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Abstract:

Representing transformations

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Operational theories

An operational theory has three categories of objects corresponding to three types of experimental procedures.[1]

- Preparations $\mathcal{P}$;
- Transformations $\mathcal{T}$; and
- Measurements $\mathcal{M}$.

[1] ...s, Phys. Rev. A **71**, 052108 (2005).

Quantum theory

- Preparations: pure quantum states ψ, mixed states ρ
- Transformations: unitary operators U, quantum channels $\mathcal{T}$
- Measurements: projectors $\mathbf{P}$, POVM elements E, Quantum instruments $\mathcal{L}$

Probability assignment is the Born rule,

$$\Pr(k|\mathcal{P}, \mathcal{T}, \mathcal{M}) = \mathrm{tr}[E_{k,\mathcal{M}} \mathcal{T}(\rho_{\mathcal{P}})].$$

Ontological models[3]

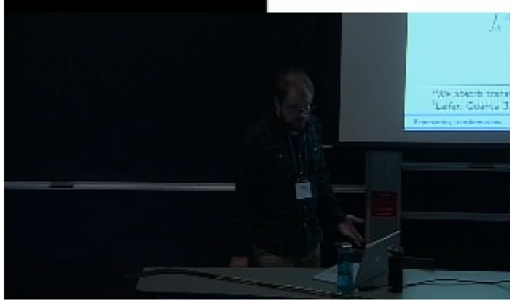
Operational theories should match observations insofar as they describe physics.

An ontological model of an operational theory[2] is a set of ontic states Λ and (conditional) probability measures such that:

$$\mu(\lambda|\mathcal{P}) = 1 \quad \forall \mathcal{P}$$

$$\sum_k \eta(k|\lambda, \mathcal{M}) = 1 \quad \forall \mathcal{M}, \lambda$$

$$\int_\Lambda \mathrm{d}\mu(\lambda|\mathcal{P})\eta(k|\lambda, \mathcal{M}) = \mathrm{Pr}(k|\mathcal{P}, \mathcal{M}) \quad \forall k, \mathcal{P}, \mathcal{M}$$



orb transformations into preparations for notational simplicity.

Quanta **3**, 67 (2014)

Frame representations[4]

Let $\mathbb{F} = \{F_\lambda : \lambda \in \Lambda\}$ and $\mathbb{D} = \{D_\lambda : \lambda \in \Lambda\}$ be frames for a vector space V with inner product $\langle ., . \rangle$ satisfying

$$M = \int d\lambda \, \langle F_\lambda, M \rangle \, D_\lambda \quad \forall M \in V.$$

By linearity,

$$\langle E, \rho \rangle = \int d\lambda \, \langle F_\lambda, \rho \rangle \, \langle E, D_\lambda \rangle = \int d\lambda \mu(\lambda|\rho)\mu(E|\lambda) = \langle \mu(*|\rho), \mu(E|*) \rangle \, .$$

Prepare a state $\rho \Leftrightarrow$ sample $\lambda \in \Lambda$ with quasiprobability $\mu(\lambda|\rho)$

nd Emerson, NJP **11**, 063040 (2009)

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Measure $\{E_1, \ldots, E_k\} \Leftrightarrow$ return k with quasiprobability $\mu(E_k|\lambda)$

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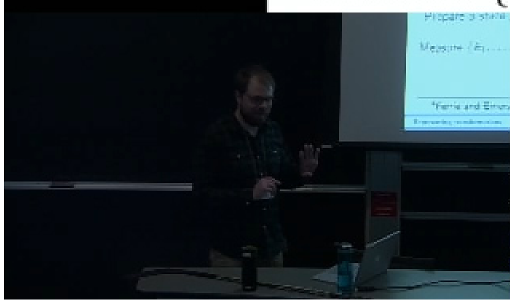

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Ontological models from quasiprobability representations

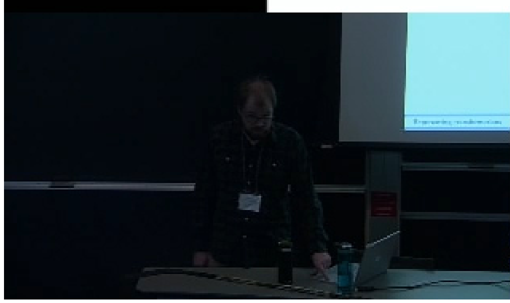
A fixed pair of dual frames $\mathbb{F}, \mathbb{D}$ gives an ontological model for:

