Title: PSI 2017/2018 - Scattering Amplitudes in QFT & String Theory - Lecture 16

Date: Apr 26, 2018 11:30 AM

URL: http://pirsa.org/18040048

Abstract:

Review:

- monodromy relations $\rightsquigarrow (n-3)! \times (n-3)!$ basis of $Z(\beta|\rho)$

$$0 = \sum_{j=2}^{n-1} \sin\left(\pi\left(s_{12}+s_{13}+\ldots+s_{1j}\right)\right) Z(2,3,\ldots,j,1,j+1,\ldots,n|\rho)$$

- tree amplitudes of closed (type II A/B) superstring

$$M_{\text{closed}}(1,2,\ldots,n;4\alpha') = \left(\frac{2\alpha'}{\pi}\right)^{n-3} \int_{\mathbb{C}^n \setminus \{z_i = z_j\}} \frac{d^2z_1 \ldots d^2z_n}{\text{vol}\, SL_2(\mathbb{C})} |J_n|^2$$

$$\left| \sum_{\alpha,\beta \in S_{n-3}} A_{YM}(1,\alpha,n-1,n)\, S_0[\alpha|\beta]_1\, \frac{1}{(1,\beta,n,n-1)} \right|^2$$

- closed-string counterpart of $Z(\beta|\rho)$

$$W(1,2,\dots,n\,|\,\rho(1,2,\dots,n)) = \left(\frac{2\alpha'}{\pi}\right)^{n-3} \times \int_{\mathbb{C}^n\setminus\{z_i=z_j\}} \frac{d^2z_1\cdots d^2z_n}{\mathrm{vol}\,Sl_2(\mathbb{C})} \frac{|\mathcal{T}_n|^2}{(1,2,\dots,n)\times(\rho(1,2,\dots,n))}$$

$$\int_{\mathbb{C}^n \setminus \{z_i = z_j\}} \frac{d^2 z_1 \cdots d^2 z_n}{\mathrm{vol}\, Sl_2(\mathbb{C})} \frac{|J_n|^2}{(1,2,\ldots,n) \times (\rho(1,2,\ldots,n)}$$

