Title: A Kazhdan-Lusztig algorithm for Whittaker modules

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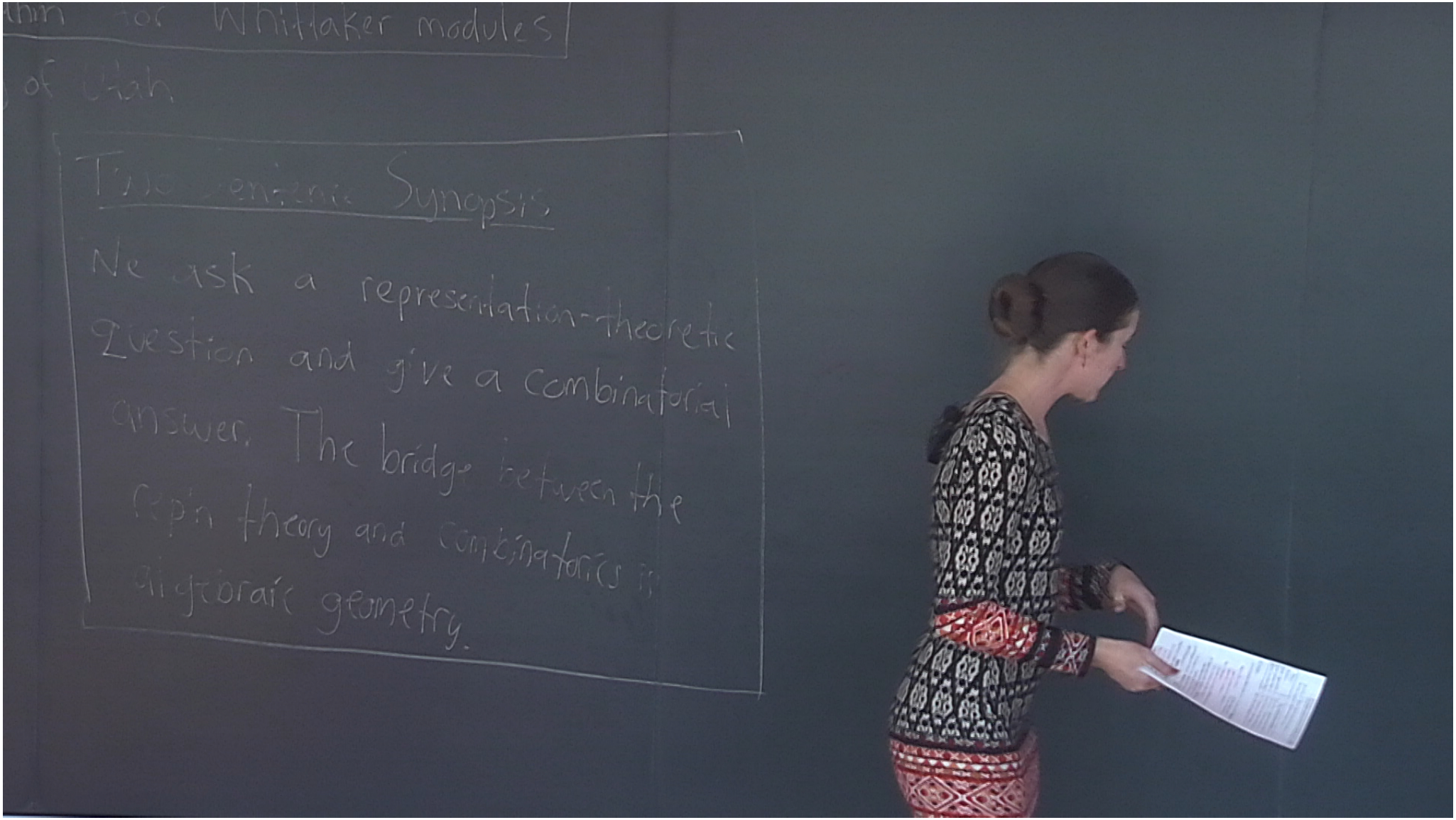
Abstract: This project examines the structure of a certain category of representations of a Lie algebra called Whittaker modules. Whittaker modules generalize highest weight modules, and the structure of the category is similar to that of Bernstein-Gelfand-Gelfandâ€™s category O. In particular, Whittaker modules have finite length composition series and all irreducible Whittaker modules appear as quotients of standard Whittaker modules, which are generalizations of Verma modules. Using the localization theory of Beilinson-Bernstein, one obtains a beautiful geometric description of Whittaker modules as twisted sheaves of D-modules on the associated flag variety. I use this geometric setting to develop an algorithm for computing the multiplicities of irreducible Whittaker modules in the composition series of standard Whittaker modules. This algorithm establishes that the multiplicities are determined by parabolic Kazhdan-Lusztig polynomials.

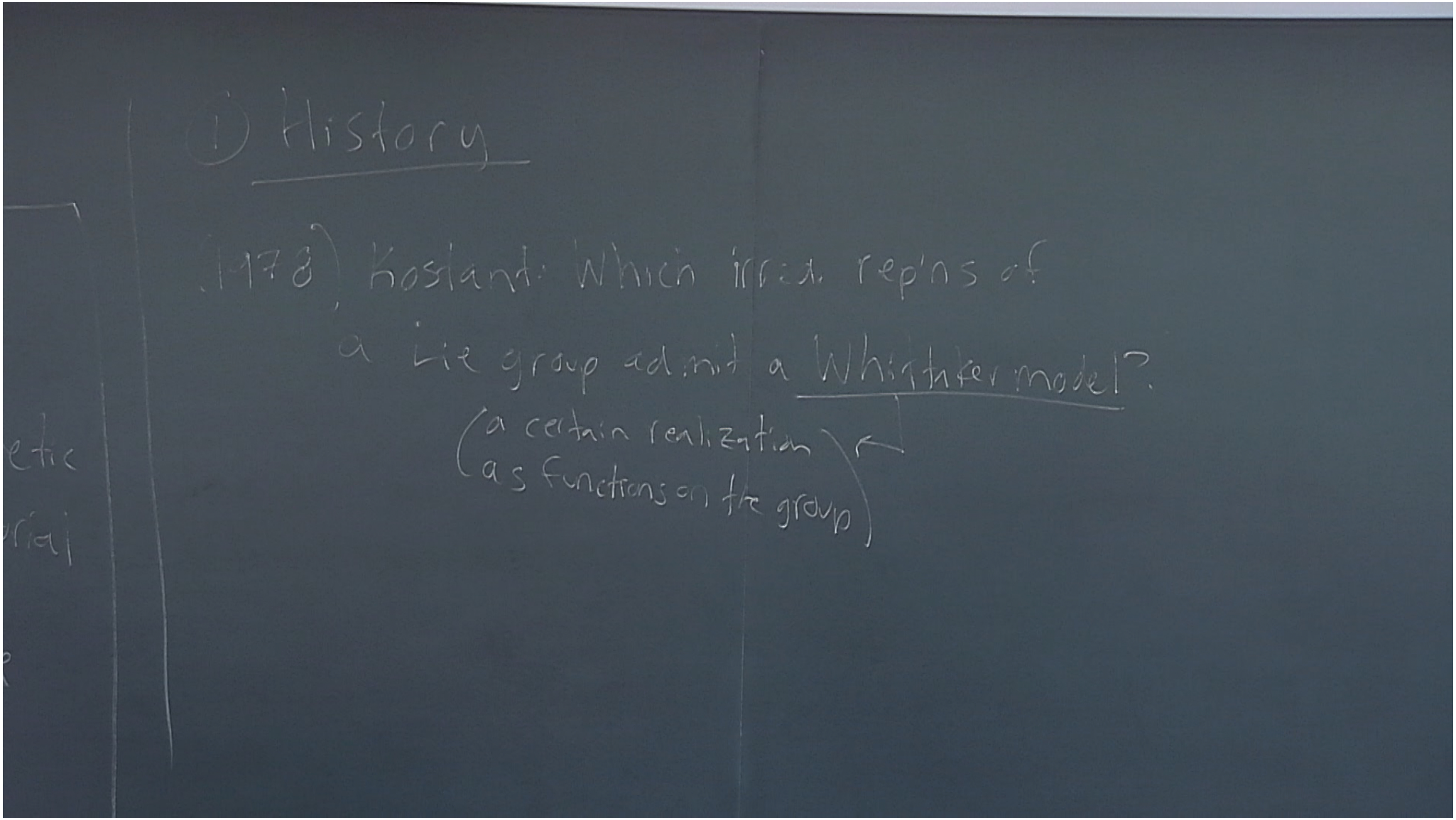
A Kazhdan-Lusztig algorithm for Whittaker modules

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Plan

1. History
2. The algebraic story
3. The geometric story
4. The combinatorial story


for Whittaker modules
of Utah
Synopsis:
We ask a representation-theoretic question and give a combinatorial answer. The bridge between the rep'n theory and combinatorics is algebraic geometry.


① History
1978 Kostant: Which irred. rep'ns of
a Lie group admit a Whittaker model?
(a certain realization
as functions on the group)

1) History

1978) Kostant: Which irr. reps of a Lie group admit a Whittaker model?

(a certain realization as functions on the group)

- of interest: vectors which transform according to a nondegenerate character of π

→ Whittaker vectors, Whittaker modules

in the process: A new class of reps of Lie g/g.

(1979): Block: all irreps of $\mathfrak{sl}(2,\mathbb{C})$

are:

1) highest wt modules
2) Whittaker modules
3) other family

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model?

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modules

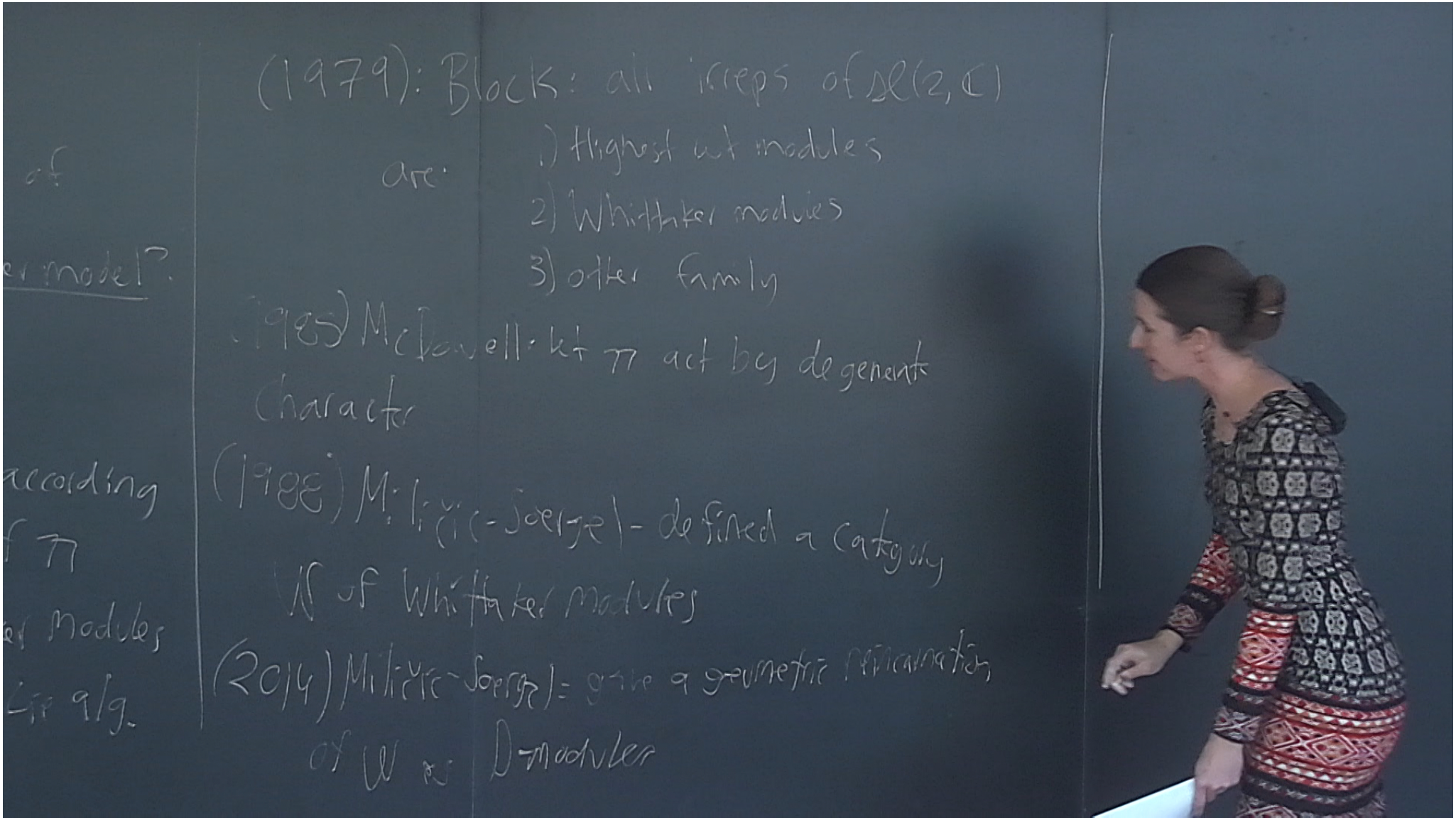
g/g.

