Title: Agents, Subsystems, and the Conservation of Information

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Abstract: Dividing the world into subsystems is an important component of the scientific method. The choice of subsystems, however, is not defined a priori. Typically, it is dictated by our experimental capabilities, and, in general, different agents may have different capabilities. Here we propose a construction that associates every agent with a subsystem, equipped with its set of states and its set of transformations. In quantum theory, this construction accommodates the traditional notion of subsystems as factors of a tensor product, as well as the notion of classical subsystems of quantum systems. We then restrict our attention to systems where all physical transformations act invertibly. For such systems, the future states are a faithful encoding of the past states, in agreement with a requirement known as the Conservation of Information. For systems satisfying the Conservation of Information, we propose a dynamical definition of pure states, and show that all the states of all subsystems admit a canonical purification. This result extends the purification principle to a broader setting, in which coherent superpositions can be interpreted as purifications of incoherent mixtures. As an example, we illustrate the general construction for subsystems associated with group representations.

AGENTS, SUBSYSTEMS, AND THE CONSERVATION OF INFORMATION

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WHAT IS A SUBSYSTEM?

The notion of subsystem is fundamental in physics.
But how are subsystems defined in a general theory?

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But how are subsystems defined in a general theory?

Subsystems are often taken as a **primitive notion:**
cf.

- categorical quantum mechanics
- operational-probabilistic theories

In these frameworks, there is a basic operation that forms composite systems from subsystems: $(A, B) \mapsto A \otimes B$

Composite systems come with a
preferred decomposition into subsystems.

QUANTUM SUBSYSTEMS

In quantum theory, subsystems are associated to **operator algebras.**

Examples:

- local observables on a tensor product Hilbert space

$$\mathcal{A} = \left\{ O \in L(\mathcal{H}_A \otimes \mathcal{H}_B)\,, O = O_A \otimes I_B\,, O_A \in L(H_A) \right\}$$

- noise algebra generated by Kraus operators of a CPTP map

- local algebras associated to spatial regions in QFT

WHAT ABOUT GENERAL THEORIES?

Problem: the quantum notion of subsystem relies on the fact that observables are linear operators, and therefore can be multiplied. **In general theories, the multiplication of observables is not defined.**

Reasons for going beyond quantum theory:

- quantum axiomatizations.
  Broader definition of subsystems likely to yield *more powerful axioms.*
  Some old axioms could become *easy consequences of the definition.*
- unifying perspective on different physical theories.

PREVIOUS WORKS

Previous works:

- Barnum, Ortiz, Somma, Viola: generalized entanglement
  International Journal of Theoretical Physics **2005**, *44*, 2127–2145.
- Del Rio, Krämer, Renner: resource theories of knowledge
  arXiv:1511.08818 **2015**.
- Krämer PhD Thesis, Krämer, Del Rio:
  operational locality in global theories
  arXiv:1701.03280 **2017**.
- Brassard, Raymond-Robichaud:
  subsystem states as equivalence classes
  arXiv:1710.01380 **2017**

CREDITS

To Cabello, Kleinmann, Müller (…and FQXi),
for convincing me to embark into this approach.

Idea: derive quantum theory from the perspective of an agent that tries to organize her empirical data about a chaotic external word.
Different subsystems are different ways to organize the data.

To implement this idea, it is natural to
start from a single, undifferentiated system and define subsystems afterwards.

A PRE-OPERATIONAL FRAMEWORK

Ingredients:

- A system S (the universe of discourse)
- St(S) = set of states of the system,
  no particular structure assumed (e.g. no convexity)
- Transf(S) = set of physical transformations,
  closed under sequential composition,
  and containing the identity transformation.
  In short, a *monoid.*
- Transformations act on states

INTERPRETATIONS

Option 1: objective interpretation
St(S)/Transf(S) are the possible states/dynamics of the system,
they represent "the world as it is"

Option 2: subjective interpretation
St(S) represent our beliefs of the system
[e.g. about outcomes of possible experiments]
Transf(S) represent our beliefs on the possible evolutions.

