

Title: Effective line-elements in loop quantum gravity

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Abstract: Canonical quantization schemes often suggest modifications to classical dynamics, such as in an effective Friedmann equation. However, although often ignored, they also necessarily imply new effects for quantum space-time leading to new (quantum) symmetries. The invariant line-element, corresponding to new geometrical structures emerging in the presence of holonomy modifications in loop quantum gravity, shall be consistently derived in this talk. We shall use black-hole models to illustrate new features of this quantum space-time, going beyond standard Riemannian manifolds.

Effective line elements in loop quantum gravity

Suddhasattwa Brahma

Asia Pacific Center for Theoretical Physics

M. Bojowald, S.B. & D.-h. Yeom, [1803.01119](#)

M. Bojowald, S.B., U. Büyükçam & F. D'Ambrosio, [1610.08355](#)

M. Bojowald & S.B., [1610.08840](#), *forthcoming*
S.B., [1411.3661](#)

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Motivation

→ Canonical quantization techniques often leads to new effects in quantum gravity \Rightarrow both in the **dynamics** and the **structure of space-time**.

→ First effects: **Classical dynamical equations** are modified due to **background-independent** quantizations.

→ **Example**: The effective Friedmann equation in LQC

$$H^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_c} \right)$$

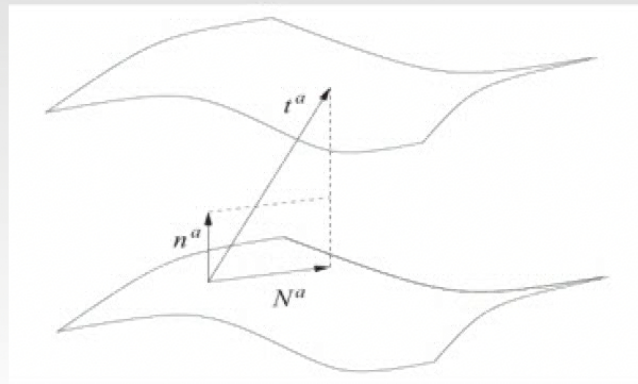
Remarkable deviations from classical dynamics \Rightarrow **Singularity resolution** due to **holonomy modifications** \Rightarrow originates from nontrivial regularizations in the LQG **Hamiltonian constraint**.

What effects do the same quantum correction have on the structure of space-time? (Different motivation: Deformed symmetries of quantum spacetimes)
Physical consequence \Rightarrow Derivation of consistent line elements?



Canonical gravity

- In the Lagrangian formulation, space and time are treated **equally** and on the same footing.
- In the Hamiltonian formulation, **split** space and time by using an arbitrary (time) function to foliate globally hyperbolic spacetime.
- $ds^2 = g_{\mu\nu} dx^\mu dx^\nu \Rightarrow$ ADM metric:
 $ds^2 = -N^2 dt^2 + q_{ab} (dx^a + N^a dt) (dx^b + N^b dt).$
- **One-to-one correspondence** between $g_{\mu\nu}$ and (q_{ab}, N, N^a) but **roles** of different components space-time line element crucially **different**.






GR as a constrained system

Hamiltonian for GR:

$$H_{\text{grav}}^{\text{tot}} = \int d^3x (NC_{\text{grav}} + N^a D_a^{\text{grav}})$$

where C_{grav} & D_a^{grav} are the ‘Hamiltonian’ and (spatial) ‘diffeomorphism’ constraints respectively. No **absolute time**.

⇒ These ‘first-class’ constraints play a dual role:

- Constraints generate the EOMs for the system, for a given choice of the Lagrange multipliers ⇒ **EOMs derived in some choice of ‘time’ (gauge)**. 
- Same constraints also generate gauge transformations, which do not change the physical solutions ⇒ **freedom in choice of ‘time’**.
- Form of both sets of equation very similar but **meaning very different**.

