Title: The Fingerprints of Black Holes - Information content in the Black Holes Shadow

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Abstract: <p>First I will introduce the concept of the shadow of a black hole and what it means for the shadows of two observers to be degenerate. I will then present
our results showing that no continuous
degenerations exist between the shadows of observers at any point in the exterior region of any Kerr-Newman black hole spacetime of unit mass. Therefore an observer can, by measuring the black holes shadow, in principle determine the angular momentum and the charge of the black hole under observation, as well as his radial distance from the black hole and his angle of elevation above the equatorial plane.</p>

The Fingerprints of Black Holes
-
Shadows and their Degeneracies

Claudio Paganini[†*]

joint work with: Marius Oancea[†], Marc Mars[‡]

[†]Albert Einstein Institut, Potsdam, Germany
[*]School of Mathematical Sciences, Monash University, Victoria, Australia
[‡]Instituto de Física Fundamental y Matemáticas, Universidad de Salamanca, Salamanca, Spain

Perimeter 05.04.2018

Event Horizon Telescope

1. South Pole Telescope **2.** Atacama Large Millimeter/submillimeter Array and Atacama Pathfinder Experiment (Chile) **3.** Large Millimeter Telescope (Mexico) **4.** Submillimeter Telescope (Arizona) **5.** James Clerk Maxwell Telescope and Submillimeter Array (Hawaii) **6.** IRAM 30-meter (Spain)

My Goal

I want to convince you that in principal [1] an observer can, by measuring the black holes shadow,

- determine the angular momentum,
- the charge of the black hole under observation,
- the observer's radial position,
- the angle of elevation above the equatorial plane.
- Furthermore, his/her relative velocity compared to a standard observer can also be measured.

Without any further measurement!

[1] The theoretical maximum of information the equations allow to be extracted.

Outline

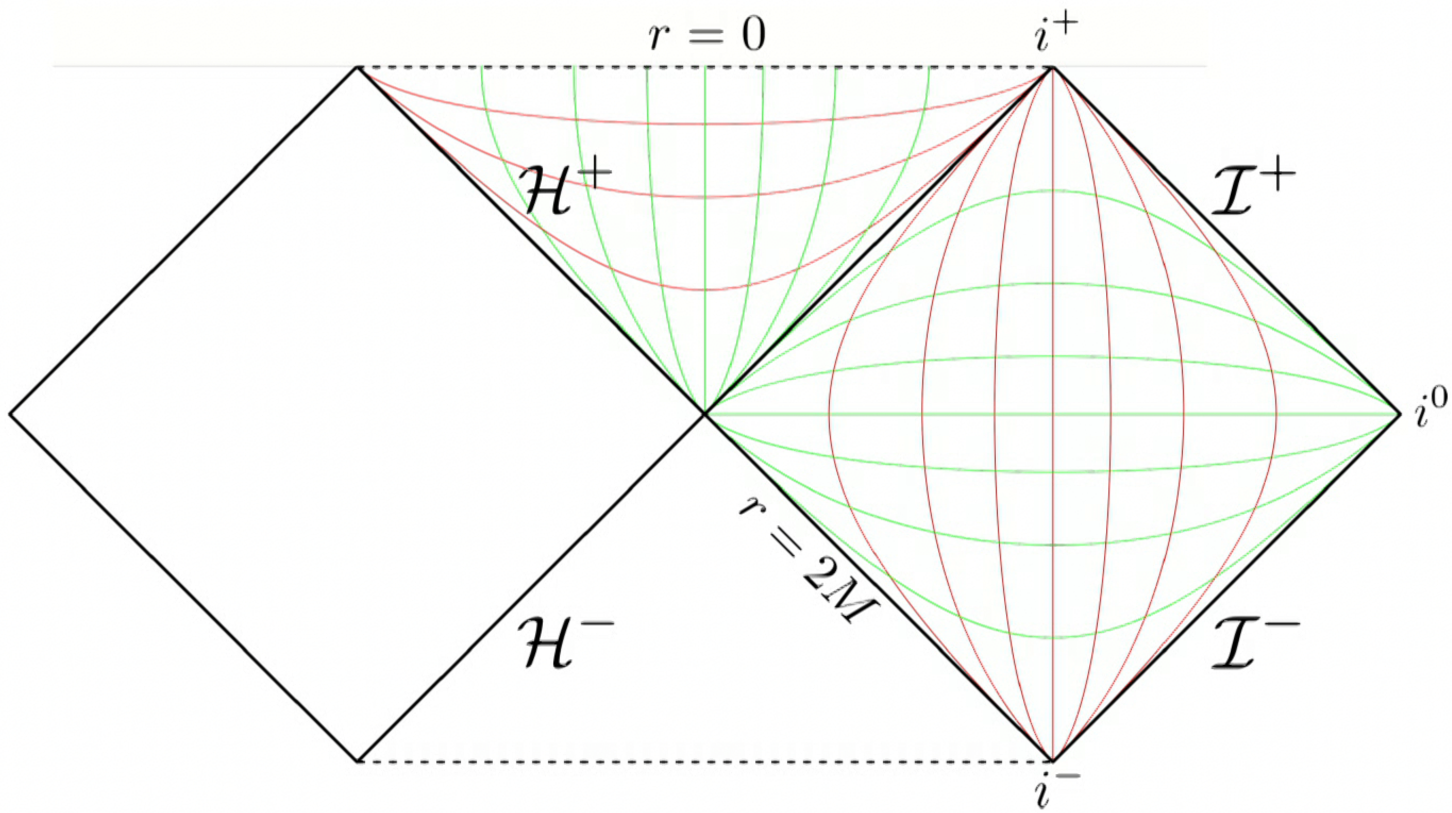
Schwarzschild Metric

In spherical coordinates (t, r, θ, ϕ) the metric is given by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \qquad (1)$$

- Spherically symmetric
- asymptotically flat
- static
- event horizon at $r = 2M$

Conformal Compactification



What is a Black Hole?

Definition (Black hole)

We define the black hole region by $\mathscr{M} \setminus J^{-}(\mathscr{I}^{+})$.

Definition (Domain of outer communication)

We define the DOC by $J^{-}(\mathscr{I}^{+}) \cap J^{+}(\mathscr{I}^{-})$

Kerr-Newman-Taub-NUT metric

In Boyer-Lindquist coordinates, (t, r, θ, ϕ), the metric is given by

$$\begin{aligned} \mathrm{d}s^2 =& \Sigma\left(\frac{1}{\Delta}\mathrm{d}r^2 + \mathrm{d}\theta^2\right) + \frac{1}{\Sigma}\left((\Sigma + a\chi)^2 \sin^2\theta - \Delta\chi^2\right)\mathrm{d}\phi^2 + \\ & \frac{2}{\Sigma}\left(\Delta\chi - a(\Sigma + a\chi)\sin^2\theta\right)\mathrm{d}t\mathrm{d}\phi - \frac{1}{\Sigma}\left(\Delta - a^2\sin^2\theta\right)\mathrm{d}t^2, \end{aligned} \tag{2}$$

where

$$\Sigma = r^2 + (l + a\cos\theta)^2, \qquad \chi = a\sin^2\theta - 2l(\cos\theta + C),$$
$$\Delta = r^2 - 2Mr + a^2 - l^2 + Q^2.$$

- stationary
- axially symmetric
- type D spacetimes
- event horizon at $r_+ = M + \sqrt{M^2 - a^2 + l^2 - Q^2}$, where r_+ is the largest root of $\Delta = 0$
- Electro-vac solutions

Parameters

- The mass M
- The charge Q
- The spin parameter a
- The NUT parameter l which can be interpreted as a gravitomagnetic charge
- Manko and Ruiz parameter C

Contains Schwarzschild ($a = Q = l = 0$), Kerr ($Q = l = 0$), Reissner-Nordström ($a = l = 0$), Kerr-Newman ($l = 0$), and Taub-NUT ($a = Q = 0$)

Constants of Motion

From metric

$$m = g_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} \tag{3}$$

from Killing vectorfield

$$E = -(\partial_t)_\mu \dot{\gamma}^{\mu}, \qquad L_z = (\partial_\phi)_\mu \dot{\gamma}^{\mu} \tag{4}$$

from Killing Tensor

$$\sigma_{\mu\nu} = \Sigma((e_1)_\mu (e_1)_\nu + (e_2)_\mu (e_2)_\nu) - (l + a\cos\theta)^2 g_{\mu\nu}, \qquad K := \sigma_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}. \tag{5}$$

Trapping in Schwarzschild

Radial equation for null geodesics in the equatorial plane

$$\frac{1}{2}E^2 = V(r) + \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2, \tag{6}$$

$$V(r) = \frac{L^2}{2r^2} - \frac{ML^2}{r^3},$$

With condition

$$V(r) = V'(r) = 0 \tag{7}$$

we get the photon sphere at $r = 3M$.

Geodesic Equation

The null geodesic equation as a system of first order ODEs

$$\dot{t} = \frac{\chi(L_z - E\chi)}{\Sigma \sin^2\theta} + \frac{(\Sigma + a\chi)((\Sigma + a\chi)E - aL_z)}{\Sigma\Delta}, \tag{8a}$$

$$\dot{\phi} = \frac{L_z - E\chi}{\Sigma \sin^2\theta} + \frac{a((\Sigma + a\chi)E - aL_z)}{\Sigma\Delta}, \tag{8b}$$

$$\Sigma^2 \dot{r}^2 = R(r, E, L_z, K) := ((\Sigma + a\chi)E - aL_z)^2 - \Delta K, \tag{8c}$$

$$\Sigma^2 \dot{\theta}^2 = \Theta(\theta, E, L_z, Q) := K - \frac{(\chi E - L_z)^2}{\sin^2\theta}. \tag{8d}$$

System homogeneous in E thus for $E \neq 0$ we have:

$$R(r, E, L_z, Q) = E^2 R(r, 1, L_E, K_E), \tag{9}$$

$$\Theta(\theta, E, L_z, Q) = E^2 \Theta(r, 1, L_E, K_E), \tag{10}$$

where $L_E = L_z/E$ and $K_E = K/E^2$.

