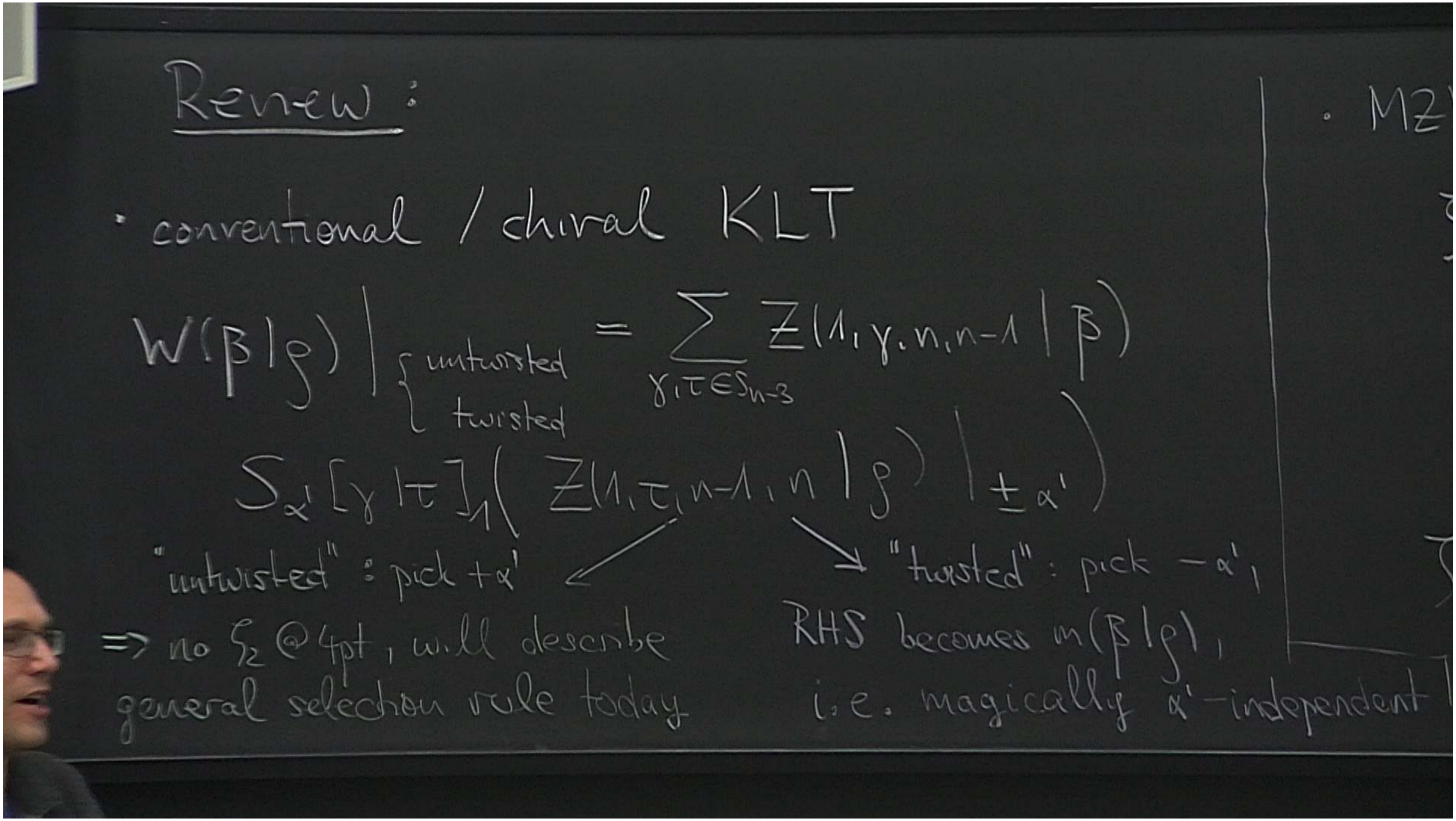
Title: PSI 2017/2018 - Scattering Amplitudes in QFT & String Theory - Lecture 17

Date: Apr 27, 2018 11:30 AM

URL: http://pirsa.org/18040049

Abstract:



Review:

- conventional / chiral KLT

$$W(\beta|\rho)\Big|_{\begin{cases}\text{untwisted}\\ \text{twisted}\end{cases}} = \sum_{\gamma,\tau\in S_{n-3}} Z(1,\gamma,n,n-1|\beta)\, S_{\alpha'}[\gamma|\tau]_1 \left(Z(1,\tau,n-1,n|\rho)\Big|_{\pm\alpha'}\right)$$

"untwisted": pick $+\alpha'$

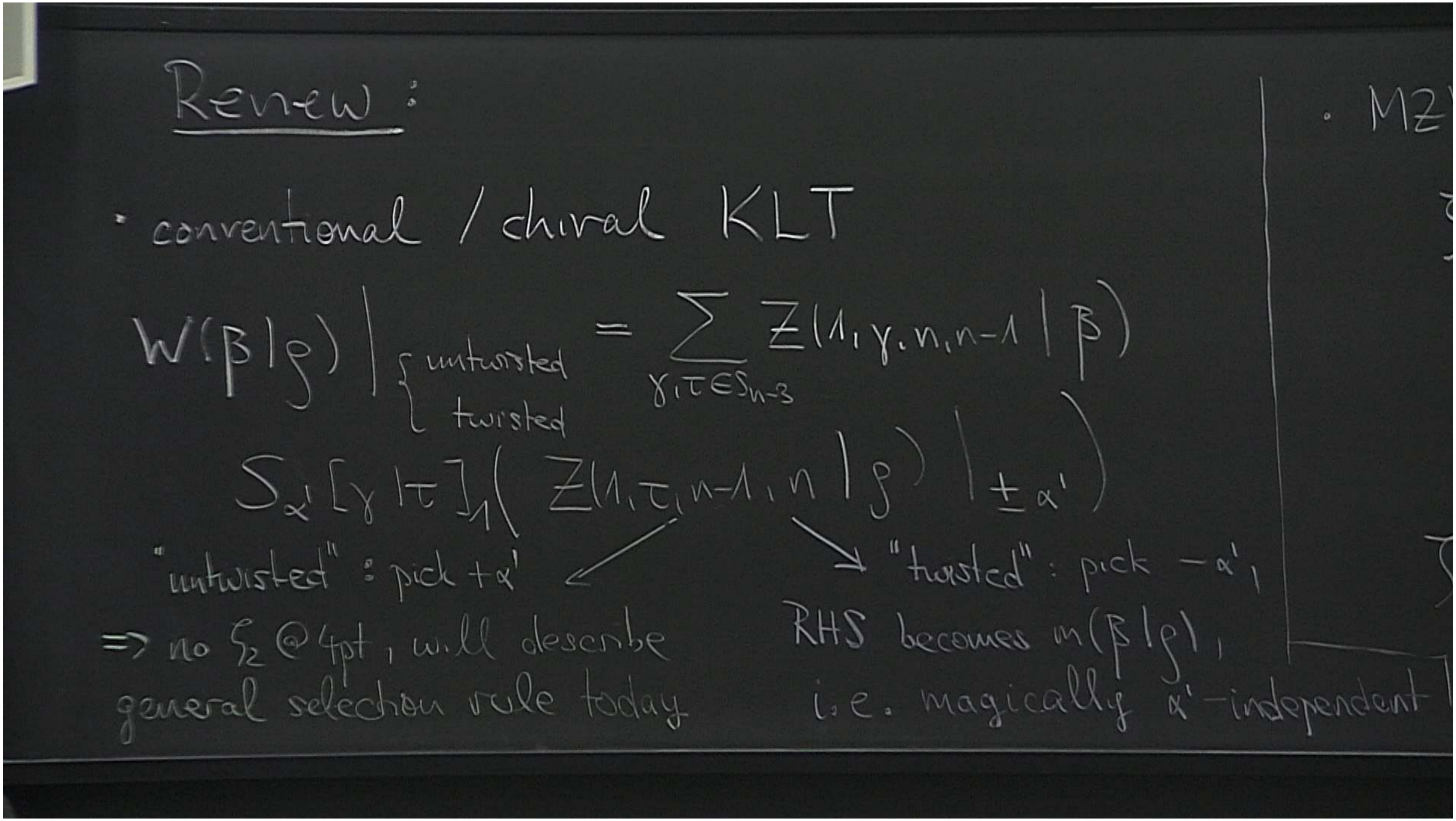
$\Rightarrow$ no ζ_2 @ 4pt, will describe general selection rule today

"twisted": pick $-\alpha'$,

RHS becomes $m(\beta|\rho)$,

i.e. magically α'-independent

- MZ



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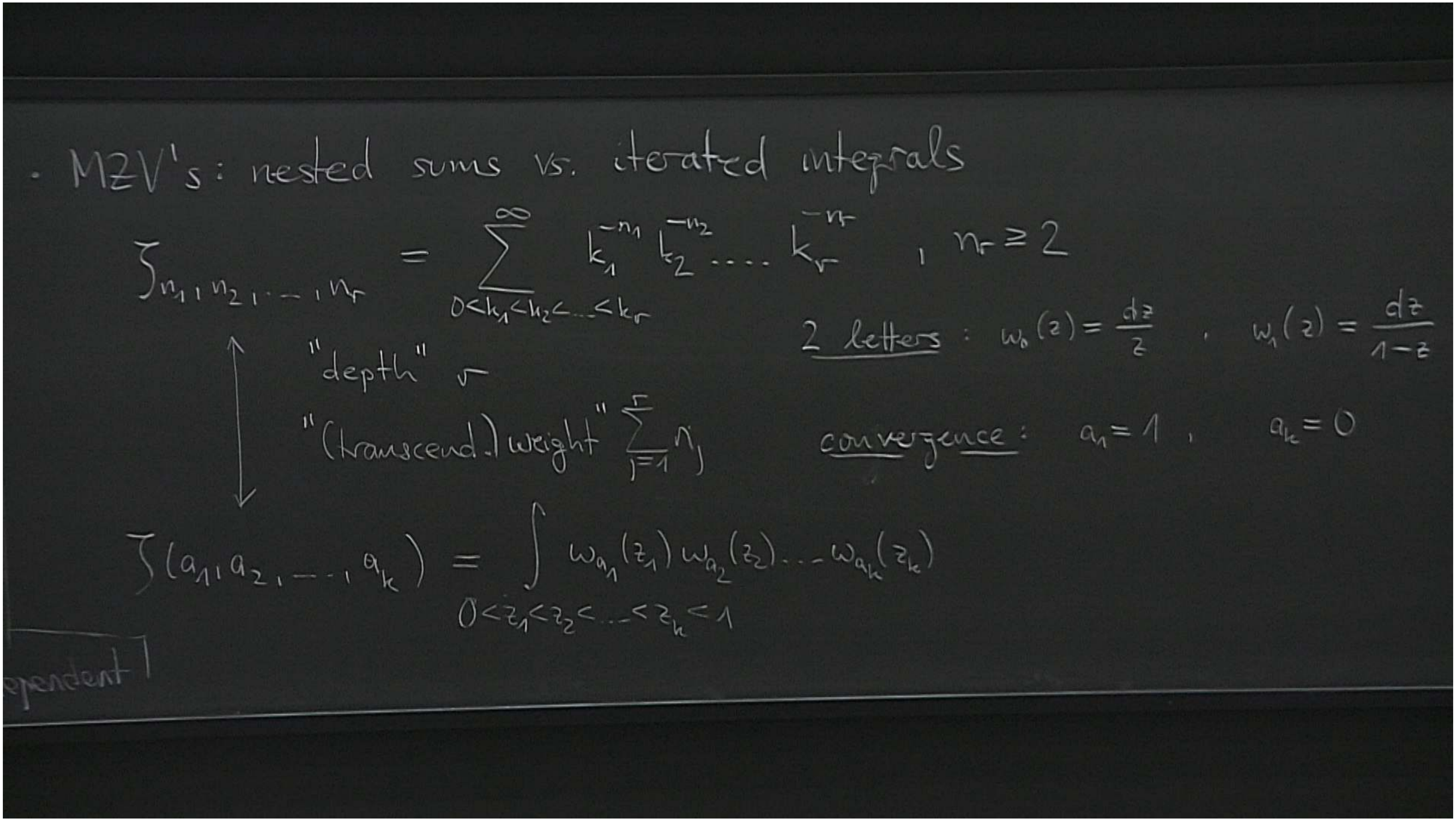


· MZV's: nested sums vs. iterated integrals

$$\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \cdots\cdots k_r^{-n_r} \quad , \quad n_r \geq 2$$

