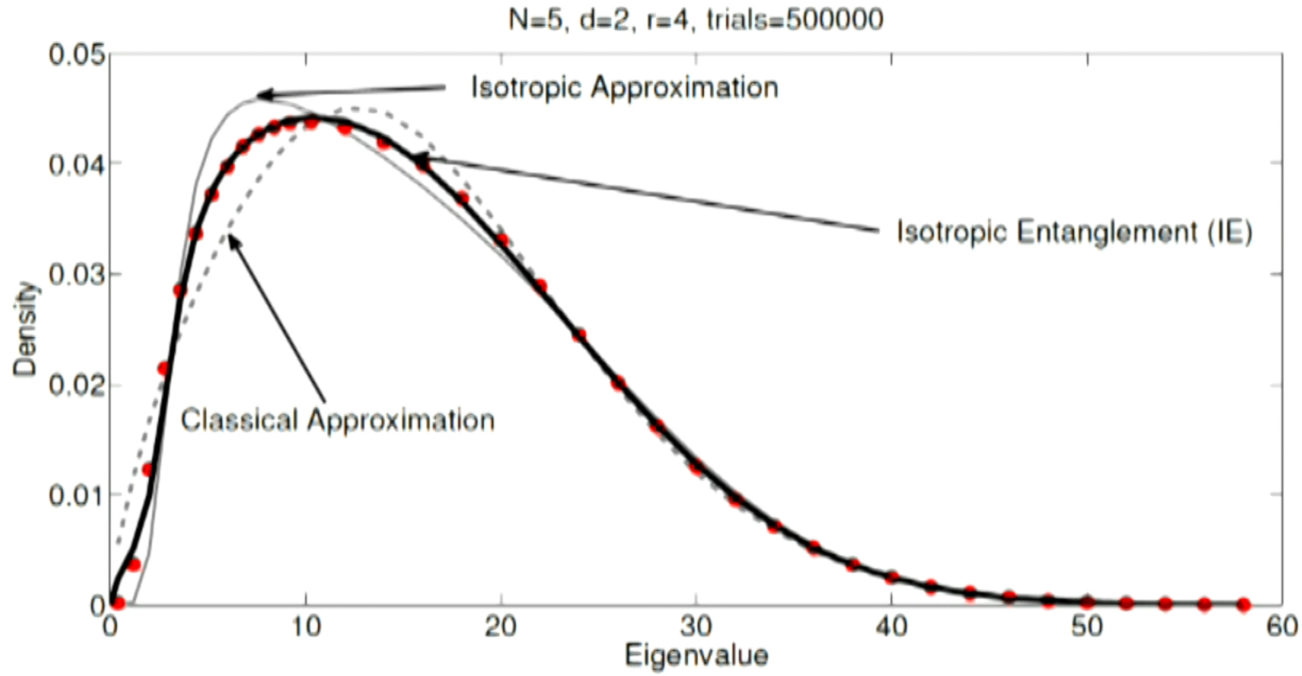
$p = 0$ $\quad$ $p = 1$

$$d\nu \approx p\,d\nu_F + (1-p)\,d\nu_c$$

$$p = \left\{1 - m\mathrm{E}\left(|q_s|^4\right)\right\}$$

Local terms: Wishart matrices



RM-, A Edelman (PRL 2010)

The action starts at the fourth moment

Theorem

(The Matching Three Moments Theorem) *The first three moments of the structured, free (iso) and classical sums are equal:*

$$\varphi\left[\Lambda_A + Q^{-1}\Lambda_B Q\right]^k = \varphi\left[\Lambda_A + Q_s^{-1}\Lambda_B Q_s\right]^k = \varphi\left[\Lambda_A + \Pi^{-1}\Lambda_B \Pi\right]^k,$$

