Title: Schroedinger's Equation for Conformal Symmetry

Date: May 11, 2018 01:00 PM

URL: http://pirsa.org/18050005

Abstract: <p>Polyakovâ€™s bootstrap programme aims at solving conformal field theories using
 unitarity and conformal symmetry. Its implementation in two dimensions has been
 highly successful and numerical studies, in particular of the 3-dimensional Ising
 model, have clearly demonstrated the potential for higher dimensional theories.
 Analytical results in higher dimensions, however, require significant insight
 into the conformal group and its representations. Surprisingly little is actually
 known about this important group theory challenge. I will explain a remarkable
 and unexpected connection with a class of Schroedinger equations that was uncovered
 in recent joint work with M. Isachenkov. The study of the relevant quantum mechanics
 systems has created an entire branch of modern mathematics whose results can now be
 put to use in the conformal bootstrap program. </p>

Schrödinger's Equation for Conformal Symmetry

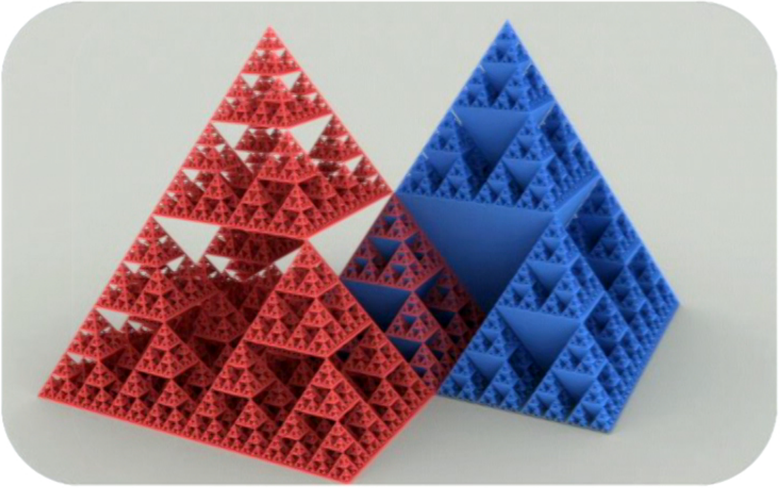
Perimeter, May 11, 2018

Volker Schomerus

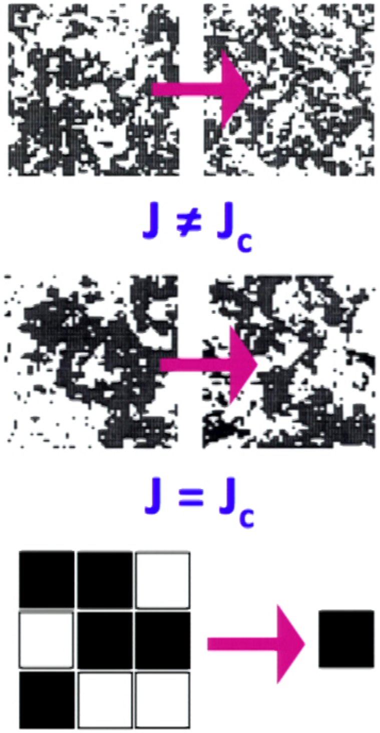
Based on work with M. Isachenkov, E. Sobko, Y. Linke, P. Liendo, I. Buric ....

50 years of

Conformal Field Theory



H 1: Block Spins and Fixed Points



Work of Kadanoff and Wilson highlighted special role of <u>scale invariant</u> statistical & quantum systems.

Examples:

- **Critical Ising model**
- **Conformal window of QCD**

  :

- **N=4 super Yang-Mills theory**

H.2 Conformal Symmetry

Many homogenous, isotropic, scale invariant systems possess symmetry

G = SO(1,1+d)

Rotations R = SO(d) **Translations T**

Dilations D = SO(1,1) **Special conformal transformations N**

States and fields (operators) transform in representations of SO(1,d+1)

(Δ, λ)

weights SO(1,1) & SO(d)

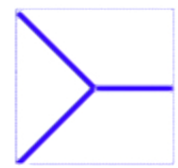
Primary fields $\Phi_{\Delta,\lambda}(x)$

