Title: Symmetric Informationally Complete Measurements Identify the Essential Difference between Classical and Quantum

Date: Apr 10, 2018 02:00 PM

URL: http://pirsa.org/18040114

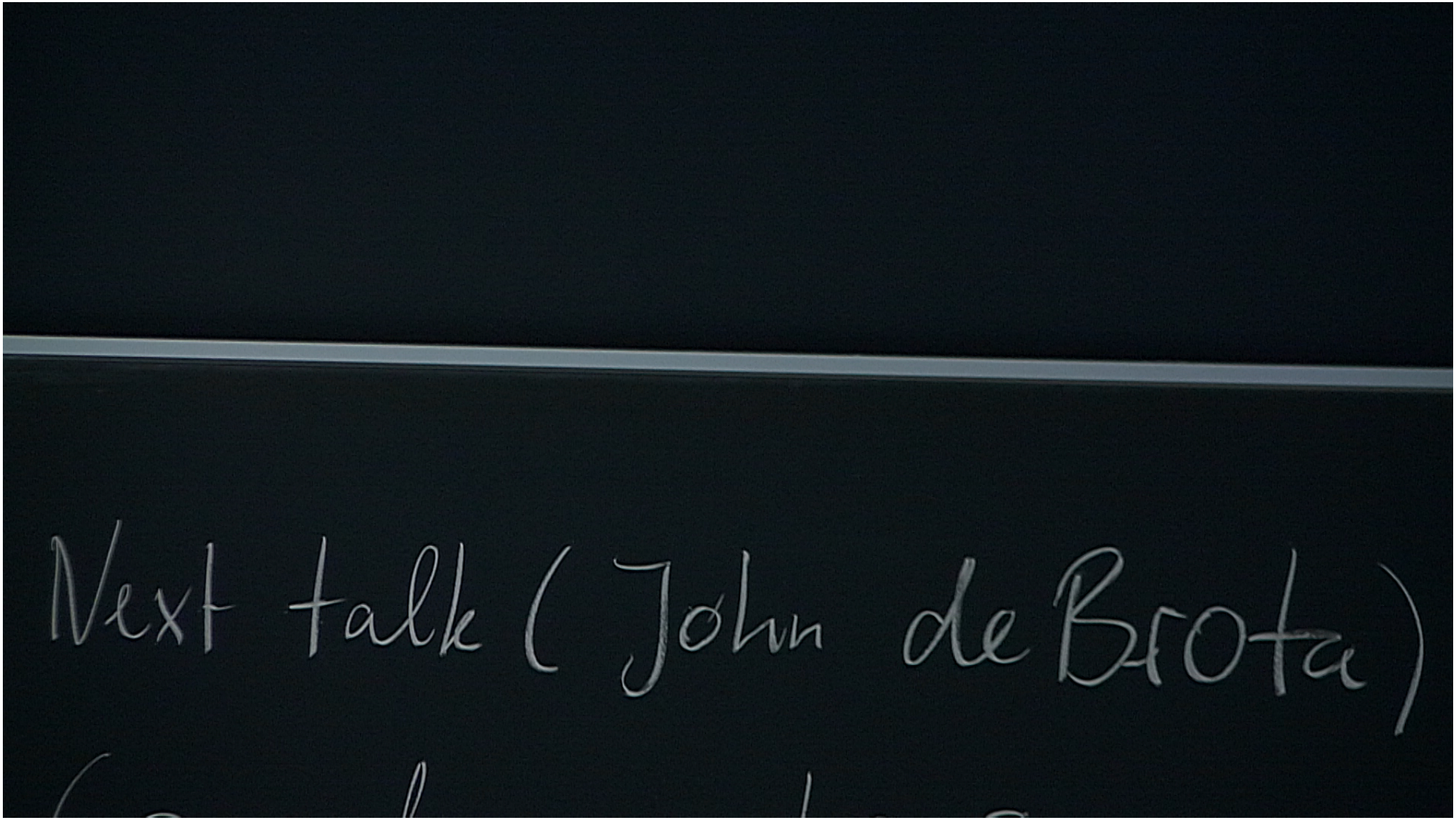
Abstract: In this talk we describe a general procedure for associating a minimal informationally-complete quantum measurement (or MIC) with a probabilistic representation of quantum theory. Towards this, we make use of the idea that the Born Rule is a consistency criterion among subjectively assigned probabilities rather than a tool to set purely physically mandated probabilities. In our setting, the difference between quantum theory and classical statistical physics is the way their physical assumptions augment bare probability theory: Classical statistical physics corresponds to a trivial augmentation, while quantum theory makes crucial use of the Born Rule. We prove that the representation of the Born Rule obtained from a symmetric informationally-complete measurement (or SIC) minimizes the distinction between the two theories in at least two senses, one functional, the other geometric. Our results suggest that this representation supplies a natural vantage point from which to identify their essential differences, and, perhaps thereby, a set of physical postulates reflecting the quantum nature of the world.

