Title: Progress in null canonical gravity in terms of free initial data

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Abstract: <p>The talk will focus on issues connected with the quantization of initial data for vacuum general relativity on null hypersurfaces. Complete and constraint free initial data for GR can be given on pairs of intersecting null hypersurfaces. These offer a route to a canonical quantization of GR which avoids having to deal with constraints. However the Poisson algebra of these data is rather intricate and it is not obvious how one might quantize it. This algebra is however almost the same in the cylindrically symmetric case as in the full theory, which suggests that the cylindrically symmetric theory ought to offer valuable insights. Here I present recent joint work with Javier Peraza and Miguel Paternain relating the Poisson algebra in the cylindrically symmetric case to well studied algebraic structures.</p>

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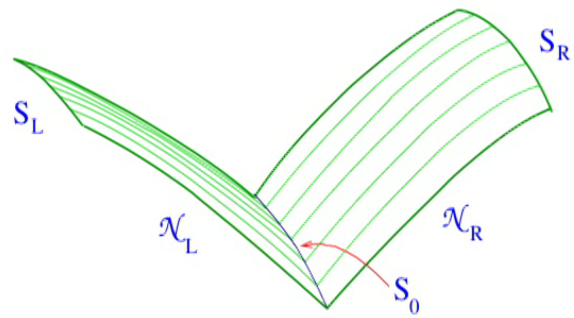
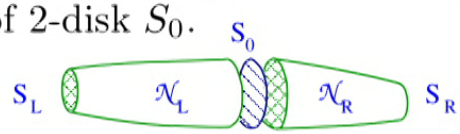
Double null sheets as initial data hypersurfaces

- A double null sheet $\mathcal{N}$ is a pair of intersecting null hypersurfaces (or "lightfronts") - like an open book in spacetime.



- $\mathcal{N}_R$, $\mathcal{N}_L$ are 3-surfaces swept out by null geodesics emerging normally from the two sides of 2-disk S_0.



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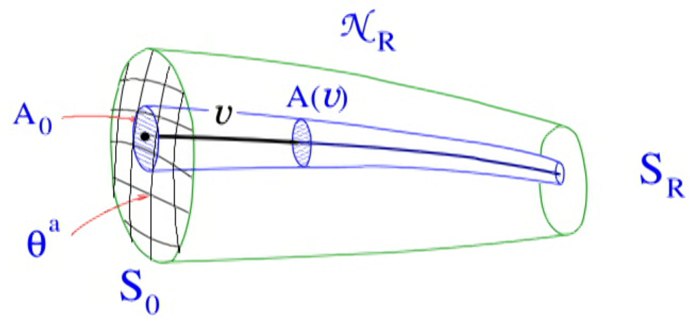
The free initial data

Coordinates adapted to $\mathcal{N}$



- θ^1, θ^2 coordinates on S_0. Held constant on generators.
- v is a parameter along each generator defined so that the cross sectional area of an infinitesimal bundle of neighboring generators is

$$A(v) = A_0 v^2$$

where A_0 is the cross sectional area at S_0. ($\mathcal{N}$ truncated so A monotonic along generators.)

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"Bulk" data on all of $\mathcal{N}$. "Surface" data on S_0.

- Bulk data = conformal 2-metric $e_{ab}(\theta^1, \theta^2, v)$
  - Induced metric on $\mathcal{N}$ degenerate because $\mathcal{N}$ is null, so
    $ds^2 = h_{ab} d\theta^a d\theta^b$ - no dv terms
  - Definition:
    $e_{ab} = h_{ab}/\rho$ with $\rho = \sqrt{\det h}$ - makes $\det e = 1$
  - Parametrize e_{ab} by a single complex number valued field μ

$$ds^2 = h_{ab} d\theta^a d\theta^b = \frac{\rho}{1 - \mu\bar{\mu}}(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz)$$

  with $z = \theta^1 + i\theta^2$.