Trapping

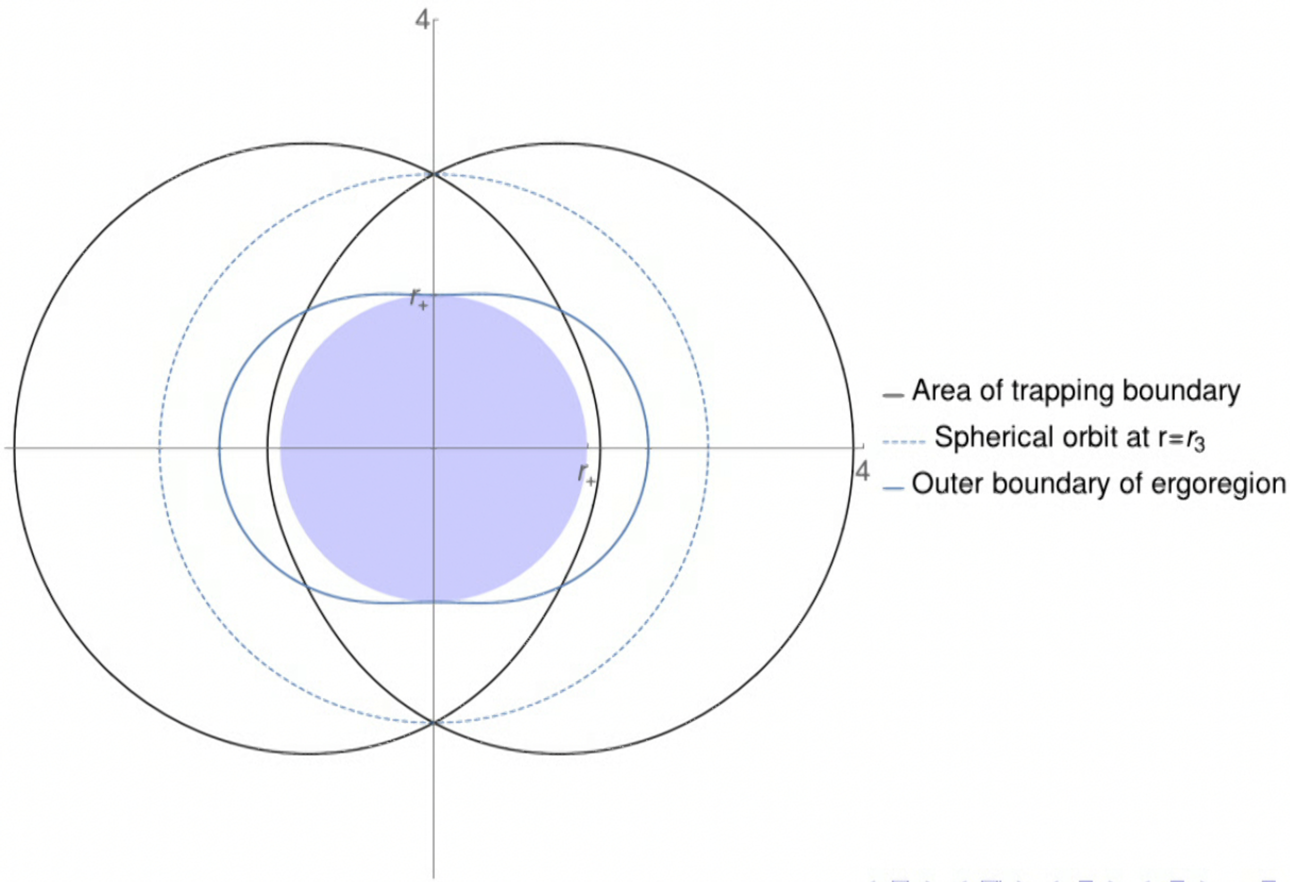
The trapped null geodesics are those which stay at a fixed value of r and hence satisfy $\dot{r} = \ddot{r} = 0$, which corresponds to:

$$R(r, L_E, K_E) = \frac{d}{dr} R(r, L_E, K_E) = 0. \tag{11}$$

These equations can be solved for the constants of motion in terms of the constant value $r = r_{trapp}$ as:

$$K_E = \left. \frac{16r^2\Delta}{(\Delta')^2} \right|_{r=r_{trapp}}, \qquad aL_E = (\Sigma + a\chi) - \left. \frac{4r\Delta}{\Delta'} \right|_{r=r_{trapp}}, \tag{12}$$

Area of Trapping, $a = 0.902$



Celestial Sphere

At any point p in $\mathscr{M}$ choose an orthonormal basis (e_0, e_1, e_2, e_3) for the tangent space, with e_0 time-like and future directed. The tangent vector to any past pointing null geodesic at p can be written as:

$$\dot{\gamma}(k|_p)|_p = \alpha(-e_0 + k_1 e_1 + k_2 e_2 + k_3 e_3), \tag{13}$$

where $\alpha = g(\dot{\gamma}, e_0) > 0$ and $k = (k_1, k_2, k_3)$ satisfies $|k|^2 = 1$, hence $k \in \mathbb{S}^2$.

Definition

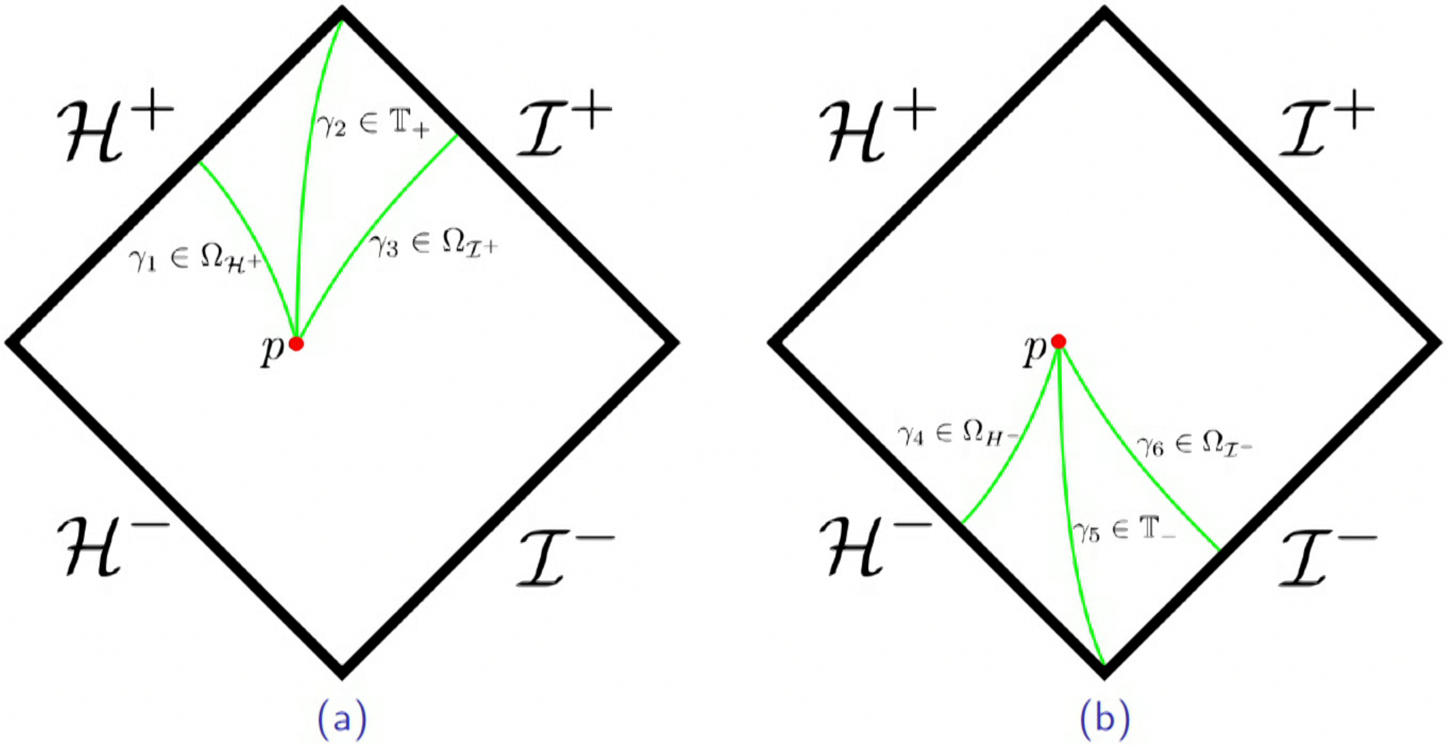
Let $\gamma(k|_p)$ denote a null geodesic through p for which the tangent vector at p is given by equation (13).