SICs Identify the Essential Difference between Classical and Quantum

John B. DeBrota

QBism Group, Physics Department
University of Massachusetts Boston

April 11, 2018


Next talk (John de Brota)

Why look for other representations?

E.T. Jaynes

Omelet

Why look for other representations?

E.T. Jaynes

Quantum Theory

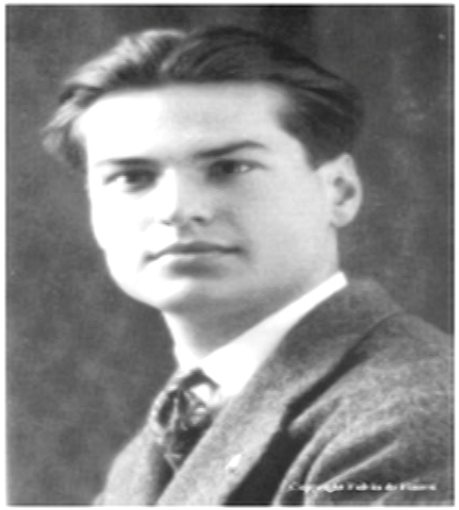
Seek a Probabilistic Representation

- Probabilities are about our expectations/information/beliefs. About us. How they fit together would be about nature.

"It is obviously *possible* to devise a formulation of quantum mechanics without probability amplitudes. One is never forced to use any quantities in one's theory other than the raw results of measurements. However, there is no reason to expect such a formulation to be anything other than extremely ugly."

Personalist/Subjective Bayesian Probabilities

- Probabilities are not given by nature. They are an agent's degrees of belief, specifically gambling commitments. Always.
- The price you are willing to buy *or* sell a ticket worth \$1 if the event occurs.
- Standard probabilistic rules arise from consistency among simultaneous gambling commitments.



- With *additional* assumptions, one can derive rules of inference (e.g., the Bayes Rule), but that's not the subject of this talk.

Eg: Sum of exhaustive probabilities cannot exceed unity

- Let A be a lottery ticket worth $\$1$ if it snows today and $\$0$ if it doesn't and $B \equiv \neg A = 1 - A$ be the reverse lottery ticket.
- Suppose you were willing to pay $P(A) = \$0.60$ *and* $P(B) = \$0.90$
- Your expected net gain for buying both A and B at prices $P(A)$ and $P(B)$ is

$$(A - P(A)) + (B - P(B)) = \$1 - \$0.60 - \$0.90 = -\$0.50$$

Figure 1: April 6th (Snow) Showers in Boston

The Law of Total Probability

$$P(D_i) = \sum_j P(D_i|H_j)P(H_j)$$

- LTP is a consequence of consistency among one's expectations for two consecutive measurements.
- Conditional probability $P(D_i|H_j)$ is price you're willing to buy *or* sell ticket worth \$1 if D_i occurs in second action which is refunded if H_j is not obtained in first action.
- Your probabilities must be *related* in this way or you're a sure loser.

But What About the Born Rule?

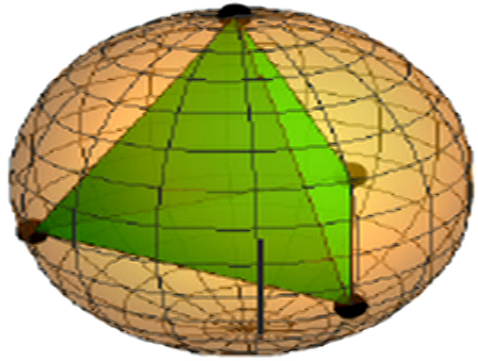
- Given quantum state ρ and POVM $\{E_i\}$, $P(E_i) = \mathrm{tr}\,\rho E_i$.
- If all probabilities are personal judgements,
  - ρ must be some expression/encoding of a personal judgement.
  - Association between experiences and the set $\{E_i\}$ must also be personal judgements.
- Nonetheless, the Born Rule is what allows us to *relate* probabilities in different contexts. Probabilities must *hold together* in a particular way.
- Candidate for probabilistic representation.
- Is it ugly?