- Surface data on S_0:
  - area density on S_0, ρ_0,
  - $\lambda = -\ln|n_L \cdot n_R|$,
  - and the twist, $\tau_a = \frac{n_L \cdot \nabla_a n_R - n_R \cdot \nabla_a n_L}{n_L \cdot n_R}$.

  $n_A = \partial_{v_A}$ is the tangent to the generators of $\mathcal{N}_A$.

The Poisson brackets for free data on $\mathcal{N}$ for classical vacuum GR

Brackets not shown vanish.

$$\begin{aligned}\{\mu(\mathbf{1}),\bar\mu(\mathbf{2})\} &= 4\pi G\,\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\,H(\mathbf{1},\mathbf{2})\Big[\frac{1-\mu\bar\mu}{v_A}\Big]_{\mathbf{1}}\\ &\quad\times\Big[\frac{1-\mu\bar\mu}{v_A}\Big]_{\mathbf{2}}\,e^{\int_{\mathbf{1}}^{\mathbf{2}}(\bar\mu d\mu-\mu d\bar\mu)/(1-\mu\bar\mu)}\end{aligned}$$

for $\mathbf{1}$, $\mathbf{2}$ in the same branch, $\mathcal{N}_A$.

$$\begin{aligned}\{\rho_0(\theta_1),\lambda(\theta_2)\} &= 8\pi G\delta^2(\theta_2-\theta_1)\\ \{\rho_0(\theta),\tau[f]\} &= -8\pi G\pounds_f\rho_0(\theta)\\ \{\lambda(\theta),\tau[f]\} &= -8\pi G\Big[\pounds_f\lambda+\frac{\pounds_f\mu}{(1-\mu\bar\mu)^2}(\partial_{v_R}\bar\mu-\partial_{v_L}\bar\mu)\Big]_\theta\end{aligned}$$

$$\begin{aligned}\{\tau[f_1],\tau[f_2]\} = -16\pi G\Big[\tau[[f_1,f_2]]-\frac{1}{2}\int_{S_0}\pounds_{[f_1,f_2]}\epsilon\\ +\int_{S_0}\big[\frac{\pounds_{f_1}\mu}{(1-\mu\bar\mu)^2}\{\epsilon\pounds_{f_2}\bar\mu-\frac{1}{2}\pounds_{f_2}\epsilon(\partial_{v_R}\bar\mu+\partial_{v_L}\bar\mu)\}-(1\leftrightarrow 2)\big]\Big].\end{aligned}$$

For $\mathbf{1}$ in $\mathcal{N}_R - S_0$

$$\{\mu(\mathbf{1}),\lambda(\theta_2)\} = 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)[v_R\partial_{v_R}\mu]_{\mathbf{1}}$$

$$\{\mu(\mathbf{1}),\tau[f]\} = -16\pi G\Big[\pounds_f\mu-\frac{1}{4}\frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\mu\Big]_{\mathbf{1}}.$$

For $\mathbf{1}$ in S_0

$$\begin{aligned}\{\mu(\mathbf{1}),\lambda(\mathbf{2})\} &= 0\\ \{\mu(\mathbf{1}),\tau[f]\} &= -8\pi G[\pounds_f\mu]_{\mathbf{1}}.\end{aligned}$$

For $\mathbf{1}$ in $\mathcal{N}_L - S_0$

$$\begin{aligned}\{\mu(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)[v_L\partial_{v_L}\mu]_{\mathbf{1}}\\ \{\mu(\mathbf{1}),\tau[f]\} &= -4\pi G\Big[\frac{\pounds_f\rho_0}{\rho_0}v_L\partial_{v_L}\mu\Big]_{\mathbf{1}}.\end{aligned}$$