$$k = \{1, 2, 3\}$$

where

$$\varphi[\bullet] \equiv \frac{1}{m}\mathbb{E}\mathrm{Tr}[\bullet]$$

The Departure Theorem

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$$m_4^{iso} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 Q^{-1}\Lambda_B Q + 4\Lambda_A^2 Q^{-1}\Lambda_B^2 Q + 4\Lambda_A Q^{-1}\Lambda_B^3 Q + \underline{\mathbf{2(\Lambda_A Q^{-1}\Lambda_B Q)^2}} + \Lambda_B^4\right]\right\}$$
$$m_4^{q} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 Q_s^{-1}\Lambda_B Q_s + 4\Lambda_A^2 Q_s^{-1}\Lambda_B^2 Q_s + 4\Lambda_A Q_s^{-1}\Lambda_B^3 Q_s + \underline{\mathbf{2(\Lambda_A Q_s^{-1}\Lambda_B Q_s)^2}} + \Lambda_B^4\right]\right\}$$
$$m_4^{c} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 \Pi^{-1}\Lambda_B \Pi + 4\Lambda_A^2 \Pi^{-1}\Lambda_B^2 \Pi + 4\Lambda_A \Pi^{-1}\Lambda_B^3 \Pi + \underline{\mathbf{2(\Lambda_A \Pi^{-1}\Lambda_B \Pi)^2}} + \Lambda_B^4\right]\right\}$$

$U \in \{Q_s, Q, \Pi\}$:

$$\Lambda_A^4 + 4\Lambda_A^3 U^{-1}\Lambda_B U + 4\Lambda_A^2 U^{-1}\Lambda_B^2 U + 4\Lambda_A U^{-1}\Lambda_B^3 U + 2(\Lambda_A U^{-1}\Lambda_B U)^2 + \Lambda_B^4$$

The Fourth moment is where they differ.

The (amazing) key theorem

Defining p by matching fourth moments (or kurtosis):

$$m_4 = p m_4^f + (1-p) m_4^c \implies p = \frac{m_4^c - m_4}{m_4^c - m_4^f}$$

Density of states from localization

Inverse Participation Ratio : Localization

$$\frac{1}{m}\sum_{i,j=1}^{m}|u_{ij}|^2 = 1$$ **Eigenvectors are normalized**

$$\frac{1}{m}\sum_{i,j=1}^{m}|u_{ij}|^4 \Leftrightarrow mE\left[|u_{ij}|^4\right]$$ **IPR: measure of localization**

$$mE\left[|u_{ij}|^4\right] = \begin{cases} m\left(\frac{1}{m^2}\sum_{i,j=1}^{m}|u_{ij}|^4\right) = \frac{1}{m} & u = \frac{1}{\sqrt{m}}(1,1,\ldots,1) \quad \textbf{Dislocalized} \\ m\left(\frac{1}{m^2}\sum_{i,j=1}^{m}|u_{ij}|^4\right) = m\left(\frac{1}{m}\right) = 1 & u = (0,\ldots,0,1,0,\ldots,0) \quad \textbf{Localized} \end{cases}$$

What if matrices are close to commuting?

The β-Orthogonal Group

0 | $o(\beta m^2)$ | $\Theta(\beta m^2)$

$M_c = \Lambda_A + \Pi^{-1}\Lambda_B \Pi$ | $M = \Lambda_A + Q_s^{-1}\Lambda_B Q_s$ | $M_f = \Lambda_A + Q^{-1}\Lambda_B Q$

$dv_c = dv_A * dv_B$ | $dv_F = dv_A \boxplus_F dv_B$

p

$p = 0$ … $p = 1$

$$dv \approx p\,dv_F + (1-p)\,dv_c$$

$$p = \left\{1 - m\mathrm{E}\left(|q_s|^4\right)\right\}$$

Density of states from localization

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$d\nu_c = d\nu_A * d\nu_B$

$d\nu_F = d\nu_A \boxplus_F d\nu_B$

p

$p = 0$ $\qquad$ $p = 1$

$$d\nu \approx p d\nu_F + (1-p) d\nu_c$$

$$p = \left\{1 - m\mathrm{E}\left(|q_s|^4\right)\right\}$$

Density of states from localization

Inverse Participation Ratio : Localization

$$\frac{1}{m}\sum_{i,j=1}^{m}\left|u_{ij}\right|^2 = 1$$

Eigenvectors are normalized

$$\frac{1}{m}\sum_{i,j=1}^{m}\left|u_{ij}\right|^4 \Leftrightarrow mE\left[\left|u_{ij}\right|^4\right]$$