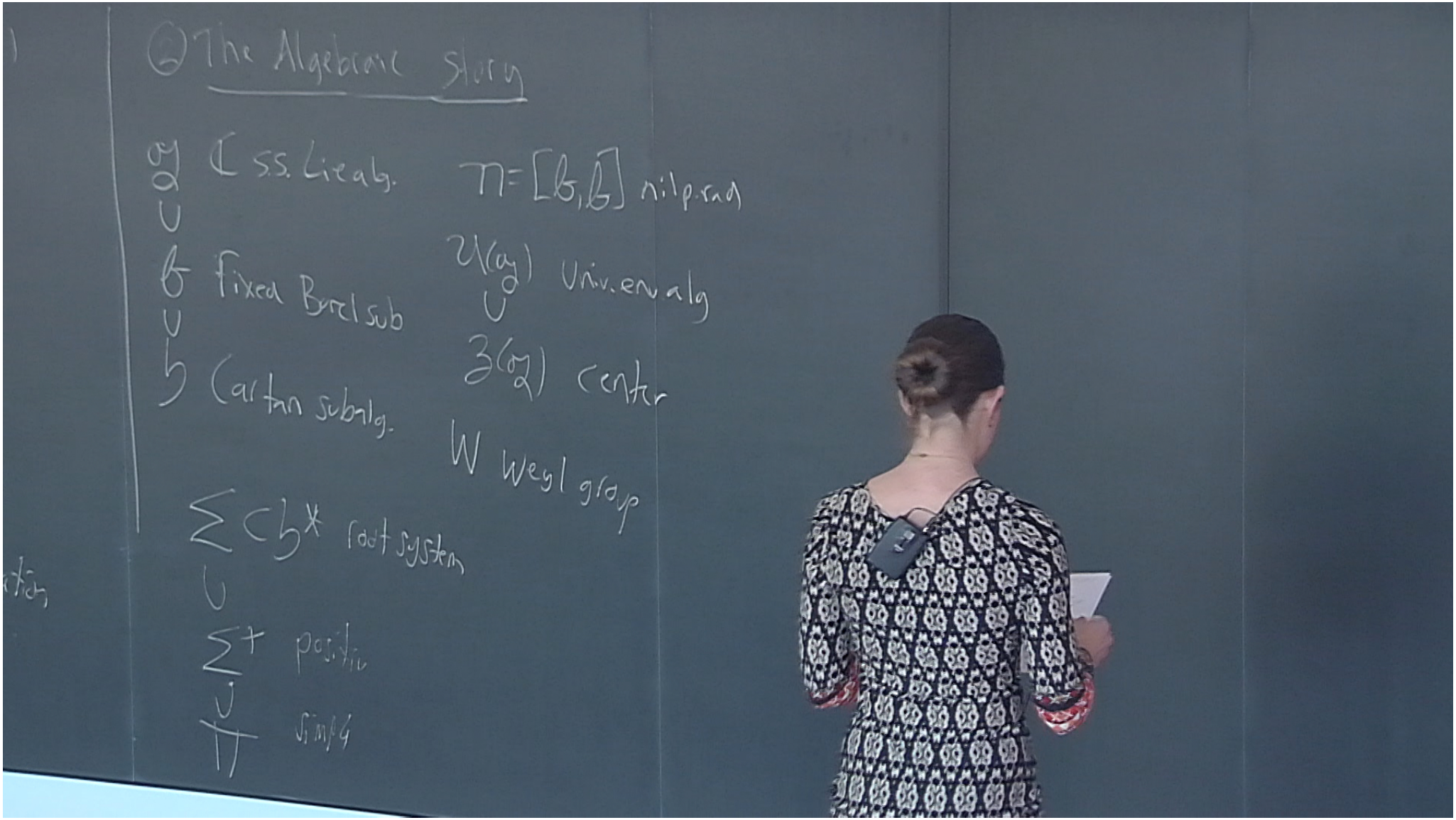
(1979): Block: all irreps of $\mathfrak{sl}(2,\mathbb{C})$ are:

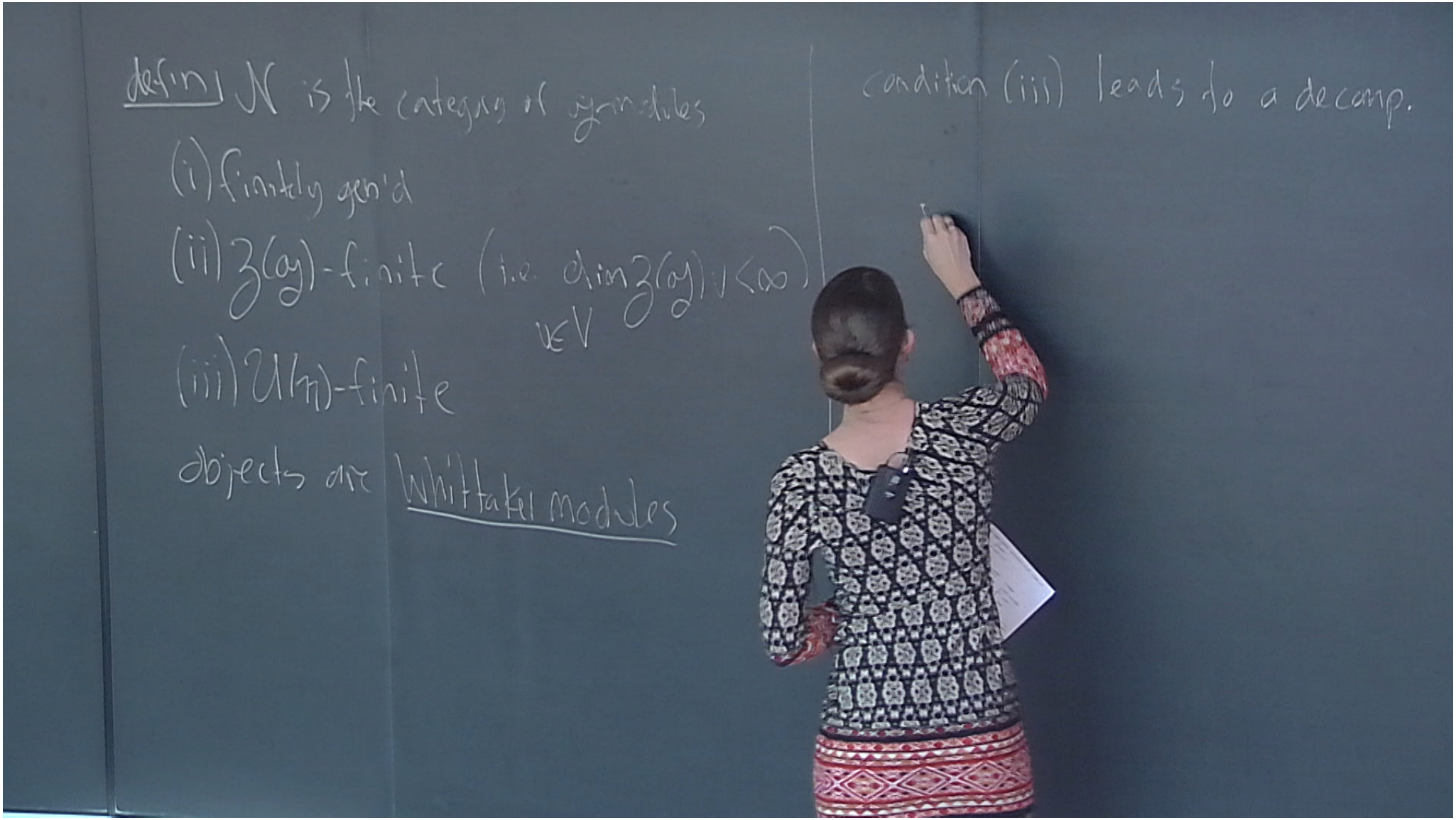
1) highest wt modules
2) Whittaker modules
3) other family

(1985) McDowell: $\mathfrak{h}$ act by degenerate character

(1988) Milicic-Soergel - defined a category $\mathcal{N}$ of Whittaker modules


(1979): Blocks: all irreps of $\mathfrak{sl}(2,\mathbb{C})$
are:
1) highest wt modules
2) Whittaker modules
3) other family
(1985) McDowell: π act by degenerate character
(1988) Milicic-Soergel: defined a category of Whittaker Modules
(2014) Milicic-Soergel: gave a geometric realization of W as D-modules


② The Algebraic Story
g
C ss. Lie alg.
n = [b, b] nilp. rad
U(g)
b
fixed Borel sub
Univ. env alg
U
3(g)
Cartan subalg.
center
W Weyl group
Σ ⊂ b*
root system
U
Σ+
positiv
U
Π
simple


defn: $\mathcal{N}$ is the category of $\mathfrak{g}$-modules
(i) finitely gen'd
(ii) $Z(\mathfrak{g})$-finite (i.e. $\dim Z(\mathfrak{g})v < \infty$ $\forall v \in V$)
(iii) $U(\mathfrak{n})$-finite
objects are Whittaker modules
condition (iii) leads to a decomp.