- the set of states with nonnegative distributions,

$$\mathbb{S}(\mathbb{F}) = \{\rho : \mu(*|\rho) \geq 0\}.$$

- the set of POVM elements with nonnegative distributions,

$$\mathbb{M}(\mathbb{D}) = \{E : \mu(E|*) \geq 0\}.$$

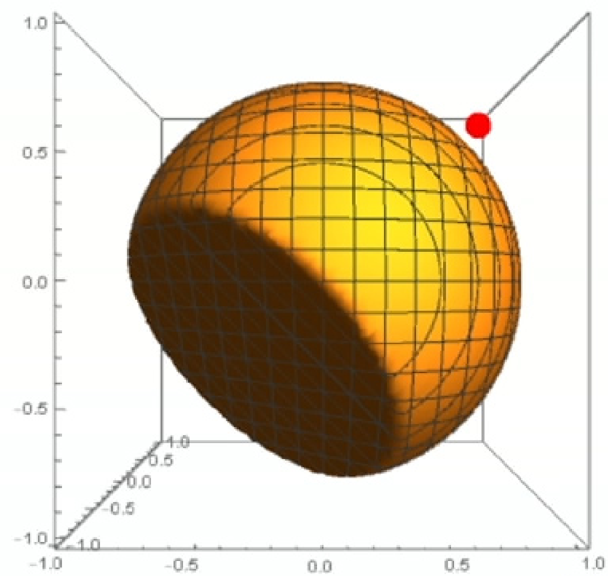


The qubit Wigner function

Let $\mathbb{D} = \{\frac{1}{2}(I + xX + xzY + zZ) : x, z = \pm\}$.

The frame has d^2 elements and so the dual is unique, $\mathbb{F} = \mathbb{D}/2$.

Nonnegative states and POVM effects



Contextuality and quasiprobabilities

Generalized contextuality: any two operational objects that generate the same statistics when varied over all other operations are ontologically identical.[5]

Any preparation and measurement noncontextual model can be obtained by restricting some quasiprobability representation to the preparations and measurements for which it is nonnegative.

s, PRA **71**, 052108 (2005)

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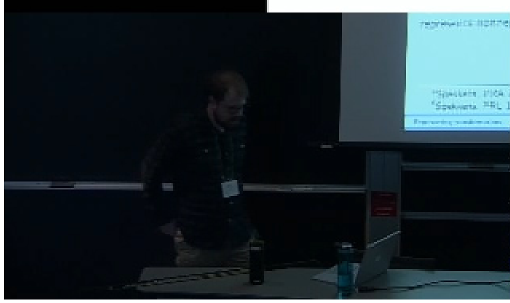
Any quasiprobability representation is a preparation and measurement noncontextual ontological model for the preparations and measurements it represents nonnegatively.[6]



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Transformations in quasiprobability representations

Transformations often folded into preparations and measurements.

Can define a quasiprobability representation of a linear map $\mathcal{T}$ by[7]

$$\mu(\lambda'|\mathcal{T},\lambda) = \langle F_{\lambda'}, \mathcal{T}(D_\lambda)\rangle \,,$$

so that

$$\mathcal{T}(\rho) = \int d\lambda \, d\lambda' \mu(\lambda|\rho)\mu(\lambda'|\mathcal{T},\lambda) D_{\lambda'}.$$

Choi-Jamiołkowski isomorphism: as $\mathrm{tr}(AB) = \mathrm{tr}\big([A \otimes B^T]\Phi\big)$, $\mu(\lambda'|\mathcal{T},\lambda)$ is a quasiprobability representation of $\mathcal{T}^\dagger \otimes I(\Phi)$.

[7] ...nd Emerson, NJP **11**, 063040 (2009)

Contexts for transformations

For preparations, the only contexts are different convex combinations.

For transformations, a channel can be implemented as a convex combination of other channels, or a composition of channels.

Convex linearity is manifestly respected.

