

Title: Construction of unitary Segal CFTs

Date: Apr 30, 2018 02:00 PM

URL: <http://pirsa.org/18040096>

Abstract: <p>In this talk I will describe joint work in progress with Andre Henriques to construct examples of Graeme Segal's functorial definition of 2d chiral conformal field theory. While Segal's definition originated in the 1980's, constructive aspects of the theory continue to be challenging, especially with regard to higher genus surfaces. I will motivate and introduce Segal's definition, and describe a new approach to constructing examples using von Neumann algebras. </p>

Construction of unitary Segal
CFTs (joint w/ André Henriques)



G compact simple simply conn

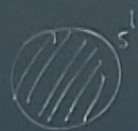
$$L G = C^\infty(S^1, G)$$

"Positive energy reps"

$$\rightarrow k \in \mathbb{Z}_{>0} \quad \text{"level"}$$

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}(n)$$

$$G \subset L G$$



$\mathcal{H}(n)$ rep of G ,

$\mathcal{H}(0)$ as a G -rep determined by $\mathcal{H}$.

$\mathcal{H}(0)$ invariant under $\text{Hol}(D, G_c)$

$$\begin{array}{c|c} \begin{array}{ccc} \bullet & \bullet & \bullet \\ \hline \end{array} & \begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \end{array} \\ \hline \text{Sp}(2)_2 & \text{Sp}(3)_2 \end{array}$$

G compact simple simply conn

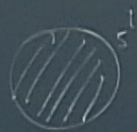
$$L G = C^\infty(S^1, G)$$

"Positive energy reps"

$\rightarrow k \in \mathbb{Z}_{\geq 0}$ "level"

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}(n)$$

$$G \subset L G$$



$\mathcal{H}(n)$ rep of G ,

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$\mathcal{H}(n)$ invariant under $\text{Hol}(D, G_n)$

$$\begin{array}{c|c} \begin{array}{ccc} \bullet & \bullet & \bullet \\ \hline \end{array} & \begin{array}{ccc} \bullet & & \bullet \\ & \bullet & \\ \bullet & & \bullet \end{array} \\ \hline \text{SU}(2)_2 & \text{SU}(3)_2 \end{array}$$

$$\text{Rep}_k = \text{PER}_5$$

is finite

conv



$\mathcal{H}(n)$ rep of G ,
 $\mathcal{H}(0)$ as a G -rep determined

by

$\mathcal{H}(0)$ invariant under $\text{Hol}(D, G_0)$

$$\begin{array}{c|c} \begin{matrix} \bullet & \bullet & \bullet \\ \hline \end{matrix} & \begin{matrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{matrix} \\ \hline S_4(2)_2 & S_4(3)_2 \end{array}$$

$$\text{Rep}_k = \text{PER}_S$$

is finite, semisimple.

S "circle", $\text{Rep}_k(S)$

G compact simple simply conn

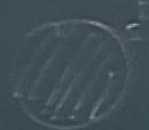
$$L_G G = C^\infty(S, G)$$

Positive energy reps

$$\Rightarrow K \in \mathbb{Z}_{>0} \text{ "level"}$$

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}(n)$$

$$G \subset L_G$$



$\mathcal{H}(0)$ rep of G

$\mathcal{H}(0)$ as a G -rep determined

by

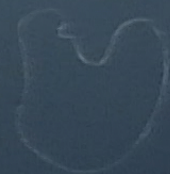
$\mathcal{H}(0)$ invariant under $H_1(D, \mathbb{Q})$

$$\begin{array}{c|c} \dots & \dots \\ \hline \text{Sym}_2 & \dots \\ & \text{Sym}_3 \end{array}$$

$$\text{Rep}_G = \overline{\text{PER}}_G$$

is finite

S "circle", $\mathbb{R}$



conn $\mathcal{H}(n)$ rep of G ,
 $\mathcal{H}(0)$ as a G -rep determinant
 $\mathcal{H}$.