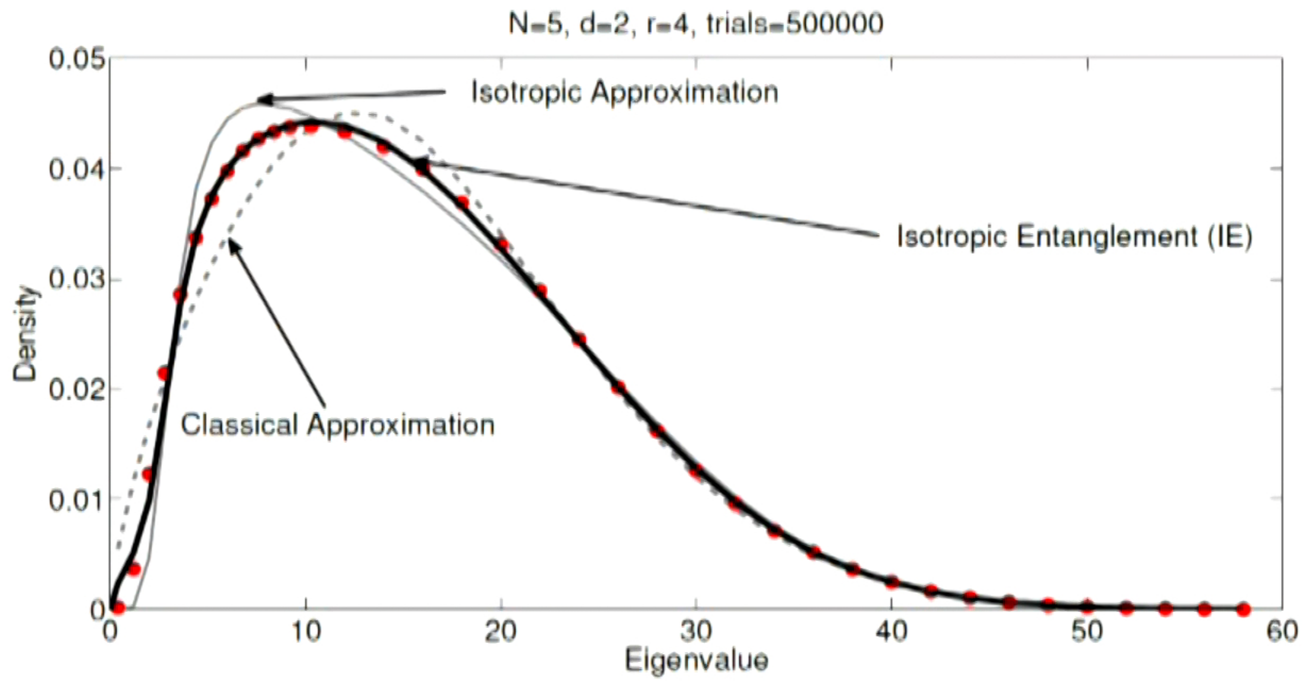
IPR: measure of localization

$$mE\left[\left|u_{ij}\right|^4\right] = \begin{cases} m\left(\frac{1}{m^2}\sum_{i,j=1}^{m}\left|u_{ij}\right|^4\right) = \frac{1}{m} & u = \frac{1}{\sqrt{m}}(1,1,\ldots,1) \\ m\left(\frac{1}{m^2}\sum_{i,j=1}^{m}\left|u_{ij}\right|^4\right) = m\left(\frac{1}{m}\right) = 1 & u = (0,\ldots,0,1,0,\ldots,0) \end{cases}$$

Dislocalized

Localized

Local terms: Wishart matrices



RM-, A Edelman (PRL 2010)

The action starts at the fourth moment

Theorem

(The Matching Three Moments Theorem) *The first three moments of the structured, free (iso) and classical sums are equal:*

$$\varphi\left[\Lambda_A + Q^{-1}\Lambda_B Q\right]^k = \varphi\left[\Lambda_A + Q_s^{-1}\Lambda_B Q_s\right]^k = \varphi\left[\Lambda_A + \Pi^{-1}\Lambda_B \Pi\right]^k,$$