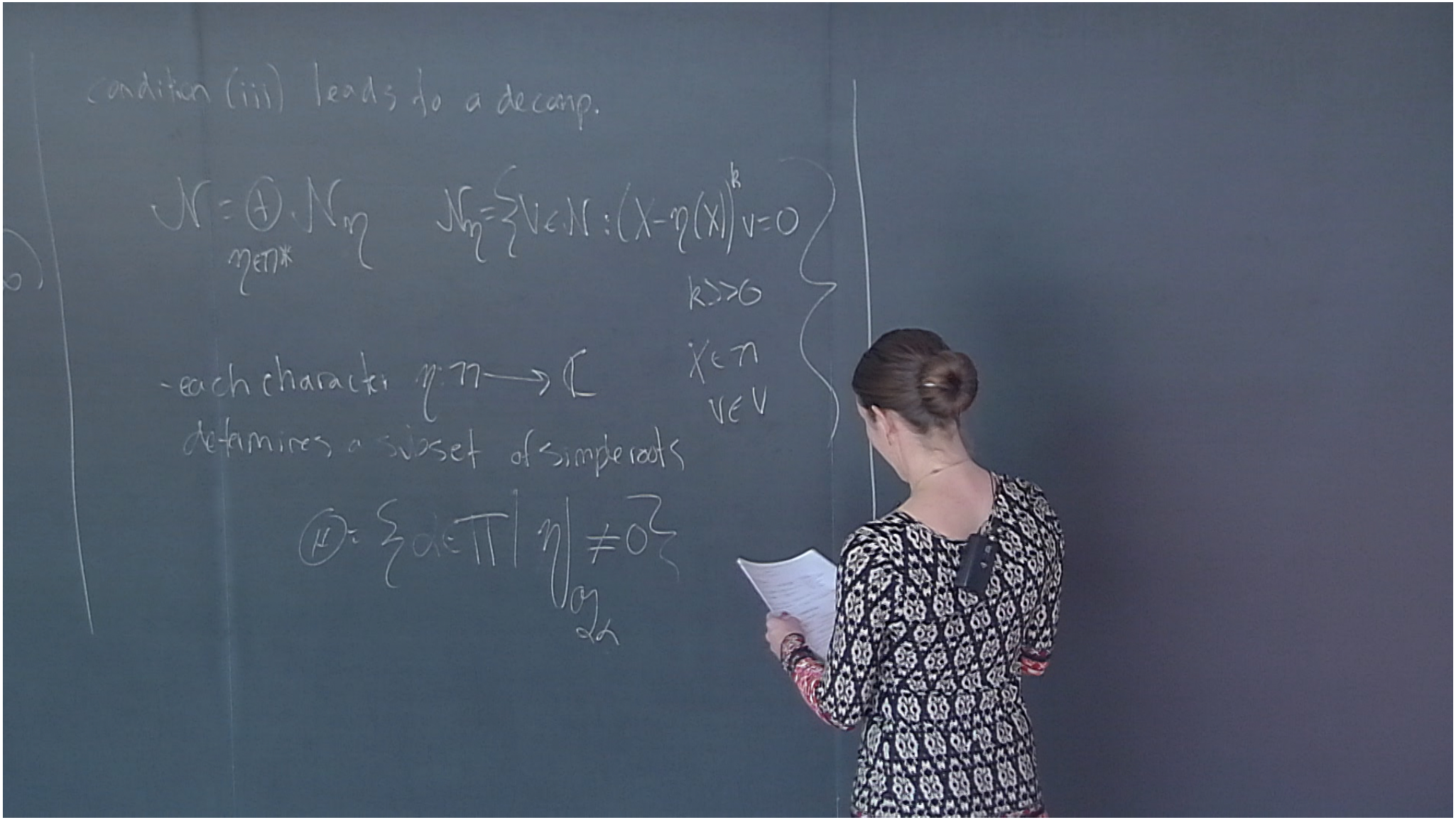
condition (iii) leads to a decomp.

$$V = \bigoplus_{\eta \in \Pi^*} V_\eta \qquad V_\eta = \{ v \in V : (X - \eta(X))^k v = 0 \}$$

$k >> 0$

$X \in \eta$

$v \in V$

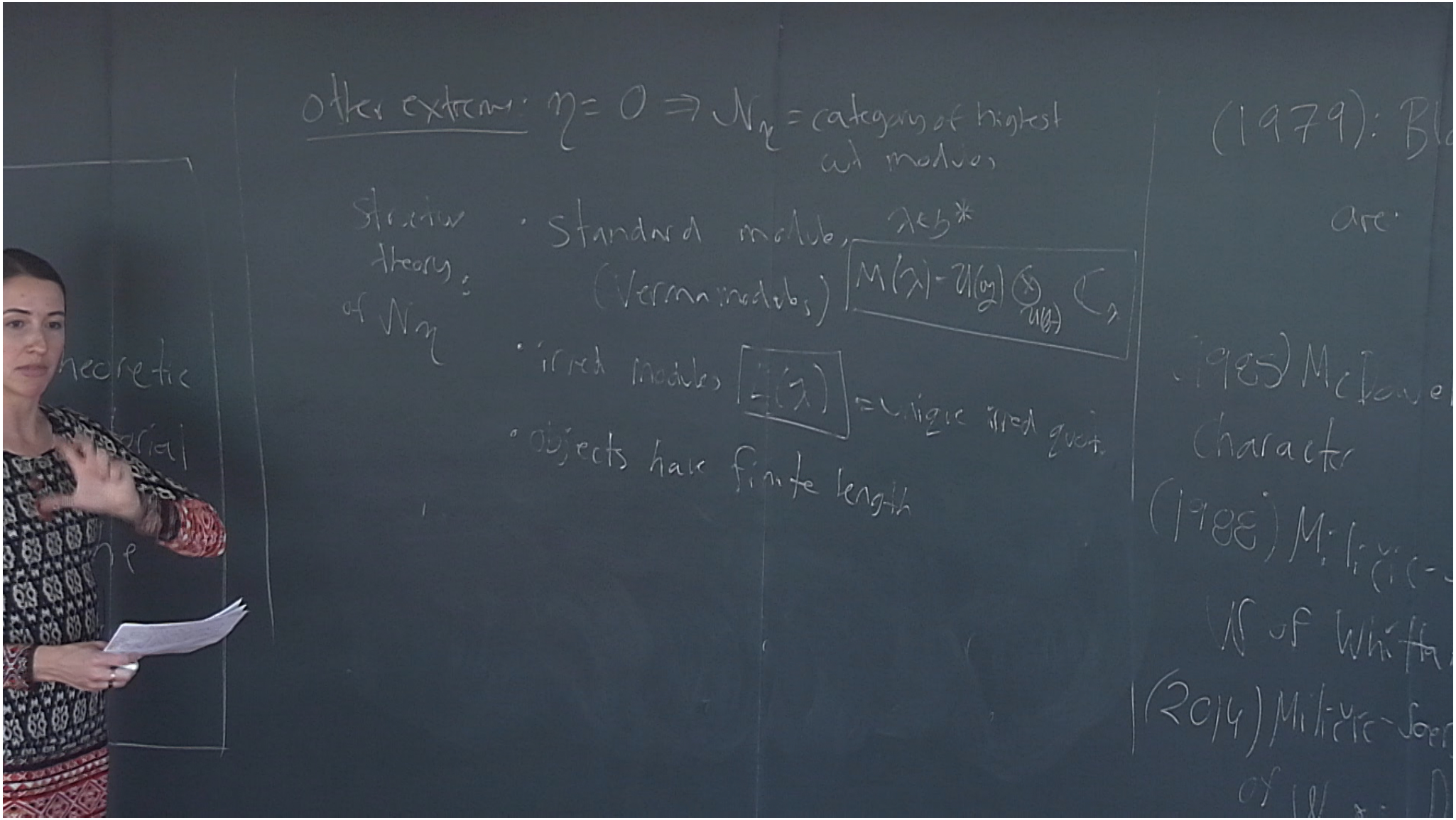

condition (iii) leads to a decomp.
$V = \bigoplus_{\eta \in \Pi^*} V_\eta$
$V_\eta = \{v \in V : (X - \eta(X))^k v = 0$
$k >> 0$
$X \in \pi$
$v \in V$
- each character $\eta : \pi \to \mathbb{C}$
determines a subset of simple roots
$\Theta = \{\alpha \in \Pi \mid \eta|_{\mathfrak{g}_\alpha} \neq 0\}$

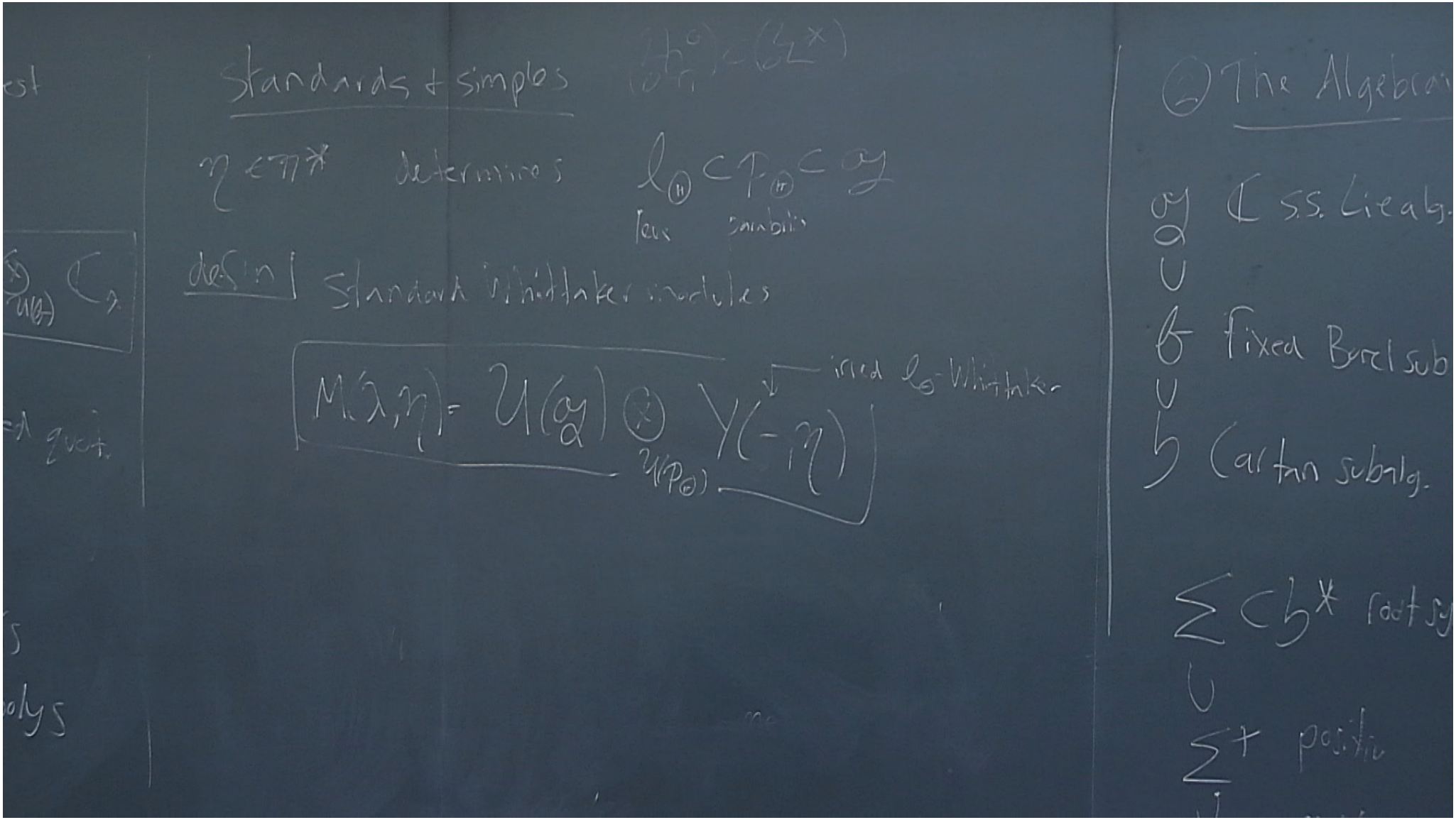
One extreme: η "nondegenerate" (i.e. $\textcircled{H} = \Pi$)