What follows is compatible with both interpretations.

AGENTS AND THEIR ADVERSARIES

Agent: agent A is specified by a set of actions Act(A;S)
Subset of Transf(S),
assumed to be a monoid.

Adversaries: intuitively, an adversary controls a
"part of the world outside the agent's lab".

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"part of the world outside the agent's lab".

Mathematical modelling: if B is an adversary of A, then

$$\mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A}$$

$$\forall \mathcal{A} \in \mathsf{Transf}(A)\,, \forall \mathcal{B} \in \mathsf{Transf}(B)$$

THE MAXIMAL ADVERSARY

Maximal adversary A′: the agent who can do **all** the operations that commute with Transf(A)

$$\mathsf{Act}(A'; S) = \mathsf{Act}(A; S)' := \left\{ \mathcal{B} \,,\, \mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A} \,,\, \forall \mathcal{A} \in \mathsf{Act}(A; S) \right\}$$

The most powerful adversary
we can conceive for agent A,
given our physical model of the system.

Caveat:
All this *looks like* the usual construction of commuting operator algebras in quantum theory, *but it is not:*
in quantum theory, the actions are CPTP maps, not operators.

THE PROBLEM

We want to define "A's subsystem",
i.e. the degrees of freedom that are *exclusively under A's control*,
i.e. the degrees of freedom that are
inaccessible to her maximal adversary.

Call the subsystem S_A.

We have to define the sets $St(S_A)$ and $Transf(S_A)$

LOCALLY IDENTICAL STATES

Intuition: sometimes, two different states of the global system S correspond to the same local state of system S_A

Question: when is it the case?

Answer: at least,

if $\psi = \mathcal{B}\phi$ for some adversarial action $\mathcal{B}$

then ϕ and ψ should correspond to the same state of S_A

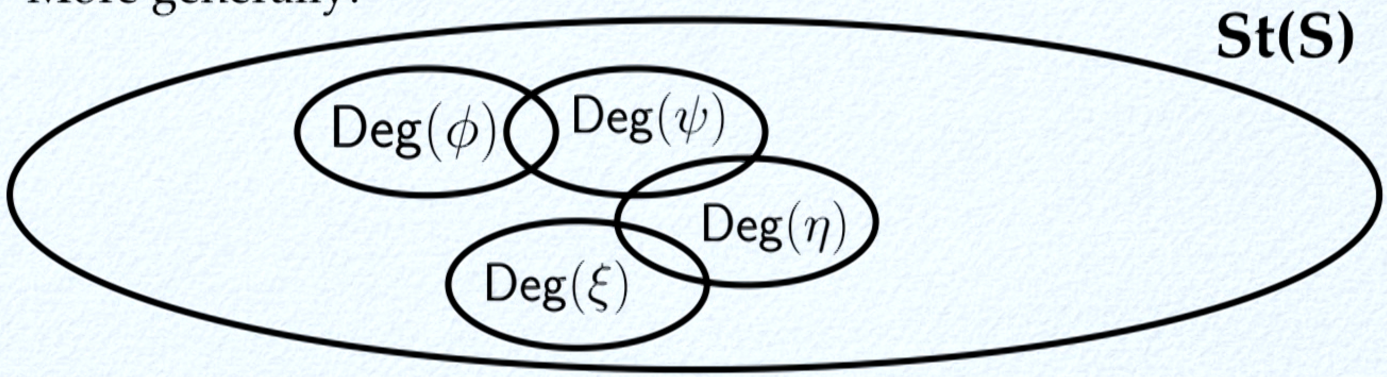
PARTITIONING THE STATE SPACE

Degraded versions of ψ: $\mathsf{Deg}(\psi) = \left\{ \mathcal{B}\psi \,, \mathcal{B} \in \mathsf{Transf}(S) \right\}$