$$k = \{1, 2, 3\}$$

where

$$\varphi[\cdot] \equiv \frac{1}{m}\mathbb{E}\mathrm{Tr}[\cdot]$$

The Departure Theorem

The Departure Theorem

$$m_4^{iso} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 Q^{-1}\Lambda_B Q + 4\Lambda_A^2 Q^{-1}\Lambda_B^2 Q + 4\Lambda_A Q^{-1}\Lambda_B^3 Q + \underline{\mathbf{2}\left(\boldsymbol{\Lambda_A Q^{-1}\Lambda_B Q}\right)^{\mathbf{2}}} + \Lambda_B^4\right]\right\}$$
$$m_4^{q} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 Q_s^{-1}\Lambda_B Q_s + 4\Lambda_A^2 Q_s^{-1}\Lambda_B^2 Q_s + 4\Lambda_A Q_s^{-1}\Lambda_B^3 Q_s + \underline{\mathbf{2}\left(\boldsymbol{\Lambda_A Q_s^{-1}\Lambda_B Q_s}\right)^{\mathbf{2}}} + \Lambda_B^4\right]\right\}$$
$$m_4^{c} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 \Pi^{-1}\Lambda_B \Pi + 4\Lambda_A^2 \Pi^{-1}\Lambda_B^2 \Pi + 4\Lambda_A \Pi^{-1}\Lambda_B^3 \Pi + \underline{\mathbf{2}\left(\boldsymbol{\Lambda_A \Pi^{-1}\Lambda_B \Pi}\right)^{\mathbf{2}}} + \Lambda_B^4\right]\right\}$$

The Fourth moment is where they differ.

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$$m_4^{iso} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 Q^{-1}\Lambda_B Q + 4\Lambda_A^2 Q^{-1}\Lambda_B^2 Q + 4\Lambda_A Q^{-1}\Lambda_B^3 Q + \mathbf{2\left(\Lambda_A Q^{-1}\Lambda_B Q\right)^2} + \Lambda_B^4\right]\right\}$$

$$m_4^{q} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 Q_s^{-1}\Lambda_B Q_s + 4\Lambda_A^2 Q_s^{-1}\Lambda_B^2 Q_s + 4\Lambda_A Q_s^{-1}\Lambda_B^3 Q_s + \mathbf{2\left(\Lambda_A Q_s^{-1}\Lambda_B Q_s\right)^2} + \Lambda_B^4\right]\right\}$$

$$m_4^{c} = \frac{1}{m}\mathbb{E}\left\{\mathrm{Tr}\left[\Lambda_A^4 + 4\Lambda_A^3 \Pi^{-1}\Lambda_B \Pi + 4\Lambda_A^2 \Pi^{-1}\Lambda_B^2 \Pi + 4\Lambda_A \Pi^{-1}\Lambda_B^3 \Pi + \mathbf{2\left(\Lambda_A \Pi^{-1}\Lambda_B \Pi\right)^2} + \Lambda_B^4\right]\right\}$$

$U \in \{Q_s, Q, \Pi\}$:

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The (amazing) key theorem

Defining p by matching fourth moments (or kurtosis):

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The (amazing) key theorem

Denote by: $U=\{Q_s,Q\}$

$$\frac{1}{m}ETr\left[\left(\Lambda_A\Pi^{-1}\Lambda_B\Pi\right)^2-\left(\Lambda_AU^{-1}\Lambda_BU\right)^2\right]=\kappa_2(\Lambda_A)\kappa_2(\Lambda_B)\left\{1-mE\left[|u_{ij}|^4\right]\right\}$$

Only the crossing partitions survive

Two Extreme Cases

Classical : $p=0$ $\quad E\left[|\pi_{ij}|^4\right]=\frac{1}{m}$ $\quad \left\{1-mE\left[|\pi_{ij}|^4\right]\right\}=0$

Free (iso) : $p=1$ $\quad E\left[|q_{ij}|^4\right]=\frac{1}{m}\left(\frac{\beta+2}{m\beta+2}\right)$ $\quad \left\{1-mE\left[|q_{ij}|^4\right]\right\}=\frac{(m-1)\beta}{m\beta+2}\to 1$

The (amazing) key theorem

Defining p by matching fourth moments (or kurtosis):

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The (amazing) key theorem

Denote by: $U = \{Q_s, Q\}$

$$\frac{1}{m} ETr\left[\left(\Lambda_A \Pi^{-1} \Lambda_B \Pi\right)^2 - \left(\Lambda_A U^{-1} \Lambda_B U\right)^2\right] = \kappa_2\left(\Lambda_A\right)\kappa_2\left(\Lambda_B\right)\left\{1 - mE\left[|u_{ij}|^4\right]\right\}$$

Only the crossing partitions survive

Two Extreme Cases

Classical : $p = 0$ $\quad E\left[|\pi_{ij}|^4\right] = \frac{1}{m}$ $\quad \left\{1 - mE\left[|\pi_{ij}|^4\right]\right\} = 0$

Free (iso) : $p = 1$ $\quad E\left[|q_{ij}|^4\right] = \frac{1}{m}\left(\frac{\beta+2}{m\beta+2}\right)$ $\quad \left\{1 - mE\left[|q_{ij}|^4\right]\right\} = \frac{(m-1)\beta}{m\beta+2} \to 1$