Sets on the Celestial Sphere

We can then define the following sets on $\mathbb{S}^2$ at every point p:

Definition

The future infalling set: $\Omega_{\mathscr{H}^+}(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap \mathscr{H}^+ \neq \emptyset\}$.
The future escaping set: $\Omega_{\mathscr{I}^+}(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap \mathscr{I}^+ \neq \emptyset\}$.
The future trapped set: $\mathbb{T}_+(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap (\mathscr{H}^+ \cup \mathscr{I}^+) = \emptyset\}$.
The past infalling set: $\Omega_{\mathscr{H}^-}(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap \mathscr{H}^- \neq \emptyset\}$.
The past escaping set: $\Omega_{\mathscr{I}^-}(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap \mathscr{I}^- \neq \emptyset\}$.
The past trapped set: $\mathbb{T}_-(p) := \{k \in \mathbb{S}^2 | \gamma(k|_p) \cap (\mathscr{H}^- \cup \mathscr{I}^-) = \emptyset\}$

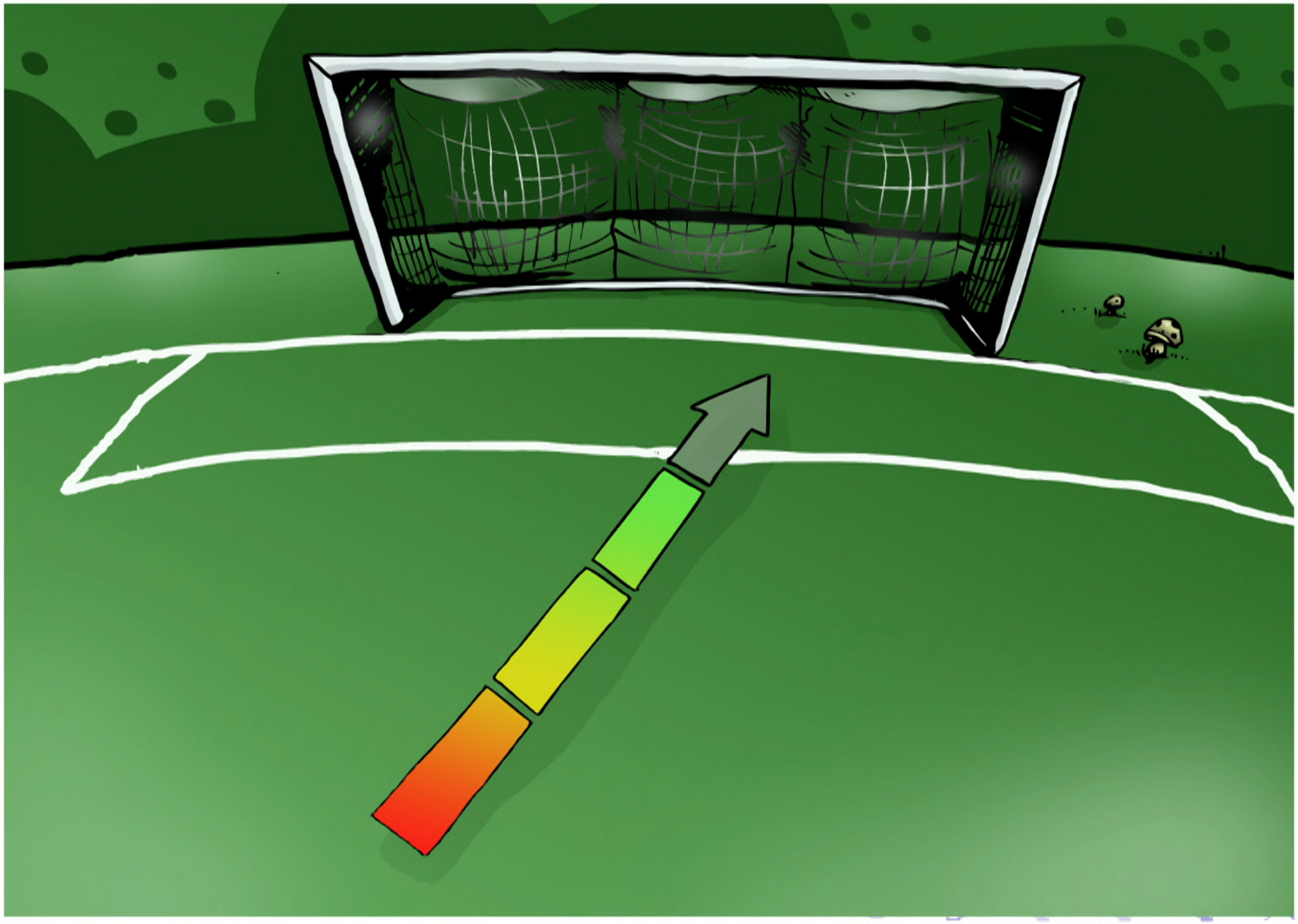


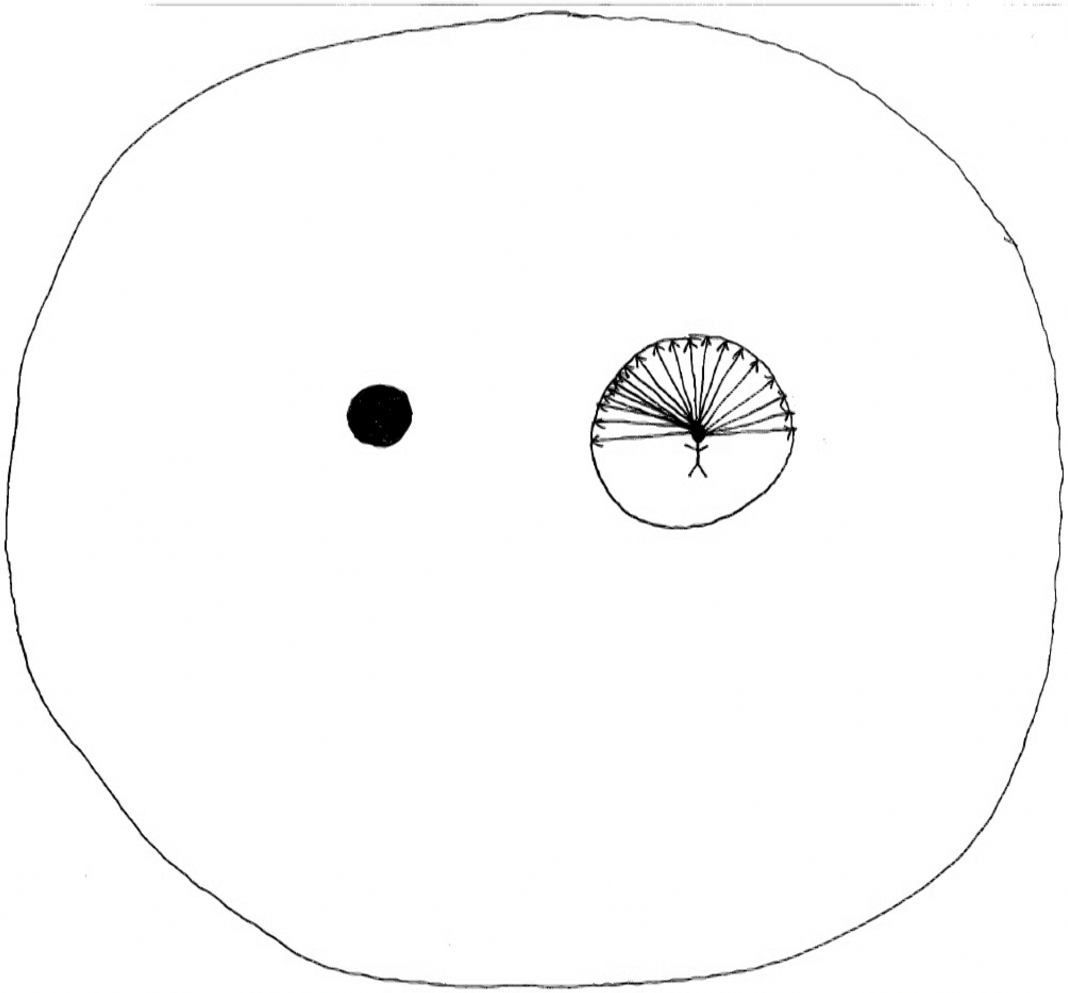
$\mathcal{H}^+$
$\mathcal{I}^+$
$\gamma_2 \in \mathbb{T}_+$
$\gamma_1 \in \Omega_{\mathcal{H}^+}$
$\gamma_3 \in \Omega_{\mathcal{I}^+}$
p
$\mathcal{H}^-$
$\mathcal{I}^-$
(a)
$\mathcal{H}^+$
$\mathcal{I}^+$
p
$\gamma_4 \in \Omega_{H^-}$
$\gamma_6 \in \Omega_{\mathcal{I}^-}$
$\gamma_5 \in \mathbb{T}_-$
$\mathcal{H}^-$
$\mathcal{I}^-$
(b)