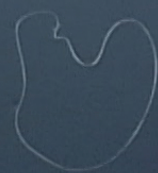
$\mathcal{H}(0)$ invariant under $H_0(D, G_c)$

$$\begin{array}{c|c} \begin{array}{ccc} \bullet & \bullet & \bullet \\ \hline \end{array} & \begin{array}{ccc} \bullet & & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \bullet \end{array} \\ \hline S_4(2)_2 & S_4(3)_2 \end{array}$$

$$\text{Rep}_k = \text{PER}_5$$

is finite, semisimple.

S "circle", $\text{Rep}_k(S)$



Segal's "Homomorphism
Induction"

Warmup G group

$\mathcal{H}_1$ G -rep

$\mathcal{H}_2$ G -rep

$\mathcal{H}_1 \otimes \mathcal{H}_2$ is a G -rep

w/ $Z: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$

s.t. $g \cdot Z(v_1, v_2) = Z(g \cdot v_1, g \cdot v_2)$

Warmup G group

$\mathcal{H}_1$ G -rep

$\mathcal{H}_2$ G -rep

$\mathcal{H}_1 \otimes \mathcal{H}_2$ is a G -rep

w/ $Z: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$

s.t. $g \cdot Z(v_1, v_2)$

$$= Z(g \cdot v_1, g \cdot v_2)$$

+ universal prop.

Warmup G group

$\mathcal{H}_1$ $\mathbb{C}S_1$ -rep

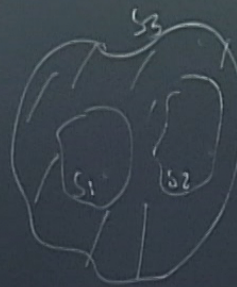
$\mathcal{H}_2$ $\mathbb{C}S_2$ -rep

$\mathcal{H}_1 \otimes \mathcal{H}_2$ is a G -rep

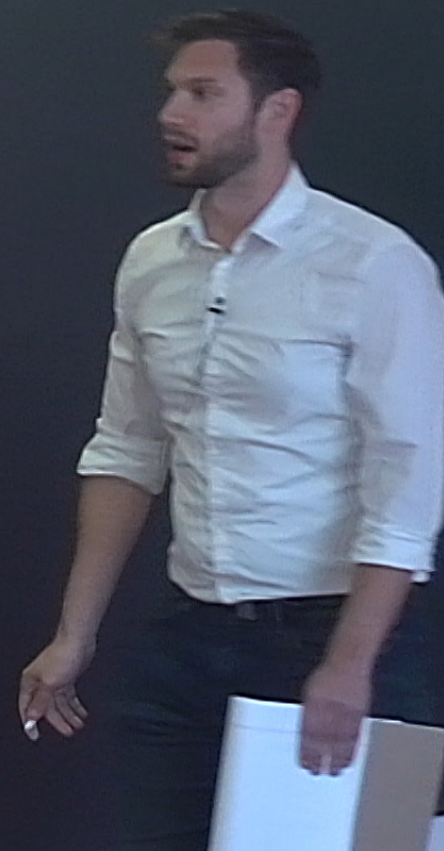
w/ $Z: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$

s.t. $g \cdot Z(v_1, v_2)$

$= Z(g \cdot v_1, g \cdot v_2)$
+ universal prop.



$= \mathbb{R}g$



Warmup G group

$\mathcal{H}_1$ $L_{S_1}^G$ -rep

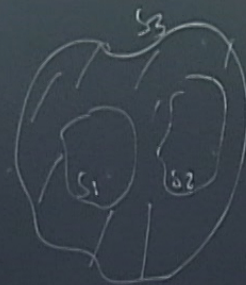
$\mathcal{H}_2$ $L_{S_2}^G$ -rep

$\mathcal{H}_1 \otimes \mathcal{H}_2$ is $L_{S_3}^G$ -rep

w/ $Z: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$

s.t. $g \cdot Z(v_1, v_2)$

$= Z(g \cdot v_1, g \cdot v_2)$
+ universal prop.