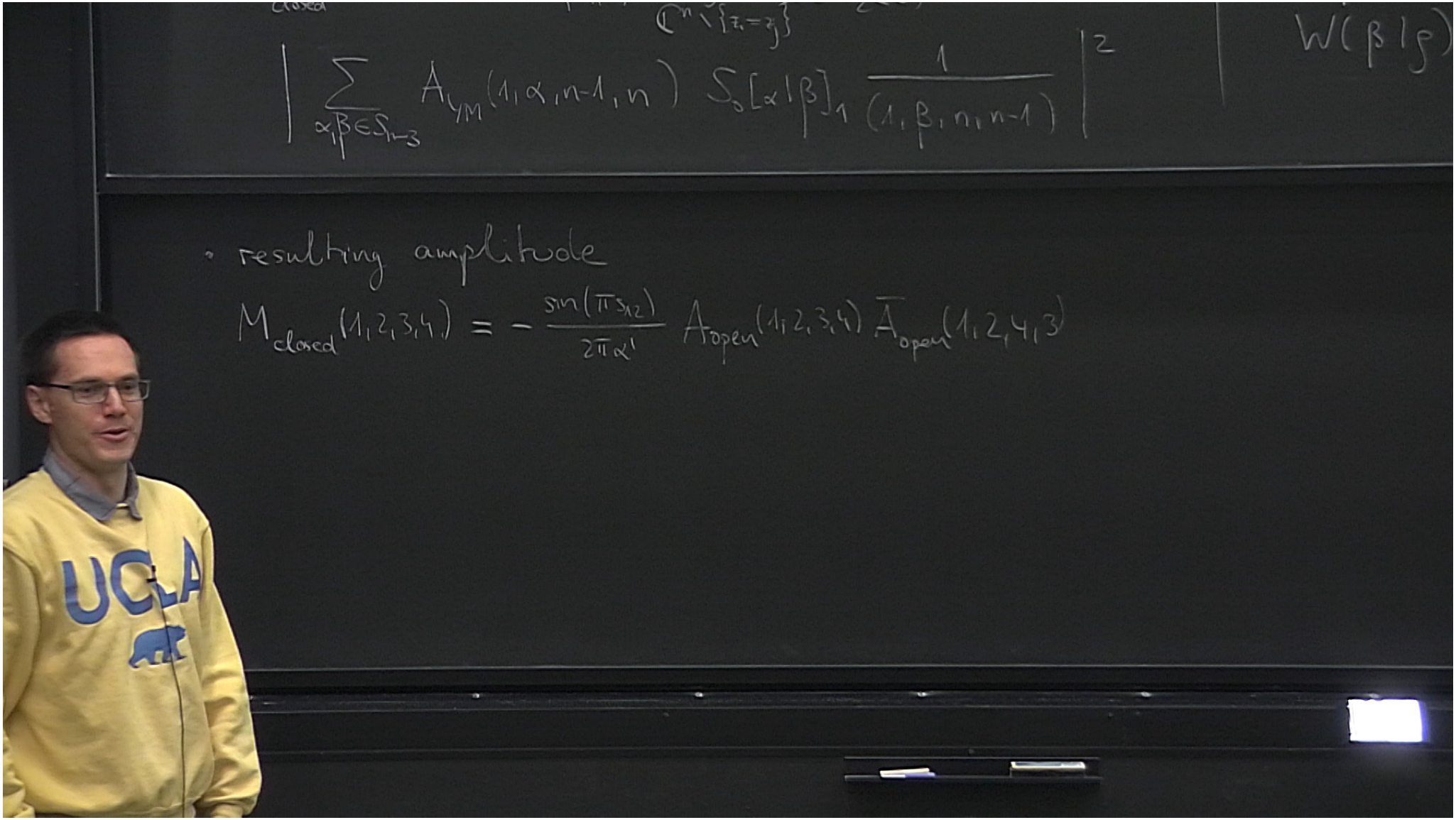
- 4pt KLT

$$W(\beta|\rho) = -\frac{\sin(\pi s_{12})}{2\pi\alpha'} Z(1,2,3,4|\beta)\, Z(1,2,4,3|\rho)$$

$$W(1,2,\ldots,n|\rho) = \int \frac{d^2z_1 \cdots d^2z_n}{\mathrm{vol}\,Sl_2(\mathbb{C})} \frac{|J_n|^2}{(1,2,\ldots,n)\times(\rho(1,2,\ldots,n)} \prod_{i<j}^{n} |z_i - z_j|^{2s_{ij}}$$

- 4pt KLT [proof : reading homework]

$$W(\beta|\rho) = -\frac{\sin(\pi s_{12})}{2\pi\alpha'} Z(1,2,3,4|\beta)\, Z(1,2,4,3|\rho)$$


closed
$\mathbb{C}^n \setminus \{z_i = z_j\}$
$\left| \sum_{\alpha,\beta \in S_{n-3}} A_{YM}(1,\alpha,n-1,n)\, S_0[\alpha|\beta]_1\, \frac{1}{(1,\beta,n,n-1)} \right|^2$
$W(\beta|\beta)$
• resulting amplitude
$M_{closed}(1,2,3,4) = -\frac{\sin(\pi s_{12})}{2\pi\alpha'} A_{open}(1,2,3,4)\, \bar{A}_{open}(1,2,4,3)$
UCLA

$\alpha, \beta \in S_{n-3}$

- resulting amplitude

$$M_{\text{closed}}(1,2,3,4) = -\frac{\sin(\pi s_{12})}{2\pi\alpha'} A_{\text{open}}(1,2,3,4)\, \bar{A}_{\text{open}}(1,2,4,3)$$

permutation symmetry $(2 \leftrightarrow 4)$

↳ holds by monodromy rel

$$\sin(\pi s_{12})\, \bar{A}_{\text{open}}(1,2,4,3) = \sin(\pi s_{23})\, \bar{A}_{\text{open}}(1,4,2,3)$$