$\updownarrow$

"depth" r

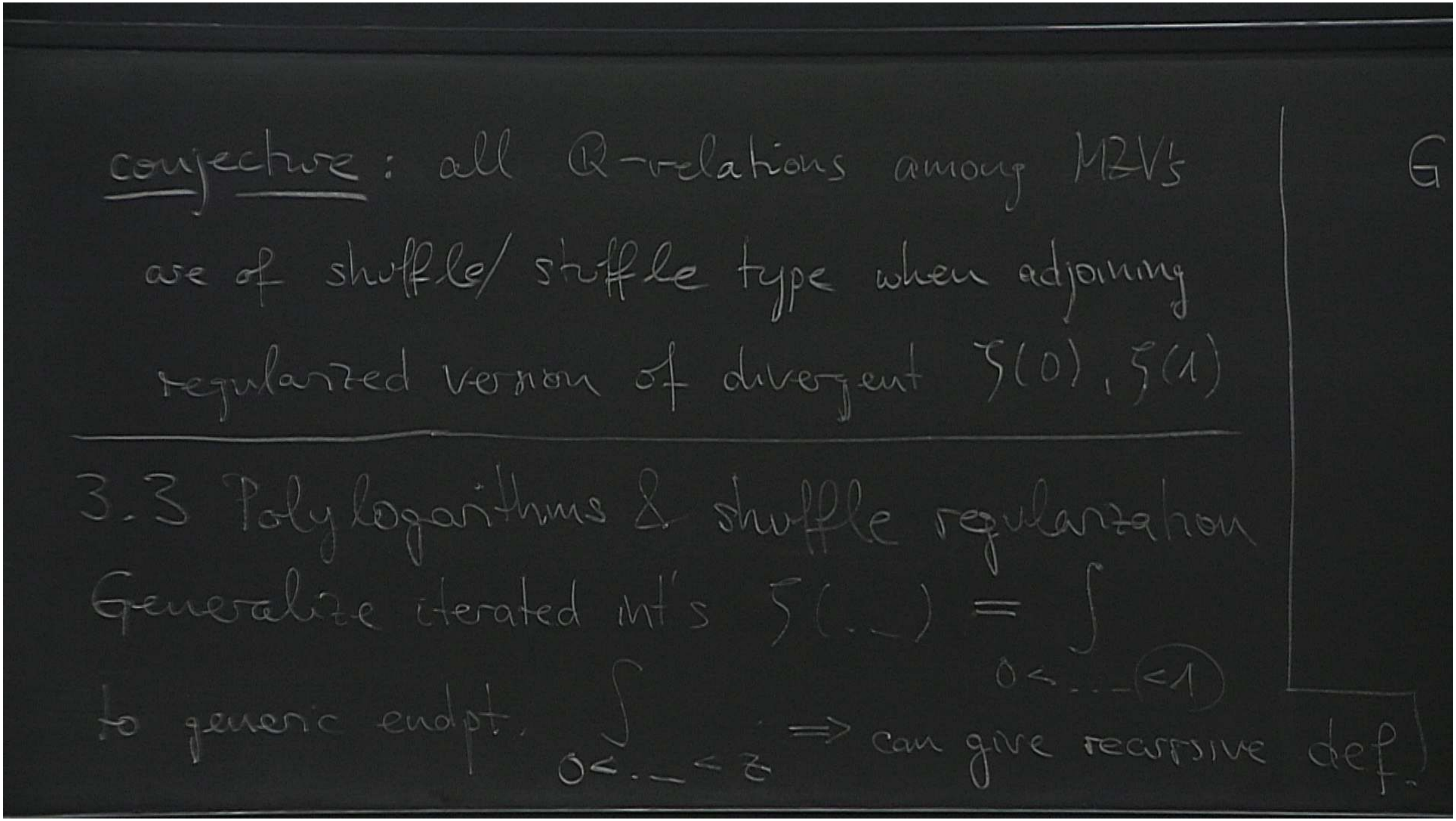
"(transcend.) weight" $\sum_{j=1}^{r} n_j$

2 letters : $\omega_0(z) = \frac{dz}{z}$, $\omega_1(z) = \frac{dz}{1-z}$

convergence : $a_1 = 1$, $a_k = 0$

$$\zeta(a_1, a_2, \ldots, a_k) = \int_{0<z_1<z_2<\ldots<z_k<1} \omega_{a_1}(z_1)\,\omega_{a_2}(z_2)\ldots\omega_{a_k}(z_k)$$

ependent



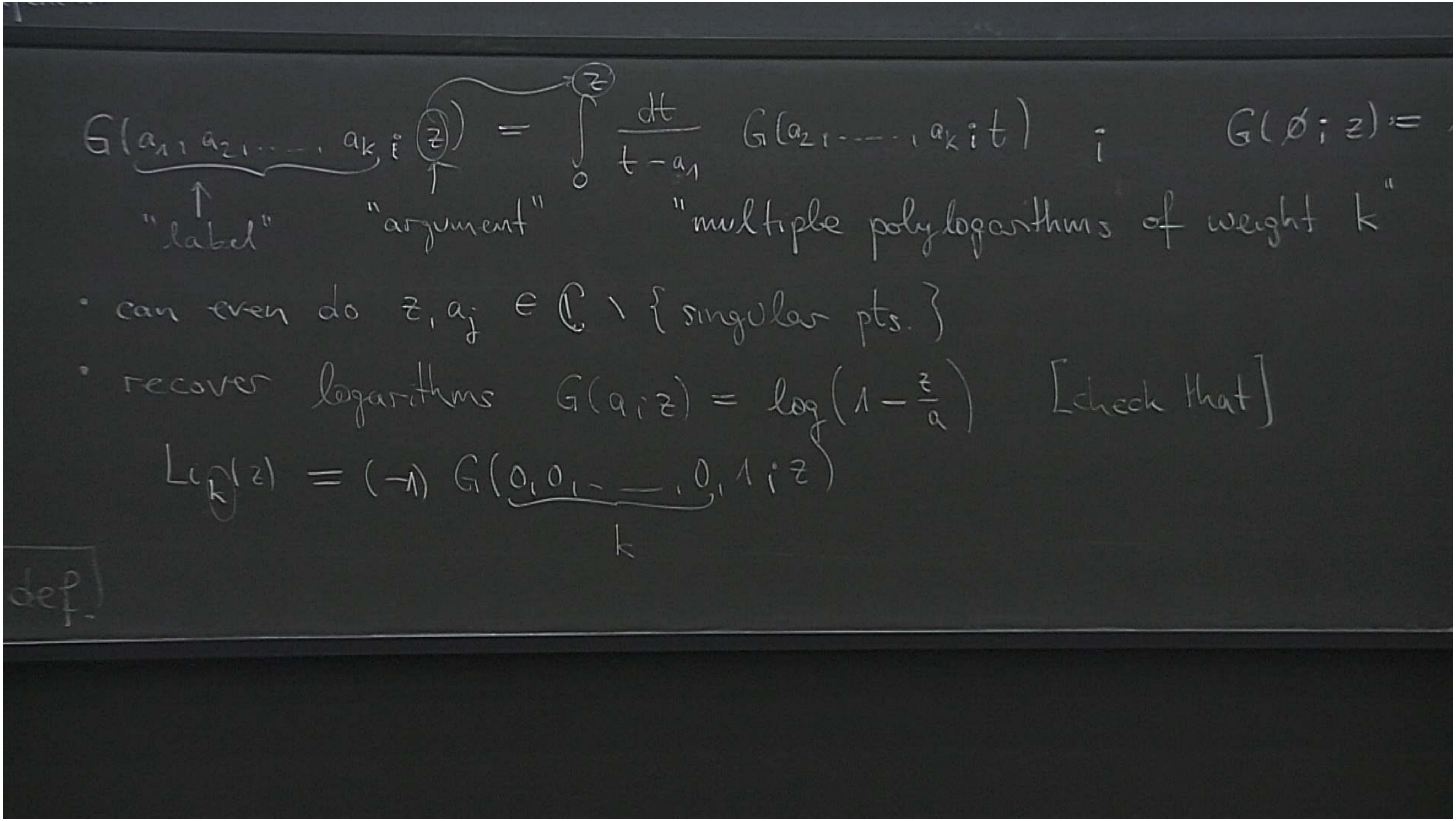
conjecture: all $\mathbb{Q}$-relations among MZV's are of shuffle/stuffle type when adjoining regularized version of divergent $\zeta(0), \zeta(1)$

3.3 Polylogarithms & shuffle regularization

Generalize iterated int's $\zeta(\ldots) = \int_{0<\ldots \enclose{circle}{<1}}$

to generic endpt. $\int_{0<\ldots<z}$ $\Rightarrow$ can give recursive def.

G



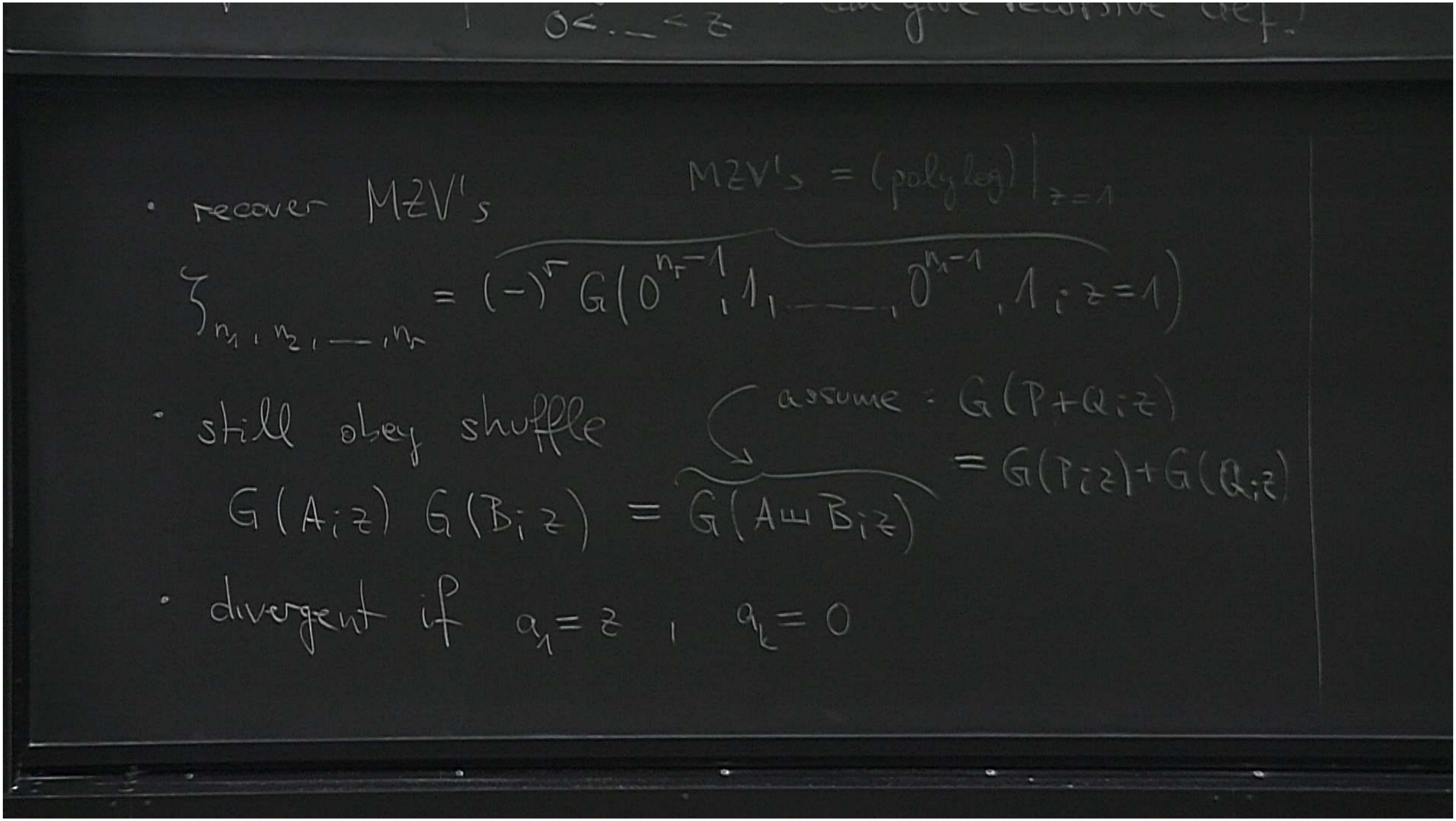
$$G(\underbrace{a_1, a_2, \dots, a_k}_{\uparrow \text{"label"}}; \underset{\uparrow \text{"argument"}}{z}) = \int_0^{z} \frac{dt}{t-a_1}\, G(a_2, \dots, a_k; t) \quad ; \quad G(\emptyset; z) =$$

"multiple polylogarithms of weight k"

- can even do $z, a_j \in \mathbb{C} \setminus \{\text{singular pts.}\}$
- recover logarithms $G(a; z) = \log\left(1 - \frac{z}{a}\right)$ [check that]

$$\mathrm{Li}_{k}(z) = (-1)\, G(\underbrace{0, 0, \dots, 0}_{k}, 1; z)$$

def.



$0 < \ldots < z$ give recursive def.)

- recover MZV's

$$\text{MZV's} = (\text{polylog})\big|_{z=1}$$

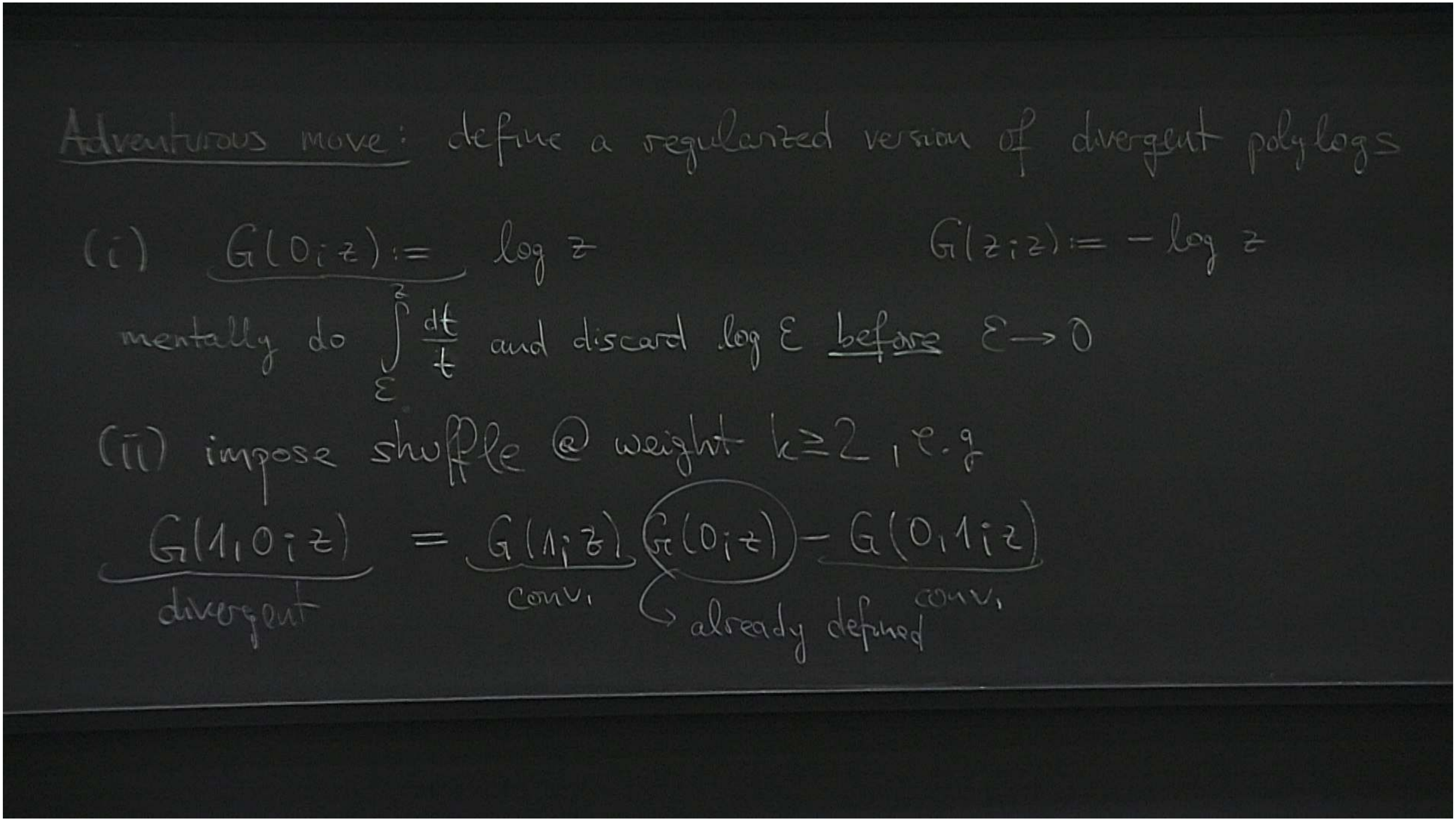
$$\zeta_{n_1, n_2, \ldots, n_r} = \overbrace{(-)^r\, G(0^{n_r - 1}, 1, \ldots, 0^{n_1 - 1}, 1; z = 1)}$$