- E_g -



Wotup G group

$\mathcal{H}_1$ $\mathcal{L}_{S_1}$ -rep

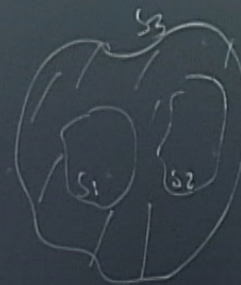
$\mathcal{H}_2$ $\mathcal{L}_{S_2}$ -rep

$\mathcal{H}_1 \otimes \mathcal{H}_2$ is $\mathcal{L}_{S_3}$ -rep

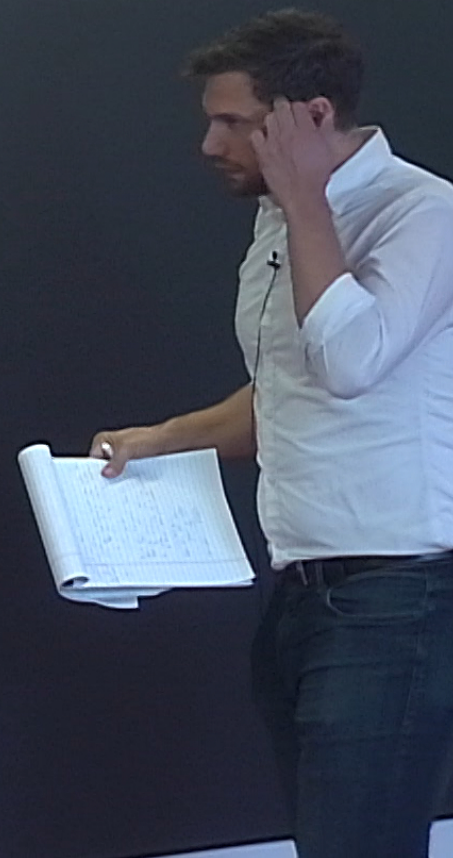
w/ $Z_2: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$

s.t. $g \cdot Z(v_1, v_2)$

$= Z(g \cdot v_1, g \cdot v_2)$
+ universal prop.



- E_8 -



Warmup G group

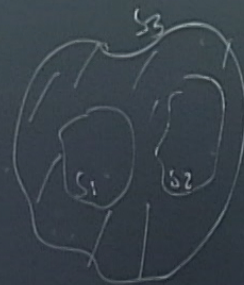
$\mathcal{H}_1$ $\mathcal{L}_{S_1}$ -rep

$\mathcal{H}_2$ $\mathcal{L}_{S_2}$ -rep

$\mathcal{H}_1 \otimes \mathcal{H}_2$ is $\mathcal{L}_{S_3}$ -rep

w/ $Z_2: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$

s.t. $f|_{S_3} Z(v_1, v_2)$

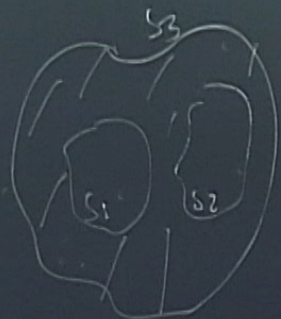


- $E_{\mathcal{L}}$

$\forall f \in \text{Hol}(\mathcal{L}, G)$

$= Z(f|_{S_1} v_1, f|_{S_2} v_2)$
+ universal prop.





- $\mathbb{E}_g$ -

$$\forall f \in \text{Hol}(\Sigma^g; G_0)$$

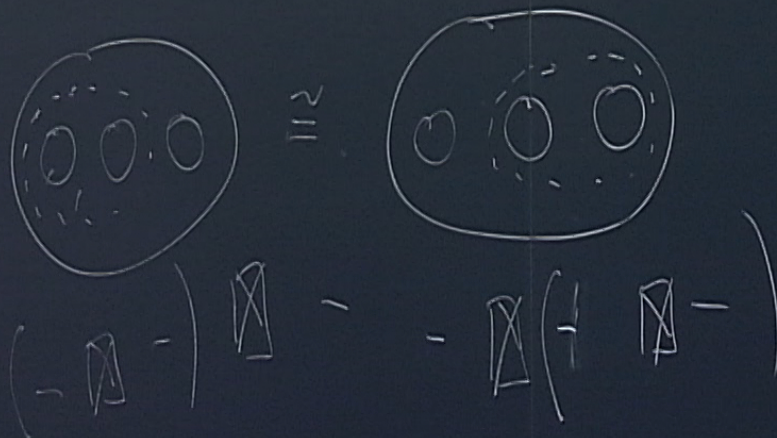
$$Z(f|_{s_1}, f|_{s_2})$$

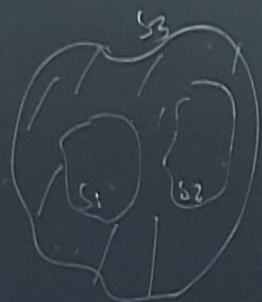
+ universal prop.