Scalar fields $\lambda = 0$

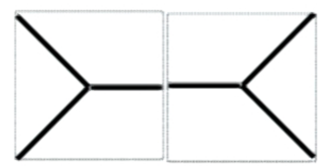
[Mack,Salam 69]

H.3 Correlation Functions and Blocks

2-point functions $\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{\delta(\Phi_1,\Phi_2)}{x_{12}^{\Delta_1+\Delta_2}}$ $\quad x_{12} = |x_1 - x_2|$

3-point functions $\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = \gamma_{123}$  **3J symbol**

Scalar fields: $= \frac{\gamma_{123}}{x_{12}^{\Delta_{12}} x_{13}^{\Delta_{13}} x_{23}^{\Delta_{23}}}$ $\quad \Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$

4-point functions $\langle \prod_{i=1}^{4} \Phi_i(x_i)\rangle = \sum_{\Phi} \gamma_{12\Phi}\gamma_{34\Phi}$  **(u_1,u_2)**

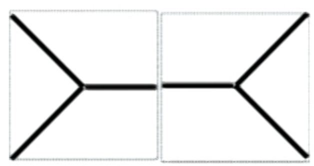
Conformal block

Conformal block expansion is ``Fourier expansion'' of correlation fcts

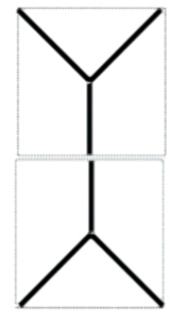
H.4 The Bootstrap Programme

... exploits compatibility condition (associativity)

$$\langle \prod_{i=1}^{4} \Phi_i(x_i) \rangle = \sum_{\Phi} \gamma_{12\Phi}\gamma_{34\Phi}$$



$$= \sum_{\Psi} \gamma_{23\Psi}\gamma_{14\Psi}$$



or equivalently

$$\gamma_{12\Phi}\gamma_{34\Phi} = \sum_{\Psi} \gamma_{23\Psi}\gamma_{14\Psi}$$

6J symbol

[Polyakov 73]

H.5 BPZ and Beyond

Bootstrap carried out for d = 2 [Belavin,Polyakov,Zamolodchikov 83]

Conformal block → Virasoro, Kac-Moody … block

→ 6J symbol of some q-deformed universal enveloping algebra

Many solutions of bootstrap eqs. (incl. boundaries, defects…) 83 ……….

How about d > 2 ? Numerical bootstrap [Rychkov et al. 2012…]

Outline of Talk

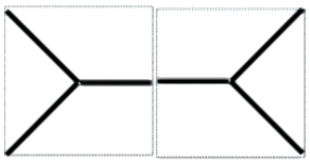
I. **Conformal blocks – three characterizations**

II. **Harmonic analysis approach to conformal blocks**

III. **Outlook: Extensions and solutions**

I.1 ...as Integrals (Shadow Formalism)

[Ferrara,Gatto,Parisi,Grillo]



= $G_{\Delta,l}(z,\bar{z}) \sim$

$$\mu = \frac{x_{10}}{x_{10}^2} - \frac{x_{20}}{x_{20}^2} \qquad \tilde{\mu} = \frac{x_{40}}{x_{40}^2} - \frac{x_{30}}{x_{30}^2}$$

$$\sim \int d^d x_0 \, x_{10}^{l+a-\Delta} x_{20}^{l-a-\Delta} x_{30}^{l-b+\Delta-d} x_{40}^{l+b+\Delta-d} \, (|\mu||\tilde{\mu}|)^l \, Y_l^d \left(\frac{\mu \cdot \tilde{\mu}}{|\mu||\tilde{\mu}|} \right)$$

Coordinates:

Zonal spherical functions

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

These are just the simplest blocks

I.2 ..as Solutions of Casimir Equation

$G = G_{\Delta,l}$ **can be characterized by** [Dolan,Osborn]

$$\mathrm{Cas}_d^2 G(z,\bar{z}) = \frac{1}{2} C_{\Delta,l} G(z,\bar{z})$$

$$C_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2)$$

where

$$\mathrm{Cas}_d^2 := D^2 + \overline{D}^2 + \epsilon \left[\frac{z\overline{z}}{\overline{z} - z} \left(\overline{\partial} - \partial \right) + \left(z^2 \partial - \overline{z}^2 \overline{\partial} \right) \right]$$