- still obey shuffle

$$G(A;z)\, G(B;z) = \overbrace{G(A \sqcup\!\sqcup B; z)}$$

assume: $G(P+Q;z) = G(P;z) + G(Q;z)$

- divergent if $a_1 = z$, $a_k = 0$



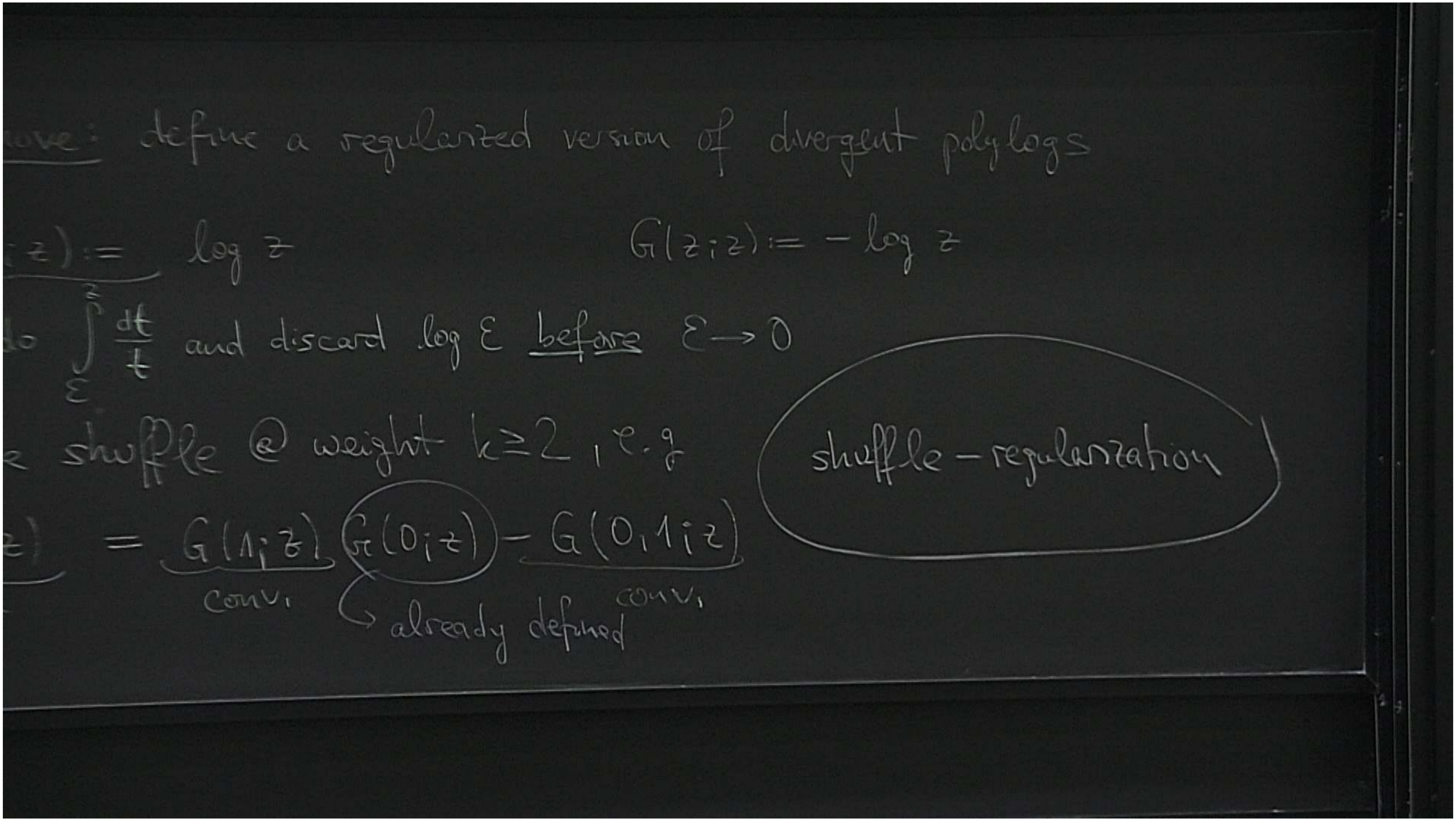
Adventurous move: define a regularized version of divergent polylogs

(i) $\underbrace{G(0;z) :=} \log z$ $\qquad G(z;z) := -\log z$

mentally do $\int_{\varepsilon}^{z} \frac{dt}{t}$ and discard $\log \varepsilon$ before $\varepsilon \to 0$

(ii) impose shuffle @ weight $k \geq 2$, e.g.

$$\underbrace{G(1,0;z)}_{\text{divergent}} = \underbrace{G(1;z)}_{\text{conv.}} \underset{\hookrightarrow \text{already defined}}{\boxed{G(0;z)}} - \underbrace{G(0,1;z)}_{\text{conv.}}$$



ove: define a regularized version of divergent polylogs

$\underline{;z) :=}$ $\log z$ $\qquad\qquad G(z;z) := -\log z$

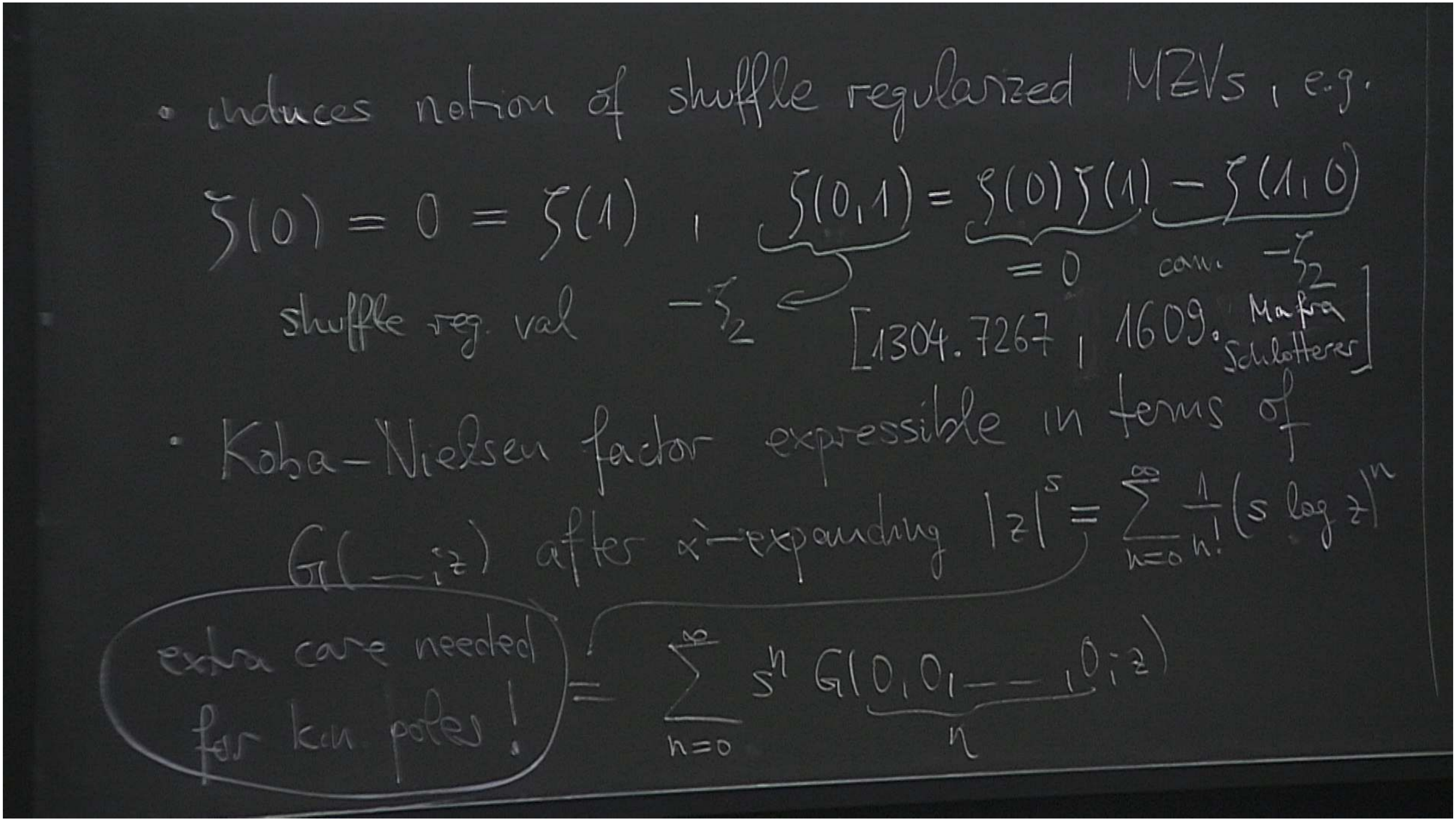
do $\int_{\varepsilon}^{z} \frac{dt}{t}$ and discard $\log \varepsilon$ $\underline{\text{before}}$ $\varepsilon \to 0$

e shuffle @ weight $k \geq 2$, e.g.

shuffle-regularization

$z) = \underbrace{G(1;z)}_{\text{conv.}} \boxed{G(0;z)} - \underbrace{G(0,1;z)}_{\text{conv.}}$

↳ already defined



- induces notion of shuffle regularized MZVs, e.g.

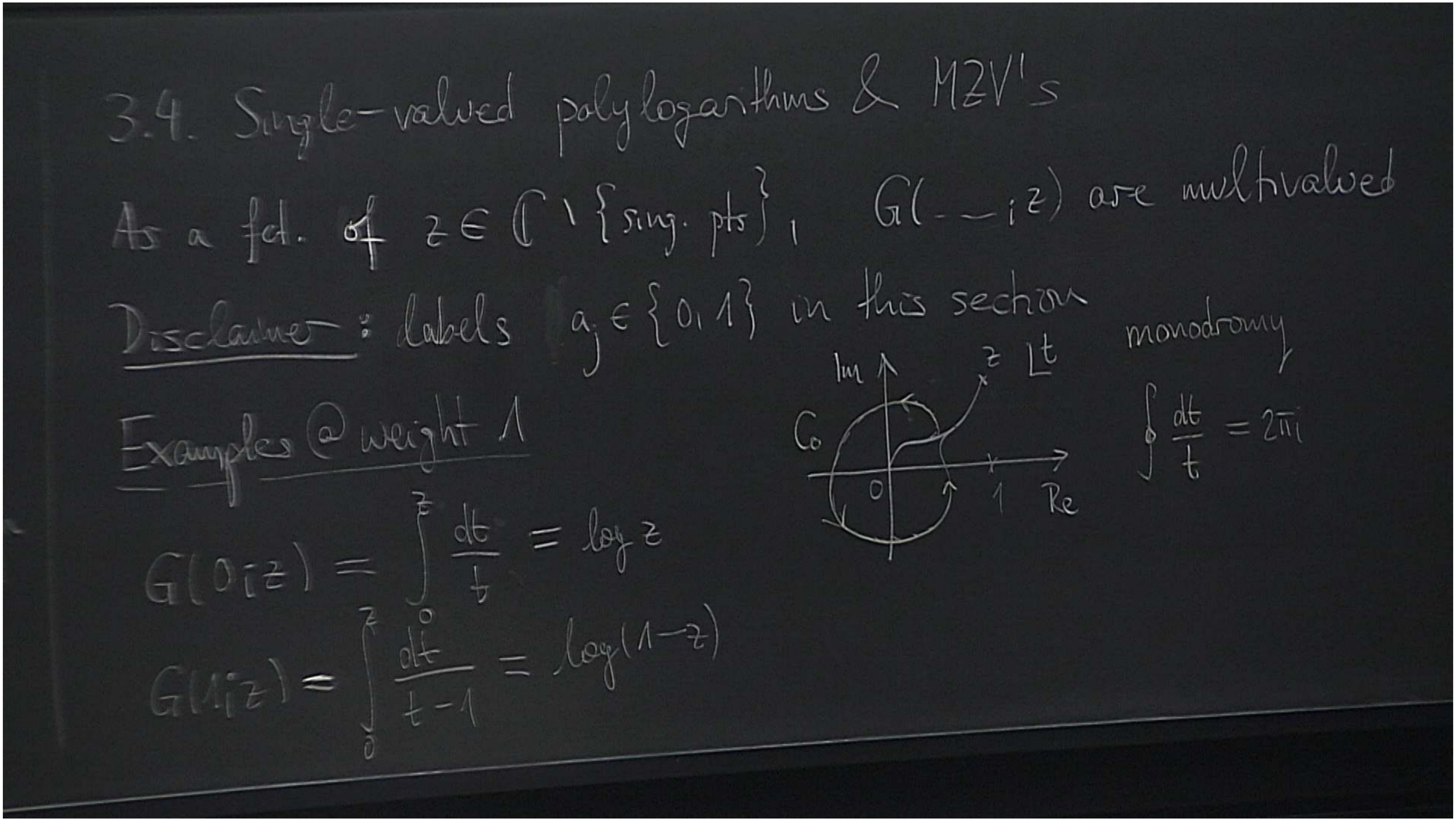
$$\zeta(0) = 0 = \zeta(1)\ , \quad \underbrace{\zeta(0,1)}_{\text{shuffle reg. val}\ \ -\zeta_2} = \underbrace{\zeta(0)\zeta(1)}_{=0} \underbrace{-\zeta(1,0)}_{\text{conv.}\ \ -\zeta_2}$$

[1304.7267, 1609. Mafra Schlotterer]

- Koba-Nielsen factor expressible in terms of $G(\_\_;z)$ after α'-expanding

$$|z|^s = \sum_{n=0}^{\infty} \frac{1}{n!}(s \log z)^n = \sum_{n=0}^{\infty} s^n G(\underbrace{0,0,\ldots,0}_{n};z)$$

extra care needed for kin. poles!