Issues: • Existence?

• PER?

• Associativity





$$- E_{\mathcal{E}} -$$

$$\forall f \in \text{Hol}(\mathcal{E}, G_0)$$

rep

$\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$

$\mathcal{H}_2$

$$= \mathbb{Z}(H_{s_1}, v_1, f|_{v_2})$$

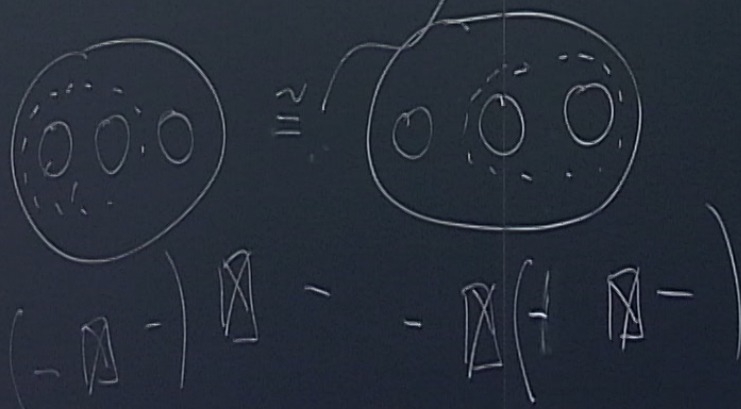
+ universal prop.

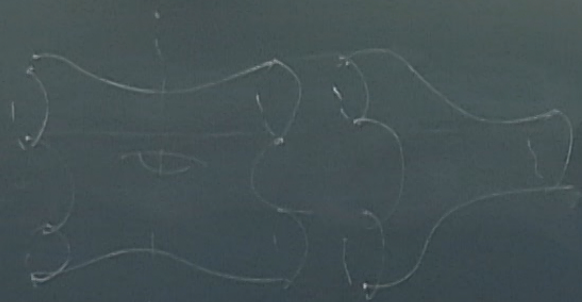
Issues: • Existence?

• PER?

• Associativity

of the pair $(\mathbb{Z}, - \boxtimes -)$





can do
holomorphic
induction

Def. (Segal)

1) S : U circles

a) $\mathcal{C}(S)$ category

b) $\mathcal{C}(S) \hookrightarrow \text{Hilb}$

2) $\mathcal{C}$: $\mathbb{C}$ -cobordism ($\partial_{\text{out}} \neq \emptyset$)

a) $F_2 : \mathcal{C}(S)$

$$a) F_2 : \mathcal{C}(\mathcal{A}_1) \rightarrow \mathcal{C}(\mathcal{A}_2)$$

$$b) Z_2 : H_{in} \rightarrow F_2(H_{in})$$

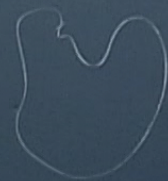
$$3) [0,1] \ni t \mapsto \mathcal{Q}_t$$

$$a) F_{\mathcal{Q}_0} \cong F_{\mathcal{Q}_1}$$

$$b) \text{Chiral, projectively flat}$$

is finite, semisimple.

S "circle", $\text{Rep}_k(S)$



Segal's
Induct

$$e_1) \rightarrow e_1(\omega_x)$$

$$n \rightarrow \overline{F}_2(H_{in})$$

$$t \mapsto \mathcal{E}_t$$

$\overline{F}_2$
1, projectively flat

Rational

•) $e(s')$ finite semisimple

•) $\text{Allan } \partial_{\text{out}} = \emptyset$.

olomorphic

$$e_1 \rightarrow e_2(\omega)$$

$$n \rightarrow F_2(H_{in})$$

$$t \mapsto \mathcal{E}_t$$

$$F_2$$

1, projectively flat

Rational

•) $e(s')$ finite semisimple

•) Allow $\partial_{out} = \emptyset$.