Introduce α'-dependent version $S_0[\alpha|\beta]_1$

$$S_{\alpha'}[2|2]_1 = -\frac{\sin(\pi s_{12})}{2\pi\alpha'}$$

$$S_{\alpha'}[A,j|B,j,C]_1 = -\frac{\sin\left(2\pi\alpha' k_j\cdot(k_1+k_B)\right)}{2\pi\alpha'} S_{\alpha'}[A|B,C]_1$$

$$S_{\alpha'}[2|2]_1 = -\frac{\sin(\pi s_{12})}{2\pi\alpha'}$$

$$S_{\alpha'}[A,j|B,j,C]_1 = -\frac{\sin\left(2\pi\alpha' k_j\cdot(k_1+k_B)\right)}{2\pi\alpha'} S_{\alpha'}[A|B,C]_1$$

KLT n-pt

$$W(\beta|\rho) = \sum_{j,\tau\in S_{n-3}}$$

Introduce α'-dependent version $S_0[\alpha|\beta]_1$

$$S_{\alpha'}[2|2]_1 = -\frac{\sin(\pi s_{12})}{2\pi\alpha'}$$

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KLT n-pt

$$W(\beta|\rho) = \sum_{\gamma,\tau\in S_{n-3}} Z(1,\gamma,n,n-1|\beta)\, S_{\alpha'}[\gamma|\tau]_1\, Z(1,\tau,n-1,n|\rho)$$

Introduce α'-dependent version $S_0[\alpha|\beta]_1$

$$S_{\alpha'}[2|2]_1 = -\frac{\sin(\pi s_{12})}{2\pi\alpha'}$$

$$S_{\alpha'}[A_{1,j}|B_{1,j,C}]_1 = -\frac{\sin(2\pi\alpha' k_j\cdot(k_1+k_B))}{2\pi\alpha'} S_{\alpha'}[A|B,C]_1$$

KLT n-pt

$$W(\beta|\rho) = \sum_{\gamma,\tau\in S_{n-3}} Z(1,\gamma,n,n-1|\beta) S_{\alpha'}[\gamma|\tau]_1 Z(1,\tau,n-1,n|\rho)$$

PT's , spectators

Introduce α'-dependent version $S_0[\alpha|\beta]_1$

$$S_{\alpha'}[2|2]_1 = -\frac{\sin(\pi s_{12})}{2\pi\alpha'}$$

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KLT n-pt [read more details in 1010.3933]

$$W(\beta|\rho) = \sum_{\gamma,\tau\in S_{n-3}} Z(1,\gamma,n,n-1|\beta) S_{\alpha'}[\gamma|\tau]_1 Z(1,\tau,n-1,n|\rho)$$

PT's . spectators

duce α'-dependent version $S_\alpha[\alpha|\beta]_1$

$$[2|2]_1 = -\frac{\sin(\pi s_{12})}{2\pi\alpha'}$$

$$[A_{1,j}|B_{1,j},C]_1 = -\frac{\sin(2\pi\alpha' k_j\cdot(k_1+k_B))}{2\pi\alpha'}\, S_{\alpha'}[A|B,C]_1$$

n-pt [read more details in 1010.3933]

$$(\beta|\rho) = \sum_{\gamma,\tau\in S_{n-3}} Z(1,\gamma,n,n-1|\beta)\, S_{\alpha'}[\gamma|\tau]_1\, Z(1,\tau,n-1,n|\rho)$$