3.4. Single-valued polylogarithms & MZV's

As a fct. of $z \in \mathbb{C} \setminus \{\text{sing. pts}\}$, $G(\_\_;z)$ are multivalued

Disclaimer: labels $a_j \in \{0,1\}$ in this section

Examples @ weight 1

$$G(0;z) = \int_0^z \frac{dt}{t} = \log z$$

$$G(1;z) = \int_0^z \frac{dt}{t-1} = \log(1-z)$$

monodromy

$$\oint \frac{dt}{t} = 2\pi i$$

polylogarithms & MZV's

$\mathbb{C} \setminus \{\text{sing. pts}\}$, $G(\ldots ; z)$ are multivalued

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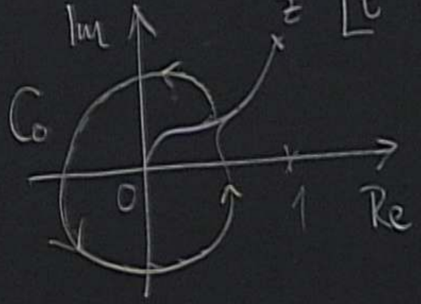

monodromy

$$\oint_{C_0} \frac{dt}{t} = 2\pi i$$

$= \log z$

$= \log(1-z)$

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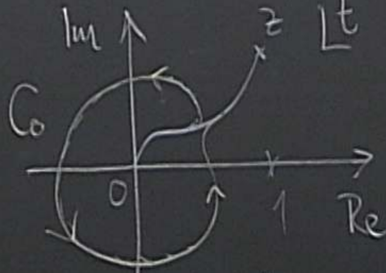

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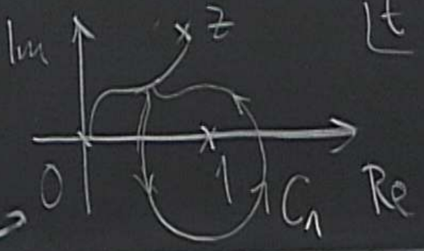
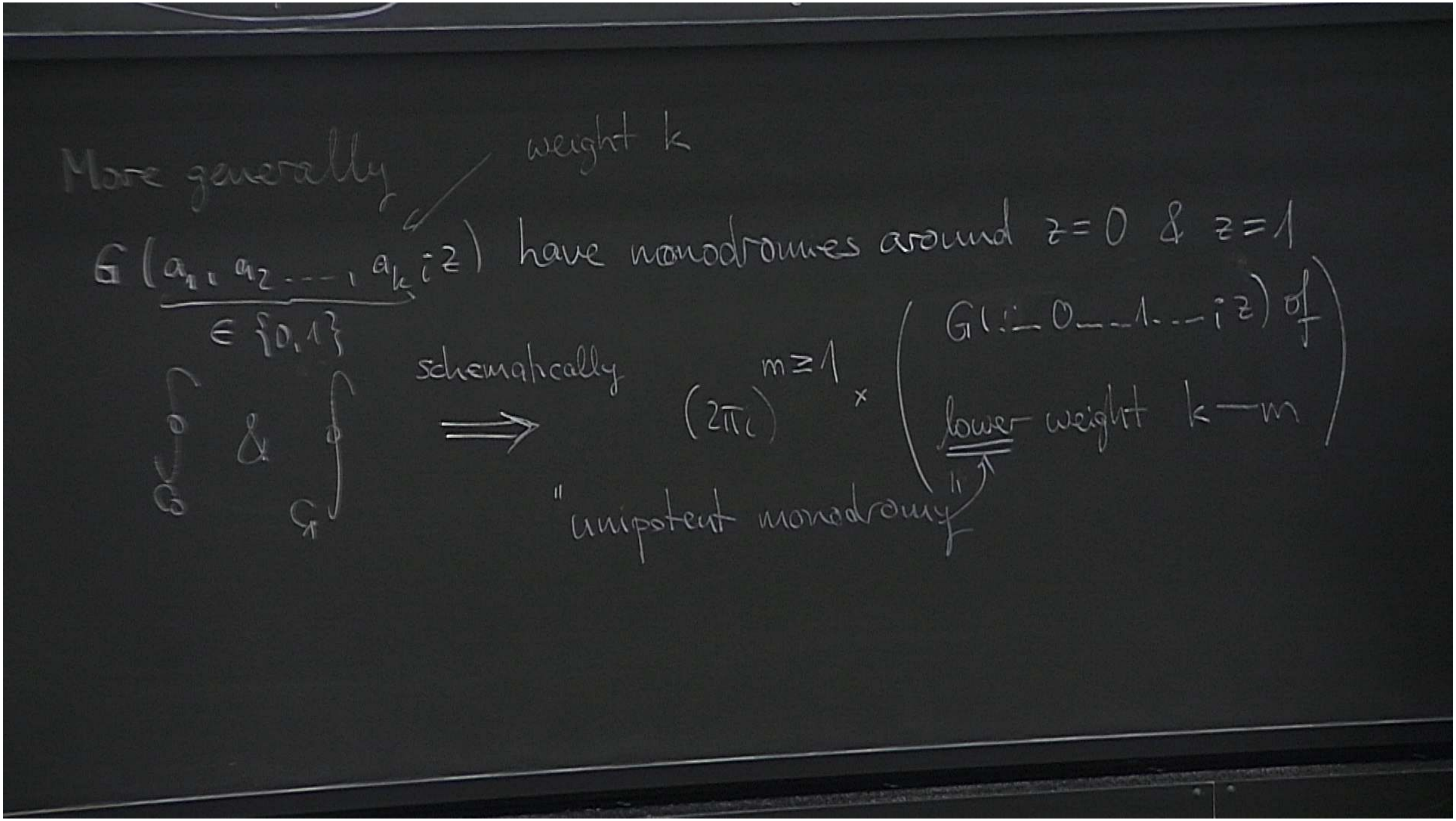


monodromy

$$\oint_{C_1} \frac{dt}{t-1} = 2\pi i$$



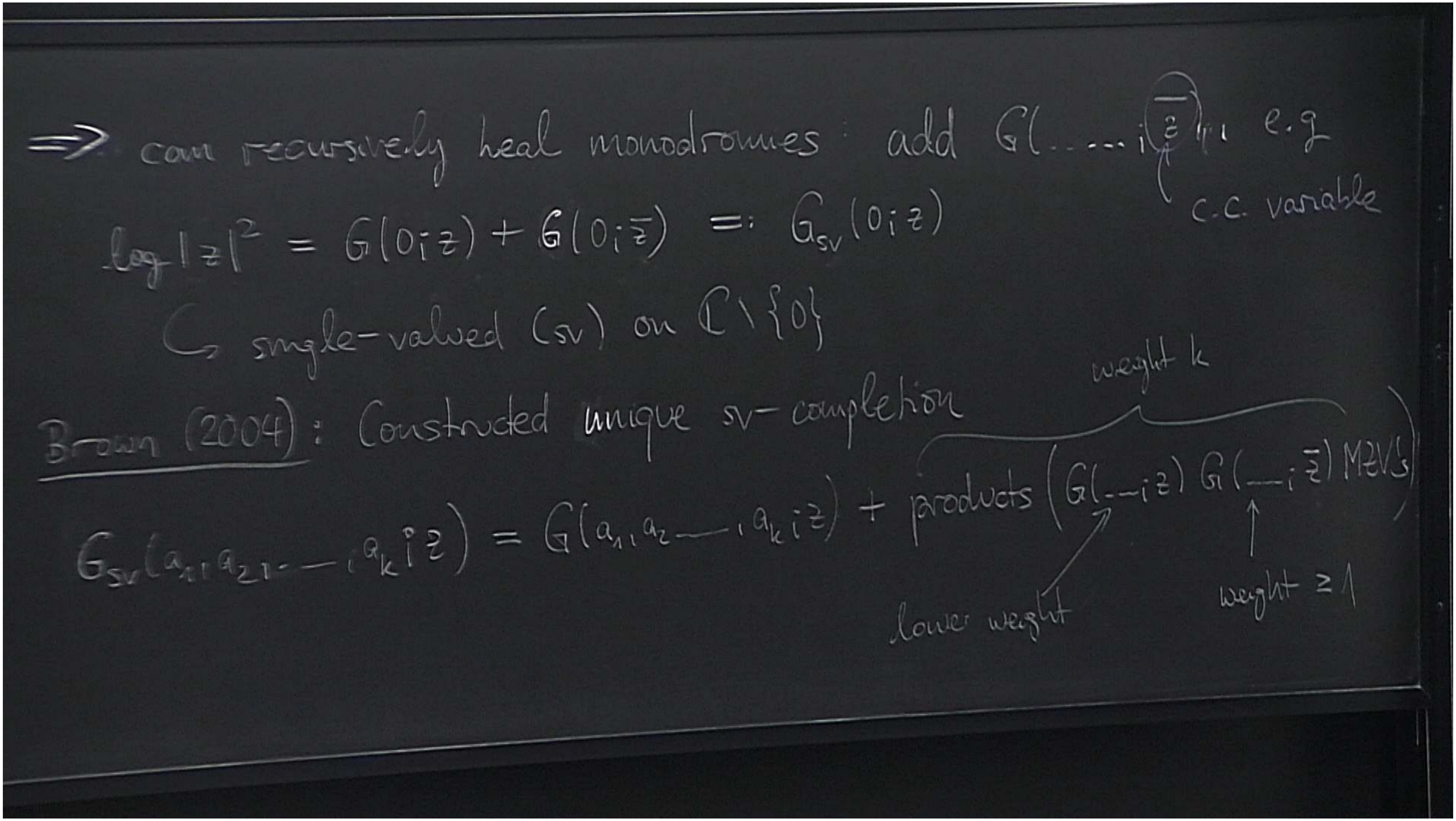
More generally

weight k

$$G(\underbrace{a_1, a_2, \ldots, a_k}_{\in \{0,1\}}; z)$$ have monodromies around $z=0$ & $z=1$

$\oint_{C_0}$ & $\oint_{C_1}$ schematically $\Longrightarrow$ $(2\pi i)^{m \geq 1} \times \left(\begin{array}{l} G(\ldots 0 \ldots 1 \ldots; z) \text{ of} \\ \underline{\underline{\text{lower}}} \text{ weight } k-m \end{array} \right)$

"unipotent monodromy"