Composition is also respected:

$$\mu(\lambda''|\mathcal{T}\mathcal{V},\lambda) = \langle F_{\lambda''}, \mathcal{T}(\mathcal{V}[D_\lambda])\rangle$$
$$= \int_\Lambda \mathrm{d}\lambda' \, \langle F_{\lambda''}, \mathcal{T}(D_{\lambda'})\rangle \, \langle F_{\lambda''}, \mathcal{V}(D_{\lambda'})\rangle \, .$$

The importance of being idle

When are *any* transformations represented nonnegatively?

Suppose you leave a system alone (for an instant), how does the ontic state evolve? It doesn't.

So for the identity channel to be represented nonnegatively, need

$$\delta(\lambda, \lambda') = \mu(\lambda' | \mathcal{I}, \lambda).$$

Satisfied if and only if Λ has d^2 points.

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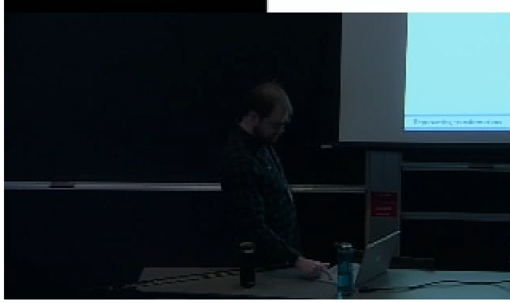
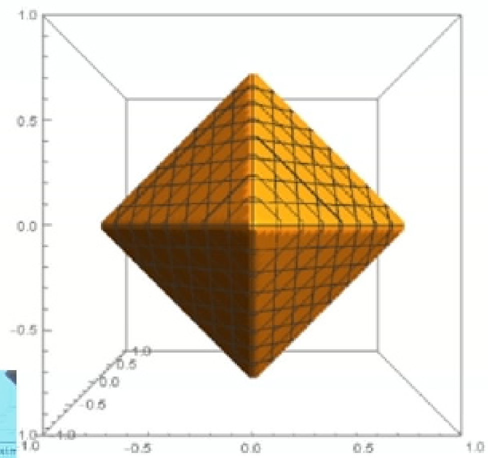


8-state model[9]

Let $\mathbb{D} = \{\frac{1}{2}(I + xX + yY + zZ) : x, y, z = \pm\}$.

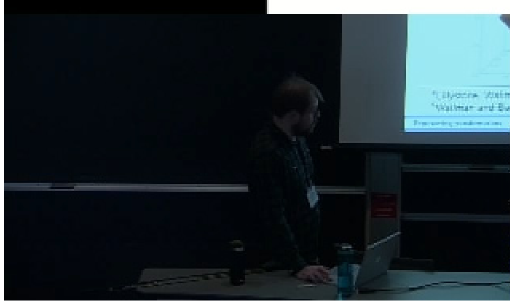
The frame has 8 elements and so the dual is not unique! Set $\mathbb{F} = \mathbb{D}/4$.

Nonnegative states and POVM effects



All Clifford transformations permute the frame elements.

Gives a noncontextual representation of the Clifford *group*, not the *semigroup*.



e, Wallman, and Emerson, arXiv:1802.06121

n and Bartlett, PRA **85**, 062121 (2012)

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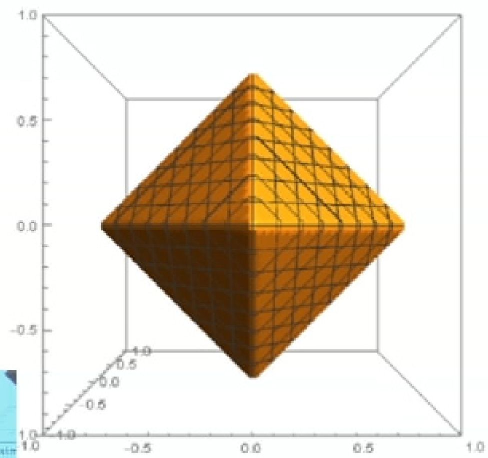

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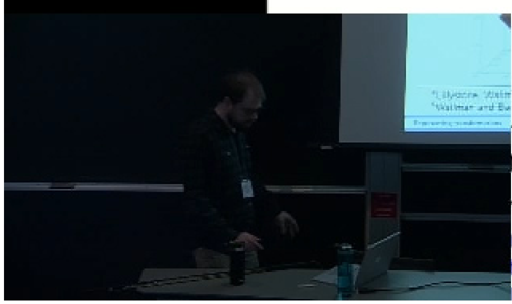
E.g., applying a random Pauli matrix vs applying a random Clifford.[8]



e, Wallman, and Emerson, arXiv:1802.06121

n and Bartlett, PRA **85**, 062121 (2012)