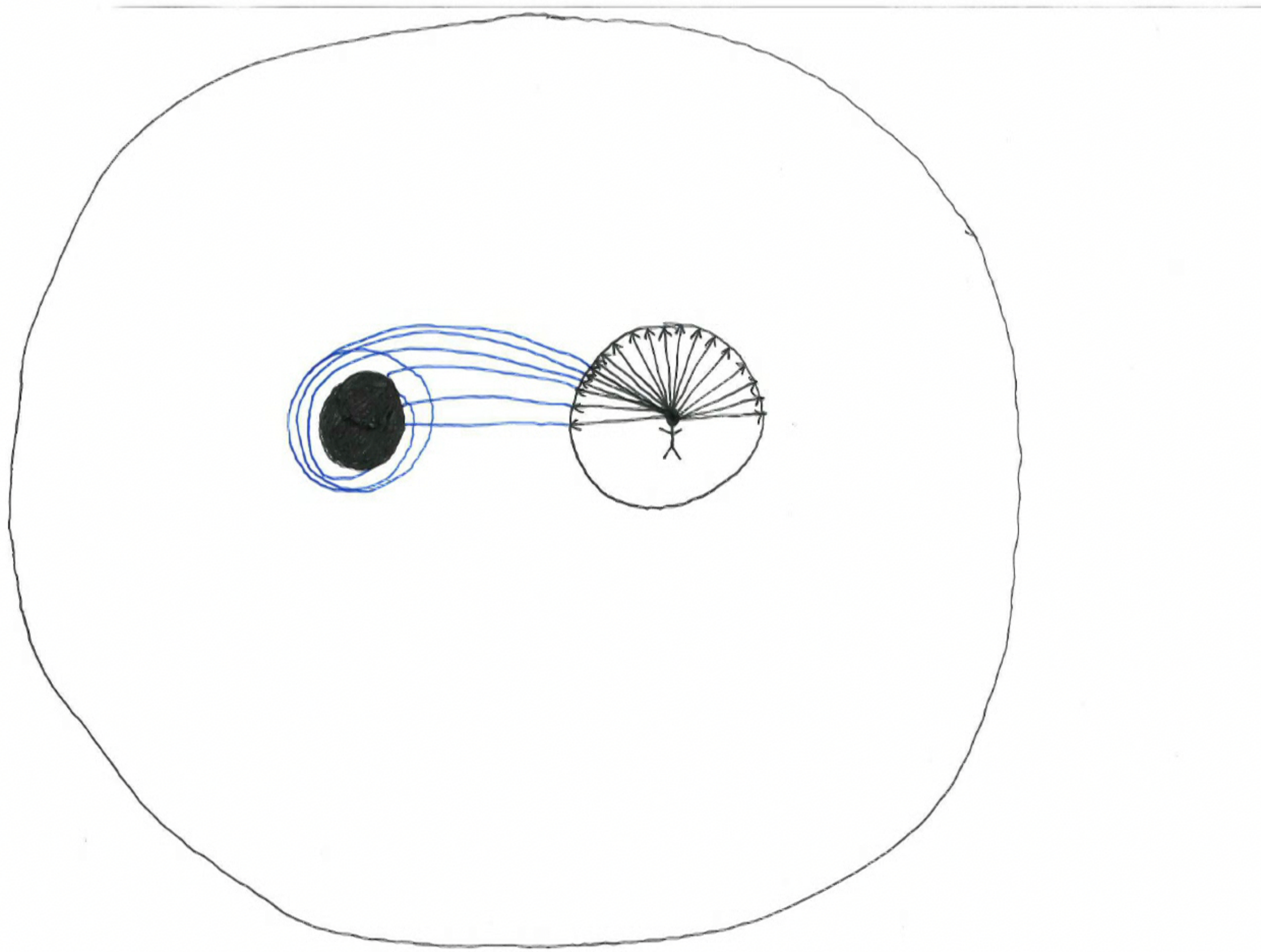
The Shadow

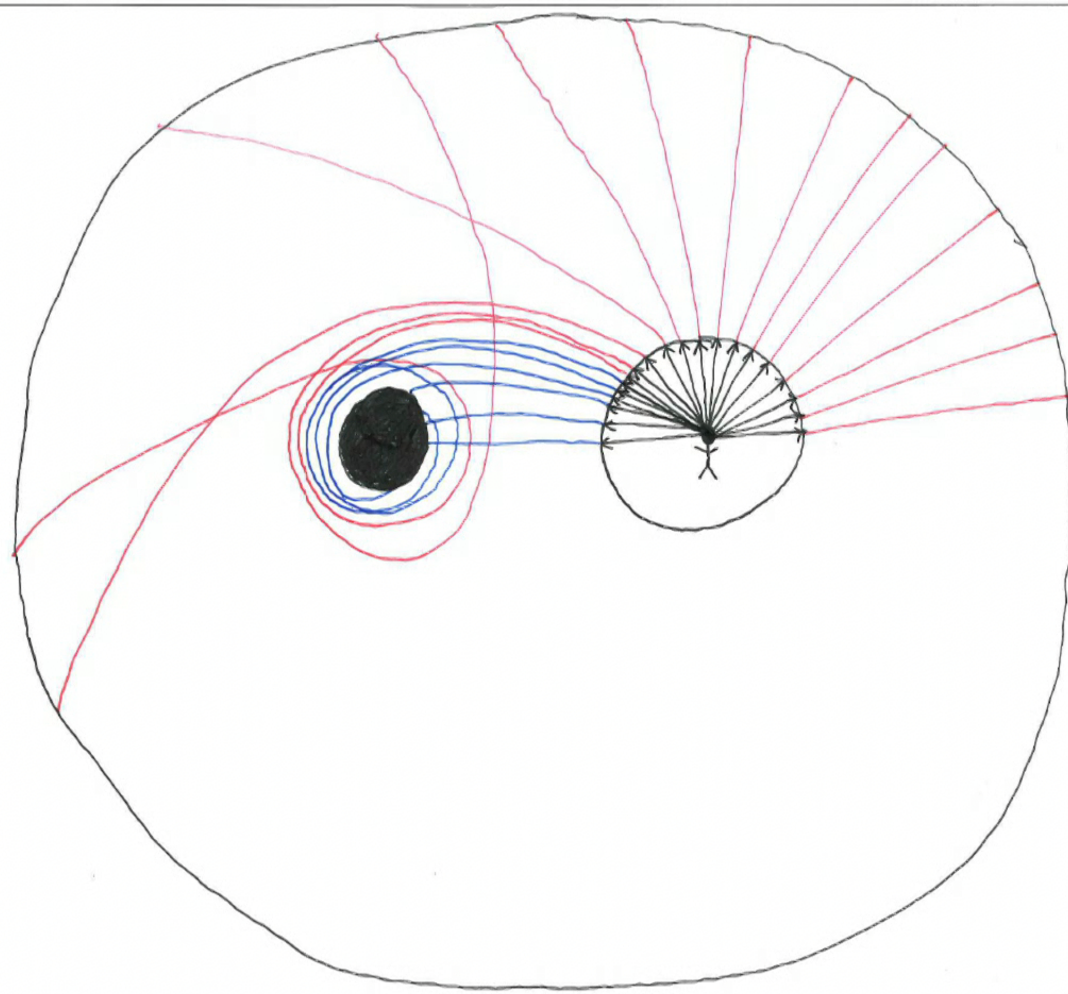
Definition

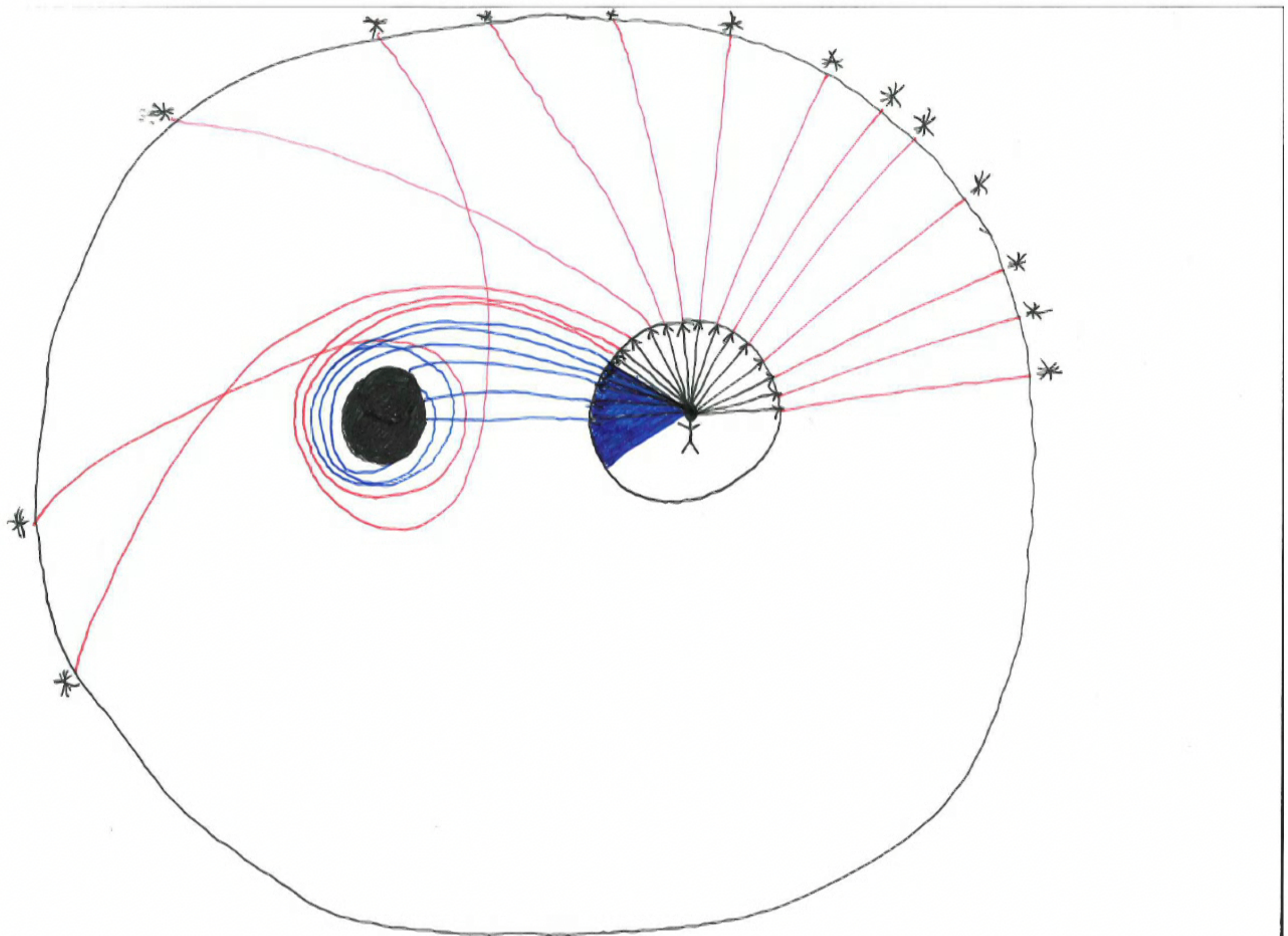
We refer to the set $\Omega_{\mathscr{H}^-}(p) \cup \mathbb{T}_-(p)$ as the shadow of the black hole.











Orthonormal Tetrad

We will use the following orthonormal tetrad at point p:

$$e_0 = \left.\frac{(\Sigma + a\chi)\partial_t + a\partial_\phi}{\sqrt{\Sigma\Delta}}\right|_p, \qquad e_1 = \left.\sqrt{\frac{1}{\Sigma}}\partial_\theta\right|_p, \tag{14}$$

$$e_2 = \left.\frac{-(\partial_\phi + \chi\partial_t)}{\sqrt{\Sigma}\sin\theta}\right|_p, \qquad e_3 = -\left.\sqrt{\frac{\Delta}{\Sigma}}\partial_r\right|_p.$$

We will refer to this particular e_0 as "standard observer".