For $\mathbf{1}\in\mathcal{N}_R$ (including $\mathbf{1}\in S_0$)

$$\begin{aligned}\{\bar\mu(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\Big[(v_R\partial_{v_R}\bar\mu)_{\mathbf{1}}\\ &\quad+\Big(\frac{1}{v_R}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar\mu)/(1-\mu\bar\mu)}(\partial_{v_L}\bar\mu)_{\mathbf{1}_0}\Big]\\ \{\bar\mu(\mathbf{1}),\tau[f]\} &= -8\pi G\Big[\Big(2\pounds_f\bar\mu-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\bar\mu\Big)_{\mathbf{1}}\\ &\quad-\Big(\pounds_f\bar\mu-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}\partial_{v_L}\bar\mu\Big)_{\mathbf{1}_0}\Big(\frac{1}{v_R}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar\mu)/(1-\mu\bar\mu)}\Big]\end{aligned}$$

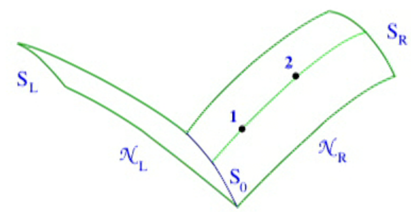
where $\mathbf{1}_0\in S_0$ is the origin of the generator through $\mathbf{1}$.
For $\mathbf{1}\in\mathcal{N}_L$

$$\begin{aligned}\{\bar\mu(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\Big[(v_L\partial_{v_L}\bar\mu)_{\mathbf{1}}\\ &\quad+\Big(\frac{1}{v_L}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar\mu)/(1-\mu\bar\mu)}(\partial_{v_R}\bar\mu)_{\mathbf{1}_0}\Big]\\ \{\bar\mu(\mathbf{1}),\tau[f]\} &= -8\pi G\Big[\Big(\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}v_L\partial_{v_L}\bar\mu\Big)_{\mathbf{1}}\\ &\quad+\Big(\pounds_f\bar\mu-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}\partial_{v_R}\bar\mu\Big)_{\mathbf{1}_0}\Big(\frac{1}{v_L}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar\mu)/(1-\mu\bar\mu)}\Big].\end{aligned}$$

M.R. Phys. Rev. Lett. 101:211101, 2008

One generator Poisson algebra

Only data on same generator have non-zero bracket. Consistent with causality: points on distinct generators spacelike separated.



- One generator algebra of μ, $\bar{\mu}$, λ, ρ_0 on $\mathcal{N}_L - S_0$.

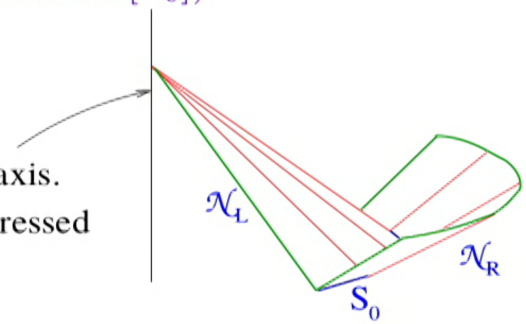
$$
\begin{aligned}
\{\mu(\mathbf{1}), \bar{\mu}(\mathbf{2})\} &= 4\pi G' \frac{1}{\sqrt{\rho_1 \rho_2}} H(\mathbf{1},\mathbf{2})[1-\mu\bar{\mu}]_{\mathbf{1}}[1-\mu\bar{\mu}]_{\mathbf{2}}\, e^{\int_{\mathbf{1}}^{\mathbf{2}} \frac{\bar{\mu}d\mu - \mu d\bar{\mu}}{1-\mu\bar{\mu}}} \\
\{\lambda, \rho_0\} &= -8\pi G' \\
\{\lambda, \mu(\mathbf{2})\} &= -4\pi G' \frac{1}{\rho_0}[v_L \partial_{v_L} \mu]_{\mathbf{2}} \\
\{\lambda, \bar{\mu}(\mathbf{2})\} &= -4\pi G' \left(\frac{1}{\rho_0}[v_L \partial_{v_L} \bar{\mu}]_{\mathbf{2}} + \frac{1}{v_L(\mathbf{2})}\, e^{2\int_{S_0}^{\mathbf{2}} \frac{\mu d\bar{\mu}}{1-\mu\bar{\mu}}} [\partial_{v_R} \bar{\mu}]_{S_0} \right)
\end{aligned}
$$