Rebit Wigner function[10]

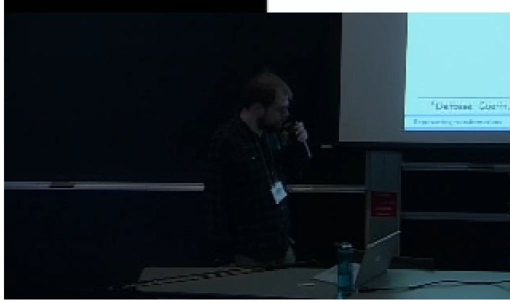
Let $\mathbb{D} = \{\frac{1}{2}(I + xX + zZ) : x, z = \pm\}$.

The frame has 4 elements and so the dual is not unique! Set $\mathbb{F} = \mathbb{D}/2$.

Applying I or Y randomly maps $(x, z) \to \pm(x, z)$

Applying X or Z randomly maps $(x, z) \to \pm(x, -z)$.

The operations are identical on rebit states!



, Guerin, Bian, and Raussendorf, Phys. Rev. X 5, 021003 (2015).

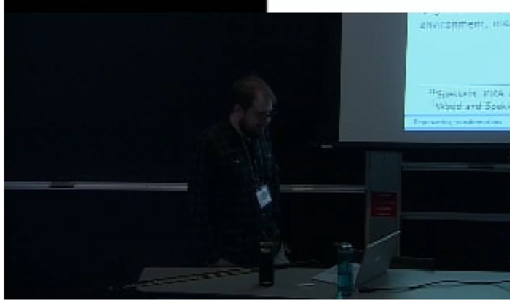
A foundational principle

Generalized contextuality (Spekkens): any two operational objects that have the same operational statistics when varied over all other operations are ontologically identical.[11]

Leibniz's principle: any two distinct objects have a distinct property.

Empiricist's version: any two distinct objects can be distinguished by measuring some property.

Objection: if operational objects are mixed due to information loss to the environment, measure the environment.



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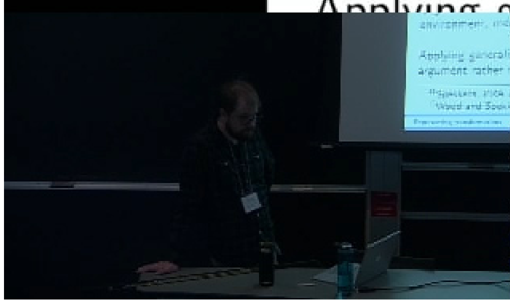
Applying generalized contextuality to mixed operations is a no-fine-tuning rather than an appeal to Leibniz's principle.[12]



s, PRA **71**, 052108 (2005)

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Noncontextuality for unitary transformations

No operational way to distinguish between applying U_1 then U_2 and applying $U_2 U_1$.

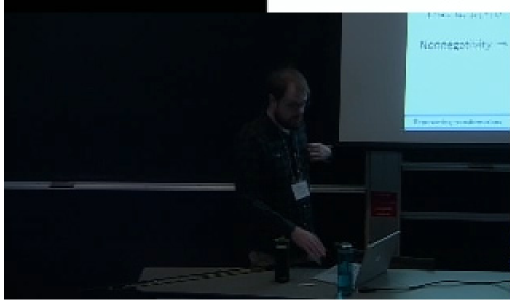
Contexts for unitary operations: composition.

Transformation noncontextuality for a group $\mathbb{G}$:

$$\mu(\lambda''|U_2 U_1, \lambda) = \int_\Lambda \mathrm{d}\lambda' \mu(\lambda''|U_2, \lambda')\mu(\lambda'|U_1, \lambda)$$

That is, $\mu(*|U, *)$ is a representation of $\mathbb{G}$.

Nonnegativity $\Rightarrow$ $\mu(*|U, *)$ is a permutation representation.