Enter the SICs

Consider set of rank-1 projectors such that

$$\mathrm{tr}\Pi_i\Pi_j = \frac{d\delta_{ij}+1}{d+1} \ .$$

These form a MIC $H_i = \frac{1}{d}\Pi_i$ called a Symmetric Informationally-Complete POVM (SIC).



Coefficients are particularly simple in this basis.

$$\rho = \sum_{i=1}^{d^2} \left[(d+1)P(H_i) - \frac{1}{d} \right] \Pi_i \ .$$

Consider an arbitrary measurement D_j. The probability for outcome j would be

$$Q(D_j) = \mathrm{tr}\rho D_j = \sum_{i=1}^{d^2} \left[(d+1)P(H_i) - \frac{1}{d} \right] \mathrm{tr}\Pi_i D_j \ .$$

If one performed a SIC measurement on ρ and obtained outcome i, they would update to the state Π_i (Lüders' rule). So $\mathrm{tr}\Pi_i D_j = P(D_j|H_i)$ and

$$Q(D_j) = \sum_{i=1}^{d^2} \left[(d+1)P(H_i) - \frac{1}{d} \right] P(D_j|H_i) \ .$$

Probabilites for one measurement in terms of the probabilities and conditional probabilities for another.

More ways SICs are special

In dimensions where a SIC exists (and they seem to in all finite dimensions), they

- are optimal measurements for quantum state tomography, even when complete set of MUBs doesn't exist.
- are most sensitive to quantum eavesdropping.
- may be used as "magic states" in quantum computation.
- are a defininig feature of the Singapore protocol for quantum key distribution.

Our representation of the Born Rule was aesthetically motivated, but perhaps the SIC representation is optimal as well?

The General Scenario

- Consider a reference process consisting of a MIC measurement $\{H_i\}$ and the subsequent preparation of a state σ_i from a linearly independent set $\{\sigma_j\}$.
- In the special case where H_i and σ_i are rank-1 and proportional, the post-measurement state is what would be obtained from an application of Lüders' rule.

Coefficients are particularly simple in this basis.