- Brackets with $\delta^2(\theta_2 - \theta_1)$ removed. μ, $\bar{\mu}$ functions on single line. λ, ρ_0 single numbers.
- All other brackets of these data vanish.
- $H(\mathbf{1},\mathbf{2}) = 1$ if $\mathbf{2}$ follows $\mathbf{1}$ along the generator, 0 otherwise.

Cylindrically symmetric vacuum gravity

- One generator algebra is Poisson algebra of cylindrically symmetric GR, on $\mathcal{N}$ swept out by radial light rays from axis, provided $G' = G/(\theta \text{ coordinate area}[S_0])$.

Worldsheet of symmetry axis.
Dimension along axis supressed

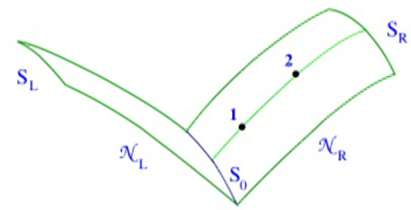


  - Obtained as brackets of averages of data over symmetry orbits at symmetric solutions, or from symmetry reduced action.

- Cylindrically symmetric GR with $\tau = 0$ is an integrable system. Quantized on algebraic level in asymptotically flat case [Korotkin and Samtleben 1998].

One generator Poisson algebra

Only data on same generator have non-zero bracket. Consistent with causality: points on distinct generators spacelike separated.

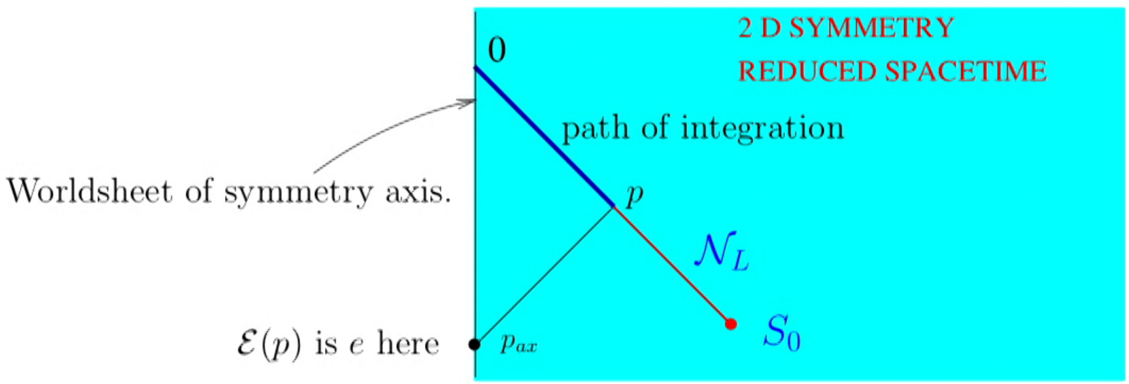


- One generator algebra of μ, $\bar{\mu}$, λ, ρ_0 on $\mathcal{N}_L - S_0$.