Structured as a convex combination of *classical* and *iso*

Theorem

(universality) p is independent of the eigenvalues and is given by

$$p = \frac{m_4^c - m_4}{m_4^c - m_4^f} = \frac{\left\{1 - m\mathbb{E}\left(|q_s|^4\right)\right\}}{\left\{1 - m\mathbb{E}\left(|q|^4\right)\right\}} \to \left\{1 - m\mathbb{E}\left(|q_s|^4\right)\right\},$$

Structured as a convex combination of *classical* and *iso*

In conclusion:

1. Either the free probability answer provides a good approximation, or
2. The $p-$weighted convex combination of free and classical

Structured as a convex combination of *classical* and *iso*

Theorem

(Slider Theorem) $0 \leq p \leq 1$.

Proof.

Since by normality of eigenvectors $\sum_{i=1}^{m} |q_s^i|^2 = 1$, we have that $0 \leq \sum_{i=1}^{m} |q_s^i|^4 \leq 1$. Now $m\mathbb{E}\left(|q_s|^4\right) = m\left(\frac{1}{m}\sum_{i=1}^{m} |q_s^i|^4\right)$. So we have that $0 \leq 1 - m\mathbb{E}\left(|q_s|^4\right) \leq 1$. □

Structured as a convex combination of *classical* and *iso*

In conclusion:

1. Either the free probability answer provides a good approximation, or
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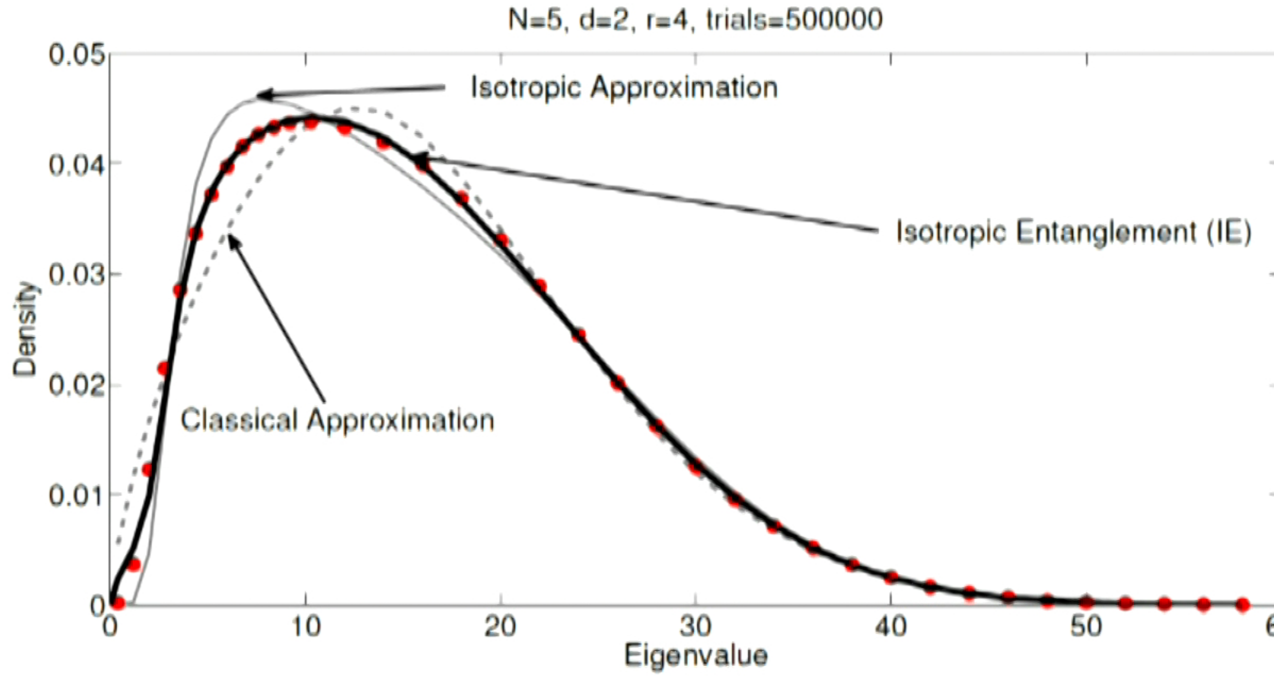
Local terms: Wishart matrices



RM-, A Edelman (PRL 2010)

Lastly

Thank you