Pure operations are generalized noncontextual[13]

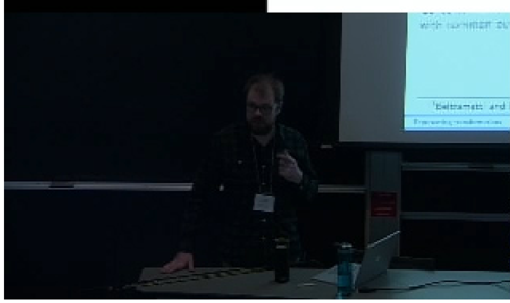
Pure operations consist of pure states, unitary transformations and projection-valued measures.

Preparations: $\phi \to \delta(\phi')$ over $\mathbb{CP}^{d-1}$

Unitaries: $U : \delta(\phi) \to \delta(U\phi U^\dagger)$

PVMs: $\Pi_k : \delta(\phi) \to \delta(\Pi_k \phi \Pi_k / \Pr(k))$ with probability $\Pr(k)$

Contexts: composition of unitaries, repeated measurements, measurements with common outcomes.



etti and Bugajski, J. Phys. A **28**, 3329 (1995).

Constructing noncontextual models for unitary groups

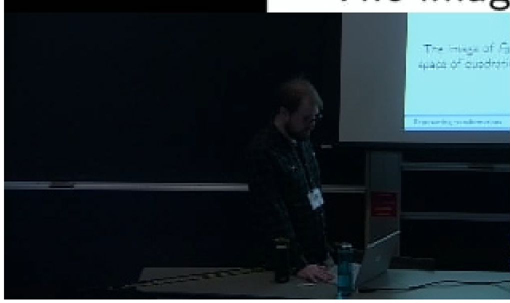
Let $\mathbb{G} \subset \mathcal{U}(d)$ be a (finite) group. Any noncontextual ontological model of $\mathbb{G}$ is equivalent to some permutation representation τ.

Can associate the elements of the representation space V to operators (quasiprobability representation) or pure states under the corresponding action of $\mathbb{G}$.

Example: let $\mathbb{G}\langle X_j, H_j, CZ_{j,k}\rangle$ and

$$F_\eta = 2^{-n} \sum_{z,x \in \mathbb{Z}_2^n : x \cdot z = 0} (-1)^{\eta(z,x)} Z[z] X[x].$$

The image of F_0 under $\mathbb{G}$ is a frame, the ontic space is a subset of the



uadratic functions.

Contextuality is a resource... for classical simulations

Best-case simulation cost is $O(\log \dim V)$

Can simulate an additional unitary h by:[14]

- Finding a map such that $h(D_\lambda) = \int_\Lambda d\lambda' \mu(\lambda'|h,\lambda) D_\lambda$.
- Adding an additional variable $\kappa \in \mathbb{C}$ (initialized to 1).
- Mapping $\lambda \to \lambda'$ with probability $\Pr(\lambda'|\lambda)$ and $\kappa \to \kappa\mu(\lambda'|h,\lambda)/\Pr(\lambda'|\lambda)$.

A natural probability distribution is

$$\Pr(\lambda'|\lambda) = \frac{|\mu(\lambda'|h,\lambda)|}{\int_\Lambda d\lambda' |\mu(\lambda'|h,\lambda)|}.$$

[14] ...n, Wallman, and Bartlett, PRL **115**, 070501 (2015).

Example

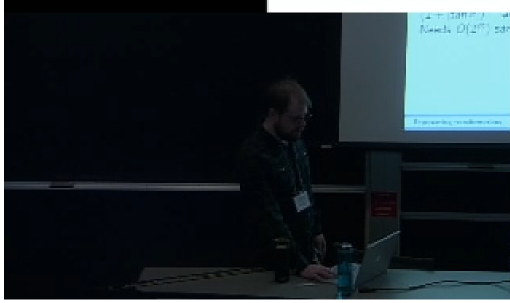
All stabilizer states can be written as

$$\psi \propto \sum_{x\in A} i^{l(x)}(-1)^{q(x)}\,|x\rangle$$

Let $\Lambda = \{A, l, q\}$.

Applying a Clifford gate permutes the (A, l, q).

$\cos\theta I + i\sin\theta P$ can be implemented by applying I or P with probabilities $(1+|\tan\theta|)^{-1}$ and $(1+|\cot\theta|)^{-1}$ respectively (or sometimes trivially). Needs $O(2^m)$ samples to converge for m $\pi/8$ gates.