→ LQG-modified constraints ⇒ Crucial to understand **interplay** between **evolution** and **gauge**.

→ **Consistent** interplay between EOMs and GTs rely on the **off-shell** algebra of the **(modified)** constraints.





Dirac algebra

→ Hypersurface deformation algebra (HDA) of classical space-time

(generalization of local Poincaré algebra):

$$\{D(w_1^a), D(w_2^b)\} = D(\mathcal{L}_{w_1} w_2^b)$$

$$\{H(N), D(w^a)\} = -H(\mathcal{L}_w N)$$

$$\{H(N_1), H(N_2)\} = D(q^{ab} (N_1 \partial_b N_2 - N_2 \partial_b N_1))$$

with (lapse) N : function on space,
metric on spatial slice.

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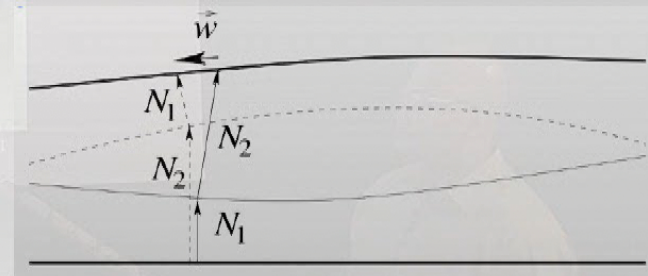
• Invariance under HDA implies
general covariance

with (lapse) N : function on space, (shift) w : vector field and q :
metric on spatial slice.

• Second-order field equations
invariant under HDA must equal

• **Invariance** under HDA implies
general covariance. [Dirac, 1951]

• **Second-order field equations**
invariant under HDA must **equal**
GR. [Hojman, Kukař & Teitelboim, 1974-76]



→ **Modified** constraints, including LQG corrections, still form a **closed**
algebra avoiding **gauge anomalies**. But **deformations** appear.

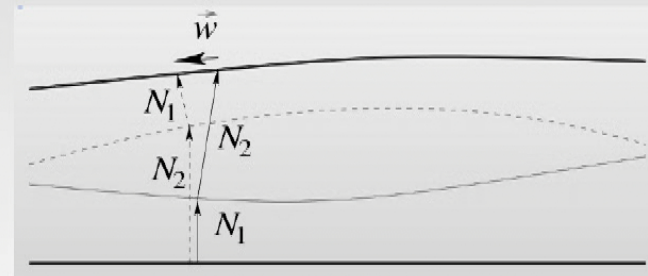
Dirac algebra

→ Hypersurface deformation algebra (HDA) of classical space-time (generalization of local Poincaré algebra):

$$\begin{aligned} \{D(w_1^a), D(w_2^b)\} &= D(\mathcal{L}_{w_1} w_2^a) \\ \{H(N), D(w^a)\} &= -H(\mathcal{L}_w N) \\ \{H(N_1), H(N_2)\} &= D(q^{ab} (N_1 \partial_b N_2 - N_2 \partial_b N_1)) \end{aligned}$$

with (lapse) N : function on space, (shift) w : vector field and q : metric on spatial slice.

- **Invariance** under HDA implies **general covariance**. [Dirac, 1951]
- **Second-order field equations** invariant under HDA must **equal GR**. [Hojman, Kukař & Teitelboim, 1974-76]



→ **Modified** constraints, including LQG corrections, still form a **closed algebra** avoiding **gauge anomalies**. But **deformations** appear.