$(u) \iff$ Modular functor

$\mathbb{Q}$: Forget (b) is $\omega \rightarrow 1$

olomorphic

$$a) F_{\mathcal{L}} : \mathcal{L}(\mathcal{D}_u) \rightarrow \mathcal{L}(\mathcal{D}_w)$$

$$b) Z_{\mathcal{L}} : H_{in} \rightarrow F_{\mathcal{L}}(H_{in})$$

$$3) [0,1] \ni t \mapsto \mathcal{L}_t$$

$$a) F_{\mathcal{L}_0} \cong F_{\mathcal{L}_1}$$

$$b) \text{Chiral, projectively flat}$$

Rational

$$\bullet) \mathcal{L}(s^1) \text{ finite semisimple}$$

$$\bullet) \text{Allow } \mathcal{D}_{out} = \emptyset.$$

$$(4) \iff \text{Modular functor}$$

$$\text{Forget (b) is } \infty \rightarrow 1$$

Rational

•) $e(s')$ finite semisimple

•) Allow $\mathcal{Z}_{\text{out}} = \emptyset$.

(4) $\iff$ Modular functor

Forget (b) is $\mathcal{D} \rightarrow \mathcal{I}$

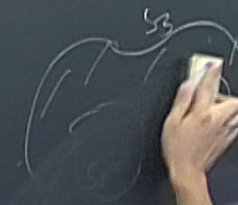
Q Given (a), $\exists$? (b)

ectively flat

isomorphism

Theorem (Henriques-T)

For every conformal net w/ nice analytic property, there is a Segal CFT.



- E_g -

$\forall$

(G_0)

- $Z(\mathcal{H}_1, \mathcal{H}_2)$

+ universal

Issues:



$(-A)$

Theorem (Henriques-T)

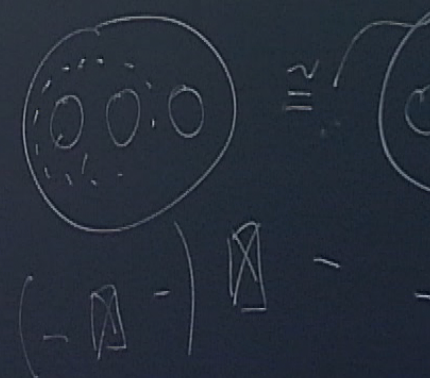
For every conformal
net w/ nice analytic
property, there is
a Segal CFT.

"nice" includes all loop
group models, lattices,
minimal, Moonshine, many

Vir_{CZ1}

Issues:

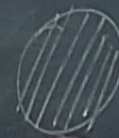
- Existence?
- PER?
- Associativity



Theorem (Henriques-T)

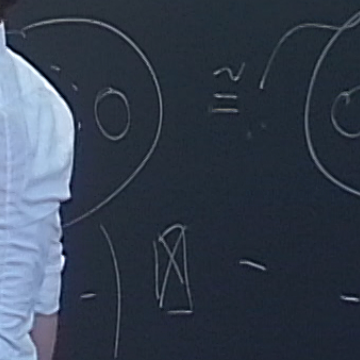
For every conformal net w/ nice analytic property, there is a Segal CFT.

Building blocks:

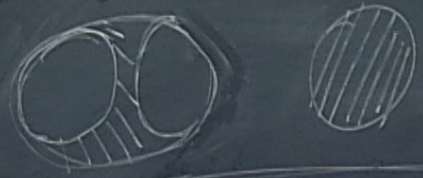


Issues:

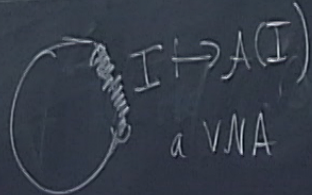
- Existence?
- PER?
- Associativity



Building blocks:



Conformal net



H PER of LG

$$L^I G = \{g \mid \text{supp}(g) \subseteq I\}$$

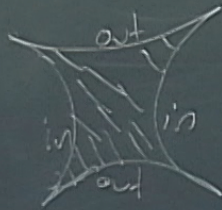
$$A(I) = \overline{\mathbb{C} L^I G} \quad \text{Ptwise convergence}$$

Conj: $A(I)$ independent of H .