Observation: if $\mathsf{Deg}(\phi) \cap \mathsf{Deg}(\psi) \neq \emptyset$

then ϕ and ψ correspond to the same state of system S_A

More generally:



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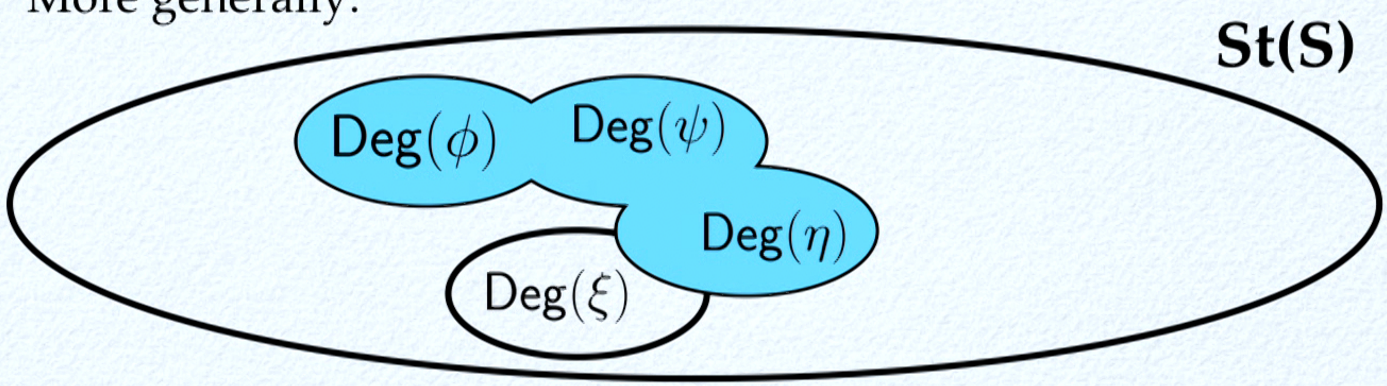

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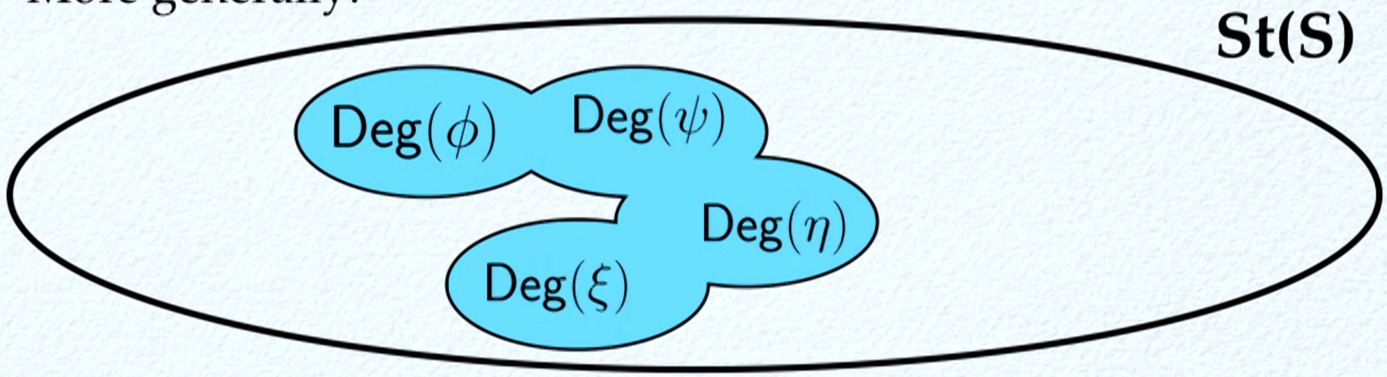

THE STATES OF S_A

States of S_A: equivalence classes of states of S
under the relation

$\phi \simeq_{A'} \psi$ if exists finite sequence $(\psi_1, \psi_2, \ldots, \psi_n)$

such that $\psi_1 = \phi$, $\psi_n = \psi$, and $\mathsf{Deg}(\psi_i) \cap \mathsf{Deg}(\psi_{i+1})$

cf. Krämer and Del Rio: convergence through a monoid
arXiv:1701.03280 **2017**.