$$
\begin{aligned}
\{\mu(\mathbf{1}), \bar{\mu}(\mathbf{2})\} &= 4\pi G' \frac{1}{\sqrt{\rho_1 \rho_2}} H(\mathbf{1},\mathbf{2})[1-\mu\bar{\mu}]_{\mathbf{1}}[1-\mu\bar{\mu}]_{\mathbf{2}}\, e^{\int_{\mathbf{1}}^{\mathbf{2}} \frac{\bar{\mu}d\mu - \mu d\bar{\mu}}{1-\mu\bar{\mu}}} \\
\{\lambda, \rho_0\} &= -8\pi G' \\
\{\lambda, \mu(\mathbf{2})\} &= -4\pi G' \frac{1}{\rho_0}[v_L \partial_{v_L} \mu]_{\mathbf{2}} \\
\{\lambda, \bar{\mu}(\mathbf{2})\} &= -4\pi G' \left(\frac{1}{\rho_0}[v_L \partial_{v_L} \bar{\mu}]_{\mathbf{2}} + \frac{1}{v_L(\mathbf{2})}\, e^{2\int_{S_0}^{\mathbf{2}} \frac{\mu d\bar{\mu}}{1-\mu\bar{\mu}}} [\partial_{v_R} \bar{\mu}]_{S_0} \right)
\end{aligned}
$$

- Brackets with $\delta^2(\theta_2 - \theta_1)$ removed. μ, $\bar{\mu}$ functions on single line. λ, ρ_0 single numbers.
- All other brackets of these data vanish.
- $H(\mathbf{1},\mathbf{2}) = 1$ if $\mathbf{2}$ follows $\mathbf{1}$ along the generator, 0 otherwise.

- We treat only the branch $\mathcal{N}_L$ which touches symmetry axis.
- Define new variable: deformed conformal metric $\mathcal{E}$. $\mathcal{E}(p)$ is non-linear integral transform of e on interval in $\mathcal{N}_L$ from axis to p.
- Captures all information in e that propagates off $\mathcal{N}_L$.
- On solutions $\mathcal{E}_{ab}(p) = e_{ab}(p_{ax})$.



Poisson bracket of new variable

- A calculation yields [Fuchs and Reisenberger, 2017]

$$\{\mathcal{E}_{ab}(w_1), \mathcal{E}_{cd}(w_2)\} =$$
$$p.v.(\frac{32\pi G'}{w_1 - w_2})\,\mathrm{Sym}_{(ab),(cd)}\left(\mathcal{E}_{ad}(w_1)\mathcal{E}_{cb}(w_2) - \frac{1}{2}\mathcal{E}_{ab}(w_1)\mathcal{E}_{cd}(w_2)\right)$$

  - $p.v.(1/x)$ is Cauchy principal value distribution of $1/x$.
  - w is the "spectral parameter". On $\mathcal{N}_L$ can be identified with twice area density, 2ρ, or with $-2\tilde{\rho}$, where $\tilde{\rho}$ is time coordinate so that $ds^2 \propto -d\tilde{\rho}^2 + d\rho^2$.

- Equivalently

$$\{\overset{1}{\mathcal{E}}, \overset{2}{\mathcal{E}}\} = p.v.(\frac{32\pi G'}{w_1 - w_2})\left(\Omega\overset{1}{\mathcal{E}}\overset{2}{\mathcal{E}} + \overset{1}{\mathcal{E}}{}^t\Omega\overset{2}{\mathcal{E}} + \overset{2}{\mathcal{E}}\Omega^t\overset{1}{\mathcal{E}} + \overset{1}{\mathcal{E}}\overset{2}{\mathcal{E}}{}^t\Omega^t\right)$$

where $\Omega = \overset{12}{\Omega} = \frac{1}{8}(\overset{1}{\sigma}_x\overset{2}{\sigma}_x + \overset{1}{\sigma}_y\overset{2}{\sigma}_y + \overset{1}{\sigma}_z\overset{2}{\sigma}_z)$ is the Casimir tensor (inverse Killing form).