⇒ can recursively heal monodromies: add $G(\dots;\bar{z})$'s e.g

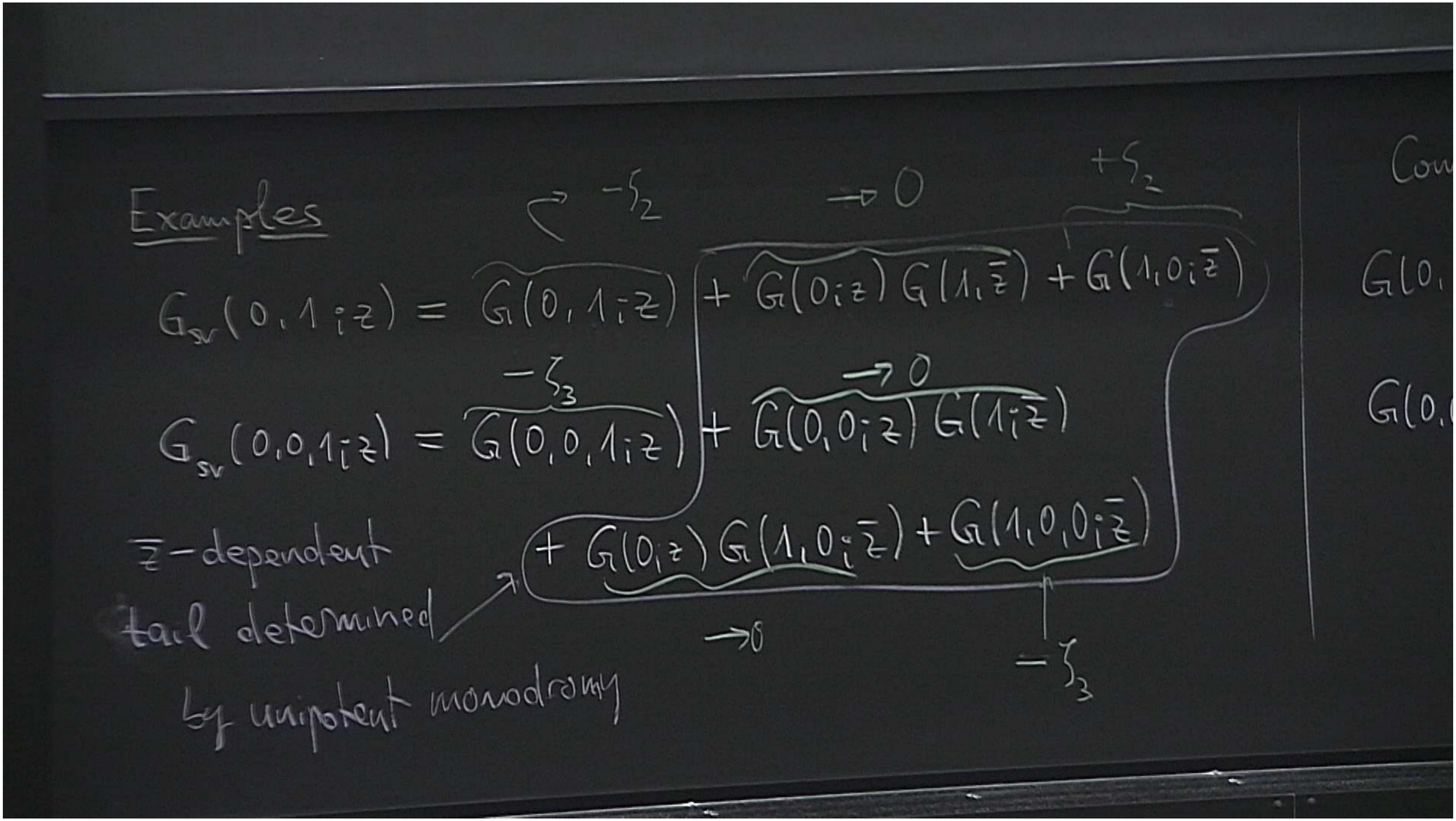
c.c. variable

$$\log|z|^2 = G(0;z) + G(0;\bar{z}) =: G_{sv}(0;z)$$

↳ single-valued (sv) on $\mathbb{C}\setminus\{0\}$

Brown (2004): Constructed unique sv-completion

$$G_{sv}(a_1,a_2,\dots,a_k;z) = G(a_1,a_2,\dots,a_k;z) + \text{products}\overbrace{\Big(\underset{\text{lower weight}}{G(\dots;z)}\;\underset{\text{weight} \geq 1}{G(\dots;\bar{z})}\;\text{MZV's}\Big)}^{\text{weight } k}$$



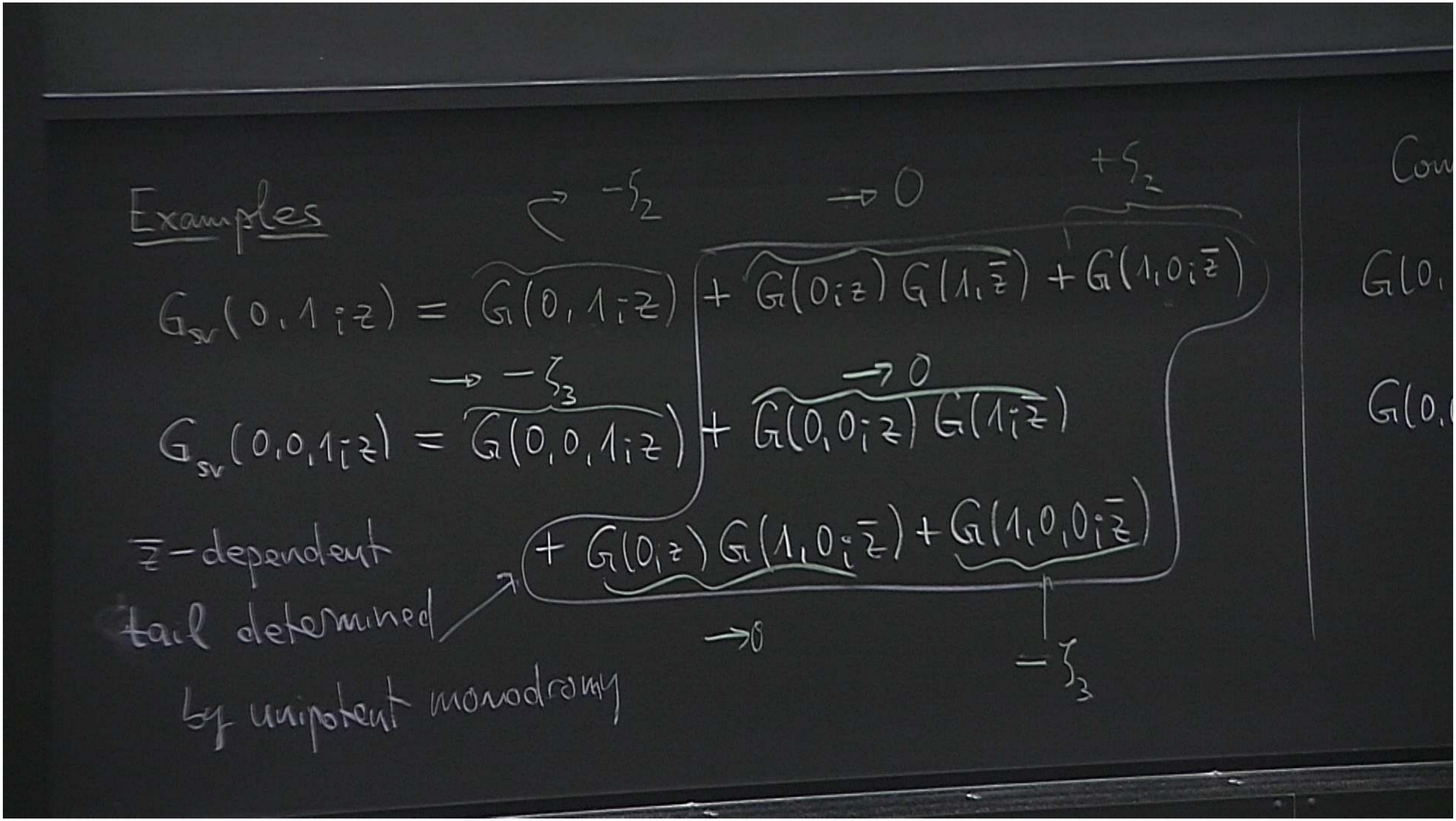
Examples

$$G_{sv}(0,1;z) = \overbrace{G(0,1;z)}^{-\zeta_2} + \overbrace{G(0;z)\,G(1;\bar{z})}^{\to 0} + \overbrace{G(1,0;\bar{z})}^{+\zeta_2}$$

$$G_{sv}(0,0,1;z) = \overbrace{G(0,0,1;z)}^{-\zeta_3} + \overbrace{G(0,0;z)\,G(1;\bar{z})}^{\to 0}$$

$$+ \underbrace{G(0;z)\,G(1,0;\bar{z})}_{\to 0} + \underbrace{G(1,0,0;\bar{z})}_{-\zeta_3}$$

$\bar{z}$-dependent tail determined by unipotent monodromy



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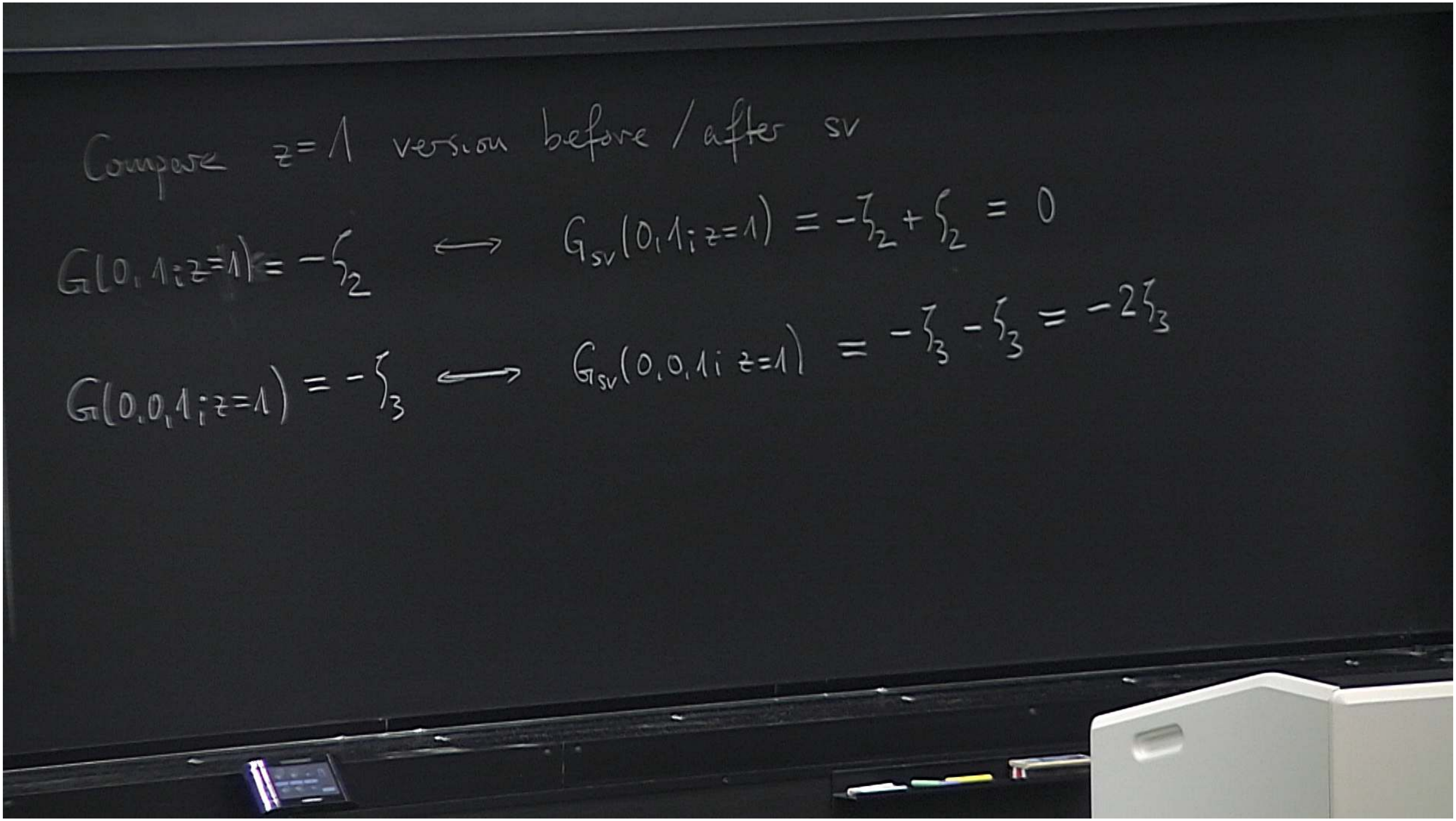
$\bar z$-dependent tail determined by unipotent monodromy

Com

G(0,

G(0,



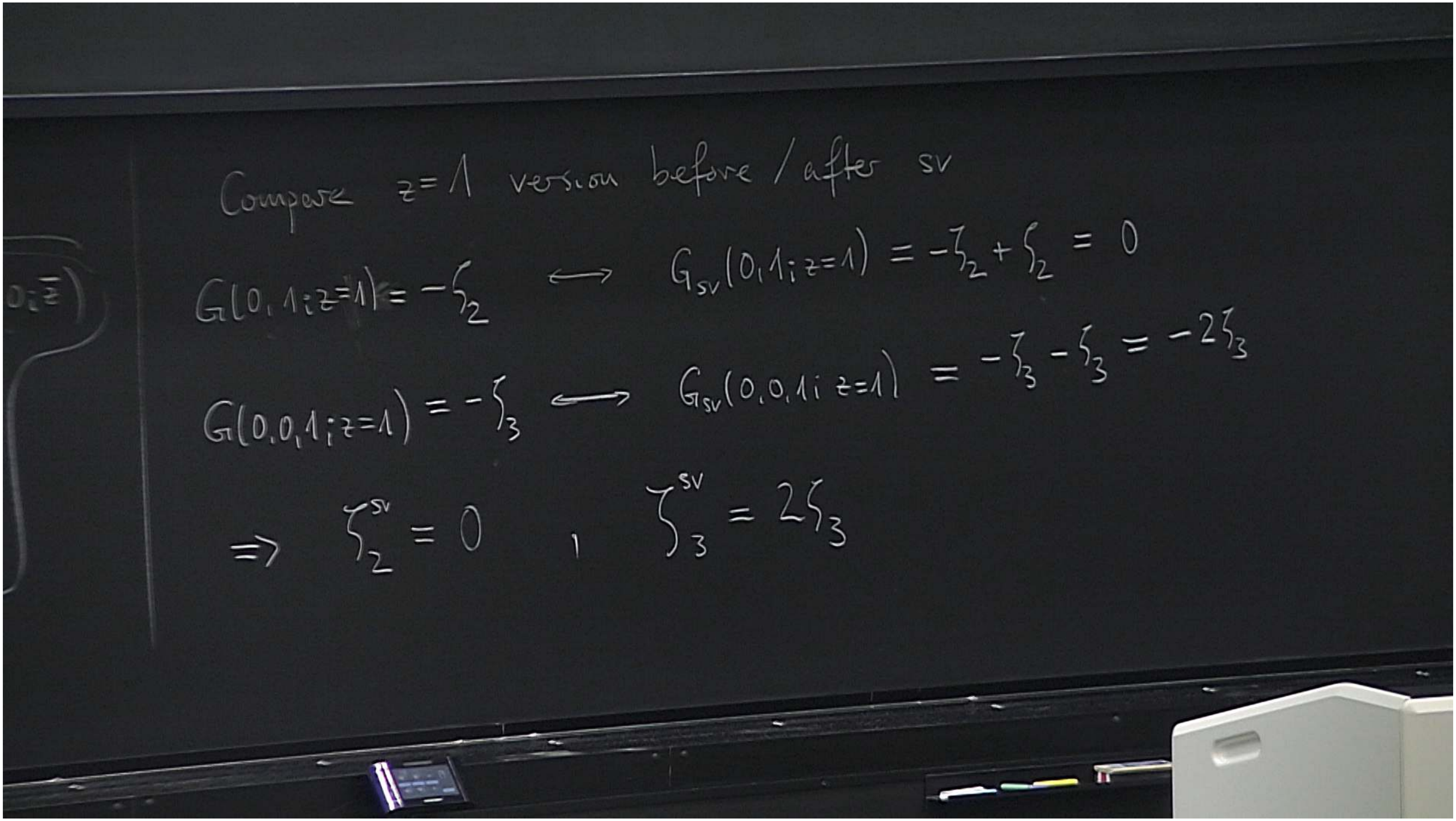
Compare $z=1$ version before / after sv

$$G(0,1;z=1) = -\zeta_2 \quad \longleftrightarrow \quad G_{sv}(0,1;z=1) = -\zeta_2 + \zeta_2 = 0$$

$$G(0,0,1;z=1) = -\zeta_3 \quad \longleftrightarrow \quad G_{sv}(0,0,1;z=1) = -\zeta_3 - \zeta_3 = -2\zeta_3$$