A sector of A

is a family of reps

$$\pi_I: A(I) \rightarrow \mathcal{R}(H_\pi)$$



$$I \rightarrow \mathcal{C}(I) = \text{Rep}(A(I))$$

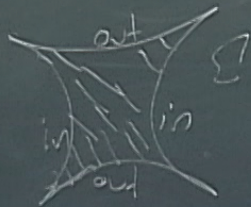
Def (Segal)

1) (S) $\cup$ circles

a) $\mathcal{C}(S)$ category

b) $\mathcal{C}(S) \hookrightarrow \text{Hilb}$

2) Σ $\mathcal{C}$ -cobordism ($\partial_{\text{out}} \neq \emptyset$)



$$I \mapsto \mathcal{C}(I) = \text{Rep}(A(I))$$

$$\Sigma \mapsto F_\Sigma: \mathcal{C}(I_{in}) \rightarrow \mathcal{C}(I_{out})$$

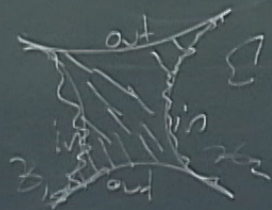
Def (Segal)

1) (S) $\cup$ circles

a) $\mathcal{C}(S)$ category

b) $\mathcal{C}(S) \hookrightarrow \text{Hilb}$

2) Σ $\mathbb{C}$ -cobordism



$$I_1 \mapsto \mathcal{C}(I) = \boxed{\text{Rep}(A(I))}$$

$$\Sigma \mapsto F_g: \mathcal{C}(I_{1,n}) \rightarrow \mathcal{C}(I_{\text{out}})$$

To do that:-

Hint a rep of $A(I_n)$

$\rightarrow$ construct $X_g \in A(I_n)$

$$\langle \gamma_1 \otimes \eta_1, \gamma_2 \otimes \eta_2 \rangle_{\text{out}} :=$$

$$\langle \boxed{X_g} \gamma_1 \otimes \eta_1, \gamma_2 \otimes \eta_2 \rangle$$

To do that:

H_{in} a rep of $A(I_n)$

→ construct $X_g \in A(I_n)$

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{out} :=$$

$$\langle X_g \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle$$

$$H_{out} = \overline{H_{in}}^{out}$$

$Z_g = \text{Projection } H_{in} \rightarrow H_{out}$

↑
This needs to be continuous

Rational

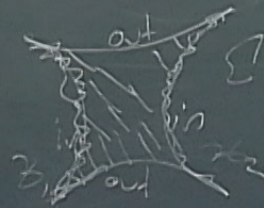
•) $e(s')$

•) Allow 2π

(4) $\iff$ Mod

Forget (b)

① Given (a), I ?

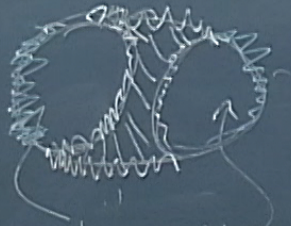


$$\begin{aligned}
 I &\mapsto \mathcal{C}(I) = \boxed{\text{Rep}(A(I))} \rightarrow \text{construct} \\
 \Sigma &\mapsto F_\Sigma: \mathcal{C}(I_{in}) \rightarrow \mathcal{C}(I_{out}) \quad \left(\begin{array}{c} \boxed{\times_\Sigma \tilde{\gamma}_i} \end{array} \right)
 \end{aligned}$$

To do +
 H_{in} a v

→ H_{out}

uous



Given H_1, H_2

get H_{out} action $A(\{ \}$

by declaring they

commute w/ Z_n .

isomorphic

Theorem (Henn)

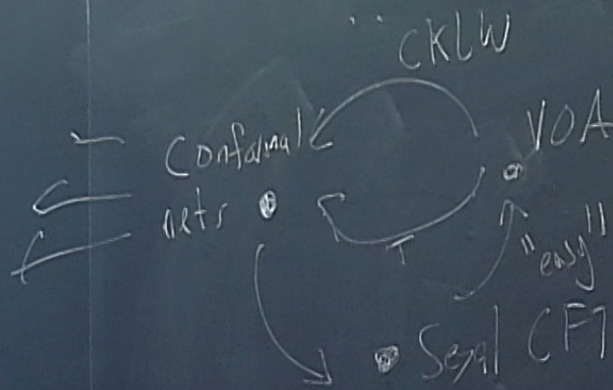
For every

(net w/ n)

