Title: TBA

Date: May 14, 2018 02:00 PM

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Abstract: <p>Abstract TBA</p>

Curved complexes,
deformation of Khovanov-Rozansky
homology, and Hilbert Schemes.

joint w/ Eugene Gorsky.

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deformation of Khovanov-Rozansky homology, and Hilbert Schemes.

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(1) Intro

Motivation:

Conjecture

zansky

emes.

GNR

Conjecture (Gorsky-Negut-Rasmussen)

$\exists$ adj. functors

$$\begin{array}{ccc} \text{cx's of Soergel bim for } S_n & \overset{i_*}{\underset{i^*}{\rightleftarrows}} & D^b(\ldots) \text{ Sheaves on } \mathrm{Hilb}^n(\mathbb{C}^2, y=0) \end{array}$$

s.t.

(a) $i_*(X \otimes Y) = i_*(Y \otimes X)$

(b). $\mathrm{Hom}(1, X) \xrightarrow[\cong]{i_*} R\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, i_*(X))$

(c) i^* identifies $D^b\mathrm{Coh}\,\mathrm{Hilb}^n(\mathbb{C}^2, y=0)$ with Drinfeld center of $K^b(\mathrm{SBim}_n)$.

Goal: D

Goal: Reform $K^b(\mathbb{S}Bim_n)$, and

Show a relationship with full Hilbert scheme $Hilb^n(\mathbb{C}^2)$.

② Heuristic: from categories to sheaves.

③ Deformation of complexes.

④ Application to $\mathbb{S}Bim_n$.

② $\mathcal{C}, \otimes, \mathbb{1}$ $\Bbbk$-linear, additive monoidal cat.

Pick $F \in \mathcal{C}$. $A_F := \bigoplus_{k\geq 0} Hom(\mathbb{1}, F^{\otimes k})$

$\mathcal{C} \longrightarrow A_F\text{-gmod}.$

$X \longmapsto \bigoplus_{k\geq 0} Hom(\mathbb{1}, F^{\otimes k} \otimes X)$

Lemma (Nicest possible situation).

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Goal: Reform $K^b(\mathbb{S}\mathrm{Bim}_n)$, and

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(2) Heuristic: from categories to sheaves.

(3) Deformation of complexes.

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(2) $\mathcal{C}, \otimes, \mathbb{1}$ $\mathbb{k}$-linear, additive monoidal cat.

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Lemma (Nicest possible situation).

Spse $F \xrightarrow{\Psi} \mathbb{1}$ is such that

$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, F^{\otimes k}) \to \mathrm{End}_{\mathcal{C}}(\mathbb{1})$

$\alpha \longmapsto \Psi^{\otimes k} \circ \alpha$.

is injective for $k \geq 0$.

Then A_F is commutative

X)

situation).

h that

$\overline{\ }n\ \mathcal{O}_{\mathcal{C}}(1)$)

α.

In fact

$$A_F \cong \bigoplus_{k\geq 0} \overline{J}_k,$$

for some ideals

$J_k \subset \mathrm{End}_{\mathcal{C}}(1)$

$J_k J_\ell \subset \overline{J}_{k+\ell}$.

$\mathcal{C} \to \mathrm{QCoh}(\mathrm{Proj}\, A_F)$.

(GNR).

3. Deformations of complexes.

Let $\mathcal{A}$ be an additive cat, $K(\mathcal{A})$ - homotopy cat of cx's.

Given $z \in Z(\mathcal{A})$,

$X^\bullet \in K(\mathcal{C})$, we obtain

$$\begin{array}{ccccc} X^\bullet = \cdots \to & X^k & \to & X^{k+1} & \to \cdots \\ \downarrow z & \downarrow z & & \downarrow z & \\ X^\bullet = \cdots \to & X^k & \to & X^{k+1} & \to \cdots \end{array}$$

If Z

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\end{array}$$

If $z = 0$ on X,

$z = dh + hd$.

Then have 1^{st} order def.

$X \otimes_k k[y]/y^2$

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$\Delta = d \otimes 1 + h \otimes y$.

$\Delta^2 = z \otimes y$.

$G X$,

d

order defn

$1 + h \otimes y$.

If $h^2 = dh_2 + h_2 d$, $\quad [d, h^2] = [d,h]h - h[d,h] = 0.$

then have order 2 def.