$$\rho = \sum_{i=1}^{d^2} \left[(d+1)P(H_i) - \frac{1}{d} \right] \Pi_i \ .$$

Consider an arbitrary measurement D_j. The probability for outcome j would be

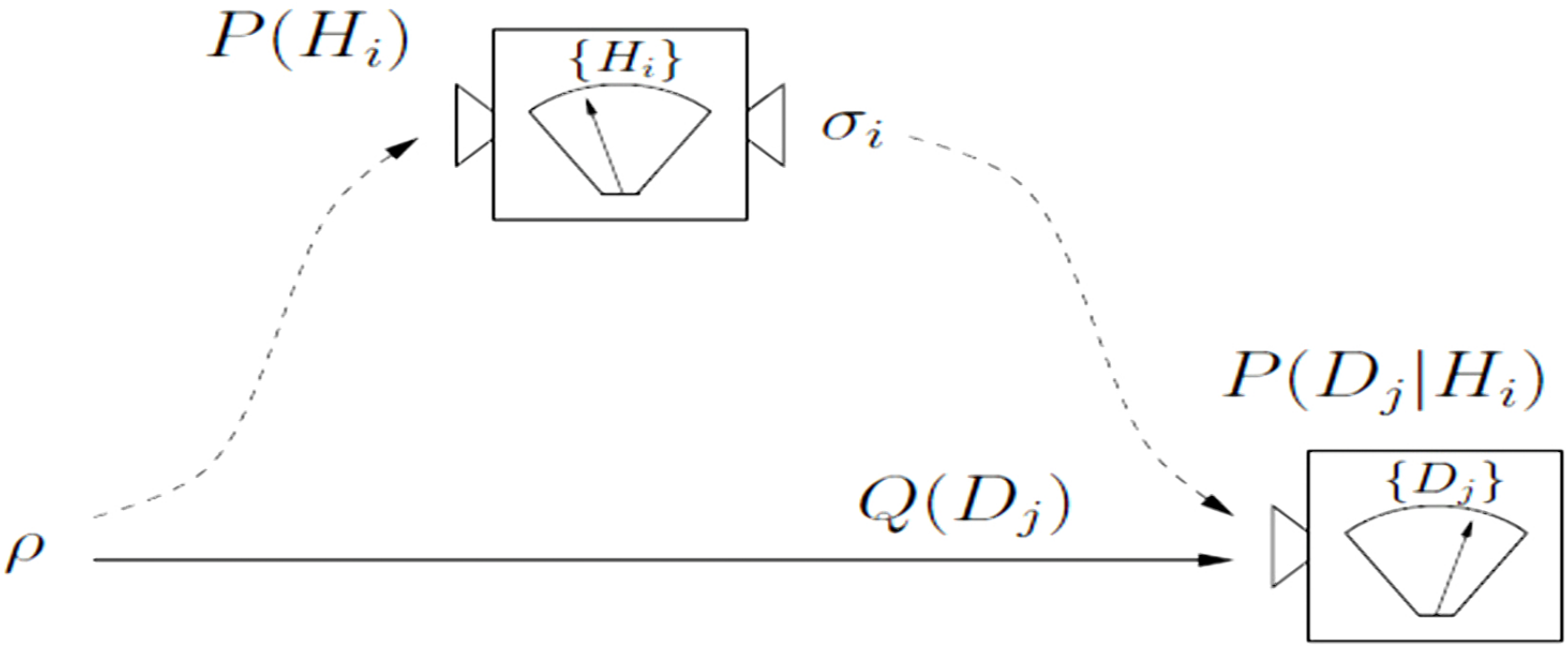
$$Q(D_j) = \mathrm{tr}\rho D_j = \sum_{i=1}^{d^2} \left[(d+1)P(H_i) - \frac{1}{d} \right] \mathrm{tr}\Pi_i D_j \ .$$

If one performed a SIC measurement on ρ and obtained outcome i, they would update to the state Π_i (Lüders' rule). So $\mathrm{tr}\Pi_i D_j = P(D_j|H_i)$ and

$$Q(D_j) = \sum_{i=1}^{d^2} \left[(d+1)P(H_i) - \frac{1}{d} \right] P(D_j|H_i) \ .$$

Probabilites for one measurement in terms of the probabilities and conditional probabilities for another.

The General Scenario



Now repeat the derivation from earlier.

The General Scenario

$$\rho = \sum_j \alpha_j \sigma_j \ .$$

The probability of outcome H_i given this quantum state is

$$P(H_i) = \mathrm{tr}\, \rho H_i = \sum_j \alpha_j \,\mathrm{tr}\, \sigma_j H_i = \sum_j \left[\Phi^{-1}\right]_{ij} \alpha_j \ ,$$