⇒ notion of single-valued MZV's

$$\begin{tikzcd}
\zeta_{n_1,n_2,\ldots,n_r} \arrow[r, dashed, "\text{obscure: no z-var}", "\text{which could cause monodromy}"'] \arrow[d, equal] & \zeta^{sv}_{n_1,n_2,\ldots,n_r} \arrow[d, equal, ":"'] \\
(-1)^r\, G(0^{n_r-1},1,\ldots,0^{n_1-1},1;z=1) \arrow[r, "\text{kill monodromies}", "\text{UNIQUE}"'] & (-)^r\, G_{sv}(0^{n_r-1},1,\ldots,0^{n_1-1},1;z=1)
\end{tikzcd}$$



$0;\bar{z})$

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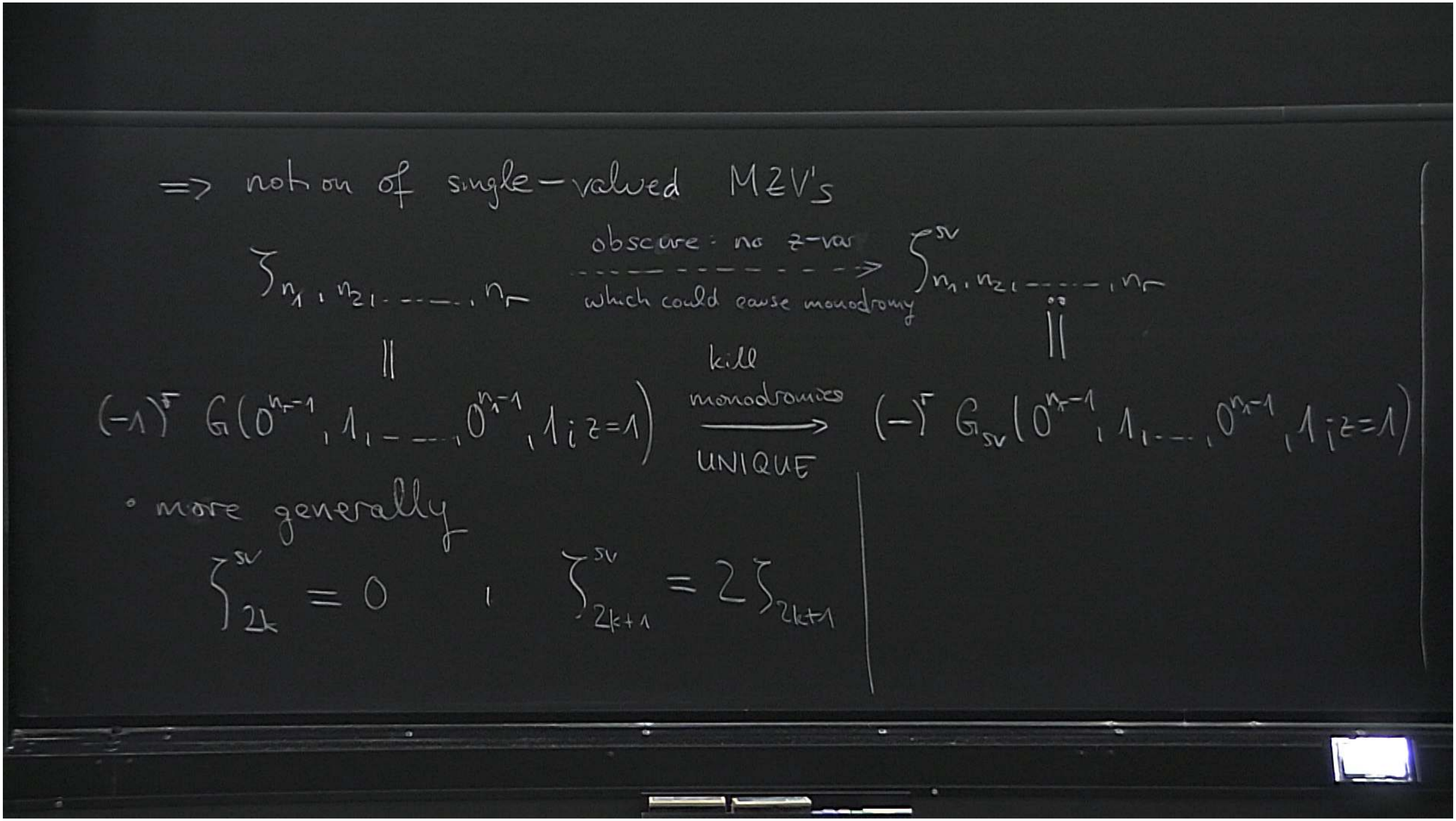
$$G(0,0,1;z=1) = -\zeta_3 \longleftrightarrow G_{sv}(0,0,1;z=1) = -\zeta_3 - \zeta_3 = -2\zeta_3$$

$$\Rightarrow \quad \zeta_2^{sv} = 0 \quad , \quad \zeta_3^{sv} = 2\zeta_3$$



$\Rightarrow$ notion of single-valued MZV's

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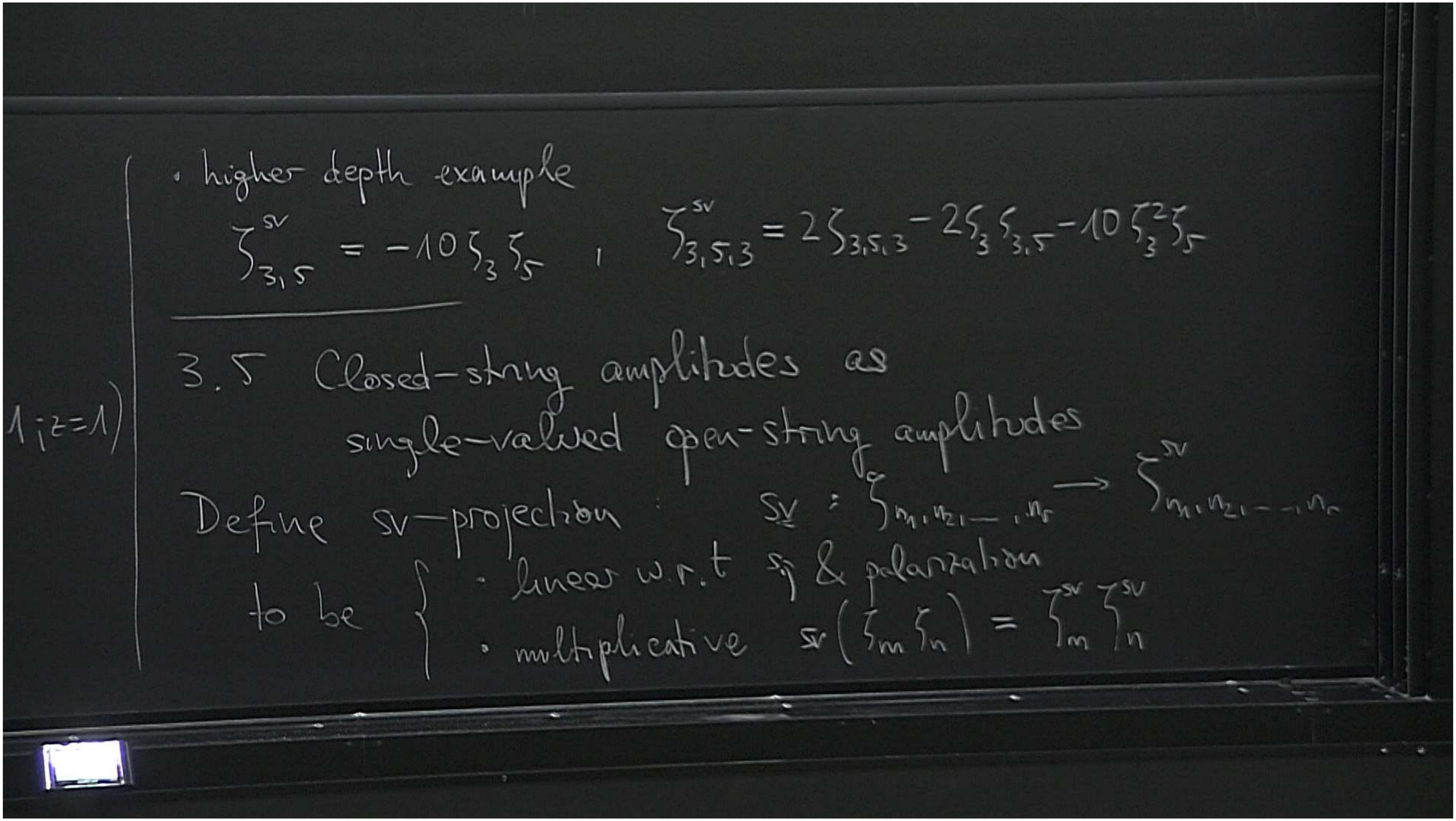
\end{tikzcd}$$

◦ more generally

$$\zeta^{sv}_{2k} = 0 \quad , \quad \zeta^{sv}_{2k+1} = 2\zeta_{2k+1}$$



- higher depth example

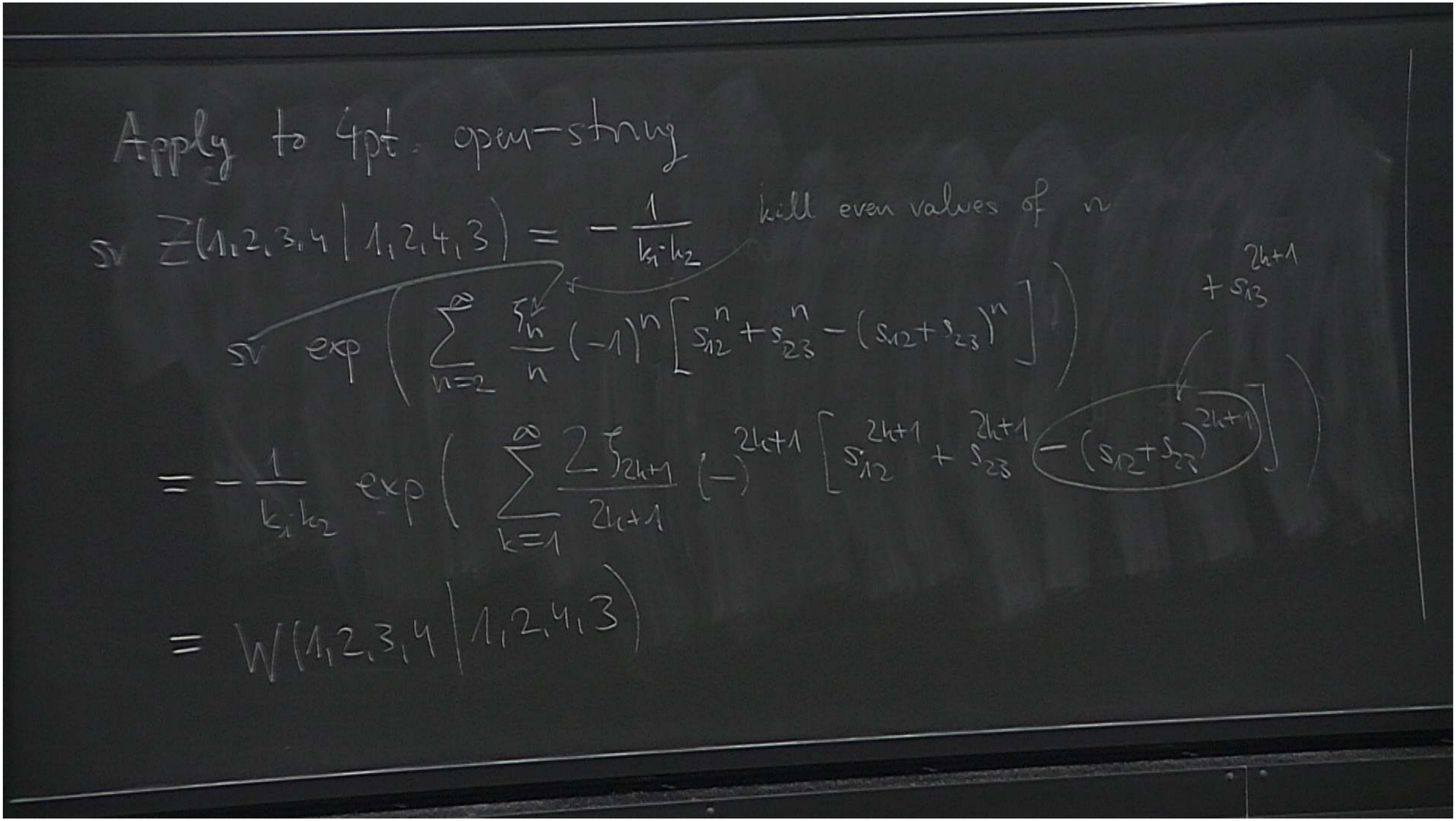
$$\zeta^{sv}_{3,5} = -10\,\zeta_3\,\zeta_5 \;, \quad \zeta^{sv}_{3,5,3} = 2\zeta_{3,5,3} - 2\zeta_3\,\zeta_{3,5} - 10\,\zeta_3^2\,\zeta_5$$