Gauge transformations and the HDA

- The phase space is given by the spatial metric (q_{ab}) and its conjugate momenta (π^{ab}). The algebra of classical constraints calculated on this phase space.
- **Gauge transformations represent coordinate freedom:** space-time Lie derivative of any (phase-space) function, $f(q_{ab}, \pi^{ab})$, given by $\mathcal{L}_{(\epsilon^0/N, \epsilon^i - \epsilon^0 N^i/N)} f = \{f, H[\epsilon^0] + D[\epsilon^i]\}$ if constraints are satisfied (time direction $t^a = Nn^a + N^a$).
- **EOMs:** $\dot{f} := \{f, H[N] + D[N^a]\}$ & **GTs:** $\delta_\epsilon f := \{f, H[\epsilon^0] + D[\epsilon^i]\}$.
- Commutation property: **Evolution of gauge-transformed initial data = Gauge transformation of evolved initial data.**
- Since a commutation relation is involved \Rightarrow **Interplay between evolution and gauge relies on Dirac algebra.**

\rightarrow This is what happens classically. For the LQG scenario, the constraints are **modified** and, as a result, the HDA gets **deformed**.



A small puzzle?

- GTs, generated by $H[\epsilon^0] + D[\epsilon^i]$, act on q_{ab} and π^{ab} but not on (N, N^a) .
- However, a **generic** coordinate transformation can clearly change both q_{ab} as well as g_{0a} components of the metric
 $ds^2 = -N^2 dt^2 + q_{ab} (dx^a + N^a dt) (dx^b + N^b dt) \Rightarrow$ Role of (N, N^a) **crucially different** from q_{ab} .

There are two ways to resolve this puzzle:

- Consider the extended phase space with p_N and p_{N^a} (primary constraints not solved). [J. M. Pons, D. C. Salisbury, and L. C. Shepley, 1997]
- **Require canonical EOMs are gauge-covariant** \Rightarrow This approach explicitly shows the **important role** played by the Dirac algebra.

\rightarrow Once the classical case is demonstrated, extend results for the **LQG-deformed HDA**.



Transformation of the Lapse and Shift

→ Canonical EOMs, in a particular gauge, must transform consistently with the GTs of the canonical variables ⇒ Possible only if $N^A := (N, N^i)$ transform properly!

- Evolution of any phase space variable: $\dot{q} = \{q, C[N^A]\}$.
- Coordinate transform $\tilde{q} = q + \delta_{\epsilon^A} q$ with $\delta_{\epsilon^A} q = \{q, C[\epsilon^A]\}$.
- Transformed q satisfy the same EOMs $\dot{\tilde{q}} = \{\tilde{q}, C[\tilde{N}^A]\}$, if there is a nontrivial transformation $\tilde{N}^A = N^A + \delta_{\epsilon^B} N^A$.
- Using the constraint algebra (HDA) and equating the LHS and RHS of the above equation, possible to calculate $\delta_{\epsilon^B} N^A$.

→ Given $\{C_A, C_B\} = F_{AB}^D C_D$, we get $\delta_{\epsilon} N^A = \dot{\epsilon}^A + N^B \epsilon^C F_{BC}^A$. Explicitly,

$$\delta_{\epsilon} N = \dot{\epsilon}^0 + \epsilon^i \partial_i N - N^i \partial_i \epsilon^0 \quad (1)$$

$$\delta_{\epsilon} N^i = \dot{\epsilon}^i + \epsilon^j \partial_j N^i - N^j \partial_j \epsilon^i - q^{ij} (N \partial_j \epsilon^0 - \epsilon^0 \partial_j N) \quad (2)$$

crucially depend on the HDA (above, we use the classical HDA).



Suddhasattwa Brahma

Effective line elements in LQG

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The Big Picture

Say, we have some (infinitesimal) coordinate transformation
 $t' \rightarrow t + \xi^0$ & $x' \rightarrow x^a + \xi^a$.

The spatial metric transforms as $\tilde{q}_{ab} = \{q_{ab}, H + D\}$.

For the line element

$ds^2 = -\tilde{N}^2 dt'^2 + \tilde{q}_{ab} \left(dx'^a + \tilde{N}^a dt' \right) \left(dx'^b + \tilde{N}^b dt' \right)$ to be meaningful,

i.e. **invariant** (co-ordinate independent), the lapse \tilde{N} and shift \tilde{N}^a must transform in an appropriate manner, which depends on the HDA, as demonstrated. 