where

$$\left[\Phi^{-1}\right]_{ij} := \mathrm{tr}\, H_i \sigma_j \ .$$

With this we get,

$$\rho = \sum_i \left[\sum_k [\Phi]_{ik} P(H_k) \right] \sigma_i \ .$$

The General Scenario

Similar to before,

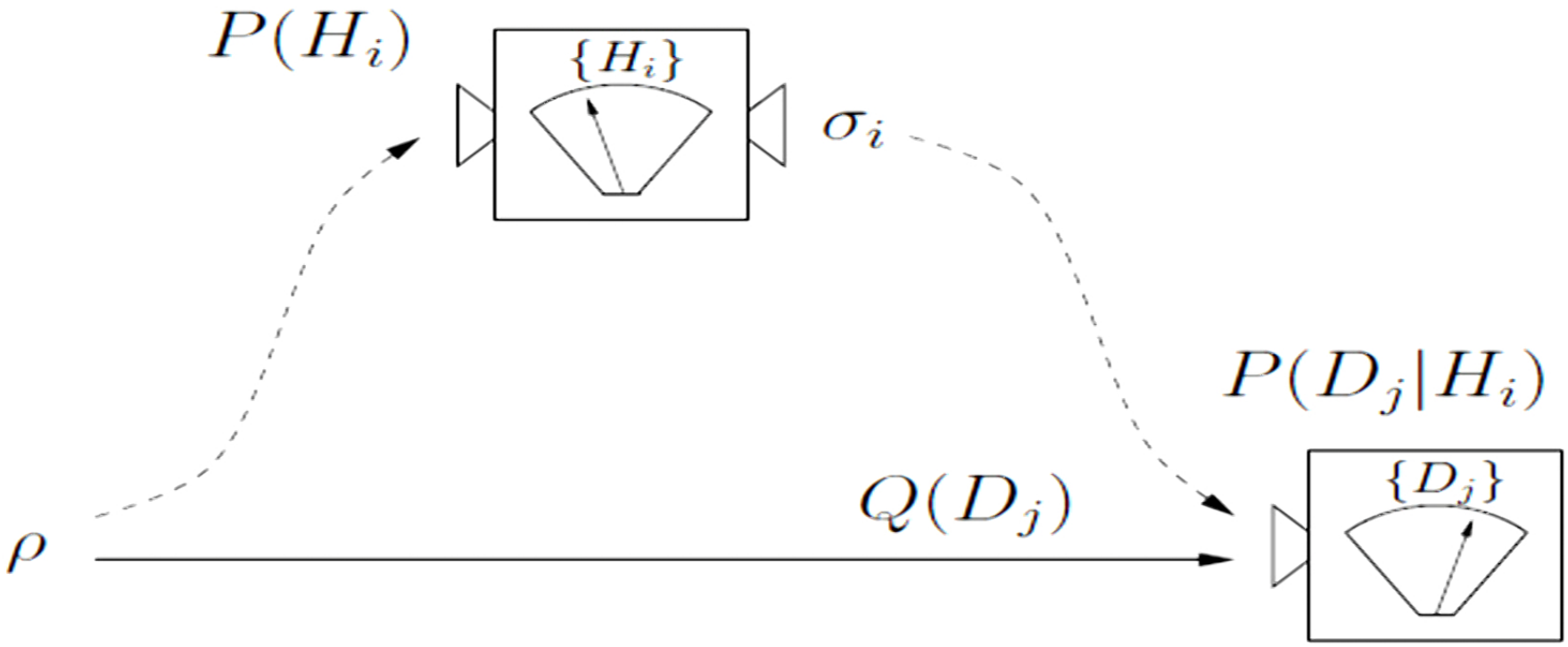
$$Q(D_j) = \mathrm{tr}\,\rho D_j = \sum_i P(D_j|H_i) \left[\sum_k [\Phi]_{ik} P(H_k) \right] ,$$

where $P(D_j|H_i) = \mathrm{tr}\,\sigma_i D_j$.

In more compact vector notation, we might write

$$Q(D) = P(D|H)\,\Phi\,P(H) ,$$

where $P(D|H)$ is a matrix of conditional probabilities.



$$Q(D) = P(D|H)\,\Phi\,P(H)$$

Φ captures deviation from LTP

LTP:

$$P(D) = P(D|H)\,P(H)\ ,$$

Born Rule probability meshing:

$$Q(D) = P(D|H)\,\Phi\,P(H)\ .$$

- Φ is column quasistochastic: real-valued matrix with columns summing to 1.
- If Φ could equal $\mathbb{I}$, then with respect to the corresponding reference process, there'd be no strange probability meshing—we could behave as though there were ontic states.
- All reference processes are valid, but some choices will "scramble subjective and objective into a messier omelette."

What about the No-Go Theorems?

- In MIC form, the Born rule automatically differs from the LTP. Not within ontological models framework (OMF).
- No-go theorems all invoke LTP conditioning on ontic variables. We don't.

$$P(\vec{a}, \vec{b}) = \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda) \qquad (2)$$

Bell (1964)

$$\int_\Lambda \xi_{M,k}(\lambda)\mu_i(\lambda)d\lambda = \langle\psi_i| E_{M,k} |\psi_i\rangle . \qquad (B1)$$

PBR (2012)

$$\int_\Lambda \Pr(E|M,\lambda)d\mu(\lambda) = \mathrm{Tr}\,(E\rho) . \qquad (9)$$

Leifer (2014)

$$P^E_\psi(k) = \langle\, p^E_k \,\rangle_\psi , \qquad (2)$$

Myrvold (2018)

How close can Φ and $\mathbb{I}$ be?