$$\epsilon = d - 2 \qquad 2a = \Delta_2 - \Delta_1 \qquad 2b = \Delta_3 - \Delta_4$$

$$D^2 = z^2(1 - z)\partial^2 - (a + b + 1)z^2\partial \; - abz$$

& boundary condition at $z, \bar{z}$ **= 0 ...**

I.3 ..as Schrödinger Wave Functions I: d=2 [Isachenkov,VS]

e.g. chiral d=2: $D^2 G(z) = h(h-1)G(z)$

$$\psi(x) := \frac{(z-1)^{\frac{a+b}{2}+\frac{1}{4}}}{\sqrt{z}}\, G(z) \qquad z = -\sinh^{-2}\frac{x}{2}$$

$G = G_h$ satisfies chiral d=2 Casimir equation $\leftrightarrow$ $\psi = \psi_e$ is eigenfunction of 1D Schrödinger equation with potential

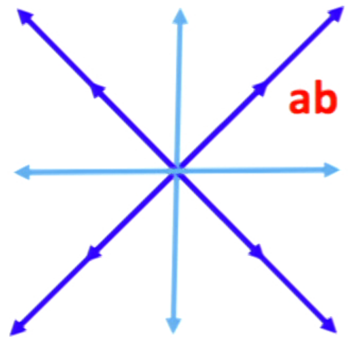
$$V_{\rm PT}^{(a,b)}(x) = \frac{(a+b)^2 - \frac{1}{4}}{\sinh^2 x} - \frac{ab}{\sinh^2(x/2)}$$ [Poeschl,Teller]

$$e = 2mE/\hbar^2 = -(2h-1)^2/4$$

Aside: Calogero-Sutherland Potential

Integrable interacting multiparticle Hamiltonians ↔ root systems

$BC_1 \times BC_1 \rightarrow BC_2$



$$V_{\mathrm{CS}}^{(a,b,\epsilon)}(x_1,x_2) = V_{\mathrm{PT}}^{(a,b)}(x_1) + V_{\mathrm{PT}}^{(a,b)}(x_2) + \frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x_1-x_2}{2}} + \frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x_1+x_2}{2}}$$

Calogero-(Moser)-Sutherland

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[Poeschl,Teller]

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I.4 … as Calogero-Sutherland Wave Functions

$G = G_{\Delta,l}$ **satisfies d-dimensional Casimir equation** → $\psi = \psi_e$ **is an eigenfunction of the** BC_2 **Calogero-Sutherland with**

$$\psi(x_1, x_2) := \prod_i \frac{(z_i - 1)^{\frac{a+b}{2}+\frac{1}{4}}}{z_i^{\frac{1}{2}+\frac{\epsilon}{2}}} |z_1 - z_2|^{\frac{\epsilon}{2}} G(z_1, z_2)$$

$$z_1 = z \qquad z_2 = \bar{z}$$

$$z_i = -\sinh^{-2}\frac{x_i}{2}$$

[Isachenkov,VS]

$$e = -d(d-2)/4 - (C_{\Delta,l} + 1)/2$$

II.1 Representations of Conformal Group

Irreps $\pi_{\Delta,\lambda}$ induced from finite dim irrep of K = SO(1,1) x SO(d) on $V_{\Delta,\lambda}$

$$\Gamma^{\Delta,\lambda}_{G/H} = \{ f : G \longrightarrow V_{\Delta,\lambda} \,|\, f(gnk) = \pi(k^{-1}) f(g) \}$$

V-valued functions on right coset space $G/H \cong \mathbb{R}^d$ $\quad H = K \ltimes N$

The tensor product $\pi_{\Delta_1,\lambda_1} \otimes \pi_{\Delta_2,\lambda_2}$ can be realized on [Dobrev et al]

$$\Gamma^{a;\lambda_1,\lambda_2}_{G/K} = \{ f : G \longrightarrow V_{\Delta_1,\lambda_1} \otimes \tilde{V}_{\Delta_2,\lambda_2} \,|\, f(gk) = \pi(k^{-1}) f(g) \}$$