$1;z=1)$

3.5 Closed-string amplitudes as single-valued open-string amplitudes

Define sv-projection : $\underline{\underline{sv}} : \zeta_{n_1,n_2,\ldots,n_r} \longrightarrow \zeta^{sv}_{n_1,n_2,\ldots,n_r}$

to be
- linear w.r.t s_{ij} & polarization
- multiplicative $sv(\zeta_m\,\zeta_n) = \zeta^{sv}_m\,\zeta^{sv}_n$



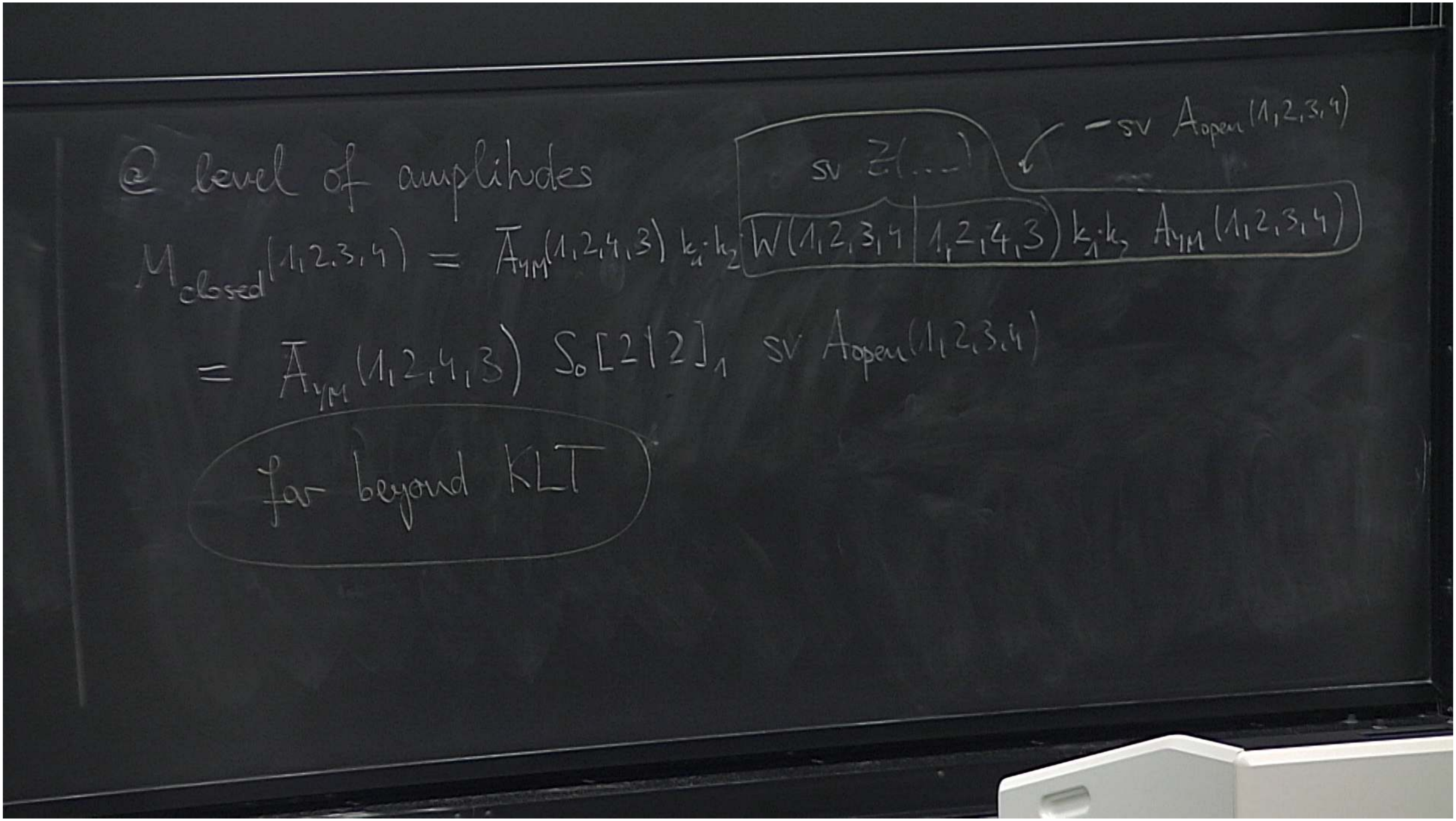
Apply to 4pt. open-string

kill even values of n

$$\mathrm{sv}\, Z(1,2,3,4\,|\,1,2,4,3) = -\frac{1}{k_1\cdot k_2}\; \mathrm{sv}\, \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} (-1)^n \left[s_{12}^n + s_{23}^n - (s_{12}+s_{23})^n \right] \right)$$

$$= -\frac{1}{k_1 k_2}\, \exp\left(\sum_{k=1}^{\infty} \frac{2\zeta_{2k+1}}{2k+1} (-)^{2k+1} \left[s_{12}^{2k+1} + s_{23}^{2k+1} \overset{+s_{13}^{2k+1}}{\boxed{-(s_{12}+s_{23})^{2k+1}}} \right] \right)$$

$$= W(1,2,3,4\,|\,1,2,4,3)$$

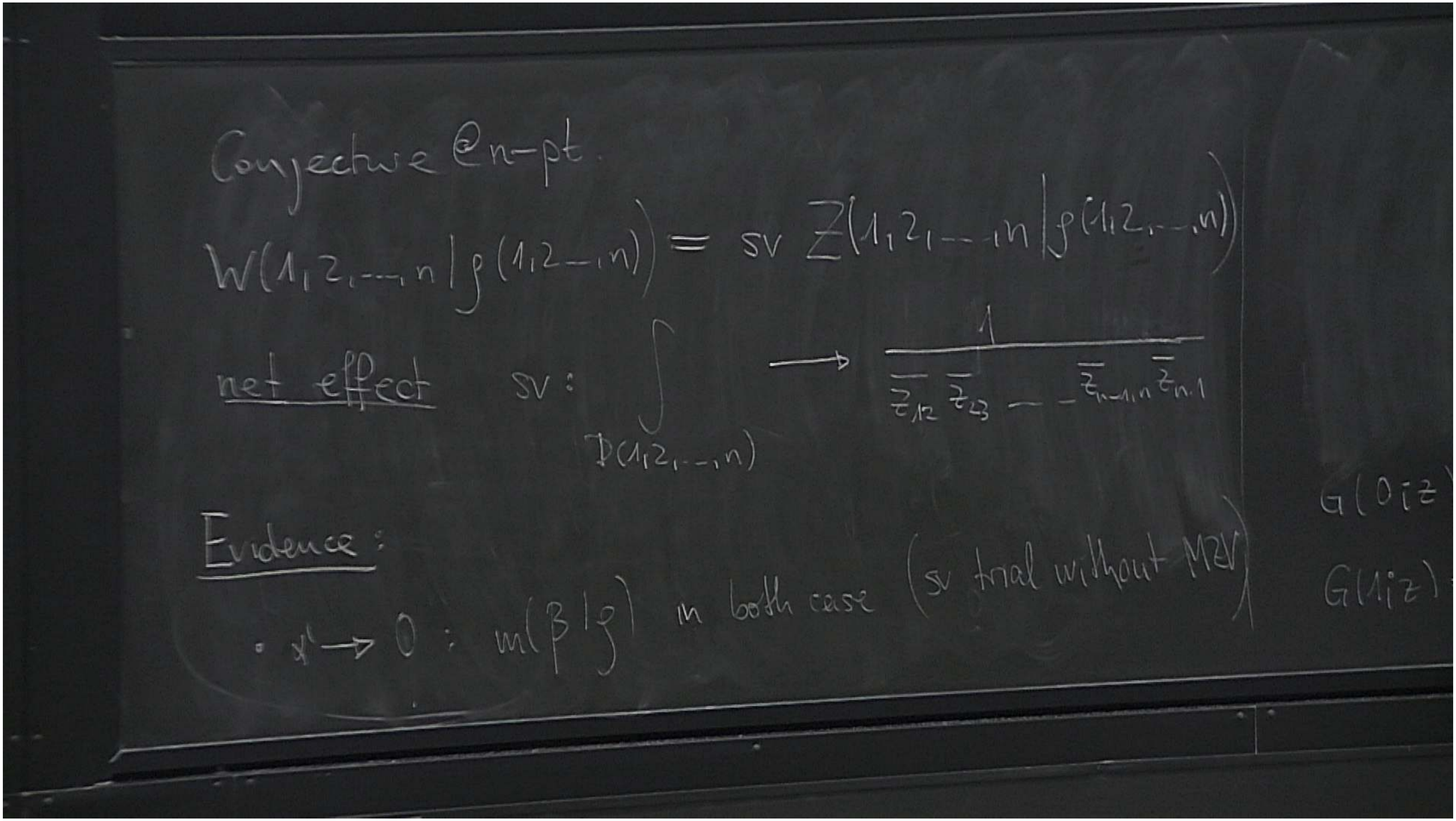


@ level of amplitudes

$$M_{closed}(1,2,3,4) = \bar{A}_{YM}(1,2,4,3)\, k_4\cdot k_2 \overset{-\text{sv } A_{open}(1,2,3,4)}{\boxed{\overbrace{W(1,2,3,4|1,2,4,3)}^{\text{sv } Z(\ldots)}\, k_1\cdot k_2\, A_{YM}(1,2,3,4)}}$$

$$= \bar{A}_{YM}(1,2,4,3)\, S_0[2|2]_1\, \text{sv } A_{open}(1,2,3,4)$$

far beyond KLT



Conjecture @ n-pt.

$$W(1,2,\dots,n|\rho(1,2\dots,n)) = \text{sv}\, Z(1,2,\dots,n|\rho(1,2,\dots,n))$$

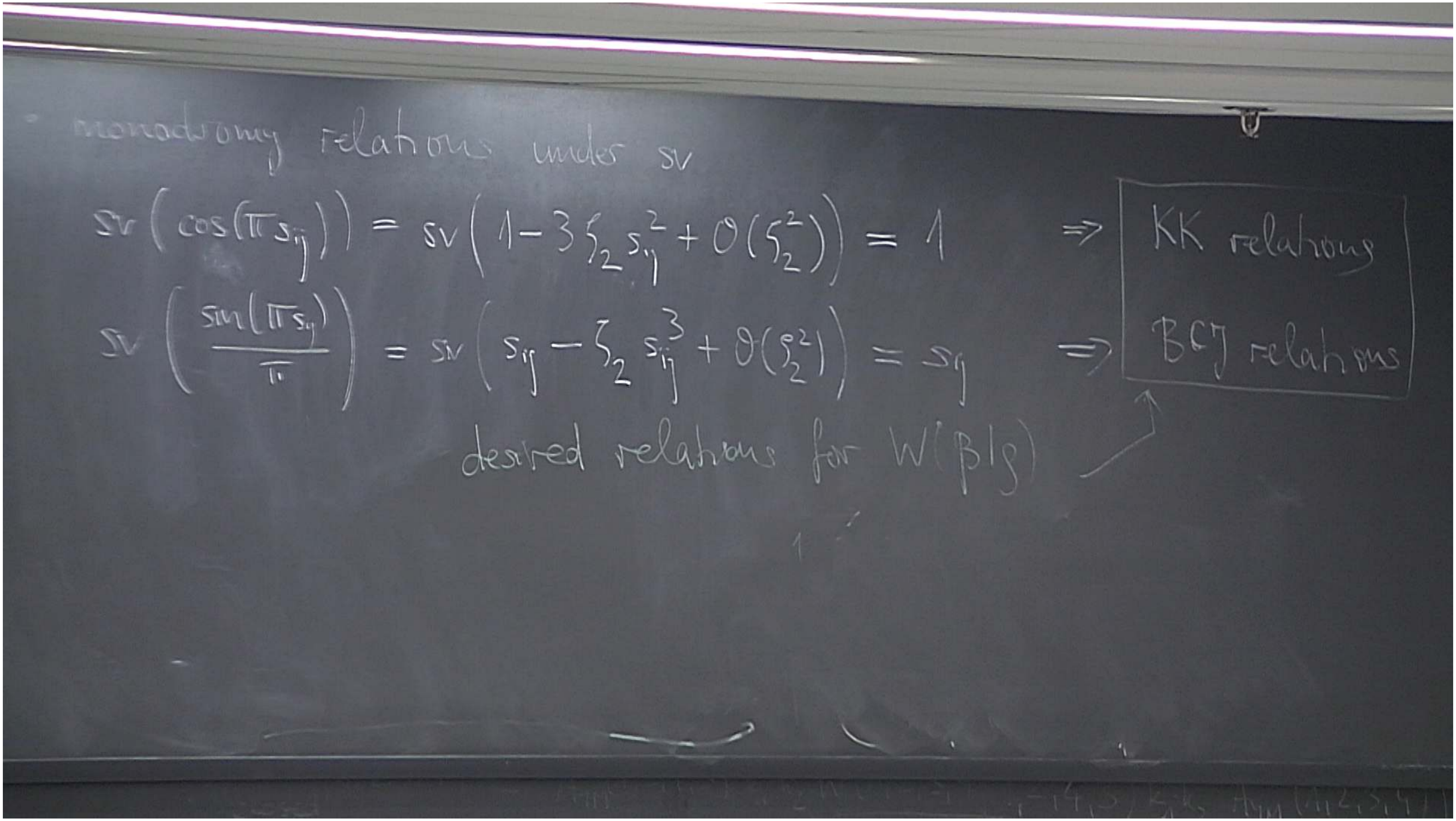
net effect sv : $\int_{D(1,2,\dots,n)} \longrightarrow \frac{1}{\bar z_{12}\,\bar z_{23} \cdots \bar z_{n-1,n}\,\bar z_{n,1}}$