Shadow Parametrization

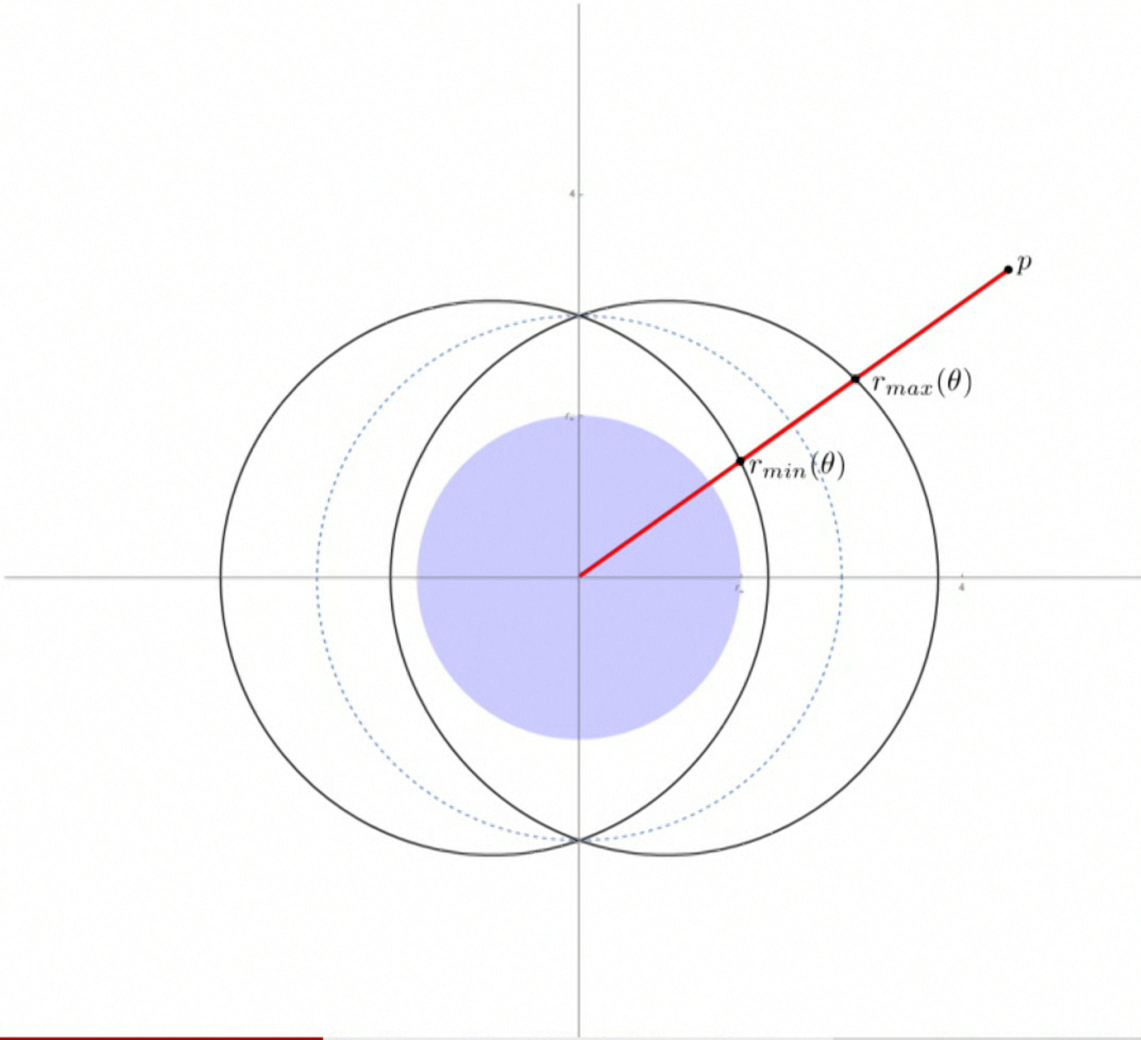
At any point p in the exterior region of a Kerr-Newman-Taub-NUT black hole away from the symmetry axis the curve $\mathbb{T}_{-}(p)$ that defines the shadow is given by the parametric expression[2]:

$$\sin\psi = \frac{\Delta'(x)\{x^2 + (l + a\cos[\theta(p)])^2\} - 4x\Delta(x)}{4ax\sqrt{\Delta(x)}\sin[\theta(p)]} \tag{16a}$$
$$:= f(x, \theta, M, a, Q, l),$$

$$\sin\rho = \frac{4x\sqrt{\Delta(r(p))\Delta(x)}}{\Delta'(x)(r(p)^2 - x^2) + 4x\Delta(x)} \tag{16b}$$
$$:= h(x, r, M, a, Q, l),$$

where the parameter x takes values in the compact interval $[r_{min}(\theta(p)), r_{max}(\theta(p))]$.

[2] A. Grenzebach, V. Perlick, C. Lämmerzahl. "Photon regions and shadows of Kerr-Newman-NUT black holes with a cosmological constant." Physical Review D, 2014



p
$r_{max}(\theta)$
$r_{min}(\theta)$

Smoothness of Shadow

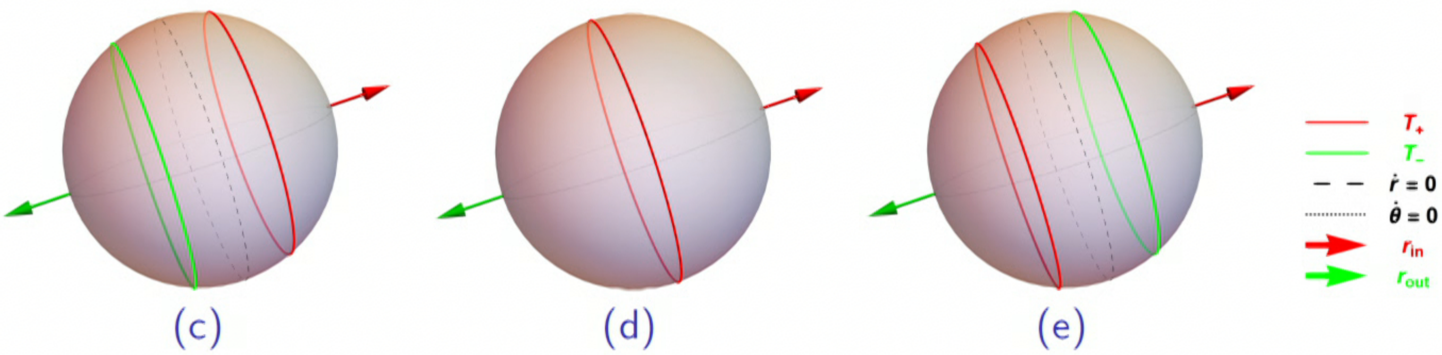
Lemma

The sets $\mathbb{T}_+(p)$ and $\mathbb{T}_-(p)$ are circles on the celestial sphere of any timelike observer at any regular point of symmetry in the exterior region of any subextremal Kerr-Newman-Taub-NUT spacetime.

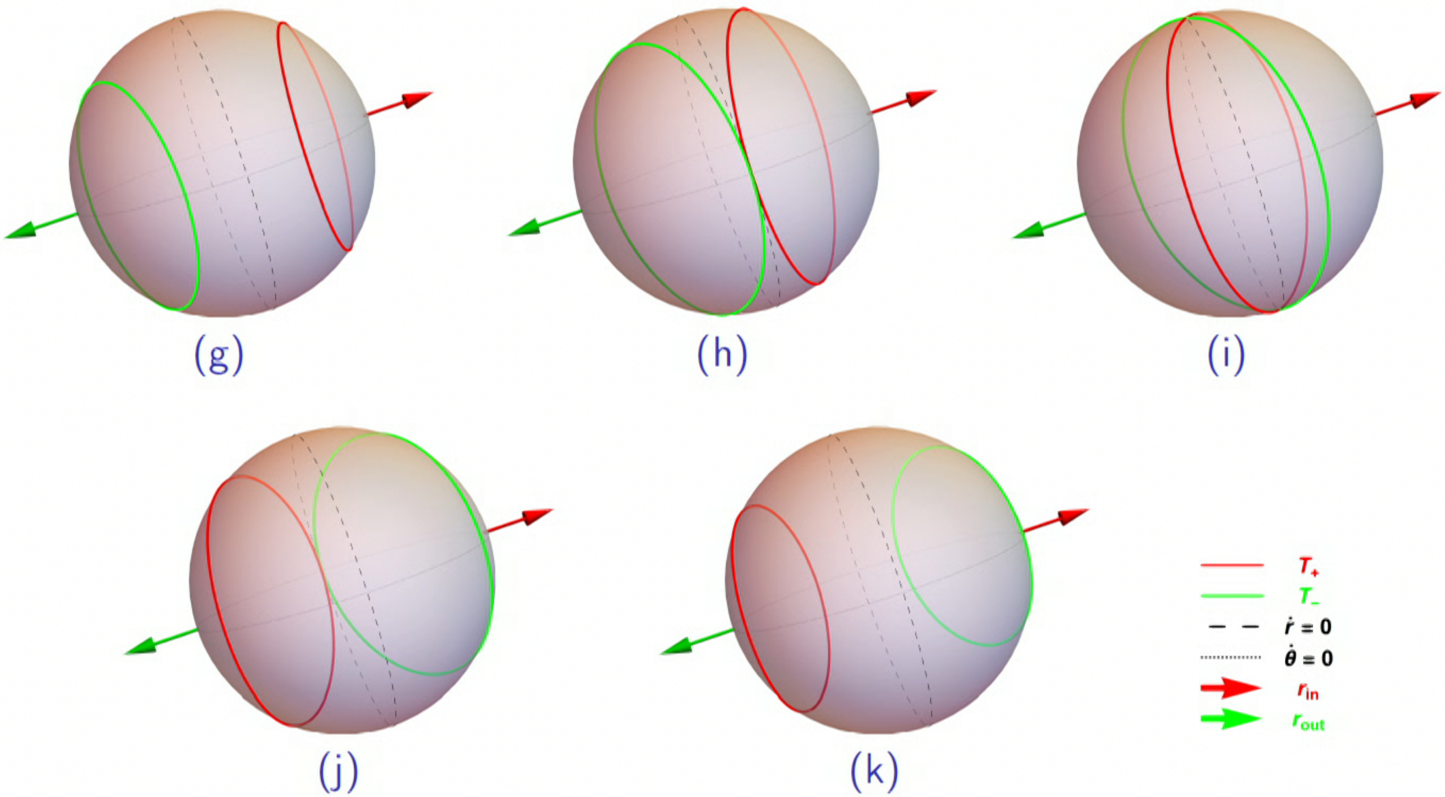
Theorem

The sets $\mathbb{T}_+(p)$ and $\mathbb{T}_-(p)$ are simple, closed, smooth curves on the celestial sphere of any timelike observer at any point in the exterior region of any subextremal Kerr-Newman-Taub-NUT spacetime.