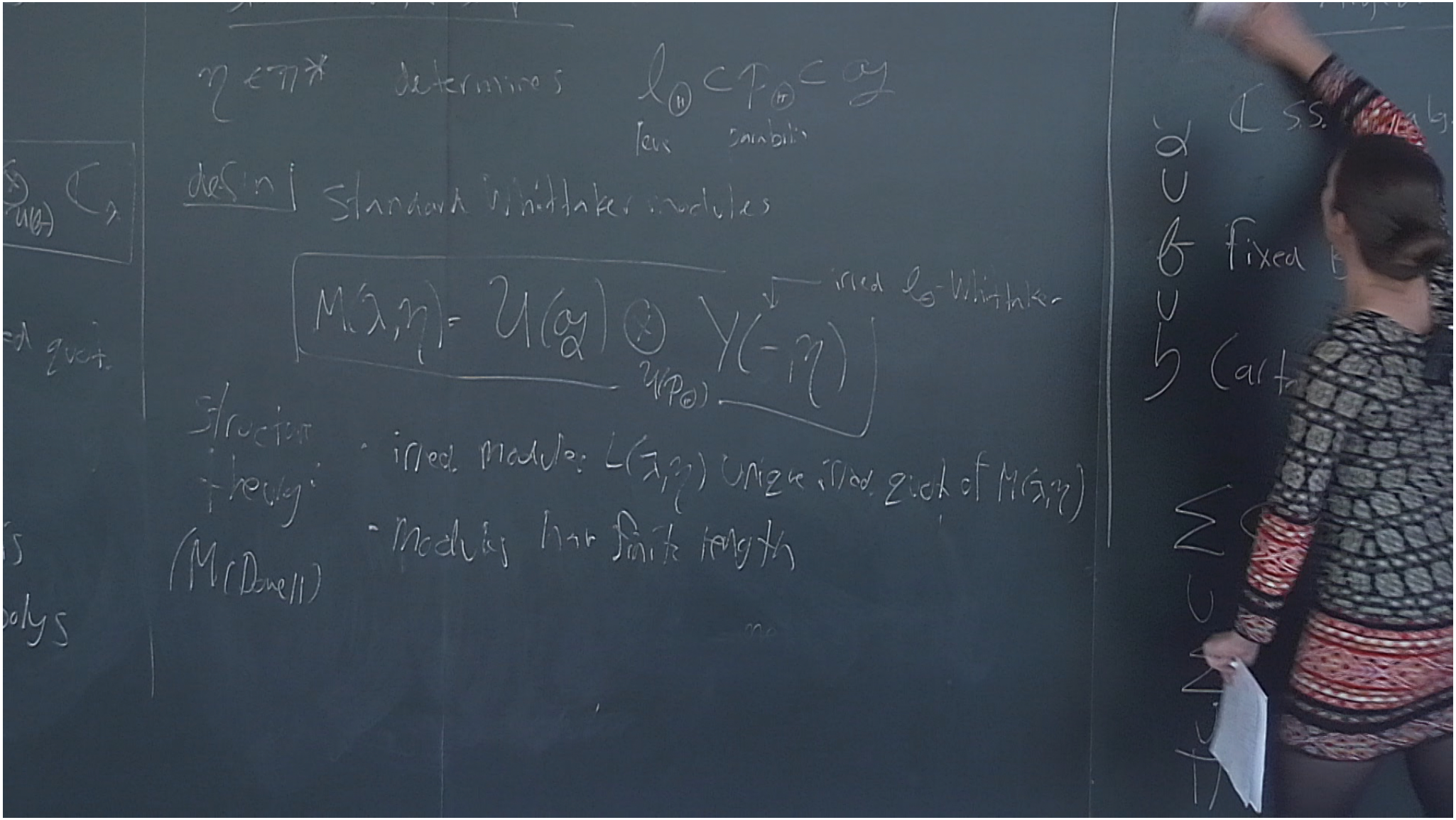
Structure of $\mathcal{N}_\eta$:

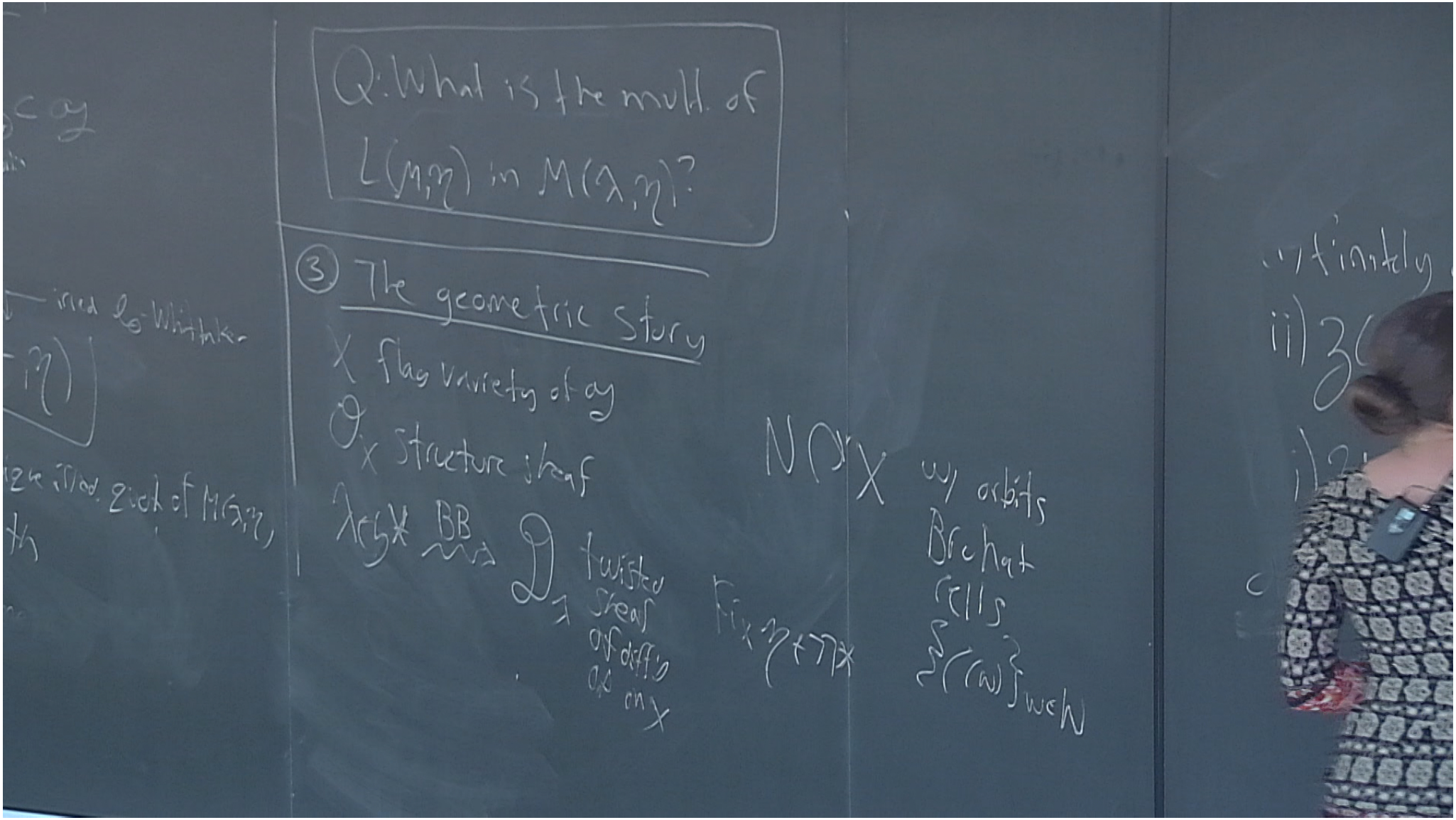
- standard modules $\lambda \in \mathfrak{h}^*$
- $Y(\lambda, \eta)$ irreducible
- $\mathcal{N}_\eta \simeq$ category of f.d. $\mathfrak{z}(\mathfrak{g})$-mod

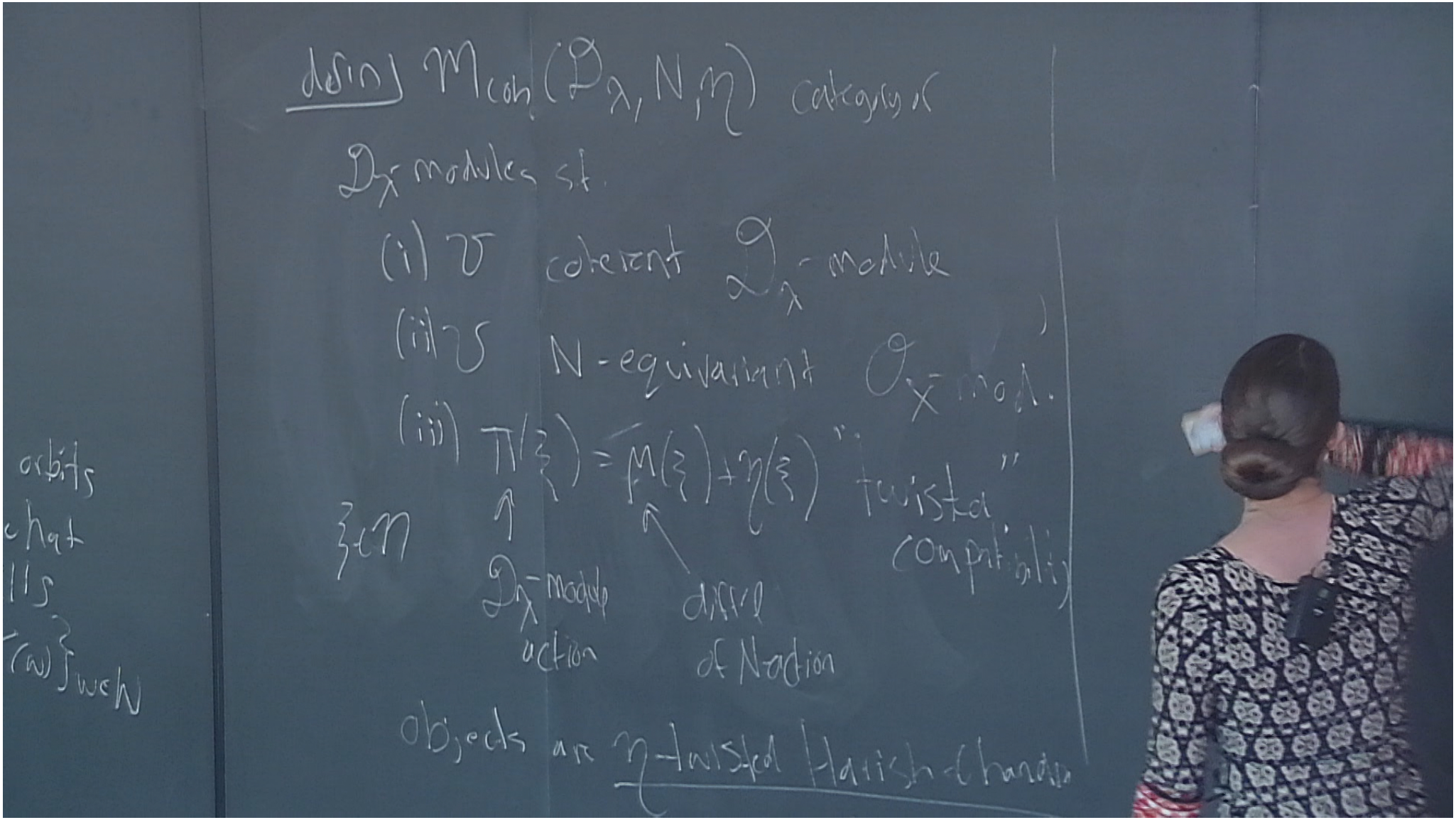
$Y(\lambda,\eta) = \mathbb{C}_{\lambda,\eta} \otimes_{\mathfrak{z}(\mathfrak{g}) \otimes \mathfrak{z}(\mathfrak{g}(\mathcal{O}_{\mathbb{R}}(\eta)))} \mathfrak{z}(\mathfrak{g})$