Evidence:

- $x' \to 0$: $m(\beta|\rho)$ in both case (sv trial without MZV)

$G(0;z$

$G(1;z)$



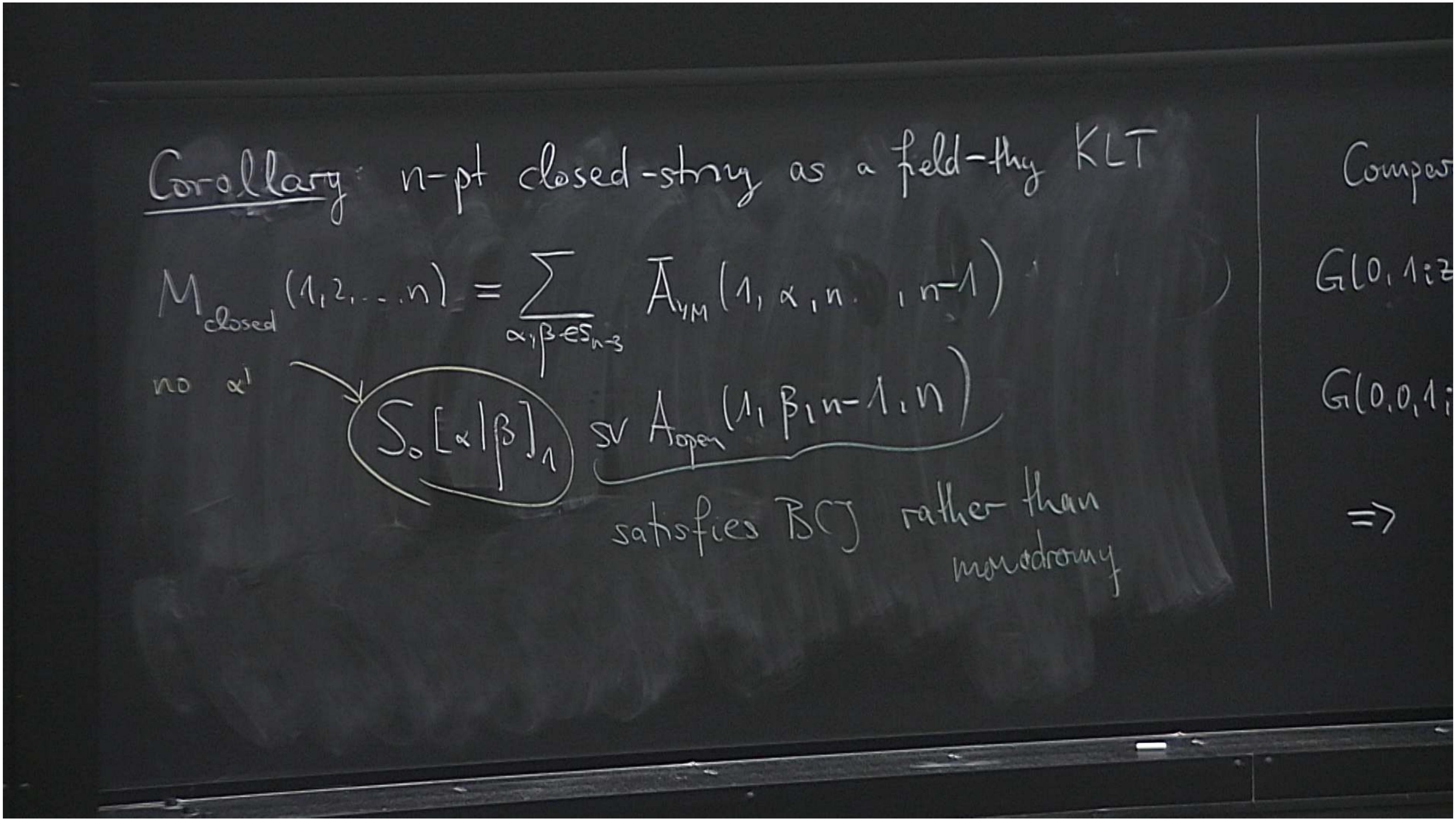
- monodromy relations under sv

$$\mathrm{sv}\left(\cos(\pi s_{ij})\right) = \mathrm{sv}\left(1 - 3\zeta_2 s_{ij}^2 + \mathcal{O}(\zeta_2^2)\right) = 1 \Rightarrow$$

$$\mathrm{sv}\left(\frac{\sin(\pi s_{ij})}{\pi}\right) = \mathrm{sv}\left(s_{ij} - \zeta_2 s_{ij}^3 + \mathcal{O}(\zeta_2^2)\right) = s_{ij} \Rightarrow$$

$$\boxed{\begin{array}{l}\text{KK relations}\\ \text{BCJ relations}\end{array}}$$

desired relations for $W(\beta|\rho)$



Corollary: n-pt closed-string as a field-thy KLT

$$M_{\text{closed}}(1,2,\ldots n) = \sum_{\alpha,\beta\in S_{n-3}} \bar{A}_{YM}(1,\alpha,n,n-1)\; \boxed{S_0[\alpha|\beta]_1}\; \underbrace{\text{sv}\, A_{\text{open}}(1,\beta,n-1,n)}_{\text{satisfies BCJ rather than monodromy}}$$

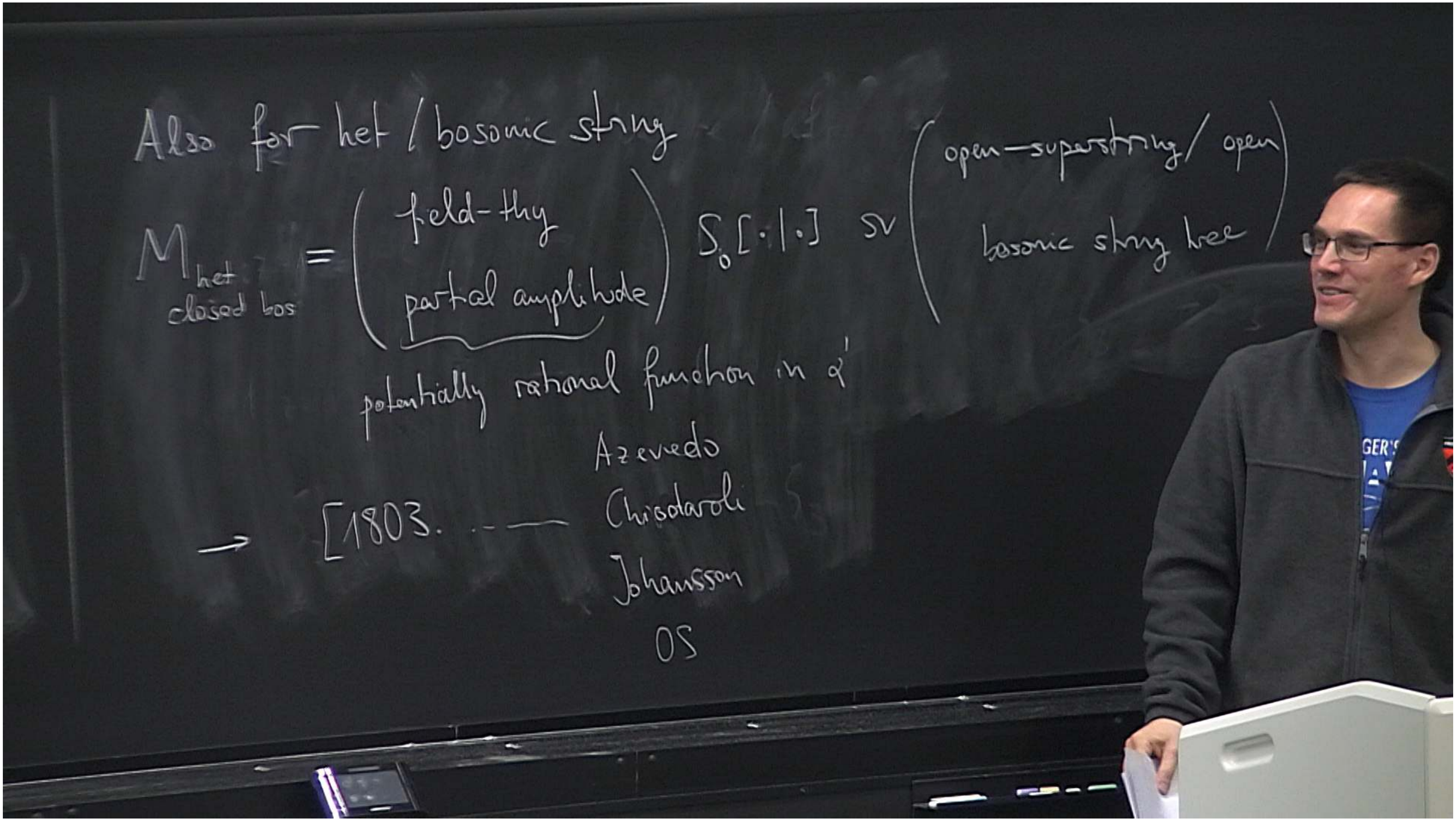
no α'

Compar

$G(0,1;z$

$G(0,0,1;$

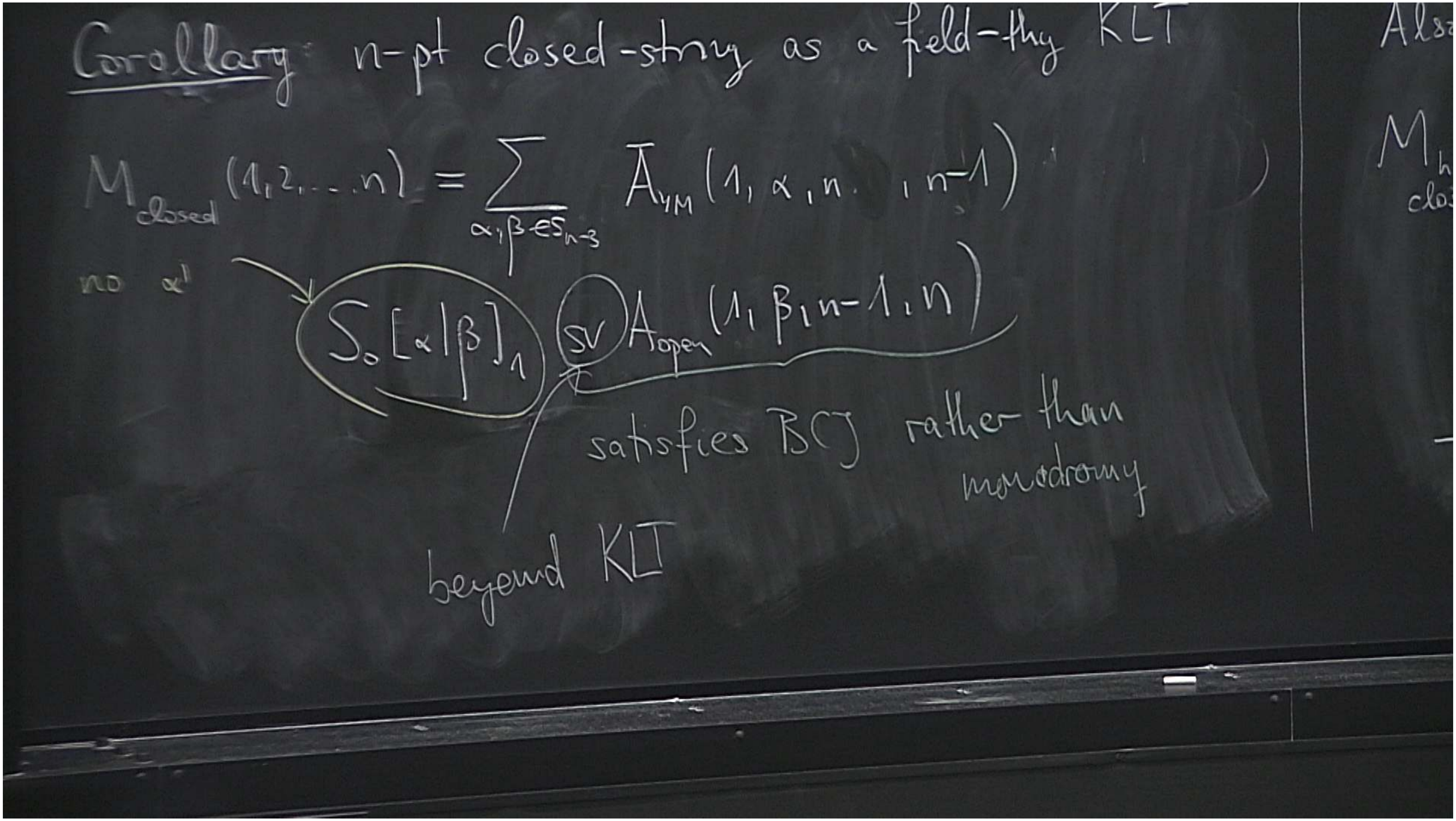
$\Rightarrow$



Also for het / bosonic string

$$M_{\substack{\text{het}\\ \text{closed bos}}} = \begin{pmatrix} \text{field-thy} \\ \underbrace{\text{partial amplitude}}_{\text{potentially rational function in } \alpha'} \end{pmatrix} S_0[\cdot|\cdot] \ \text{sv} \begin{pmatrix} \text{open-superstring / open} \\ \text{bosonic string tree} \end{pmatrix}$$

→ [1803. ---- Azevedo
Chiodaroli
Johansson
OS



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$$M_{closed}(1,2,\dots n) = \sum_{\alpha,\beta \in S_{n-3}} \bar{A}_{YM}(1,\alpha,n,\ n-1)\ \boxed{S_0[\alpha|\beta]_1}\ \underbrace{\boxed{SV}\, A_{open}(1,\beta,n-1,n)}_{\text{satisfies BCJ rather than monodromy}}$$

no α'

beyond KLT

Als

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