$V_1 \otimes V_2$ valued function on right coset space G/K **2d – dimensional**

II.2 Conformal Blocks revisited

Conformal blocks: G-invariants in 4-fold tensor product $V = \bigotimes_{i=1}^{4} V_{\Delta_i,\lambda_i}$

$$\left(\bigotimes \Gamma_{G/H}^{\Delta_i,\lambda_i}\right)^G = \{\, f : G \longrightarrow V \mid f(k_l g k_r) = \pi(k_l \otimes k_r^{-1}) f(g) \,\}$$

Sections of vector bundle on 2-sided coset space K\G/K w. fiber $V^{SO(d-2)}$

2 – dimensional [cross ratios]

Scalar blocks for $\lambda_i = 0$, otherwise spinning blocks

II.3 The Casimir Equation

Eigenvalue equation for Casimir elements of G on space Γ of blocks:

$$m^{1/2}(u)\mathcal{D}_2 m^{-1/2}(u) = -\frac{1}{2}\frac{d^2}{du_1^2} - \frac{1}{2}\frac{d^2}{du_2^2} + V(u_1, u_2)$$

m is volume of K x K orbit through u

Scalar blocks:

[M. Isachenkov, VS, E. Sobko]

$$V(u_1,u_2) = \sum_{i=1}^{2}\left(\frac{(a+b)^2 - 1/4}{2\sinh^2 u_i} - \frac{ab}{2\sinh^2 \frac{u_i}{2}}\right)$$

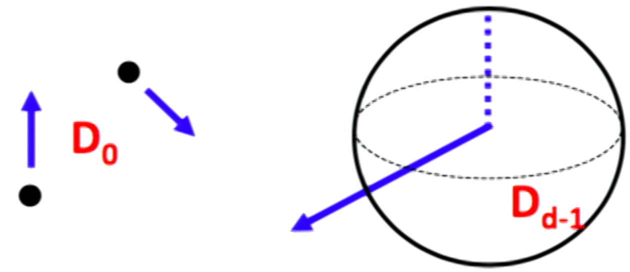
$$+\frac{(d-2)(d-4)}{16\sinh^2\frac{u_1-u_2}{2}} + \frac{(d-2)(d-4)}{16\sinh^2\frac{u_1+u_2}{2}} + \frac{d^2-2d+2}{8}$$

III.1 Extensions: e.g. Defect Blocks

p-dim conformal defect preserves $G_p = SO(1, p+1) \times SO(d-p)$

possess dim G/G_p **= (p+2)(d-p) parameters**

e.g. D_0**:2d parameters;** D_{d-1}**:d+1 parameters**



Defect 1-point function: $\langle D_p(\mathcal{X}) \Phi(x) \rangle = \frac{\alpha_\Phi^{(p)}}{|x_\perp|^{\Delta_\Phi}}$

Defect block expansion for 2-point function of defects D_p **and** D_q

$$\langle D_p(\mathcal{X}_p)\, D_q(\mathcal{X}_q) \rangle = \sum_\Phi \alpha_\Phi^{(p)} \alpha_\Phi^{(q)} G_\Phi^{pq}(u)$$ **Defect blocks**

III.4 Outlook: Solution Theory

[Heckman,Opdam] ...
...[Cherednik] ...

Calogero-Sutherland Hamiltonian is superintegrable

and surprisingly similar to free Hamiltonian

Dunkl operators generalize role of derivatives for free particle (FP)

$H^{FP} = \sum \partial_i^2$ $\qquad [\partial_i, \partial_j] = 0$ → Knizhnik-Zamolodchikov eqs for conformal blocks in any d

Hyperbolic Calogero-Sutherland ↔ rational Ruijsenaars-Schneider

$\Psi_p(u) = e^{ipu}$ $\qquad \boldsymbol{u} \leftrightarrow \boldsymbol{p}$ $\qquad$ 2nd order difference equation

Deformation to self-dual trigonometric Ruijsenaars-Schneider

Conclusions

The Calogero-Sutherland approach to conformal blocks is

powerful — by embedding into modern theory of multivariate hypergeometric functions [Isachenkov,VS]

degenerate Koornwinder-Macdonald functions

flexible — Can be applied to spinning blocks (↔ matrix CS)
Superblocks, defects etc. [VS,Sobko]

[Isachenkov,Liendo,Linke,VS]

It finally provides control over the kinematical skeleton of Conformal Field Theory & raises hopes for analytic results …