In the presence of LQG (holonomy) modifications, the HDA is deformed. What is the corresponding invariant line-element for such LQG-modified (effective) space-times?



Spherically symmetric gravity

- Ashtekar-Barbero variables:
 $\{K_\phi(x), E^\phi(y)\} = G\delta(x, y) = \frac{1}{2}\{K_x(x), E^x(y)\}$.
- Line-element: $ds^2 = -N^2 dt^2 + q_{xx} (dx + N^x dt)^2 + q_{\varphi\varphi} d\Omega^2$ with
 $q_{xx} = (E^\phi)^2/E^x$, $q_{\varphi\varphi} = E^x$.
- Spherically symmetric coordinate transformation
 $t' = t + \xi^0$, $x' = x + \xi^x$, with $(\xi^0, \xi^x) = (\epsilon^0/N, \epsilon^x - (N^x/N)\epsilon^0)$.
- Insert transformed coordinates $(x^a + \xi^a)$ in the line element directly to collect coefficients of dx^2 , dt^2 & $dxdt$. E.g.,
 $dt' = d(t + \epsilon^0/N) = dt + (\epsilon^0/N) \bullet dt + (\epsilon^0/N)' dx$.

From this one can evaluate δq_{xx} and δN^x directly, using which, one can also get δN .

- $\delta q_{xx} = \{q_{xx}, H[\epsilon^0] + D[\epsilon^x]\}$
- $\delta N^x = \dot{\epsilon}^x + \epsilon^x (N^x)' - N^x (\epsilon^x)' - q^{xx} (N(\epsilon^0)' - \epsilon^0 N')$
- $\delta N = \dot{\epsilon}^0 + N' \epsilon^x - N^x (\epsilon^0)'$

The only structure function for this system:

$$\{H[N_1], H[N_2]\} = D[q^{xx} (N_1 N_2' - N_2 N_1')].$$



Holonomy modifications

- Regularize the constraint via holonomies of extrinsic curvatures $K_\phi \rightarrow f(K_\phi)$ (Keep (bounded) function arbitrary for our purposes to allow for quantization ambiguities).
- The constraint algebra is closed, but deformed (only $[H, H]$).


$$[H[N_1], H[N_2]] = D [\beta q^{ab} (N_1 N_2' - N_2 N_1')] ;$$

$$\beta = d^2 f / dK_\phi^2 \Rightarrow \beta \rightarrow 1 \text{ classical limit}$$

- Typically, $f(K_\phi) = \sin^2(\delta K_\phi) / \delta^2 \Rightarrow \beta = \cos(2\delta K_\phi)$.
- Given (deformed) HDA, derivation follows the classical case to get $\delta N^x = \dot{\epsilon}^x + \epsilon^x (N^x)' - N^x (\epsilon^x)' - \beta q^{xx} (N(\epsilon^0)' - \epsilon^0 N')$.
- But transformation of the lapse keeps the **same form** (since no structure **functions** are involved).
- Since the term in δN^x relevant to derive δN is multiplied by β , to get required cancellations, we need coordinate transformations **not** of the **classical** line element, but of an **effective** line element $ds^2 = -\beta N^2 dt^2 + q_{xx} (dx + N^x dt)^2 + q_{\varphi\varphi} d\Omega^2$



Recap

- For the **classical HDA**, coordinate transformations $t' = t + \xi^0$, $x' = x + \xi^x$ imply that the spatial metric transforms as $\{q_{xx}, H[\epsilon^0] + D[\epsilon^x]\}$ and the **invariant** line element is **coordinate independent** only when lapse and shift transform in a specific way.
- When **corrections** are introduced in the LQG-regularized constraints, the HDA is **deformed**. Consequently, the line element which remains **invariant** under coordinate transformations has to be **modified by a factor**. Transformation of $(\delta q_{xx}, \delta N^x, \delta N)$ **consistent only** with this effective line element.
- We see how space-time structures are **strongly affected** by the **same** **corrections** which alter **dynamics** in LQG. 