- The remaining differences between the LTP and a representation as "close" as possible would be due to genuinely nonclassical properties of physical reality.
- One way to measure a representation's difference from the LTP is with an operator distance.
- Let's look at the Frobenius distance in the special case that the post-measurement states are proportional to the MIC.

SICs are the closest!

Theorem

Let Φ be the column-quasistochastic matrix associated with a MIC and a proportional set of post-measurement states. Then

$$\|I - \Phi\|_2 \geq d\sqrt{d^2 - 1}\ ,$$

with equality iff the MIC is a SIC.

One direction of proof

Via Schwarz inequality:

$$\|\mathbb{I} - \Phi\|_2^2 \geq \frac{(\mathrm{tr}\,\Phi - d^2)^2}{d^2 - 1} \ .$$

Note that

$$\sum_i \frac{1}{\lambda_i(\Phi)} = \mathrm{tr}\,\Phi^{-1} = \sum_i h_i \mathrm{tr}\,\rho_i^2 \leq \sum_i h_i = d \ .$$

The largest eigenvalue of a stochastic matrix is 1 which means the smallest eigenvalue of Φ is 1, so

$$\sum_{i<d^2} \frac{1}{\lambda_i(\Phi)} \leq d - 1 \ .$$

One direction of proof

Recall the ordering of the Pythagorean means:

$$\frac{1}{n}\sum_i x_i \geq \left(\prod_i x_i\right)^{1/n} \geq \left(\frac{1}{n}\sum_i \frac{1}{x_i}\right)^{-1} .$$

Then

$$\frac{1}{d^2-1}\sum_{i<d^2}\lambda_i(\Phi) = \frac{\mathrm{tr}\,\Phi - 1}{d^2-1} \geq \left(\frac{1}{d^2-1}\sum_{i<d^2}\frac{1}{\lambda_i(\Phi)}\right)^{-1} \geq d+1 ,$$

can be rearranged into

$$\mathrm{tr}\,\Phi \geq 1 + (d^2-1)(d+1) ,$$

with equality iff $\lambda_{i<d^2}(\Phi) = d+1$. This is the SIC value.

SICs are the closest!

Theorem

Let Φ be the column-quasistochastic matrix associated with a MIC and a proportional set of post-measurement states. Then

$$\|I - \Phi\|_2 \geq d\sqrt{d^2 - 1}\ ,$$

with equality iff the MIC is a SIC.

Majorization

We can prove a much stronger result. For this, it is helpful to review the theory of majorization.

We say that x *weakly majorizes* y, denoted $x \succ_w y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \geq \sum_{i=1}^{k} y_i^{\downarrow} , \quad \text{for } k = 1, \ldots, N .$$

If the last inequality is an equality, we say x majorizes y, denoted $x \succ y$.

Some notation

- The Φ matrix associated with the special case of a SIC measurement and proportional SIC update is denoted Φ_{SIC}.
- $s(A)$ is the vector of singular values of a matrix A sorted in descending order.

Majorization Lemma

Lemma

Let Φ be the column-quasistochastic matrix associated with an arbitrary reference process. Then

$$s(\Phi) \succ_{w\log} s(\Phi_{\rm SIC}) \ ,$$

with equality iff the MIC and post-measurement states are SICs.

Sketch of part of the proof

After significant amount of work, reason that

$$|\det \Phi| \geq (d+1)^{d^2-1} = \det \Phi_{\text{SIC}} \ .$$

For any matrix, $s(A) \succ_{\log} |\lambda(A)|$ and

$$\begin{aligned}
\log|\lambda(\Phi)| &\succ \left(\frac{\sum_{i=1}^{d^2-1} \log|\lambda_i(\Phi)|}{d^2-1}, \ldots, \frac{\sum_{i=1}^{d^2-1} \log|\lambda_i(\Phi)|}{d^2-1}, 0 \right) \\
&= \left(\frac{\log|\det \Phi|}{d^2-1}, \ldots, \frac{\log|\det \Phi|}{d^2-1}, 0 \right) \\
&\succ_w \left(\frac{\log \det \Phi_{\text{SIC}}}{d^2-1}, \ldots, \frac{\log \det \Phi_{\text{SIC}}}{d^2-1}, 0 \right) \\
&= (\log(d+1), \ldots, \log(d+1), 0) = \log \lambda(\Phi_{\text{SIC}}) \ , \\
\implies \ s(\Phi) &\succ_{\log} |\lambda(\Phi)| \succ_{w \log} \lambda(\Phi_{\text{SIC}}) = s(\Phi_{\text{SIC}}) \ .
\end{aligned}$$

All unitarily invariant norms!