Schwarzschild



Kerr, $a = 0.9$



Instability of Trapping

We immediately get the following Corollary:

Corollary

For any observer at any regular point p in the exterior region of a subextremal Kerr-Newman-Taub-NUT spacetime we have that for any $k \in \mathbb{T}_+(p)$ *and any* $\epsilon > 0$

- $B_\epsilon(k) \cap \Omega_{\mathscr{H}^+}(p) \neq \emptyset$
- $B_\epsilon(k) \cap \Omega_{\mathscr{I}^+}(p) \neq \emptyset$.

Existence of Trapping

Theorem

Let $\mathscr{M}$ be a C^1 spacetime. Let $\mathscr{M}$ be compactifiable with complete $\mathscr{I}^{\pm}$ in C^1. Let p be any point in $J^{-}(\mathscr{I}^{+}) \cap J^{+}(\mathscr{I}^{-})$. If $\Omega_{\mathscr{H}^{\pm}}(p)$ is non-empty then

- $\mathbf{T}_{\pm}(p)$ *is non-empty*
- *let $w(\lambda)$ be any continuous path on $S^2(p)$ with $\lambda \in [0,1]$ such that $w(0) \in \Omega_{\mathscr{H}^{\pm}}(p)$ and $w(1) \in \Omega_{\mathscr{I}^{\pm}}(p)$, we have that $w(\lambda) \cap \mathbf{T}_{\pm}(p) \neq \emptyset$.*

Observers at the same Point

A change of observer (i.e. an orthochronous Lorentz transformation of the tetrad) corresponds to a conformal transformation on the celestial sphere, and vice versa. Restricting oneself to orientation preserving transformations, they are isomorphic to Möbius transformations.

Change of observer

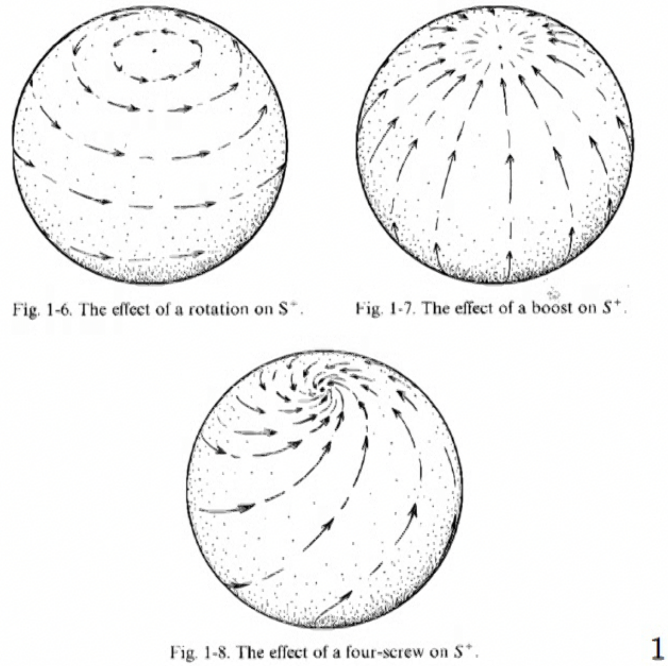


Figure: Flow on the celestial sphere under change of tetrad.

[1] Penrose, Roger, and Wolfgang Rindler. Spinors and space-time: Volume 2, Spinor and twistor methods in space-time geometry. Vol. 2. Cambridge university press, 1988.

Stereographic projection

The stereographic projection of the celestial sphere

$$c(x) = \frac{X(x) + iY(x)}{1 - Z(x)}, \tag{17a}$$

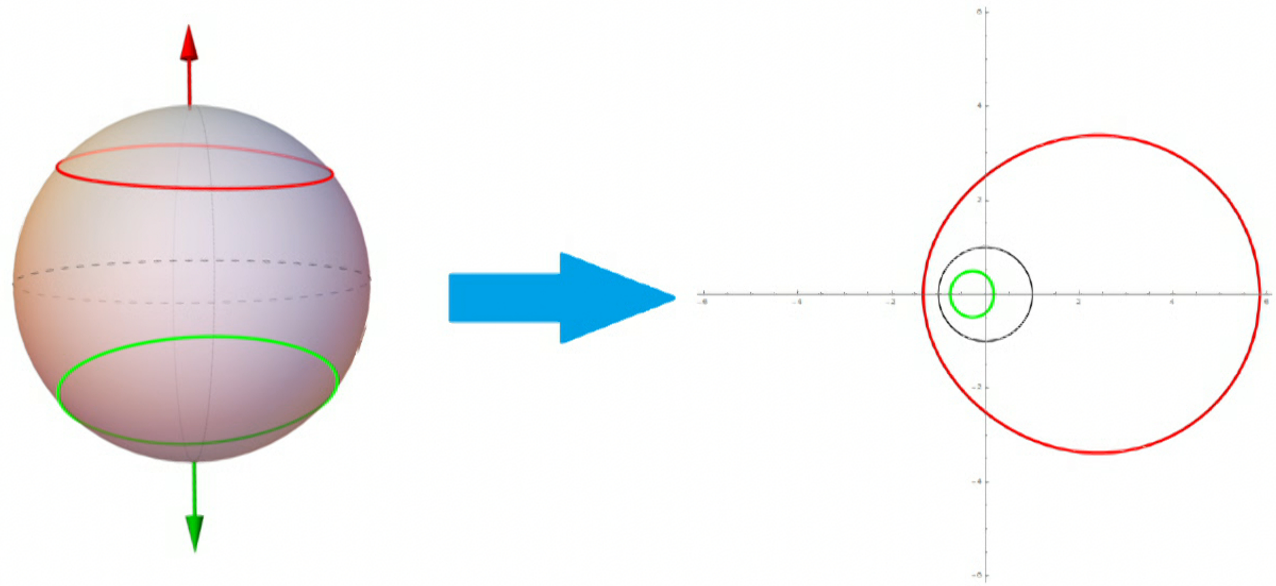
$$X(x) = \sin(\rho)\sin(\psi) = h(x) \cdot f(x), \tag{17b}$$

$$Y(x) = \sin(\rho)\cos(\psi) = \pm h(x) \cdot \sqrt{1 - f^2(x)}, \tag{17c}$$

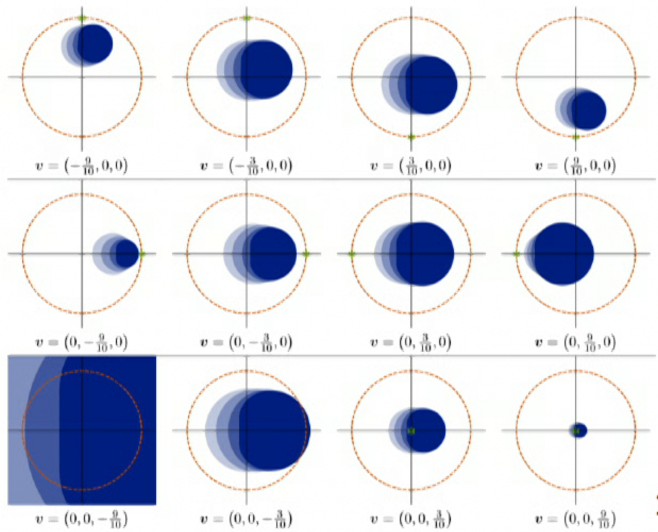
$$Z(x) = \cos(\rho) = -sgn\left(\frac{\partial h}{\partial x}\right)\sqrt{1 - h^2(x)}. \tag{17d}$$

The sign in Z makes the curve C^1 and is the right choice to describe $\mathbb{T}_-$

Degeneracies



Aberration



[2]

Figure: Aberration effect for Kerr black hole with $a = \{0, 2/5, 4/5, 1\}$

[2] Grenzebach, Arne. "Aberrational effects for shadows of black holes." Equations of Motion in Relativistic Gravity. Springer, Cham, 2015. 823-832.

Question

We have two mechanisms that deform the shape of the shadow, a change in parameters and a change in observers. Can these two effect cancel each other?

Degeneracy for Points of Symmetry

A fundamental property of conformal transformations on $\mathbb{S}^2$ is that they map circles into circles. As a consequence if p_1 and p_2 are points in (possibly different) spacetimes in the family under consideration, and both lie on an axis of symmetry then, upon identification of the two celestial spheres by a respective choice of time oriented orthonormal basis, there exists a Lorentz transformation (**LT**) of the observer such that $\mathbb{T}_-(p_1) = \mathbf{LT}[\mathbb{T}_-(p_2)]$.

Definition

The shadows at two points p_1, p_2 are called degenerate if, upon identification of the two celestial spheres by the orthonormal basis, there exists an element of the conformal group on $\mathbb{S}^2$ that transforms $\mathbb{T}_-(p_1)$ into $\mathbb{T}_-(p_2)$.