PARTITIONING THE TRANSFORMATIONS

For a transformation $\mathcal{T}$, the degraded versions are

$$\mathsf{Deg}(\mathcal{T}) = \Big\{ \mathcal{B}_1 \circ \mathcal{T} \circ \mathcal{B}_2 \,, \quad B_1, \mathcal{B}_2 \in \mathsf{Act}(S;A) \Big\}$$

Interpretation:
all these transformations act "in the same way" on A's system.

THE TRANSFORMATIONS OF S_A

Question: which transformations can be interpreted as acting "only on A's system"?

Answer: the transformations that commute with the actions of the adversary

$$\mathsf{Act}(A'; S)' = \mathsf{Act}(A; S)''$$

Transformations of the subsystem:
equivalence classes of transformations in $\mathsf{Act}(A; S)''$

EXAMPLE 1:

BIPARTITE SYSTEMS
IN
QUANTUM THEORY

BIPARTITE QUANTUM SYSTEMS

Global system

$$\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\mathsf{St}(S) = \left\{\rho \in L(H_S)\,, \rho \geq 0\,, \mathsf{Tr}[\rho] = 1]\right\}$$

$$\mathsf{Transf}(S) = \left\{\mathcal{C} : L(\mathcal{H}_S) \to \mathcal{L}(H_S)\,, \mathcal{C} \text{ is CPTP}\right\}$$

Agent

$$\mathsf{Act}(A; S) = \left\{\mathcal{A} \otimes \mathcal{I}_B\,, \quad \mathcal{A} : L(\mathcal{H}_A) \to L(\mathcal{H}_A)\,, \mathcal{A} \text{ is CPTP}\right\}$$

EXAMPLE 2:

A
SUBSYSTEM
OF
A *SINGLE* QUANTUM SYSTEM

SUPERPOSITIONS VS MIXTURES

Global system

$$\mathsf{St}(S) = \left\{ |\psi\rangle = \sum_n \psi_n |n\rangle, \quad \sum_n |\psi_n|^2 = 1 \right\}$$

$$\mathsf{Transf}(S) = \left\{ U \in L(\mathcal{H}), U^\dagger U = I_S \right\}$$

Agent

$$\mathsf{Act}(A; S) = \left\{ U_{\boldsymbol{\theta}} = \sum_n e^{i\theta_n} |n\rangle\langle n|, \qquad \boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in [0, 2\pi)^{\times d} \right\}$$

THE COMMUTANT

Theorem

The commutant of the *phase-covariant channels* are the *basis-preserving channels*,
i.e. the channels satisfying

$$\mathcal{B}(|n\rangle\langle n|) = |n\rangle\langle n| \qquad \forall n \in \{1, \ldots, d\}$$

The monoid of the phase covariant channels and the monoid of the basis-preserving channels are the commutant of one another.

THE CLASSICAL SUBSYSTEM

Heuristically, an agent without phase reference has only access to a classical system.