A unitarily invariant norm is one such that $\|A\| = \|UAV\|$.

Theorem

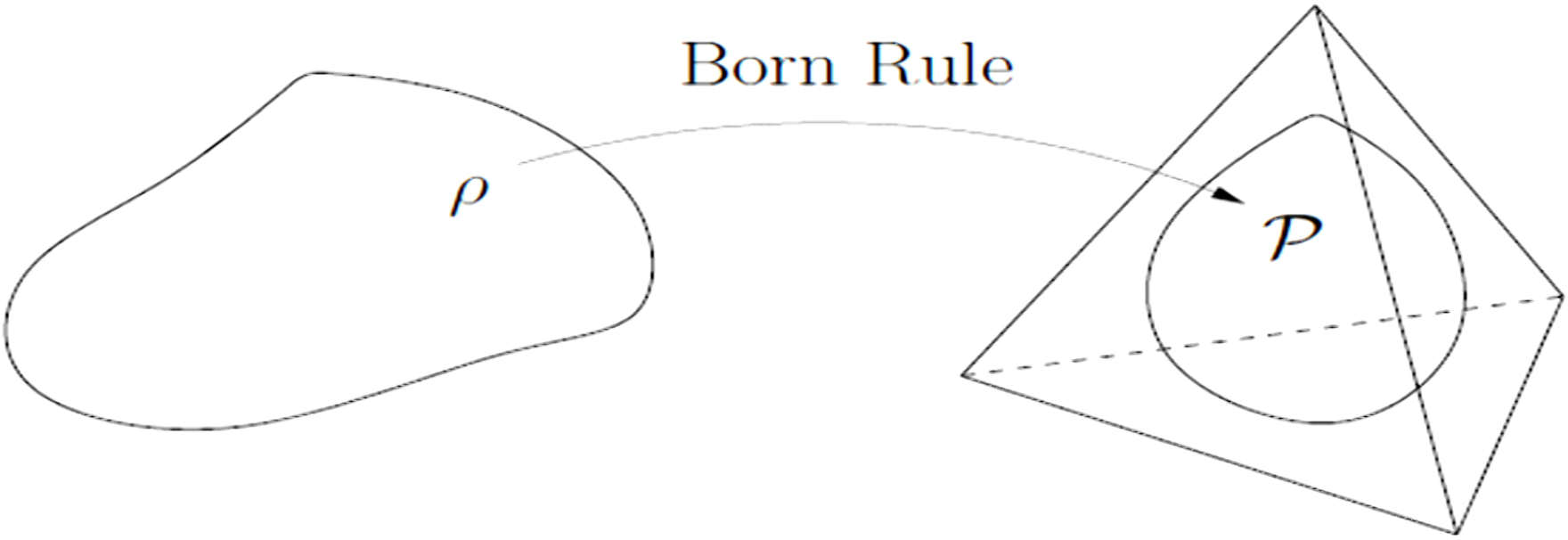
Let Φ be the column-quasistochastic matrix associated with an arbitrary reference process. Then for any unitarily invariant norm $\|\cdot\|$,

$$\|\mathbb{I} - \Phi\| \geq \|\mathbb{I} - \Phi_{\mathrm{SIC}}\| \ ,$$

with equality iff the MIC and post-measurement states are SICs.

- Schatten p-norms (including trace norm, Frobenius norm, operator norm).
- Ky Fan k-norms.
- Any norm depending only on singular values.