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Algebraic structures: Geroch group

Geroch group: $\mathcal{E}(w) \to g(w)\mathcal{E}(w)g^t(w)$

- $g(w) \in G \equiv \mathcal{L}SL(2,\mathbb{R})$ - group of $SL(2,\mathbb{R})$ valued functions of w.
- action transitive in space $\mathcal{F}$ of $\mathcal{E}$ fields - real, symmetric, positive definite, $\det \mathcal{E} = 1$

Is Poisson bracket of $\mathcal{E}$s invariant under Geroch? - Is Geroch symplectic?
No

.... unless G itself carries non-trivial Poisson bracket.
Then $G \times \mathcal{F}$ is the phase space of a composite system. And

$$\{\overset{1}{g}\overset{1}{\mathcal{E}}\overset{1}{g}^t, \overset{2}{g}\overset{2}{\mathcal{E}}\overset{2}{g}^t\}(\gamma, x) = \{\overset{1}{\mathcal{E}}, \overset{2}{\mathcal{E}}\}(\gamma \cdot x)$$

- $\gamma \in G$, $x \in \mathcal{F}$, and $\gamma \cdot x$ is the point in $\mathcal{F}$ resulting from acting with γ on x.

iff

$$\begin{aligned} \{\overset{1}{g}(w_1), \overset{2}{g}(w_2)\} &= p.v.(\frac{32\pi G'}{w_1 - w_2})[\Omega, \overset{1}{g}(w_1)\overset{2}{g}(w_2)] \\ &= [r^{\pm}, \overset{1}{g}\overset{2}{g}] \end{aligned}$$

- This is called the Sklyanin bracket.
- $r^{\pm} = \frac{32\pi G'\Omega}{w_1 - w_2 \pm i\epsilon}$ - "classical r matrix". Satisfies
  - $\overset{12}{r}{}^{+} = -\overset{21}{r}{}^{-}$
  - $\overset{12}{r}{}^{+} - \overset{12}{r}{}^{-} = -64\pi i G' \delta(w_1 - w_2)\Omega \propto$ Casimir of $\mathcal{L}SL(2, \mathbb{R})$
  - classical Yang-Baxter equation
    $0 = [\overset{12}{r}{}^{\pm}, \overset{13}{r}{}^{\pm}] + [\overset{12}{r}{}^{\pm}, \overset{23}{r}{}^{\pm}] + [\overset{13}{r}{}^{\pm}, \overset{23}{r}{}^{\pm}]$.

Algebraic structures: Classical double

- Since the Geroch group G acts transitively on the space $\mathcal{F}$ of $\mathcal{E}$s, $\mathcal{F}$ ought to be contained in G somehow.
- Most obvious way, $\mathcal{E} = gg^t$, doesn't give useful results, but there is another way: the classical double.
- The Sklyanin bracket defines a natural Lie bracket on L^*, the dual of L, the Lie algebra of G.
- L^* then integrates up to dual group G^*, and Lie bracket on L defines Poisson bracket on G^* - Semenov-Tian-Shansky bracket.
- $G \times G^*$ forms classical double. $\mathcal{F}$ is more or less G^*.
- Note that Killing form identifies L^* with L as vector space, G^* with G as manifold, but different multiplication law.

- Killing form turns $r^{\pm}$ into maps $R^{\pm} : L \to L$ with $R^{+} - R^{-} = -64\pi i G' \,\mathrm{Id}$.
- Defines decomposition $a = a^{+} - a^{-}$ for all $a \in L$. $a^{\pm} = i/(64\pi G')R^{\pm}a$. Defines factorization $g = g_{-}^{-1}g_{+}$ of G. G^{*} multiplication law is $g * h = h_{-}^{-1}g_{-}^{-1}g_{+}h_{+}$. Here decomposition is in positive and negative frecuency of w dependence.
- Can define an object that represents $\mathcal{E}$ in complexification of G^{*}: Let $\mathcal{A} = g_{+}\epsilon g_{-}^{t}$ with $\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
- Require $g_{-} = g_{+}^{*}\epsilon$ ($*$ is componentwise complex conjugate, not adjoint) and $\mathcal{A}$ real, then $\mathcal{A}$ is real, symmetric, positive definite, and of determinant 1. The Semenov-Tian-Shansky bracket on g_{+}, g_{-} implies that $\mathcal{A}$ satisfies same bracket as $\mathcal{E}$.
- Korotkin and Samtleben obtained this structure, in a different way, in 1998

Including λ and ρ_0.