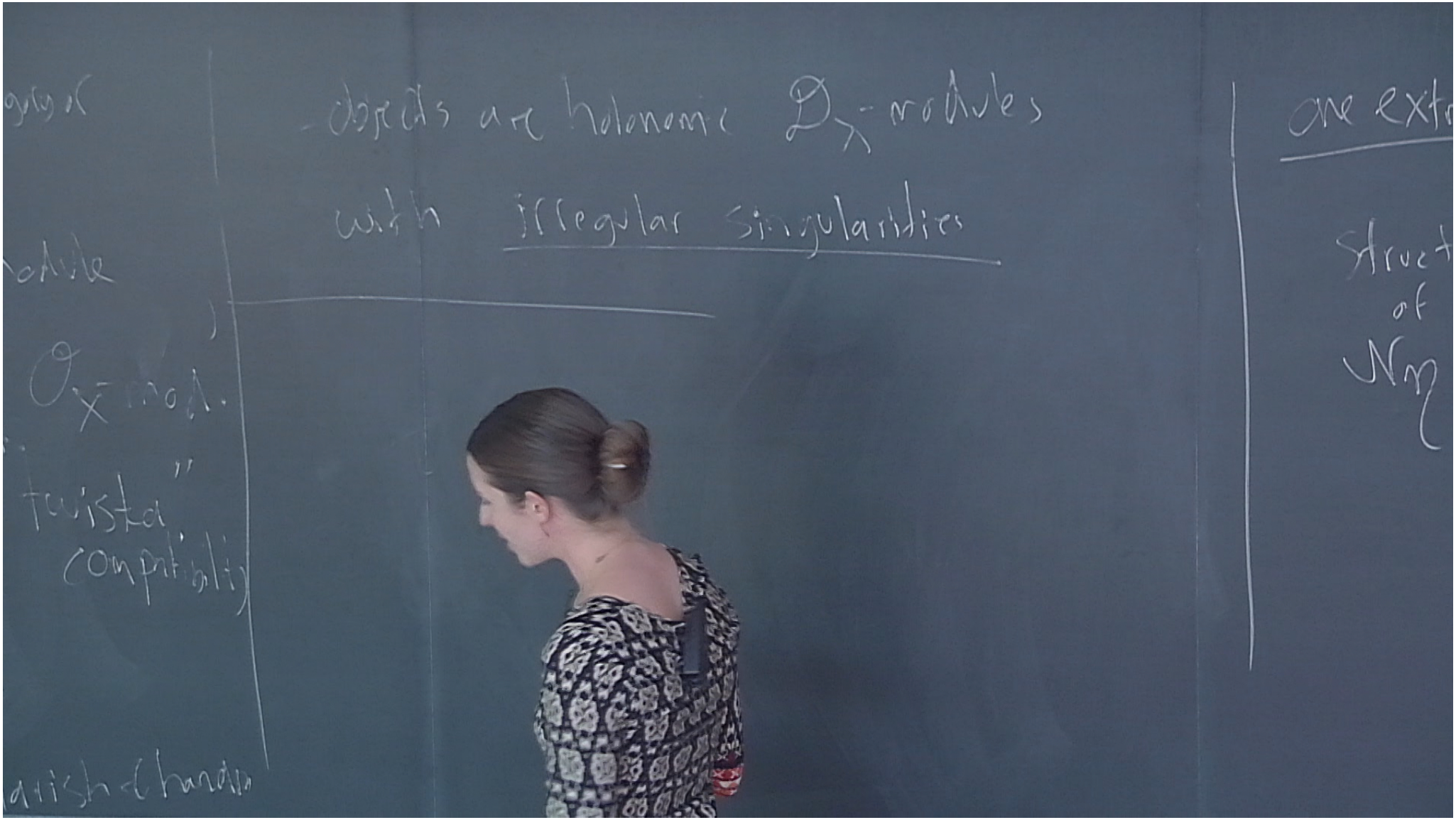

other extreme: $\eta = 0 \Rightarrow \mathcal{N}_\eta$ = category of highest wt modules
structure theory of $\mathcal{N}_\eta$
• standard modules, $\lambda \in \mathfrak{h}^*$ (Verma modules) $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
• irred modules $L(\lambda)$ = unique irred quot.
• objects have finite length
(1979): Bl
are:
(1985) M
character
(1988) M
(2014) Mili

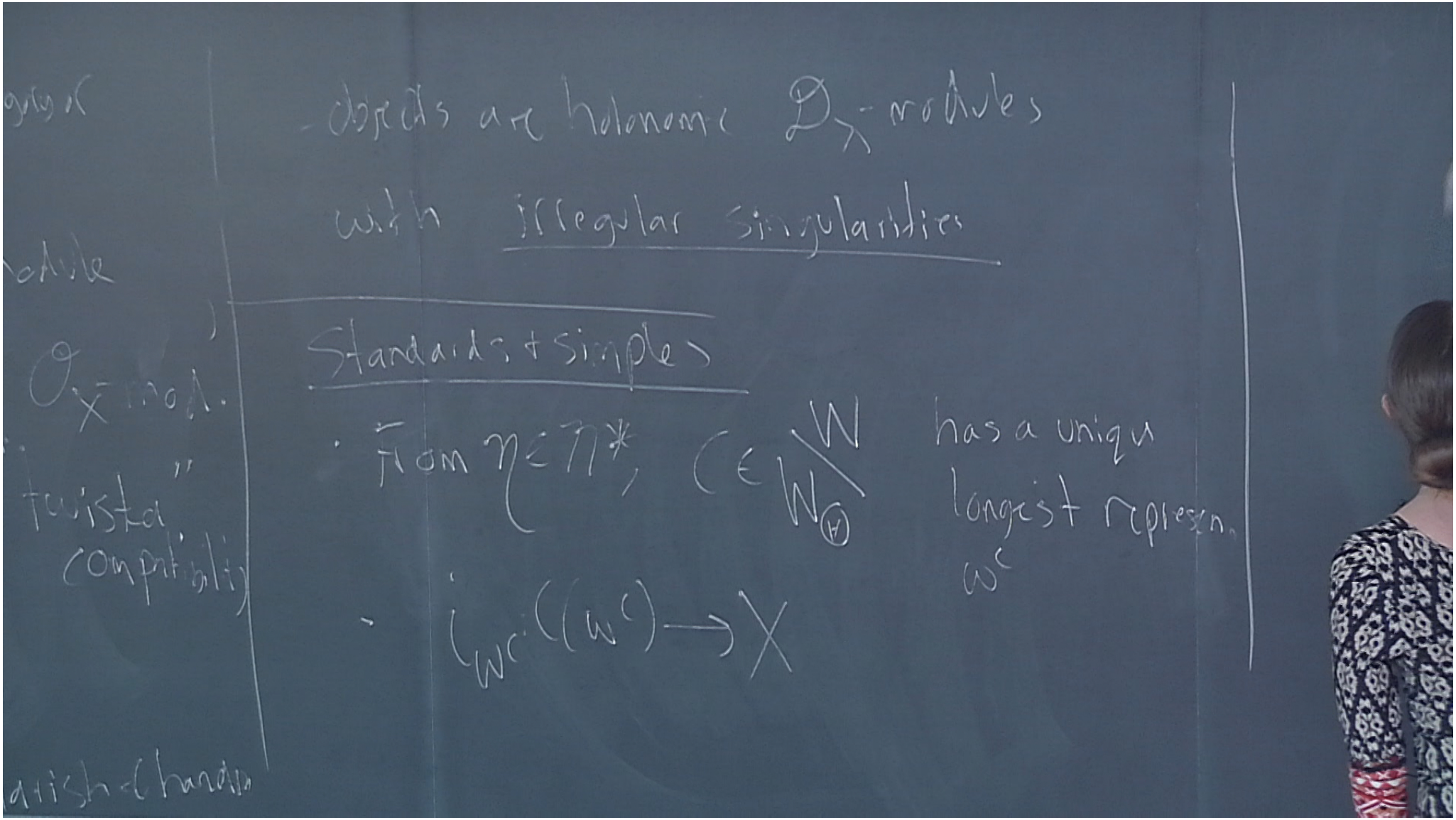

Standards + simples
$\eta \in \mathfrak{h}^*$ determines
$\mathfrak{l}_0 \subset \mathfrak{p}_\Theta \subset \mathfrak{g}$
Levi
parabolic
define Standard Whittaker modules
$M(\lambda, \eta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} Y(-\eta)$
fixed $\mathfrak{l}_0$-Whittaker
② The Algebra
$\mathfrak{g}$ C s.s. Lie alg.
$\mathfrak{b}$ fixed Borel sub
$\mathfrak{h}$ Cartan subalg.
$\Sigma \subset \mathfrak{h}^*$ root sy
Σ^+ positiv


$\eta \in \Pi^*$ determines $\mathfrak{l}_0 \subset \mathfrak{p}_\Theta \subset \mathfrak{g}$
Levi
parabolic
definition
standard Whittaker modules
$M(\lambda,\eta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} Y(-,\eta)$
irred $\mathfrak{l}_0$-Whittaker
Structure theory:
- irred. Modules $L(\lambda,\eta)$ unique irred. quot. of $M(\lambda,\eta)$
- modules have finite length
(McDowell)


Q: What is the mult. of L(η,η) in M(λ,η)?
③ The geometric story
X flag variety of g
$\mathcal{O}_X$ structure sheaf
twisted sheaves of diffl ops on X
N(T)\X
w/ orbits
Bruhat cells
finitely
ii) 3(


defin $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ category of
$\mathcal{D}_X$-modules s.t.
(i) $\mathcal{V}$ coherent $\mathcal{O}_X$-module
(ii) $\mathcal{V}$ N-equivariant
(iii) $\Pi(\xi) = M(\xi) + \eta(\xi)$ "twisted commutativity"
$\xi \in n$
$\mathcal{D}_X$-module action
differ of N-action
objects are η-twisted Harish-Chandra
orbits
chat
lls
$\Gamma(w)\}_{w \in N}$


- objects are holonomic $\mathcal{D}_X$-modules
with irregular singularities
twisted
compatibility


- objects are holonomic $\mathcal{D}_X$-modules
with irregular singularities
Standards + simples
- Fix $\eta \in \mathfrak{h}^*$, $C \in \mathfrak{h}/W$ has a unique
longest represen.
w^c
- $i_{w^c}((w^c) \to X$