And indeed, the equivalence relation yields

$$\mathsf{St}(S_A) \simeq \left\{ \boldsymbol{p} = (p_1, \dots, p_n)\,, \quad p_n \geq 0\, \forall n\,, \quad \sum_n p_n = 1 \right\}$$

$$\mathsf{Transf}(S_A) \simeq \left\{ P = [P_{mn}]\,,\, P_{mn} \geq 0\, \forall m, n \in \{1, \dots, d\}\,,\, \sum_m P_{mn} = 1\, \forall n \in \{1, \dots, d\} \right\}$$

PARTIAL TRACE
AND
NO-SIGNALLING

PARTIAL TRACE & NO-SIGNALLING

Let B be the maximal adversary of A, namely B= A′

Formally, we can define $\mathsf{Tr}_B[\psi] := [\psi]_B$

(equivalence class under the degradation relation)

Trivial fact: no signalling holds

$$\mathsf{Tr}_B[\mathcal{B}\rho] := \mathsf{Tr}_B[\rho] \qquad \forall \mathcal{B} \in \mathsf{Act}(B;S)\ \forall \rho \in \mathsf{St}(S_A)$$

In this framework,
no-signalling is due to the way subsystems are formed.
Its validity is independent on whether the subsystems are space-like or not.

THE CONSERVATION OF INFORMATION

CONSERVATION OF INFORMATION

Informally,
the conservation of information is the condition that one can always reconstruct the past from the future.

cf. Susskind, Bousso

LOGICAL VS PHYSICAL INVERTIBILITY

A transformation $\mathcal{T} \in \mathsf{Transf}(S)$ is **logically invertible** if the function

$$\widehat{\mathcal{T}}: \quad \mathsf{St}(S) \to \mathsf{St}(S), \quad \psi \mapsto \mathcal{T}\psi$$

is injective.

A transformation $\mathcal{T} \in \mathsf{Transf}(S)$ is **physically invertible** if there exists a transformation $\mathcal{T}^{-1} \in \mathsf{Transf}(S)$ such that

$$\mathcal{T}^{-1} \circ \mathcal{T} = \mathcal{I}_S$$

SYSTEMS SATISFYING INFORMATION CONSERVATION

System S satisfies the
Logical (Physical) Conservation of Information
if all transformations in Transf(S)
are logically (physically) invertible.

Fact:
If S satisfies the physical conservation of information, then the transformations of S form a group.

SUBSYSTEMS

Suppose that
system S satisfies the Physical Conservation of Information,
and that the actions of agent A form a group G_A

Adversarial actions: commutant group $G_B = G_A'$

Equivalence relations:

$$\phi \simeq_B \psi \quad \Longleftrightarrow \phi = \mathcal{U}_B \psi \quad \mathcal{U}_B \in \mathsf{G}_B$$

$$\mathcal{S} \simeq_B \mathcal{T} \quad \Longleftrightarrow \mathcal{S} = \mathcal{U}_B \circ \mathcal{T} \circ \mathcal{V}_B \quad \mathcal{U}_B, \mathcal{V}_B \in \mathsf{G}_B$$

cf. Brassard Raymond-Robichaud arXiv:1710.01380

EXAMPLE 1:

PURE STATE
QUANTUM MECHANICS
AND
CONNECTED LIE GROUPS

COMPACT LIE GROUPS

Global system

$$\mathsf{St}(S) = \left\{ |\psi\rangle\langle\psi| \,, \quad |\psi\rangle \in \mathcal{H}, \, \| |\psi\rangle \| = 1 \right\}$$

$$\mathsf{Transf}(S) = \left\{ \mathcal{U} \,, \quad \mathcal{U}(\rho) = U \rho U^\dagger \right\}$$

Agent

Suppose that agent A can perform all the unitary channels corresponding to a projective representation of a compact Lie group, such as

$$U : \mathsf{G} \to L(H) \,, \quad g \mapsto U_g$$

ISOTYPIC DECOMPOSITION

U can be decomposed into irreps, as

$$U_g = \bigoplus_{j \in \mathsf{Irr}(U)} \left(U_g^{(j)} \otimes I_{\mathcal{M}_j} \right)$$

The commutant consists of unitary operators of the form

$$V = \bigoplus_{j \in \mathsf{Irr}(U)} \left(I_{\mathcal{R}_j} \otimes V_j \right)$$

where V_J is an arbitrary unitary on the multiplicity space.