$X \otimes_{\mathbb{k}} \mathbb{k}[y]/y^3$, $\Delta = d \otimes 1 + h \otimes y + h_2 \otimes y^2$.

$\Delta^2 = Z \otimes y$.

Formal def $X \otimes_{\mathbb{k}} \mathbb{k}[[y]]$.

with curvature $Z \otimes y$.

Def'n. A deformation of $X \in K^b(\mathcal{A})$ is a $\mathbb{k}[y]$ linear endomorphism $\Delta \circlearrowleft X[y]$, s.t. $\Delta = d_X \otimes 1 \mod (y)$,

$\Delta^2 = Z \otimes y$.

$\underline{\underline{\text{Ex}}}$. $R = \mathbb{Q}[u]$. $\boxed{\mathcal{C} = K^b(\mathcal{A})}$

$\mathcal{A} = (R,R)$ bimod

$\mathcal{A}$ is linear w.r.t. $\mathbb{Q}[u,u']$.

$B = \mathbb{Q}[u] \otimes \mathbb{Q}[u]$

$\mathbb{Q}[u']$

$$\begin{tikzcd}
 & 0 & 1 & 2 \\
F = \arrow[d, "h"] & \underline{B} \arrow[r, "u-u'"] & B \arrow[r, two heads] \arrow[dl, "\mathrm{Id}"] & R \arrow[dl, "0"] \\
F & \underline{B} \arrow[r, "u-u'"] & B \arrow[r] & R
\end{tikzcd}$$

X,

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,

$+h \otimes y$.

If $h^2 = dh_2 + h_2 d$,

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$\Bbbk[y]$ linear endomorphism $\Delta \curvearrowright X[y]$.

$[d, h^2] = [d,h]h - h[d,h]$
$= 0$.

$\underline{\underline{[z, x]}}$.

$\deg_h y = 2$.

inear w.r.t.

$u, u^-]$

$\mathbb{1} = R$

$$F[y] = B[y] \underset{\mathrm{Id}\otimes y}{\overset{(u-u')\otimes 1}{\rightleftarrows}} B[y] \xrightarrow{\pi \otimes 1} R[y]$$

Claim

$$\begin{array}{ccccc} & & R & & R \\ & & \downarrow & & \downarrow \\ F[y] & = & B[y] & \underset{y}{\overset{u-u'}{\rightleftarrows}} B[y] \to & R[y] \\ \downarrow \Psi & & \downarrow & & \\ R[y] & = & R & & \end{array}$$

$-y$

$-y$.

$\Psi^{\otimes k} \circ (-)$ Identifies $\mathrm{Hom}(R, F^{\otimes k})$ with $M^k = (u, y)^k$.

GNR

(Gorsky-Negut-R

ctors

$m \underset{i^*}{\overset{i_*}{\rightleftarrows}}$

$X \otimes Y) = i_* ($

$1, X) \xrightarrow[\cong]{i_*}$.

ifies $D^b(\mathrm{Coh\,Hilb}^n(\mathbb{C}^2$

nfold center of $K^b($

$B = \mathbb{Q}[u] \otimes \mathbb{Q}[u]$ $\mathbb{Q}[u, u']$

$\mathbb{Q}[u^{\pm}]$

$$\begin{array}{ccccc} & 0 & & 1 & & 2 \\ F = & \underline{B} & \xrightarrow{u-u'} & B & \twoheadrightarrow & R \end{array}$$

$\downarrow h$ Id 0

F $\underline{B} \xrightarrow{u-u'} B \longrightarrow R$

$\mathbb{1} = R$.

$R \to B$

$\mathbb{1} \mapsto u \otimes 1 + 1 \otimes u'$.

Claim

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$\mathrm{Hom}(R,$

$m^k = (u$

$u, u']$.

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B

$u\otimes 1 + 1 \otimes u'$.

$\mathrm{Id} \otimes y$.

Claim

$$\begin{array}{ccc} F[y] & = & B[y] \underset{y}{\overset{u-u'}{\rightleftarrows}} B[y] \to R[y]. \\ \downarrow \Psi & & \downarrow \\ R[y] & = & R \end{array}$$

$R \xrightarrow{t_1} B[y]$, $R \xrightarrow{t_2} R[y]$, $-y$, $-y$.