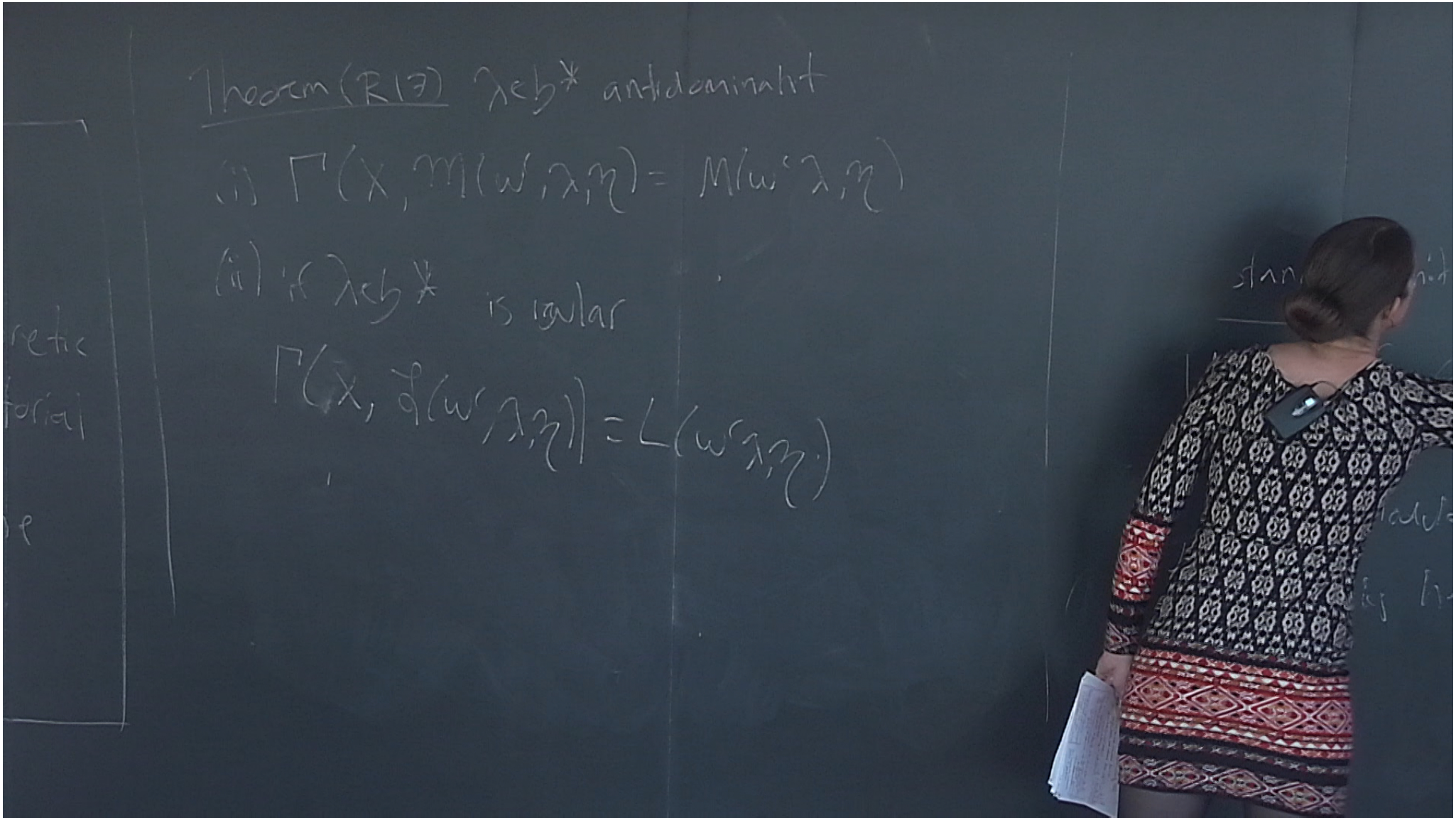
defn standard η-twisted HC sheaf

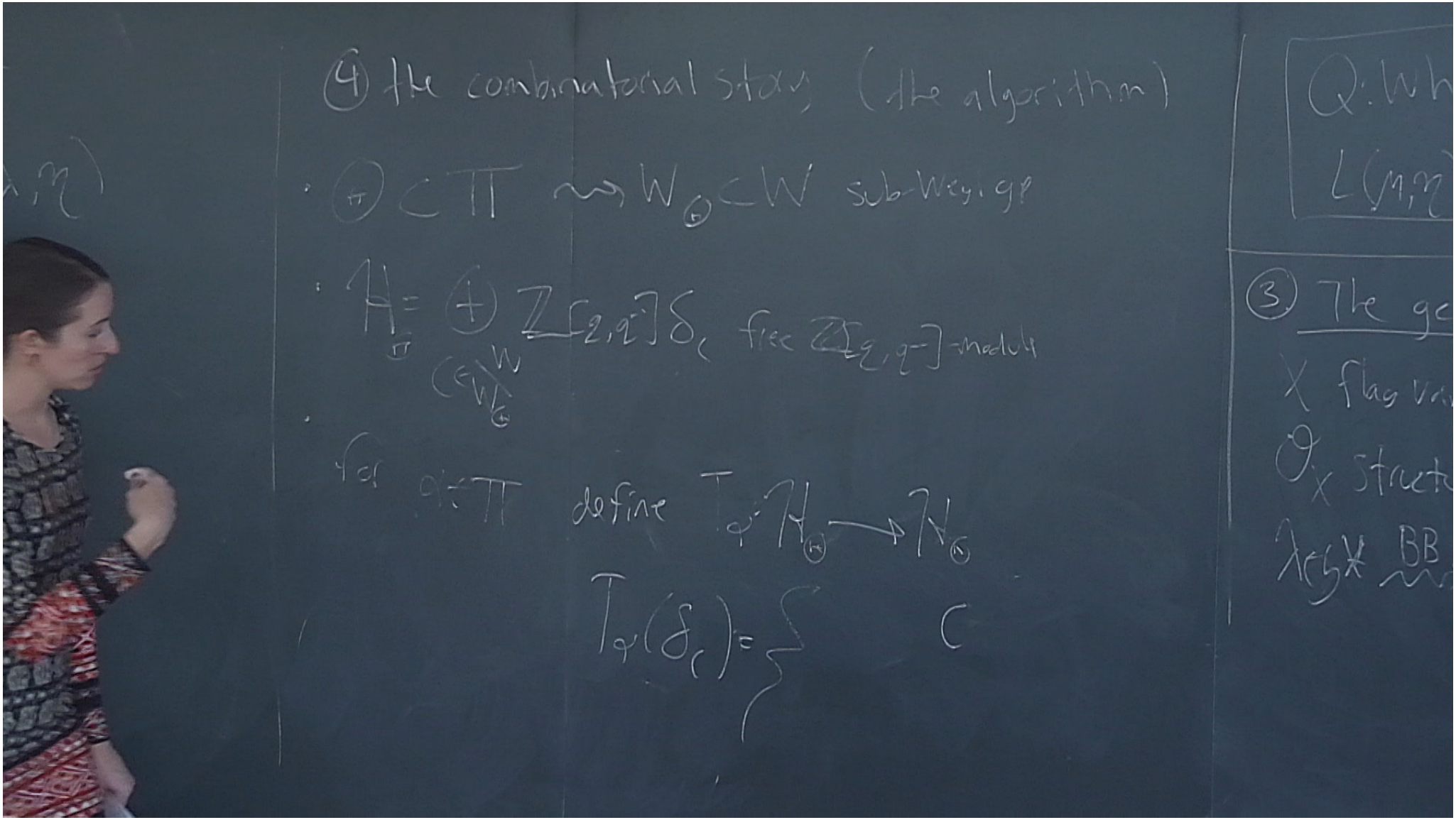
$$\mathcal{I}(w^c, \lambda, \eta) = i_{w^c +} \left(\mathcal{O}_{((w^c), \eta} \right)$$