Remark on Degeneracy

Remark

The shadow at two points p_1, p_2 being degenerate implies that for every observer at p_1 there exists an observer at p_2 for which the shadow on $\mathbb{S}^2$ is identical. Because this notion compares structures on $\mathbb{S}^2$, the two points need not be in the same manifold for their shadows to be degenerate. Just from the shadow alone an observer can not distinguish between these two configurations.

Variation Vector

The first order of the action of any member of the conformal group on $\mathbb{S}^2$ on a curve is given by:

$$\Psi_\epsilon(c) = c(x) + \epsilon\vec{\xi}|_{c(x)} + \mathscr{O}(\epsilon^2), \tag{18}$$

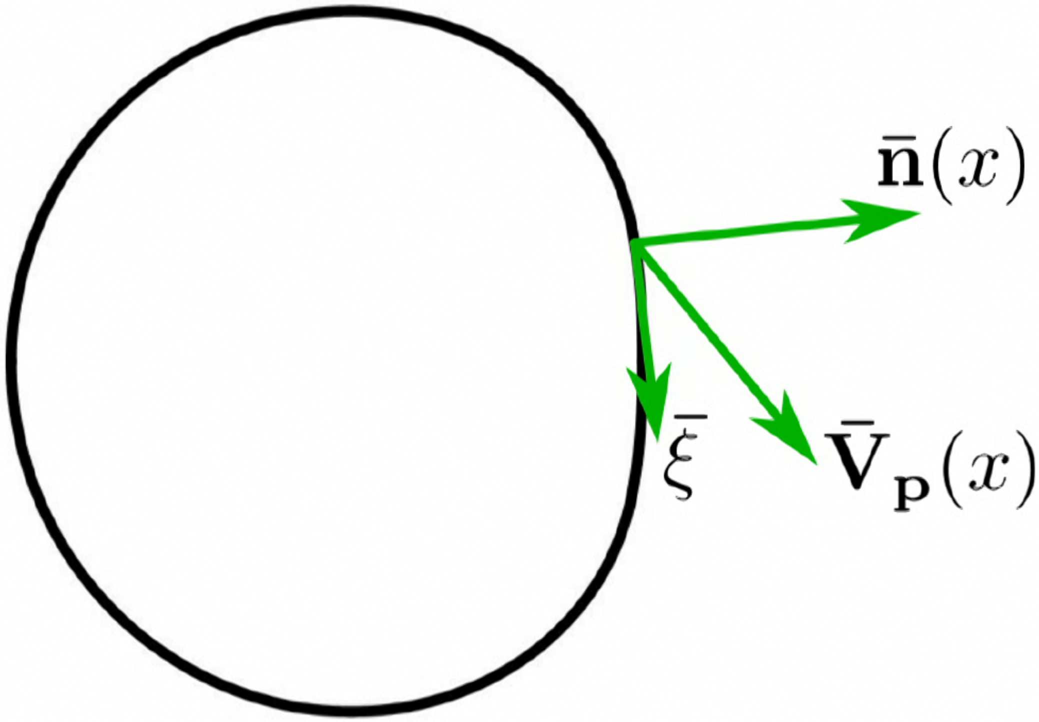
where ϵ is a small parameter and where ξ is a conformal Killing vector field on $\mathbb{S}^2$. The first variation of the curve with respect to a parameter p is given by:

$$c(x; p + dp) = c(x, p) + \vec{V}_p dp + \mathscr{O}(dp^2), \tag{19}$$

where dp is an infinitesimal change of the parameter and V_p is given by $\partial_p c(x, p)$. The most generic variation vector for a curve is then:

$$\vec{V} = \sum_{p \in \mathscr{P} = \{r,\theta,M,a,Q,l\}} \vec{V}_p dp + \sum_{\xi \in Lie(Mb)} \vec{\xi}|_{c(x)} \epsilon_\xi. \tag{20}$$

Linearization



Continuous Degeneracies

We can now formulate a necessary and sufficient condition for the curve to be invariant under a continuous deformation. This is the case if there exists a nontrivial combination of dp and ϵ_ξ such that V is tangential to the curve. Letting n be the normal to the curve $c(x,p)$, the condition is that:

$$\vec{V} \cdot \vec{n} \equiv 0 \tag{21}$$

has a nontrivial solution in terms of dp and ϵ_ξ.

Intrinsic Degenerations

Definition

A degeneration is called intrinsic when there is no need to act with a Möbius transformation to counter the deformation in the shadow due to the change in parameters.

Hence the condition reduces to

$$\sum_{p \in \mathscr{P}} \vec{V}_p \cdot \vec{n} \equiv 0 \tag{22}$$

Which in terms of the functions f and h is simply

$$\sum_{p \in \mathscr{P}} \left(\frac{\partial f(x,p)}{\partial x} \frac{\partial h(x,p)}{\partial p} - \frac{\partial f(x,p)}{\partial p} \frac{\partial h(x,p)}{\partial x} \right) dp \equiv 0, \tag{23}$$

Available Möbius Transformations

The stereographic projection of the shadow of any standard observer is reflection symmetric with respect to the real line. Only those conformal transformation that preserve the reflection symmetry can be used to "counter" the deformation from the change in parameters.
One finds that the most general such conformal Killing vector is an arbitrary linear combination of the three linearly independent vector fields given by:

$$\vec{\xi}_1 = \partial_X, \qquad \vec{\xi}_2 = X\partial_Y + Y\partial_X, \qquad \vec{\xi}_3 = (X^2 - Y^2)\partial_X + 2XY\partial_Y, \quad (24)$$

in terms of Cartesian coordinates $\{X, Y\}$ on the complex plane, i.e. $z = X + iY$.

Degeneration

The general linear combination that we required to be zero

$$\beta\vec{\xi}_1 \cdot \vec{n} + \alpha\vec{\xi}_2 \cdot \vec{n} + \gamma\vec{\xi}_3 \cdot \vec{n} + \sum_{p\in\mathscr{P}} \vec{V}_p \cdot \vec{n} dp \equiv 0, \qquad (25)$$

can be solved for β and γ

Degeneration

$$\beta = \frac{\sum_{p\in\mathscr{P}} h(x)\left(\frac{\partial f(x,p)}{\partial x}\frac{\partial h(x,p)}{\partial p} - \frac{\partial f(x,p)}{\partial p}\frac{\partial h(x,p)}{\partial x}\right) dp}{2\left((1-h^2)f(x)h(x)\frac{\partial f(x)}{\partial x} - (1-f^2(x))\frac{\partial h(x)}{\partial x}\right)} + \alpha\frac{h^2(x)\frac{\partial f(x)}{\partial x}}{2\left(f(x)h(x)\frac{\partial f(x)}{\partial x} - (1-f^2(x))\frac{\partial h(x)}{\partial x}\right)}, \tag{26}$$

$$\gamma = \frac{\sum_{p\in\mathscr{P}} h(x)\left(\frac{\partial f(x,p)}{\partial x}\frac{\partial h(x,p)}{\partial p} - \frac{\partial f(x,p)}{\partial p}\frac{\partial h(x,p)}{\partial x}\right) dp}{2\left((1-h^2)f(x)h(x)\frac{\partial f(x)}{\partial x} - (1-f^2(x))\frac{\partial h(x)}{\partial x}\right)} - \alpha\frac{h^2(x)\frac{\partial f(x)}{\partial x}}{2\left(f(x)h(x)\frac{\partial f(x)}{\partial x} - (1-f^2(x))\frac{\partial h(x)}{\partial x}\right)}. \tag{27}$$

The Non-Degenerate Case

A set of parameters $\mathscr{P}$ is said to be non-degenerate if only the trivial combination of dp_i satisfy the condition.

Theorem

Theorem

The only continuous degeneracies of the black hole shadow for observers located at coordinate position r, θ in the exterior region of Kerr-Newman-Taub-NUT black holes with parameters M, a, Q and l are given for observers such that their parameters have the same value for all the following functions:

$$\frac{a}{M} = C_1, \quad \frac{r}{M} = C_2, \quad \frac{Q}{M} = C_3, \quad \frac{l}{M} = C_4, \quad \theta = C_5. \tag{28}$$