$\Psi^{\otimes k} \circ (-)$ Identifies $\mathrm{Hom}(R, F^{\otimes k})$ with $m^k = (u, y)^k$.

$$A_{F[y]} = \mathbb{Q}[\overset{0}{u}, \overset{0}{y}, \overset{1}{t_1}, \overset{1}{t_2}] / t_2 u - t_1 y.$$

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$m \underset{i^*}{\overset{i_*}{\rightleftarrows}}$

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nfeld center of $K^b($

Fix $n \in \{1, 2, \dots\}$.

$$R := \mathbb{Q}[x_1, \dots, x_n], \quad \deg x_i = 2$$

$s = (i, i+1) \in S_n$.

$$R^s := \{ f \in R \mid s(f) = f \}.$$

$$B_s := R \otimes_{R^s} R(1)$$

(1) = downward shift

in (R,R)-gbimod.

B_i =

monoidal

(R,R)-gbi…

$B_1, \dots, B$

$) = 0)$.

$))$

$\deg x_i = 2$

$\}$.

downward shift in (R,R)-gbimod.

$\mathbb{S}\mathrm{Bim}_n$ is the smallest full monoidal subcat of (R,R)-gbimod containing

$B_1, \ldots, B_{n-1}$, closed under $\oplus$, $\overset{\oplus}{C}$, (± 1).

Rouquier. $Br_n \to K^b(\mathbb{S}\mathrm{Bim}_n)$

$\beta \longmapsto F(\beta)$.

$F(1 \cdots \underset{i \; i+1}{\times} \cdots 1) = \underline{B_i} \to R(1)$

In fact

$A_F \cong \bigoplus_{k \geq 0} J_k$,

for some ideals

$J_k \subset \mathrm{End}_{\mathcal{C}}(k)$

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Let $\mathcal{A}$ …

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$\downarrow 2$

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mod.

under $\otimes$, $\circ$,

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σ_i

$F(| \cdots \times \cdots |) = B_i \xrightarrow{\varphi} R(1)$

$i \quad i+1$

$1 \otimes 1 \longmapsto 1$

$F(| \cdots \times \cdots |) = (R(-1) \xrightarrow{b} B_i)$

$i \quad i+1$

$1 \longmapsto x_i \otimes 1 - 1 \otimes x_{i+1}$.

$\mathcal{C} \to QCoh(Proj\, A_K)$.

(GNR).

$x_1 \quad x_2$

$x_1' \quad x_2'$

$x_1 \quad x_2$

$x_1' - x_2'$

$X^\bullet \in K$

$X^\bullet$

Thm (Rouquier)

Tensoring $F(\sigma_i^{\pm})$ together yield well-def'd cx's $F(\beta)$, up to canonical iso in

$K^b(SBim_n)$.

Theorem (Khovanov)

$\mathrm{Hom}^{\bullet\bullet}_{K(SBim_n)}(R, F(\beta))$

$\cong$ A-deg zero part of KR homology of $\hat{\beta} \subset \mathbb{R}^3$.

If $h = dh_2 + h_2 d$, then have order 2 def.

$X \otimes_{\mathbb{k}} \mathbb{k}[y]/y^3$, $\Delta = d \otimes 1 + h \otimes y + h_2 \otimes y^2$

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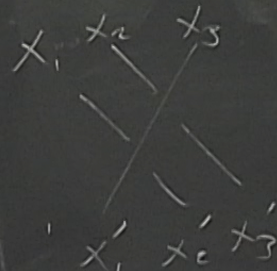
$\mathcal{T}_j Bim_n$ is linear w.r.t. $\mathbb{Q}[\underline{x}, \underline{x}']$.

Lemma If $Br_n \to S_n$, $\beta \mapsto w$,

then $x_{w(i)} - x_i' \subset F(\beta)$.



$x_3 = x_1'$

$\vdots$

$h \; -h[d,h])\; [-x$

$\deg_h y = 2$.

zero part homology

$\mathbb{R}^3$.

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, $\underline{x}'$].