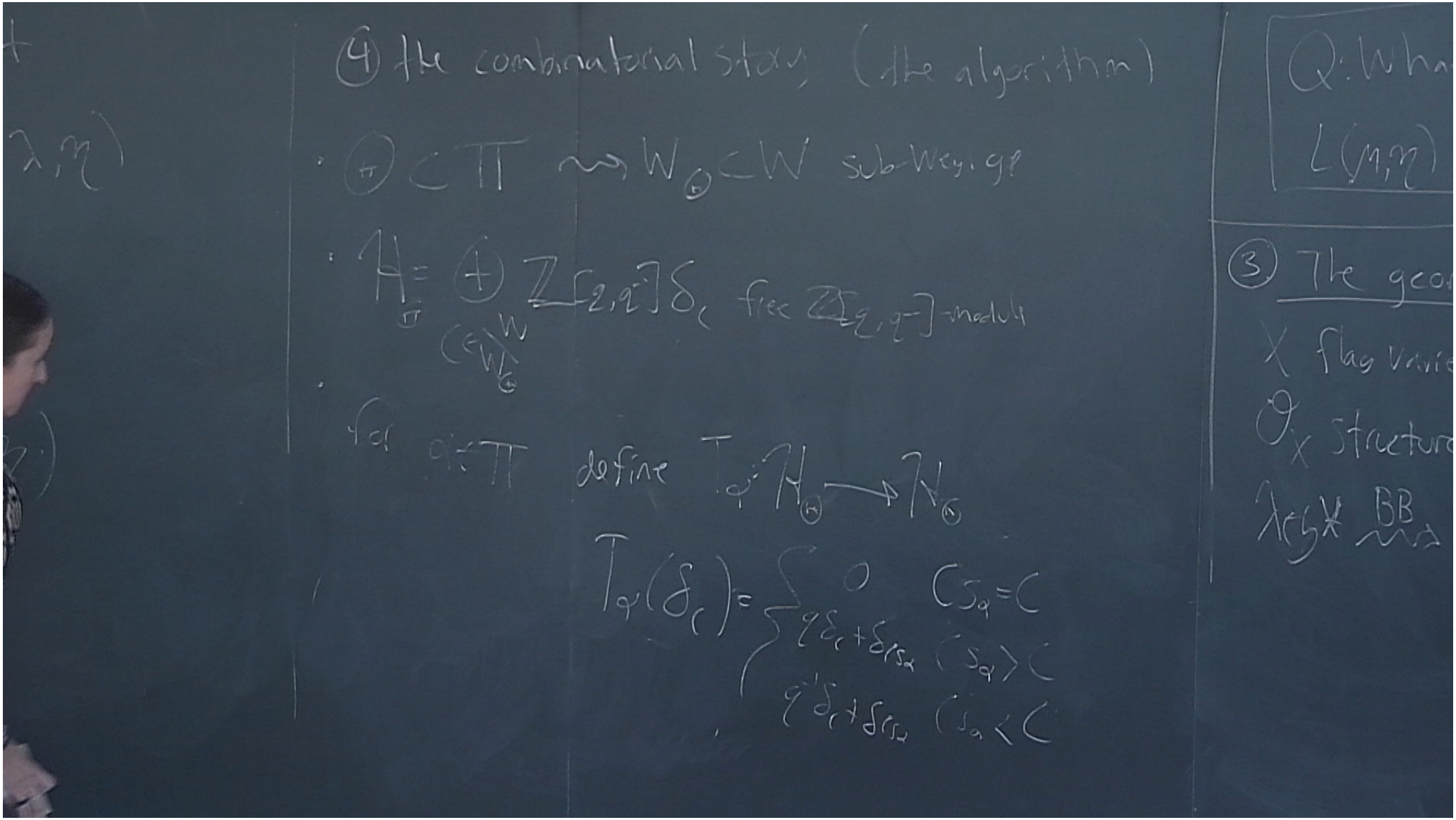
$\mathcal{D}_\lambda$-mod direct image

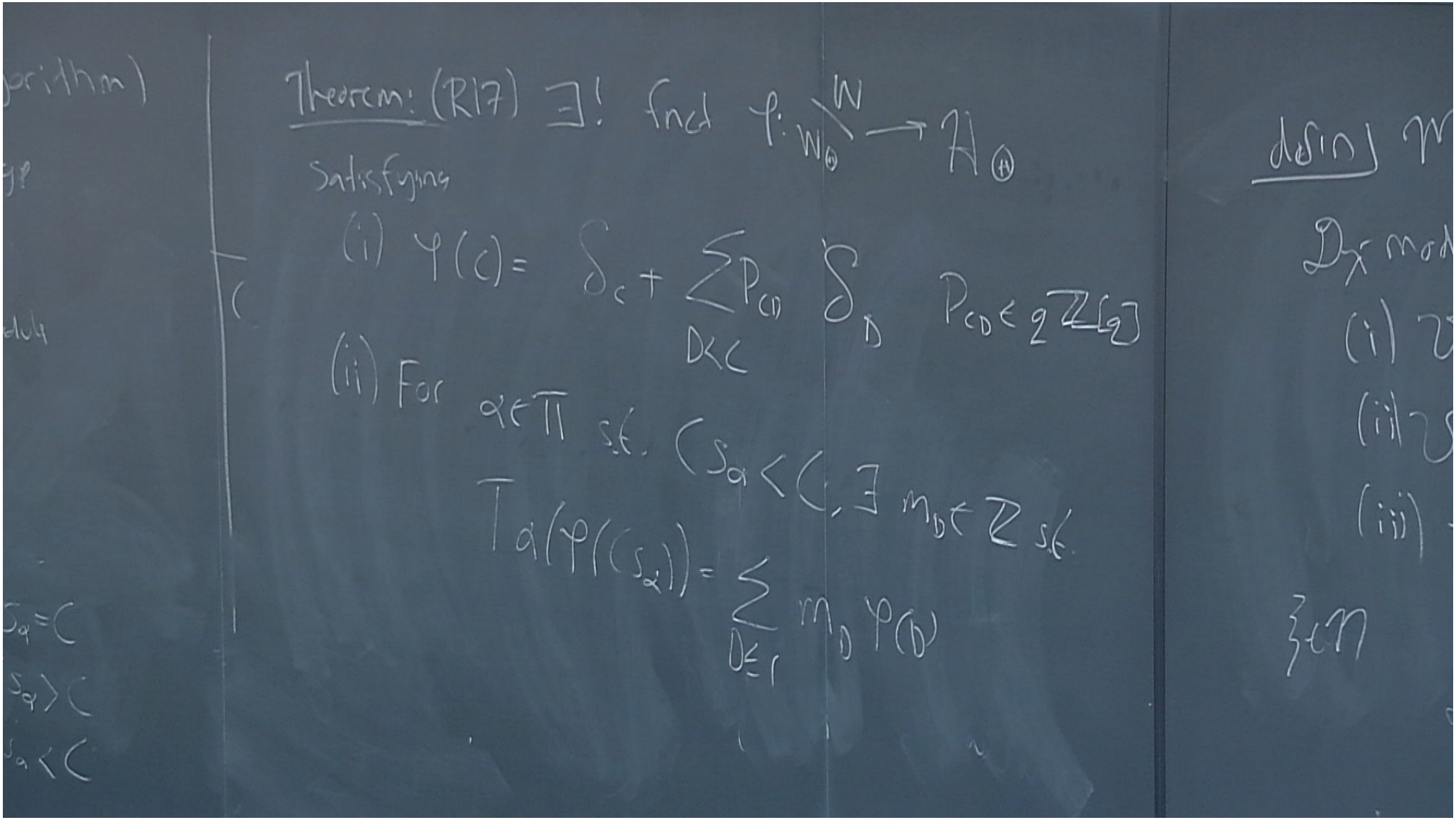
struct thm: $\mathcal{I}(w^c, \lambda, \eta)$ has unique subsheaf $\mathcal{L}(w^c, \lambda, \eta)$

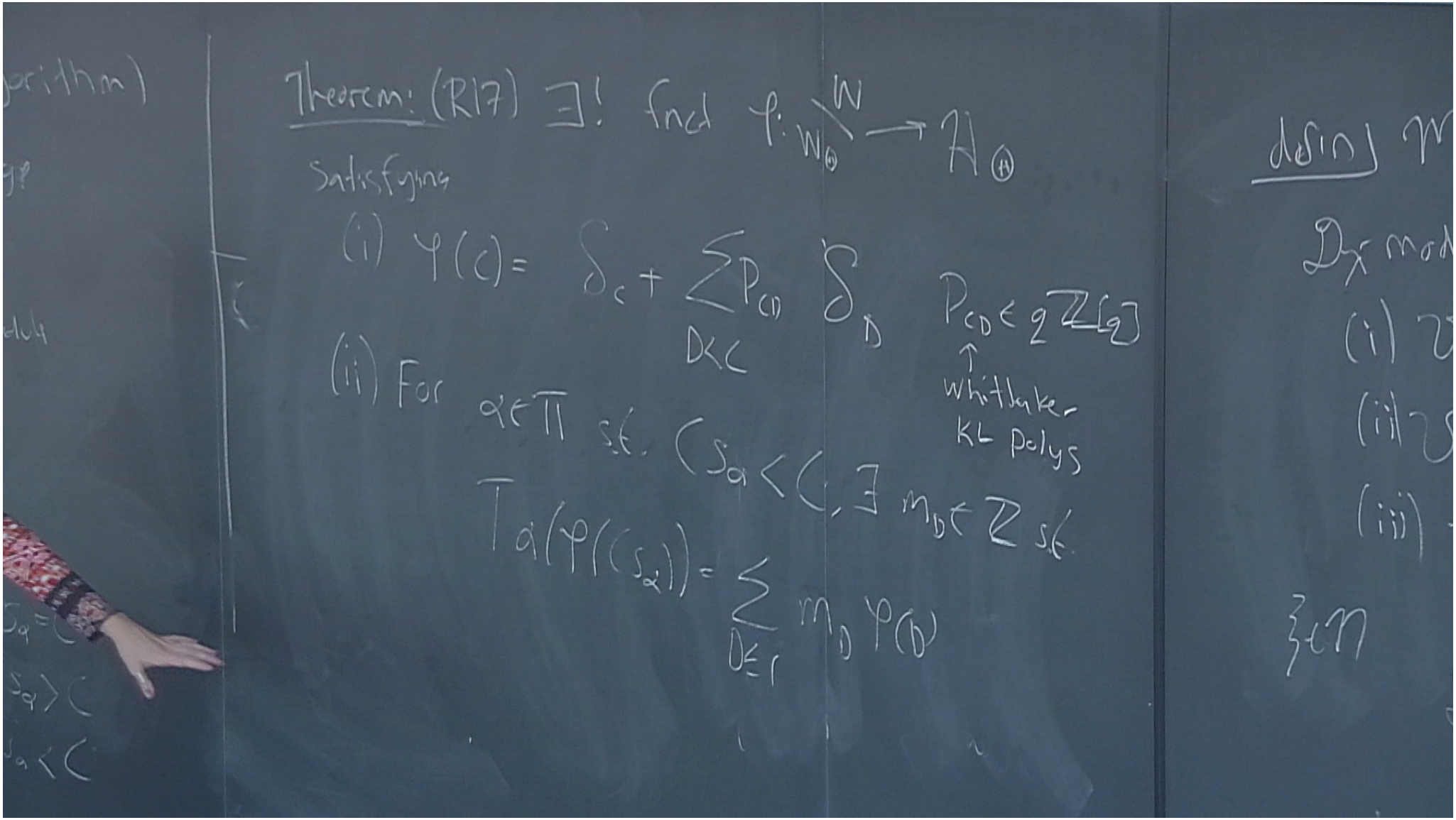
- holonomic duality, costandard $\mathcal{M}(w^c, \lambda, \eta)$

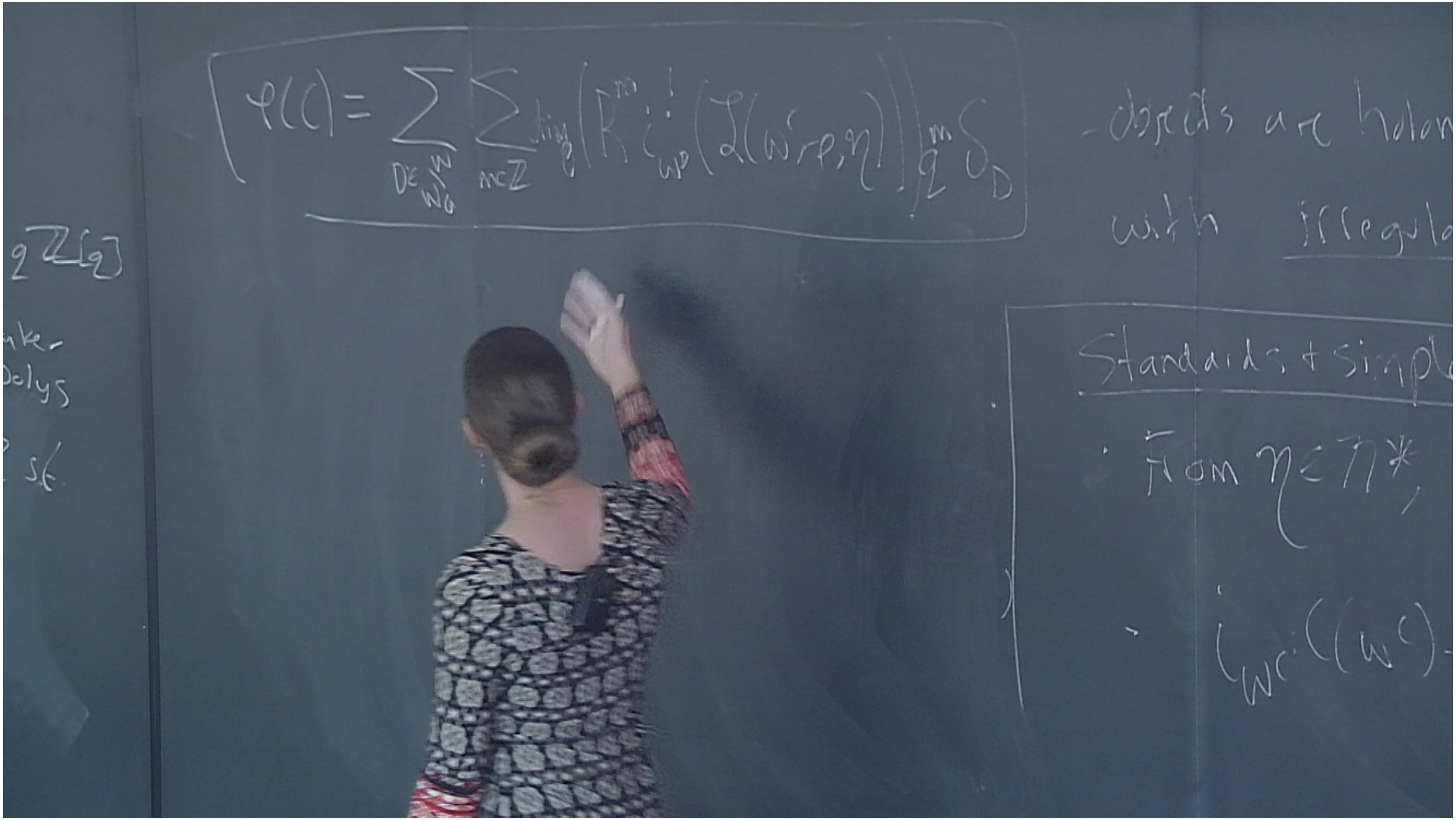

Theorem (R13) λcb* antidominant
(i) Γ(x, m(w, λ, η) = M(w, λ, η)
(ii) if λcb* is regular
Γ(x, q(w, λ, η)) = L(w, λ, η)