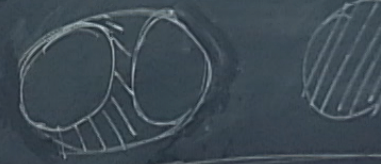
property,

a Segal

isomorphic



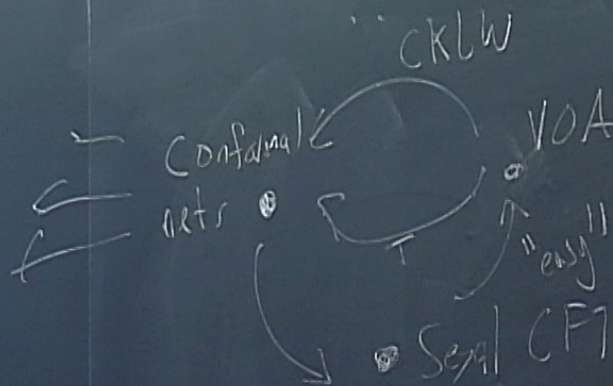
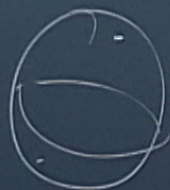
Building blocks:



Conformal net

olamorphic

$\begin{matrix} 76 & \rightarrow & 76 \\ 0' & & 17 \\ V & \rightarrow & \overline{V} \end{matrix}$



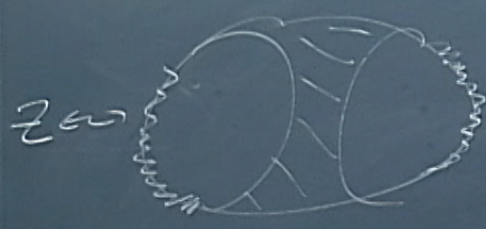
Building block



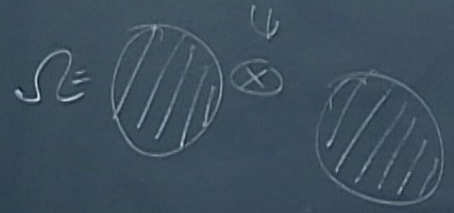
Confamil net

$\text{section } H_{in} \rightarrow H_{out}$

H_0 to be continuous



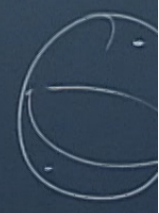
$$H_0 \otimes H_0$$



$$\omega_Z(\otimes \otimes) =$$

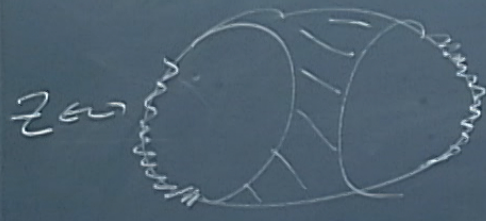
$\rightarrow Z$ commutes w/ Aff

$$\begin{array}{ccc}
 \mathbb{R} & \rightarrow & \mathbb{R} \\
 \downarrow & & \downarrow \\
 V & \rightarrow & \overline{V}
 \end{array}$$

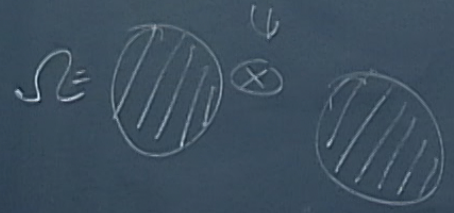


section $H_{lin} \rightarrow H_{lin}$

be continuous



$$H_0 \otimes H_0$$



$$\omega \cdot (\otimes \otimes) =$$

$\rightarrow \tau$ commutes w/ A/C

$$\begin{matrix} \mathbb{Z} & \rightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ V & \rightarrow & V \end{matrix}$$