THE COMMUTANT

Theorem

If the Lie group G is **connected**,
the commutant of the group
of unitary channels of the form

$$\mathcal{U}_g(\rho) = U_g \rho U_g^\dagger \qquad g \in \mathsf{G}$$

are the unitary channels of the form

$$\mathcal{V}(\rho) = V \rho V^\dagger \qquad V \in U'$$

THE SUBSYSTEM

States:

$$\mathsf{St}(S_A) \simeq \left\{ \rho = \bigoplus_{j \in \mathsf{Irr}(U)} p_j \rho_j \,:\, \rho_j \in \mathsf{QSt}(\mathcal{R}_j)\,,\, \mathsf{Rank}(\rho_j) \leq \min\{d_{\mathcal{R}_j}, d_{\mathcal{M}_j}\} \right\}$$

Not a convex set,
except when $d_{\mathcal{M}_j} \geq d_{\mathcal{R}_j} \qquad \forall j \in \mathsf{Irr}(U)$

Transformations:

$$\mathcal{C}_U(\rho) = \bigoplus_{j \in \mathsf{Irr}(U)} U_j \rho_j U_j^\dagger\,, \qquad U_j \in \mathsf{Lin}(\mathcal{R}_j)\,, U_j^\dagger U_j = I_{\mathcal{R}_j}$$

EXAMPLE 2:

SINGLE QUBIT
PHASE FLIPS

THE MAXIMAL ADVERSARY

$$\mathsf{Act}(A;S)' = \mathsf{S}_1 \cup \mathsf{S}_2$$

where

S_1 are the unitary channels of the form

$$\mathcal{U}_\theta(\rho) = U_\theta \rho U_\theta^\dagger\,, \qquad U_\theta = e^{i\theta}\,|1\rangle\langle 1| + |0\rangle\langle 0|$$

and

S_2 are the unitary channels of the form

$$\mathcal{V}_\theta(\rho) = U_\theta \rho U_\theta^\dagger\,, \qquad U_\theta = e^{i\theta}\,|0\rangle\langle 1| + |1\rangle\langle 0|$$

THE SUBSYSTEM

State space: a circle

Not convex, and not even a set of density matrices!

Transformations: only the identity.

EXAMPLE 3:

GENERAL COMPACT LIE GROUP

COMPACT LIE GROUPS

Global system

$$\mathsf{St}(S) = \Big\{ |\psi\rangle\langle\psi| \,, \quad |\psi\rangle \in \mathcal{H}, \, \| |\psi\rangle \| = 1 \Big\}$$

$$\mathsf{Transf}(S) = \Big\{ \mathcal{U} \,, \quad \mathcal{U}(\rho) = U \rho U^\dagger \Big\}$$

Agent

can perform all the unitary
channels corresponding to projective representation

$$U : \mathsf{G} \to L(H) \,, \quad g \mapsto U_g$$

THE MAXIMAL ADVERSARY

Theorem

The adversarial group $G_A{}'$
is isomorphic to the semidirect product $A \ltimes U'$
where A is an Abelian subgroup of the group
that permutes the irreps of U

THE SUBSYSTEM

The states are vectors of density matrices, indexed by irreps, weighted by probabilities,

$$(p_j\, \rho_j)_{j \in \mathsf{Irr}(U)}$$

and quotiented by the permutations in **A**.

The transformations are vectors of unitary channels, such as

$$(\mathcal{U}_j)_{j \in \mathsf{Irr}(U)} \qquad \mathcal{U}_j = U_j \rho U_j^\dagger \,, U_j \in L(\mathcal{R}_j)$$

quotiented by the permutations in **A**.

PURIFICATION

SUMMARY

1) Operational construction of subsystems

set of operations ⟶ commutant ⟶ quotient

This construction includes the usual subsystems, and much more.

2) Conservation of information + cyclic state

⟶ purification

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