Holonomy modifications not only imply corrections to the spatial metric by modified equations of motion generated by the Hamiltonian constraint, they also require a new factor of β of N^2 in the time-time component of the space-time line element. Signature change for $\beta < 0$ is an immediate consequence.



Implications for other physical scenarios

→ Effective line elements shall involve correction to the lapse in LQG models wherever one gets a deformed algebra of a similar form.

$$[D(w_1^a), D(w_2^b)] = D(\mathcal{L}_{w_1} w_2^a)$$

- LQC with perturbations [A. Barrau, T. Cailleteau, L. Linsefors & J. Grain, 2012; M. Bojowald & Mielczarek, 2015]

$$[H(N), D(w^a)] = -H(\mathcal{L}_w N)$$

- CGHS and Schwarzschild Black hole [M. Bojowald & S.B., 2016]

$$[H(N_1), H(N_2)] = D(\beta q^{ab} (N_1 \partial_b N_2 - N_2 \partial_b N_1))$$

- 2-dimensional dilaton gravity [M. Bojowald & S.B., 2016]

→ Holonomy modifications **necessarily lead to signature changing** deformations.

→ ‘Signature change’ **resolves** classical **singularity** ⇒ **New model of quantum spacetime** with no Riemannian structure.

→ Fluctuations and higher moments of the quantum state, related to higher curvature corrections, **cannot undo** these deformations from LQG [M. Bojowald & S.B., 2014].



'Fuzzy' Euclidean regime - I

→ Why **fuzzy**?

$$H[N] = -\frac{1}{2G} \int dx N \left(\frac{f_1(K_\varphi) E^\varphi}{\sqrt{E^x}} + 2\sqrt{E^x} K_x f_2(K_\varphi) - \frac{1}{2} E^\varphi V(E^x) - \frac{((E^x)')^2}{4E^\varphi \sqrt{E^x}} + \frac{\sqrt{E^x} (E^x)' (E^\varphi)'}{(E^\varphi)^2} - \frac{\sqrt{E^x} (E^x)''}{E^\varphi} \right)$$

- Spherically symmetry: $V(E^x) = -2/\sqrt{E^x}$
- Other choices of V for CGHS model, Gowdy model etc.
- Standard choice for holonomy modification function
 $f_1(K_\varphi) = \sin^2(\delta K_\varphi)/\delta^2$, ($f_2 = \dot{f}_1/2$ **necessarily**)
- The **maximum** is obtained for $\delta K_\varphi = \pi/2$. Expanding around this point, we write $\delta K_\varphi = \pi/2 + \delta k_E$ with small δk_E .

The **curvature-dependent** part of the Hamiltonian constraint

$$\frac{E^\varphi}{\sqrt{E^x}} f_1(K_\varphi) + 2\sqrt{E^x} f_2(K_\varphi) K_x = - \left(\frac{E^\varphi}{\sqrt{E^x}} k_E^2 + 2\sqrt{E^x} k_E K_x \right) + \frac{E^\varphi}{\delta^2 \sqrt{E^x}} + \dots$$



'Fuzzy' Euclidean regime - II

[M. Bojowald & S.B., *forthcoming*]