or

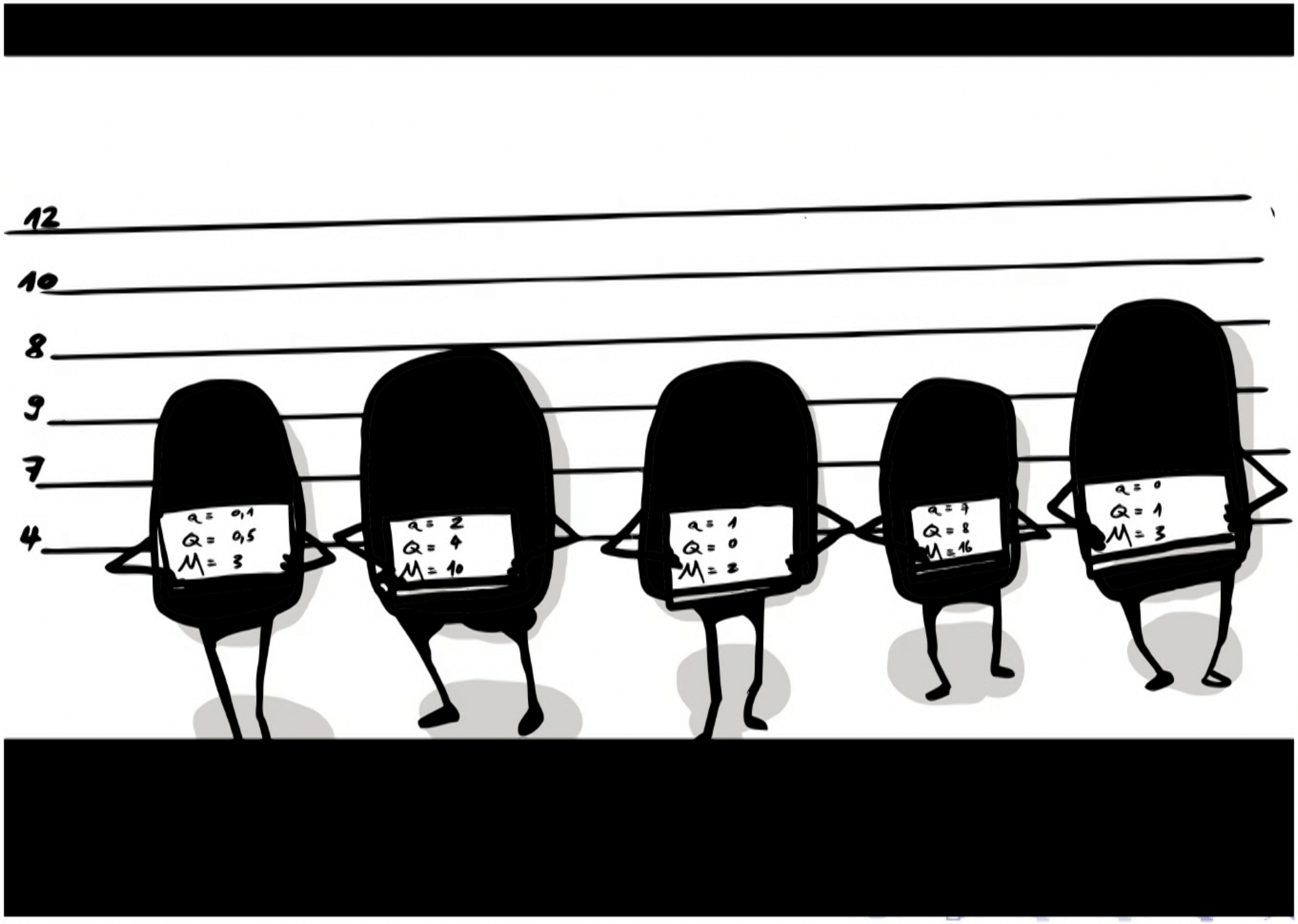
$$a \sin\theta = C_1,\ l + a\cos\theta = C_2,\ Q + 2a\cos\theta(l + a\cos\theta) = C_3,\ r = C_4\ M = C_5. \tag{29}$$

Discrete Degeneracies

There is a discrete degeneracy for two observers with:

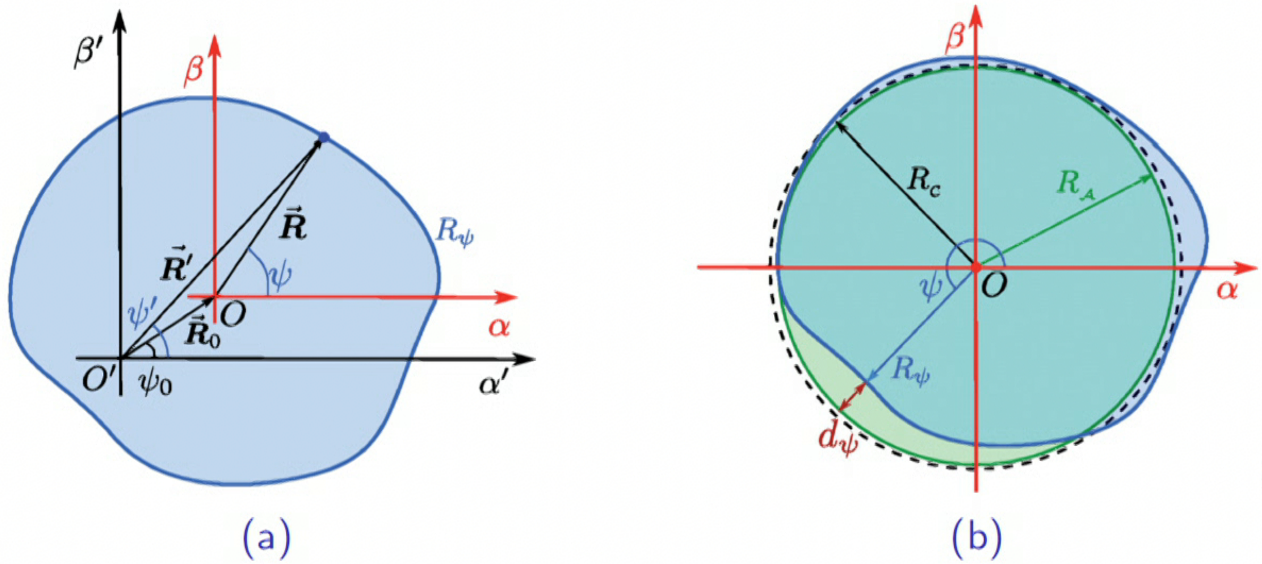
$$M_1 = M_2 \quad l_1 = -l_2 \quad r_1 = r_2 \quad a_1 = a_2 \quad Q_1 = Q_2 \quad \theta_1 = \pi - \theta_2 \qquad (30)$$

In the case $l = 0$ this corresponds to a reflection of the observers position with respect to the equatorial plane, while when $l \neq 0$ the spacetime itself changes. In either case, two observers related by this transformation are fully indistinguishable from the observation of the shadow.


12
10
8
9
7
4
a = 0,1
Q = 0,5
M = 3
a = 2
Q = 4
M = 10
a = 1
Q = 0
M = 2
Q = 8
M = 16
a = 0
Q = 1
M = 3

Access to Information

We know the information is there, now what?



[1]

[1]Abdujabbarov, A. A., L. Rezzolla, and B. J. Ahmedov. "A coordinate-independent characterization of a black hole shadow.", 2015

Shape Analysis

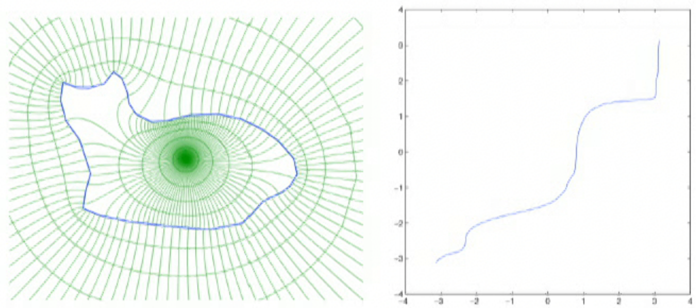
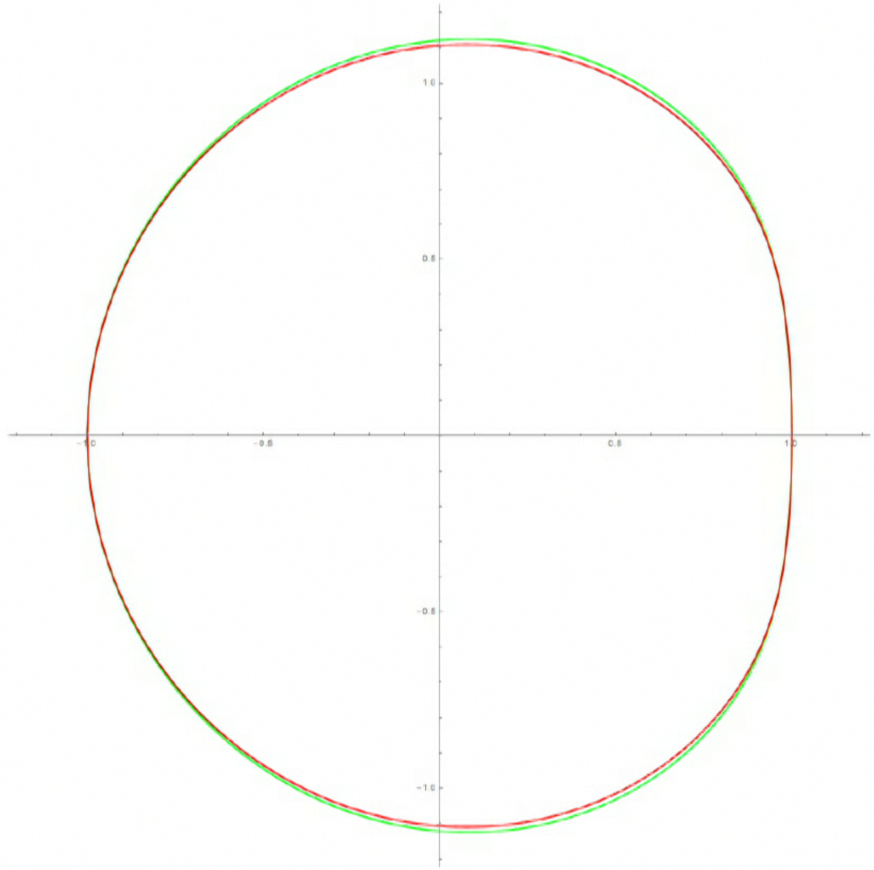
 [2]

Figure: Signature for 2D shapes.

[2] Feiszli, Matt, and David Mumford. "Shape representation via conformal mapping.", 2007
See also:
Lui, Lok Ming, Wei Zeng, Shing-Tung Yau, and Xianfeng Gu. "Shape analysis of planar objects with arbitrary topologies using conformal geometry.", 2010.

$a = 0.99$, $\theta = \pi/2$, $r = 5M$ and $r = 50M$



Conclusion & Outlook

- We showed that in principle an observer in the exterior region of a Kerr-Newman black hole can determine the parameters Q, a, r, θ.
- The necessary resolution to do so is difficult to obtain in reality.
- We assumed only background light sources exist.
- The parametrization also exists for the Kerr-Newman-de-Sitter spacetimes. Preliminary calculations turned out no intrinsic degeneracies for that case either. The proof is however more complex.
- The problem of excluding discrete degeneracies remains open.
- Finding a characterization for the shape of the curve that is invariant under Moebiustransformtions would be interesting:

$$A(c) = A(\Psi(c)) \tag{31}$$

$$e_1 \sim \partial_\theta$$

$$e_3 \sim \partial_r$$

$$\sin\psi = f(x)$$

$$\sin\zeta = h(x)$$