Is there a way to include λ and ρ_0 in this structure?
$\mathcal{L}SL(2,\mathbb{R})$ has a natural extension by two degrees of freedom to the simplest affine Lie group - exponentiation of Kac-Moody algebra A_1^1.
The extension consists of

- a central factor e^{ak}. k central generator.
- a translation operator $e^{b\partial_w}$ acting on w dependence of gs.

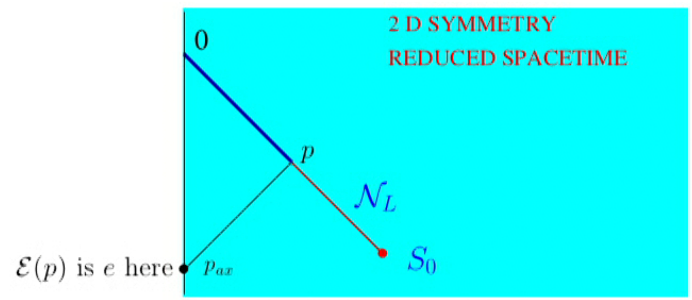
Can a and b be identified with λ and ρ_0?

Poisson algebra of $\mathcal{E}$, λ, and ρ_0.

Very simple for $\mathcal{E}$ on $\mathcal{N}_L - S_0$.

$$\begin{aligned}
\{\overset{1}{\mathcal{E}}, \overset{2}{\mathcal{E}}\} &= p.v.(\frac{32\pi G'}{w_1 - w_2}) \left(\Omega \overset{1}{\mathcal{E}} \overset{2}{\mathcal{E}} + \overset{1}{\mathcal{E}}{}^t \Omega \overset{2}{\mathcal{E}} + \overset{2}{\mathcal{E}} \Omega^t \overset{1}{\mathcal{E}} + \overset{1}{\mathcal{E}} \overset{2}{\mathcal{E}}{}^t \Omega^t \right) \\
\{\mathcal{E}, \rho_0\} &= 0 \\
\{\mathcal{E}, \lambda\} &= 0 \\
\{\lambda, \rho_0\} &= -8\pi G'
\end{aligned}$$

The surprise is $\{\mathcal{E}, \lambda\} = 0$. But $\{\mathcal{E}_{ab}(p), \lambda\} = \{e_{ab}(p_{ax}), \lambda\}$ and p_{ax} is spacelike to S_0, the home of λ.



We have two commuting subalgebras which we know (more or less) how to quantize. So maybe we should stop here.
But let's press on and see if λ and ρ_0 fit naturally into the classical double of the extended loop group.

- This is invariant under the action of the extended Geroch group $\tilde{\mathcal{E}} \to \tilde{g}\tilde{\mathcal{E}}\tilde{g}^t \qquad \tilde{g} = e^{\mu\partial_w} g(w) e^{\xi k}$ if

$$\{\overset{1}{\tilde{g}}, \overset{2}{\tilde{g}}\} = [\tilde{r}^{\pm}, \overset{1}{\tilde{g}}\overset{2}{\tilde{g}}]$$

- With

$$\begin{aligned} \tilde{r}^{+} &= 32\pi G'\left(\frac{\Omega}{w_1 - w_2 + i\epsilon} + \frac{1}{2}\overset{2}{k}\partial_{w_1}\right) \\ \tilde{r}^{-} &= 32\pi G'\left(\frac{\Omega}{w_1 - w_2 - i\epsilon} - \frac{1}{2}\overset{1}{k}\partial_{w_2}\right) \end{aligned}$$

- Quantization of extended classical doubles of this type has been studied (Reshetikhin and Semenov-Tian-Shansky 1989).