- The **sign** of the **curvature term has changed**, as it should for a Euclidean gravity model.
- Moreover, there is a **new contribution** depending only on the triad variables, which can be combined with the original dilaton potential $V(E^x)$ from the Lorentzian phase if it is changed by adding $\delta V(E^x) = -2/(\delta^2 \sqrt{E^x})$.
- For $\delta = 1$, this shift happens to be identical with the dilaton potential of spherically symmetric gravity. Holonomy-modified model, which is spherically symmetric gravity in the Lorentzian phase, has **twice the spherically symmetric potential** in the Euclidean phase. It is therefore **different from spherically symmetric Euclidean gravity**.
- Holonomy-modified CGHS model in the Lorentzian phase is equal to spherically symmetric gravity with a cosmological constant in the Euclidean phase.
- The new term makes a **huge contribution** to the potential as it is inversely proportional to the 'area-gap'.
- Similar results available for LQC models where the **new perturbative contributions** are consistent with a cosmological-constant term $\Lambda \sim 1/\ell_{Pl}^2$ added to the full Hamiltonian density.





HDA as a Lie algebroid

→ Lie algebroid: $(A, [\cdot, \cdot]_A, \rho)$ with $\rho : \Gamma(A) \rightarrow \Gamma(TB)$, such that ρ satisfies a homomorphism of Lie algebras and a Leibnitz identity.

→ Hypersurface deformation brackets form a Lie algebroid → Phase space (q_{ab}, K^{ab}) forms base manifold → Lagrangian multipliers (N, N^a) forms $(4 \times \infty)$ -dimensional fibers. [C. Blohmann, M.Fernandez & A. Weinstein, 2010]

→ Deriving HDA: “g-Gaussian” vector fields $\Rightarrow n^\mu \mathcal{L}_\nu g_{\mu\nu} = 0$, preserving Gaussian form of the metric $ds^2 = -\epsilon dt^2 + q_{ab} dx^a dx^b$.

→ Lie algebroid morphisms can **change** the deformation function $\beta(q_{ab}, K^{ab})$: [M. Bojowald, S.B., U. Büyükçam & F. D’Ambrosio, 2016]

- $q_{ab} \mapsto |\beta|^{-1} q_{ab}$ generated by **base transformations**.
- $N \mapsto \sqrt{|\beta|^{-1}} N$ generated by **fiber maps** (same as a **non-standard normal** for β spatially constant).

→ No algebroid morphisms can remove $\text{sgn}(\beta) \Rightarrow$ No Riemannian structure when β changes sign.



Specific solutions: Black holes

$$\dot{E}^x = 2N\sqrt{E^x} f_2(K_\phi) + N^x (E^x)'$$

$$\dot{E}^\phi = N\sqrt{E^x} K_x \frac{df_2(K_\phi)}{dK_\phi} + \frac{NE^\phi}{2\sqrt{E^x}} \frac{df_1(K_\phi)}{dK_\phi} + (N^x E^\phi)'$$

- EOMs are set of **coupled, non-linear, PDEs** \Rightarrow Important class of solutions easier to derive are **stationary** ones. **LHS** of \dot{E}^x eqn must be **zero**, as must be the **shift vector** \Rightarrow Looking for one of the **zeros of $f_2(K_\phi)$** $\Rightarrow K_\phi = 0, \pi/(2\delta), \dots$
- “Stationary” has to be **generalized** to imply a solution with a Killing vector **transversal** to the hypersurfaces in a $(3 + 1)$ -decomposition.
- Fixed K_ϕ can correspond to a gauge condition on the **entire spacetime region** (outside the Schwarzschild horizon) **OR** a **unique spatial slice** within a homogeneous gauge (inside the horizon).
- More alternating Lorentzian and Euclidean solutions \Rightarrow concentrate on the first Euclidean regime.