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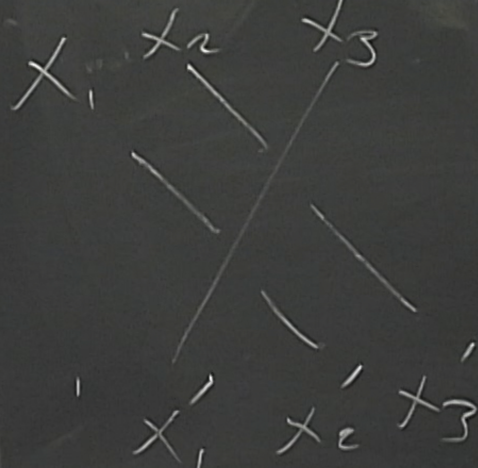

$x_3 = x_1'$.

bimod deg

(Q) hom

In fact $\exists\ h_i \in \underline{\mathrm{End}}^{2,-1}(F(\beta))$

st $d_{F(\beta)} h_i + h_i d_{F(\beta)} = x_{w(i)} - x_i'$.

ar w.r.t. In fact $\exists\ h_i$ ——,

st. $d_{F(\beta)} h_i + h_i d_{F(\beta)} = x_{w(i)} - x_i$.

. $h_i^2 = 0$, $h_i h_j + h_j h_i = 0$.

$1 \leq i, j \leq n$.

. $\sum_{i=1}^{n} h_i = 0$.

$) = (0,2)$

- homotopy cat of $\mathcal{Y}$-ifications

denoted $\mathcal{Y} = \bigoplus_{w \in S_n} \mathcal{Y}_w$.

Conv. product

$\mathcal{Y}_w \times \mathcal{Y}_v \longrightarrow \mathcal{Y}_{wv}$.

③

Fix

$R :=$

$s = ($

R^s

$B_s :=$

$B_i =$

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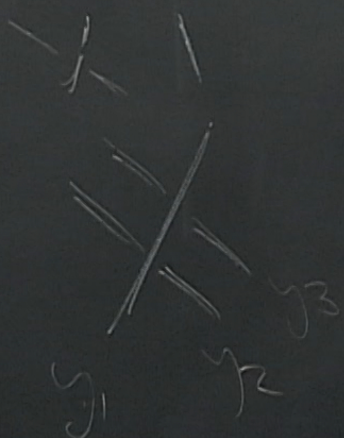


<u>Theorem</u> (Gorsky-H) Rouquier cx's

admit unique y-ifications.

$FT(\beta) \in \mathcal{Y}_w$.

$\beta \mapsto w \in S_n$.

(B) BBin

Fix $n \in \{$

$R := (\mathbb{Q}$

$s = (i, i+1)$

$R^s := \{f$

$B_s := R$

$B_i =$

$w = (i, i+1)$

r x's

ification

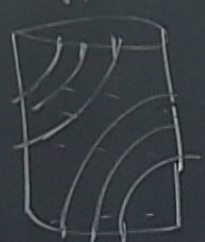
Ex $FY(Y_{i\ i+1}) = B_i[y] \underset{\iota\otimes(y_{i+1}-y_i)}{\overset{\varphi\otimes 1}{\rightleftarrows}} R[y](1)$

$)w$.

$$FY(X) = R[y](-1) \underset{\varphi\otimes(y_{i+1}-y_i)}{\overset{\iota\otimes 1}{\rightleftarrows}} B_i[y]$$

$FT_n :=$

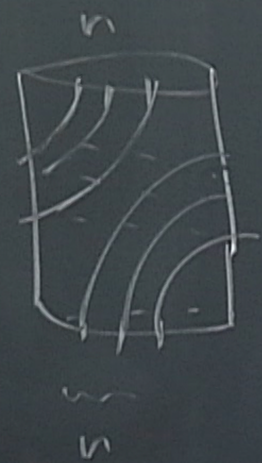
n



n

generates
center of braid gp.

$\rho \otimes (y_{i+1} - y_i)$

$FT_n :=$ 

generates center of braid gp.

$$\mathbf{\underline{Def}} \quad \overline{J} \subset \mathbb{Q}[\underline{x}, \underline{y}] \circlearrowright S_n.$$

ideal generated by sgn.

$\to R[y](1;$

$y_{i+1}-y_i)$

$\xrightarrow{\otimes 1} B_i[y]$

$\otimes(y_{i+1}-y_i)$

Theorem (Gorsky–H)

$$\mathrm{Hom}_y(\mathbb{1}[y], FT_n^k[y]) \cong J^k.$$

$$A_{FT_n[y]} \cong \bigoplus_{k\geq 0} J^k.$$

$FT_n \in K^b(SBim_n)$.

Corollary

$$A_{FT_n} = \bigoplus_{k\geq 0} J^k/(y)J^k.$$

Thm (Rouquier)

Tensoring F (

yield well-def'd

up to canonica

$K^b(SBim_n)$.