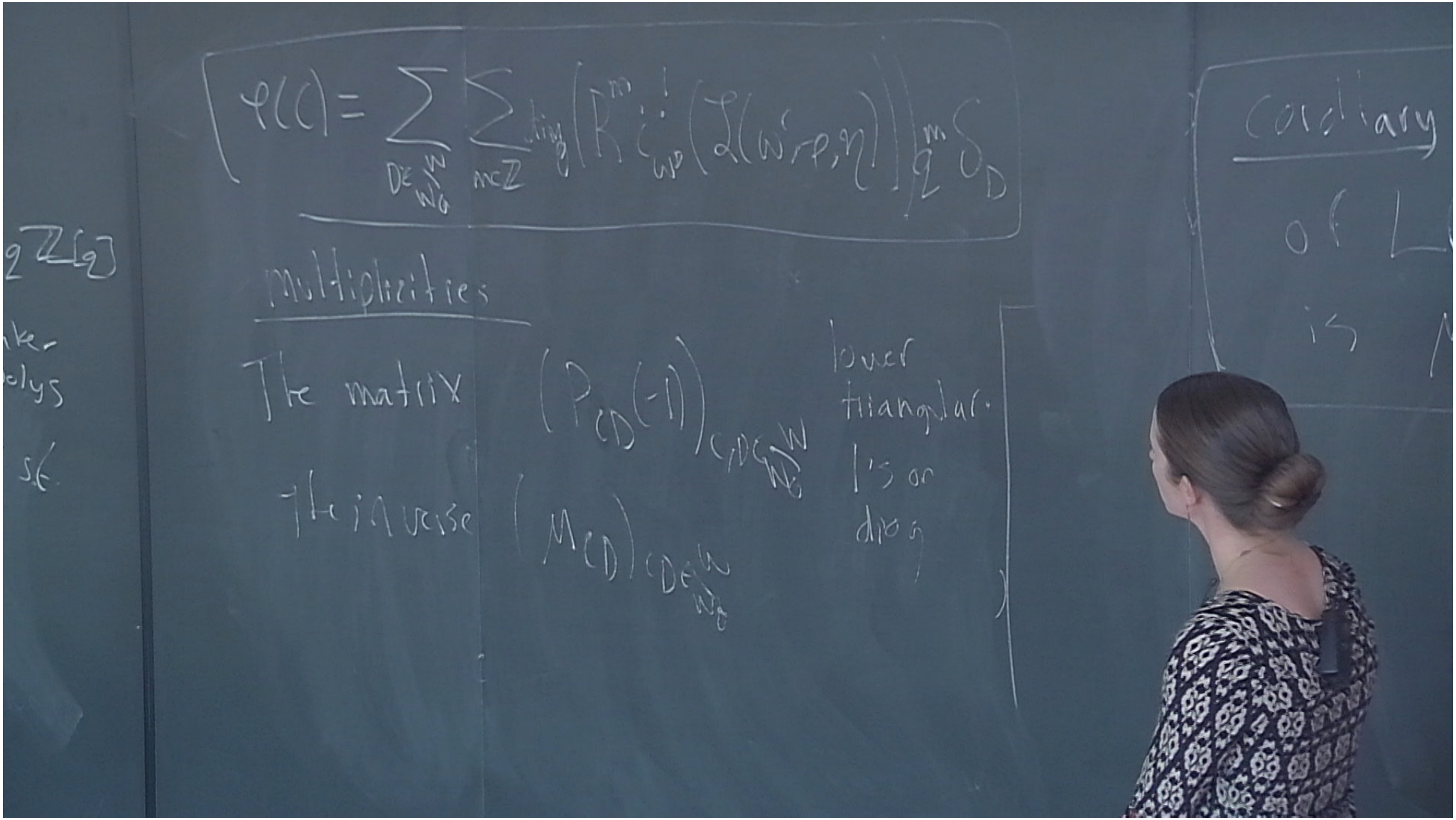

④ the combinatorial story (the algorithm)
Q: Wh
L(η,η
③ The ge
flag va
structu
BB

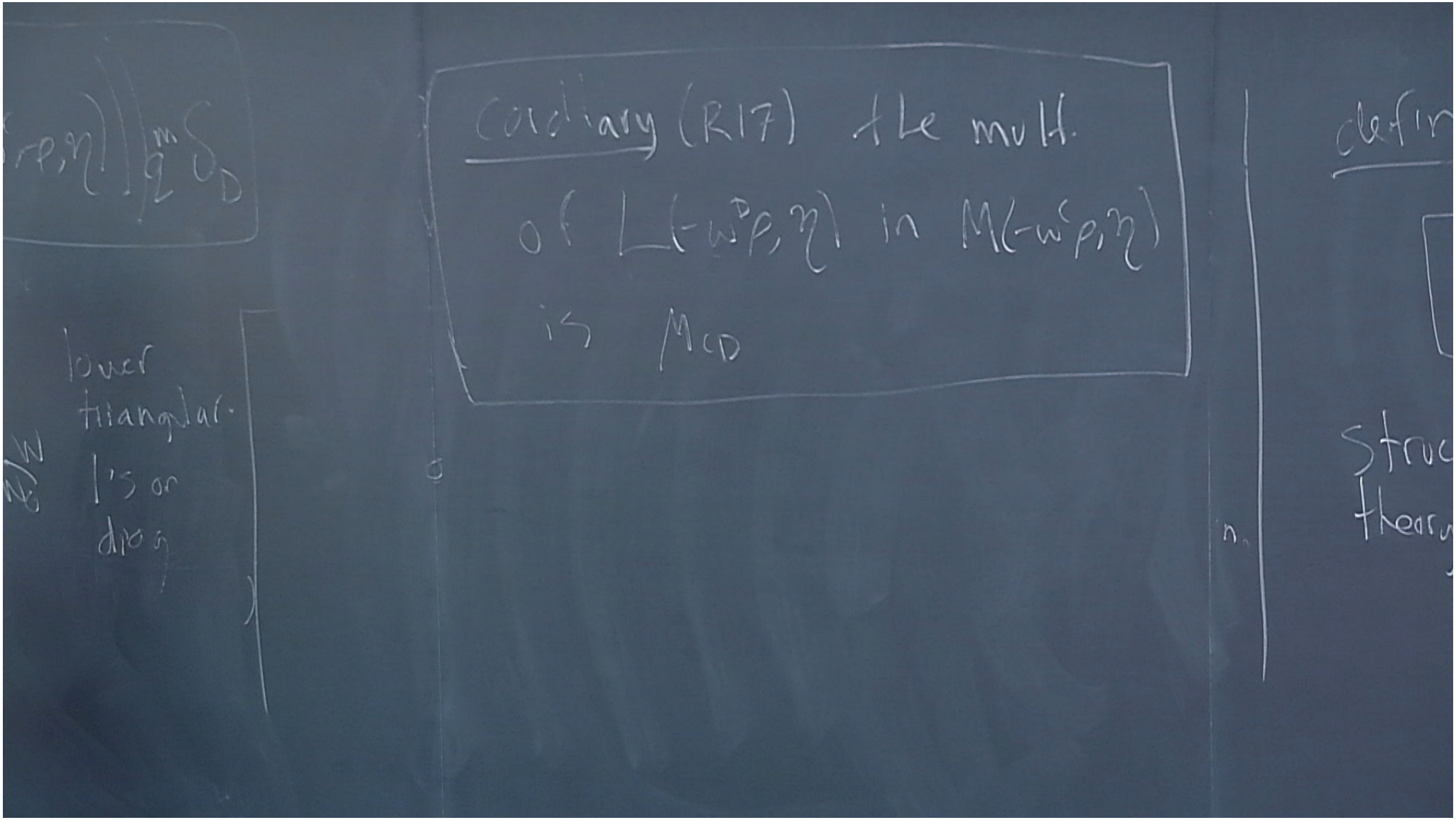

④ the combinatorial story (the algorithm)
Q: Wha
L(η,η)
③ The geo
flag varie
structure
BB


Theorem: (R17) ∃! fncl $\varphi: W_{(0)} \backslash W \to \mathcal{H}_{(1)}$
Satisfying
(i) $\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\,\delta_D \quad P_{CD} \in q\mathbb{Z}[q]$
(ii) For $a \in \Pi$ s.t. $(S_a < C, \exists\, m_D \in \mathbb{Z}$ s.t.
$T_a(\varphi(S_a)) = \sum_{D \le 1} m_D \varphi(D)$


Theorem: (R17) ∃! fncl
Satisfying
(i)
(ii) For
Whittaker-
KL polys
dsing


$\varphi(\zeta) = \sum_{D \in \frac{W}{W_0}} \sum_{n \in \mathbb{Z}} \dim_{\mathbb{C}} \left(R^n i^!_{WD} \left(\mathcal{L}(w, \rho, \eta) \right) \right) q^n \delta_D$
- objects are holon
with irregula
Standards + simple
From $\eta \in \Pi^*$,
$(w \cdot ((w^0)$
$q \mathbb{Z}[q]$
ake-
Delys
s(


multiplicities
The matrix
The inverse
lower triangular.
1's on
diag
corollary


Corollary (R17) the mult.
of $L(-w^D\rho, \eta)$ in $M(-w^C\rho, \eta)$
is M_{CD}
lower
triangular
1's or
diag
n.

(the algorithm)

sub-Weyl gp

$\mathbb{Z}[q,q^{-}]$-module

$\to \mathbb{Z}^{N}_{\Phi}$

0 $C_{S_q} = C$

$q\delta_C + \delta_{rS_\alpha}$ $(S_\alpha > C$

$\delta_C + \delta_{rS_\alpha}$ $(S_\alpha < C$

Theorem: (R17) $\exists!$ fncl $\varphi: W_\Phi^{N} \to \mathbb{Z}^{N}_{\Phi}$

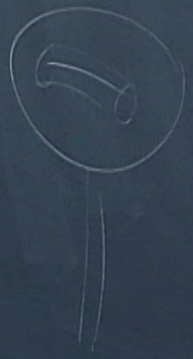
Satisfying

(i) $\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\,\delta_D$ $\quad P_{CD} \in q\mathbb{Z}[q]$

↑ Whittaker-KL polys

(ii) For $\alpha \in \Pi$ s.t. $C_{S_\alpha} < C$, $\exists\, m_D \in \mathbb{Z}$ s.t.

$$T_\alpha(\varphi(C_{S_\alpha})) = \sum_{D \le C} m_D\, \varphi(D)$$



G/P