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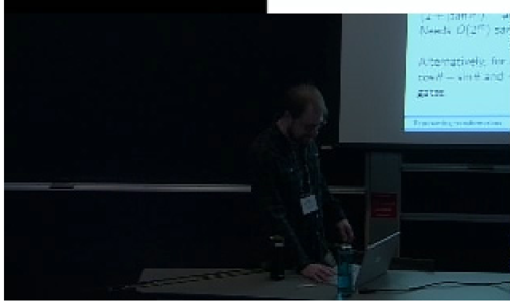
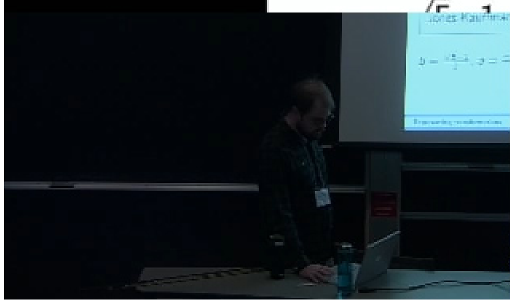


ely, for $\theta \in (0, \pi/8]$, apply I and $(I + iP)/\sqrt{2}$ with probabilities
θ and $\sqrt{2} \sin\theta$. Needs $O(1.17^m)$ samples to converge for m $\pi/8$

Optimized scalings

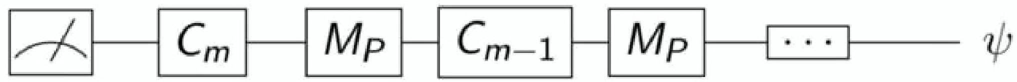
Gate set	Example gate	scaling
T-gates	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	$2^{0.23m}$
Cyclotomic Cliffords	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/5} \end{pmatrix}$	$2^{0.15m}$
n-th roots of Hadamard	$\sqrt{H}$	$2^{0.46m}$
V-basis	$(I + 2iY)/\sqrt{5}$	$2^{0.22m}$
Fibonacci anyon gate sets	$\begin{pmatrix} b & -\sqrt{b} \\ \sqrt{b} & b \end{pmatrix}$	$2^{0.12m}$
Jones-Kauffman anyon gate sets	$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$	$2^{0.077m}$

$a = \frac{-1+4\sqrt{-3}}{7}$



Sparse representations

Suppose the magic gates are $M_P = \exp(i\theta P) = \cos\theta I + i\sin\theta P$ for some Pauli P.



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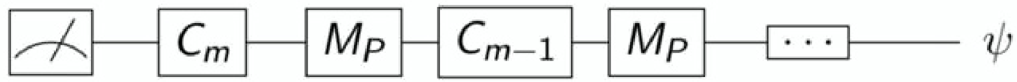
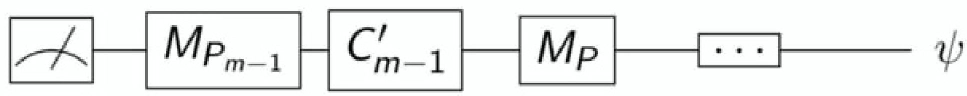


The last M_P gate is mapped to $M_{P'}$ when propagated past the last Clifford gate.



Sparse representations

"First" pass: make the l_0 norm of

$$M_{P_{m-1}} \dots M_{P_0} = \sum_i \alpha_i C_i$$

less than 2^m.

Corresponds to setting $\Pr(i) \propto \delta(\alpha_i)$.

Sufficient to make the l_0 norm of

$$M_{P_1} \dots M_{P_0} = \sum_i \alpha_i C_i$$

4.

Sparse representations

Can make the l_0 norm 3.

If P_0 and P_1 commute: $P_0 = ZI$ and $P_1 = IZ$.

Then $M_{ZI}M_{IZ} = (1 - a_- - a_+)I + a_+ CZ + a_- XXCZXX$ where $a_\pm = 1/2 - \exp(\pm 2i\theta)/2$.

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If P_0 and P_1 anticommute:

$$M_{P_1} M_{P_0} = \cos^2\theta I + i\cos\theta\sin\theta(P_0 + P_1) + \sin^2\theta P_0 P_1$$