Majorization keeps giving



Majorization keeps giving

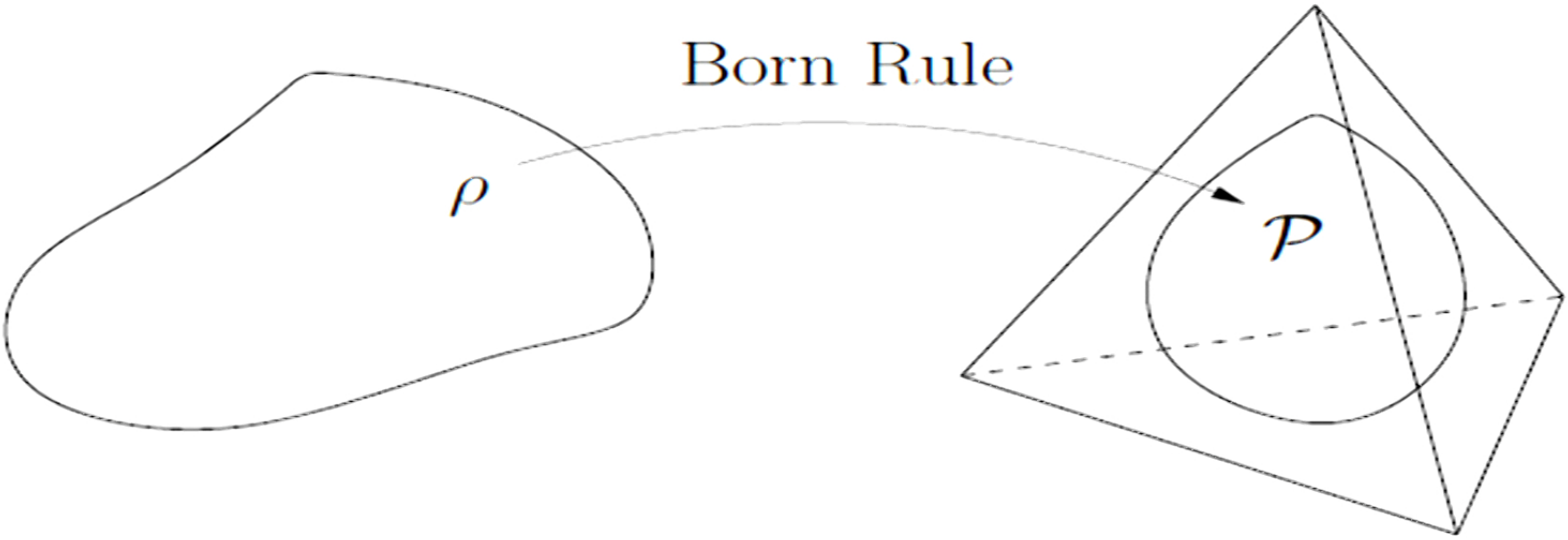
Theorem

For any MIC in dimension d, let $\mathcal{P}$ denote the image of quantum state space under the Born rule and let $\mathrm{vol}_{\mathrm{E}}(\mathcal{P})$ denote its Euclidean volume. Then

$$\mathrm{vol}_{\mathrm{E}}(\mathcal{P}) \leq \mathrm{vol}_{\mathrm{E}}(\mathcal{P}_{\mathrm{SIC}}) \ ,$$

with equality iff the MIC is a SIC.

Another notion of deviation from classicality



SICs give the largest possible volume.

Future work

- Similarly motivated search for optimal representations of general quantum time evolutions to see if results coincide.
- Full quantum reconstruction taking the SIC LTP analog as one of the axioms.
  - Progress has been made.[1] Last few assumptions stronger than we think necessary.
- Connection between quasiprobability representations and Φ, may lead to way to easily move between each.

[1] M. Appleby, C. A. Fuchs, B. C. Stacey, and H. Zhu, "Introducing the Qplex: A Novel Arena for Quantum Theory". Eur. Phys. J. D (2017) 71: 197

For more information about SICs, see:

C. A. Fuchs, M. C. Hoang, and B. C. Stacey, "The SIC Question: History and State of Play". Axioms 2017, 6(3), 21; doi:10.3390/axioms6030021.

Image credits:

- "E.T. Jaynes," Wikimedia Commons user Tomixdf. Public domain.
- "Omelette," Renee Comet. Public domain.
- "Neils Bohr," AB Lagrelius & Westphal. Public domain.
- "Werner Heisenberg," Bundesarchiv, Bild 183-R57262. CC-BY-SA 3.0.
- "William Wootters photo," Sci.Tech.Café. https://scitechcafe.files.wordpress.com/2015/12/wootters-photo.jpg.
- "Bruno de Finetti in 1928," Fulvia de Finetti. http://terzadecade.it/wp-content/uploads/2013/03/de-Finetti-nel-1928.jpg.