Outside the horizon: Classical Schwarzschild

→ Gauge conditions: $E^x = x^2$, $K_\phi = 0$

→ Solutions are given by

$$\begin{aligned} E^\phi &= \frac{x}{\sqrt{1 - \frac{2M}{x}}} \\ N &= \sqrt{1 - \frac{2M}{x}} \\ \beta &= 1 \end{aligned}$$

→ The effective line element is the classical one

$$ds^2 = - \left(1 - \frac{2M}{x}\right) dt^2 + \frac{1}{1 - 2M/x} dx^2 + x^2 d\Omega^2$$



Holonomy modified interior

- $(x \leftrightarrow t)$ exchanges role inside the horizon but metric depends on x .
- $K_\phi = 0$ not available as a gauge choice if metric depends on 'time' coordinate. $E^x = x^2$ is also not a good gauge choice.
- Classically, $E^\phi(t) = t\sqrt{\frac{2M}{t} - 1}$, $K_\phi(t) = \sqrt{\frac{2M}{t} - 1}$, $N(t) = 1/\sqrt{\frac{2M}{t} - 1}$, $E^x(t) = t^2$.

→ Suitable gauge choices: $N = \sqrt{E^x}$, $N^x = 0$, an anisotropic version of conformal time

$$\begin{aligned}
 E^\phi &= M \sin(\eta) & E^\phi &= M \sin\left(\sqrt{1 + \delta^2} \eta\right) \\
 E^x &= 4M^2 \cos(\eta/2) & E^x &= \left(\frac{M}{1 + \delta^2}\right)^2 \left[1 + 2\delta^2 + \cos\left(\sqrt{1 + \delta^2} \eta\right)\right]^2 \\
 K_\phi &= -\tan(\eta/2) & K_\phi &= \frac{1}{\delta} \arctan\left[-\frac{\delta}{\sqrt{1 + \delta^2}} \tan\left(\sqrt{1 + \delta^2} \eta/2\right)\right] \\
 K_x &= \frac{1}{8M} \sec^4(\eta/2) & K_x &= [\dots]
 \end{aligned}$$



Effective line element in deep quantum regime

→ Can calculate β inside the horizon

$$\beta(\eta) = \frac{1 - \left(\frac{\delta^2}{1+\delta^2}\right) \tan(-\sqrt{1+\delta^2}\eta/2)}{1 + \left(\frac{\delta^2}{1+\delta^2}\right) \tan(-\sqrt{1+\delta^2}\eta/2)}$$

$$\beta = -1 \Rightarrow \eta = -\frac{\pi}{\sqrt{1+\delta^2}} (\delta K_\phi = \pi/2)$$

→ “After” time $\frac{\delta}{\sqrt{1+\delta^2}} \tan(\dots) = 1$, η can be treated as a fourth spatial coordinate but not as time.

→ K_ϕ increases in the homogeneous interior and ultimately reaches $\beta = 0$. Formally, *assume* that K_ϕ keeps increasing in the direction normal to Σ . At this point, we switch to a 2-dimensional boundary value problem \Rightarrow Not clear what appropriate b.c. should be!

→ Gauge conditions: $E^x = x^2$, $\delta K_\phi = \pi/2$

→ The effective line element is the classical one

$$ds^2 = \left(1 - \frac{2M}{x}\right) d\tau^2 + \frac{\bar{\delta}}{1 - 2M/x} dx^2 + x^2 d\Omega^2$$

with $\bar{\delta} = (1 + 1/\delta^2)^{-1}$.





Summary

→ Conclusions:

- Introducing **QG corrections** gives rise to **new space-time structures**, going beyond corrections to the classical dynamics.
- **Only consistent** effective line element is **Euclidean** when holonomy effects are strong \Rightarrow only **boundary-value problems** well posed in this region.
- Important to understand the **role of quantum symmetries** \Rightarrow Deformation of general covariance. NCG: Deformations **generic** to different approaches? [M. Bojowald, S.B., U. Büyükçam & M. Ronco, 2017]

→ Looking ahead:

- Similar line-elements needs to be derived for early-universe cosmology \Rightarrow **Cannot** interpret ‘signature-change’ as *instabilities* of matter or metric perturbations on an otherwise Lorentzian manifold.
- Implications for the initial state? Smooth ‘no-boundary’ state compatible with dynamical signature-change? [M. Bojowald & S.B., *forthcoming*]