PT's . spectators

[check: consistency with $\lim_{\alpha'\to 0} W(\beta|\rho) = m(\beta|\rho)$

using $\alpha'\to 0$ beha- vior of S_α

$k_1 + k_B$)) $S_{\alpha'}[A|B,C]_1$

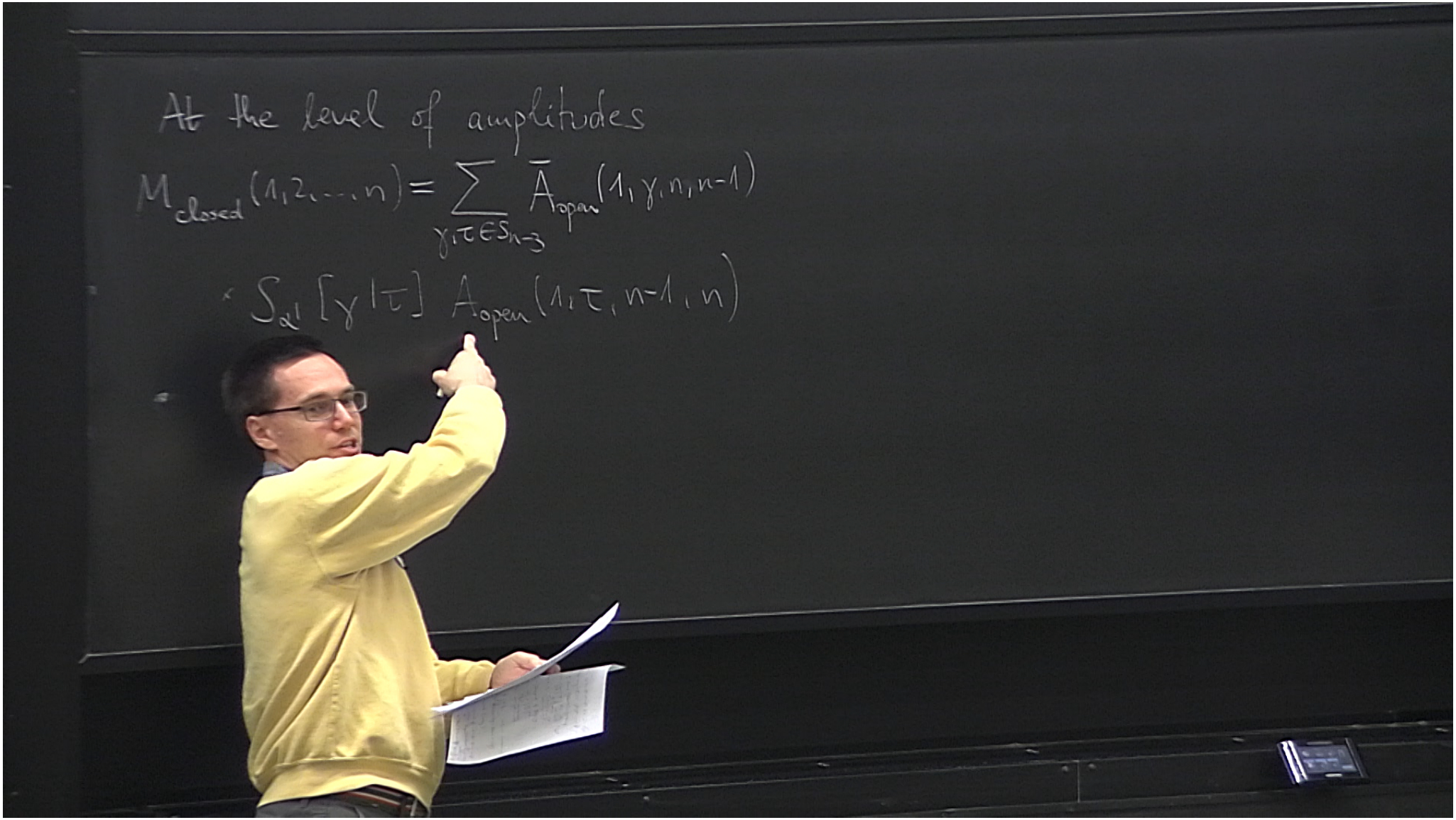
1010.3933]

$S_{\alpha'}[\gamma|\tau]_1 Z(1,\tau,n-1,n|\beta)$

PT's . spectators

[check: consistency with $\lim_{\alpha'\to 0} W(\beta|\rho) = m(\beta|\rho)$

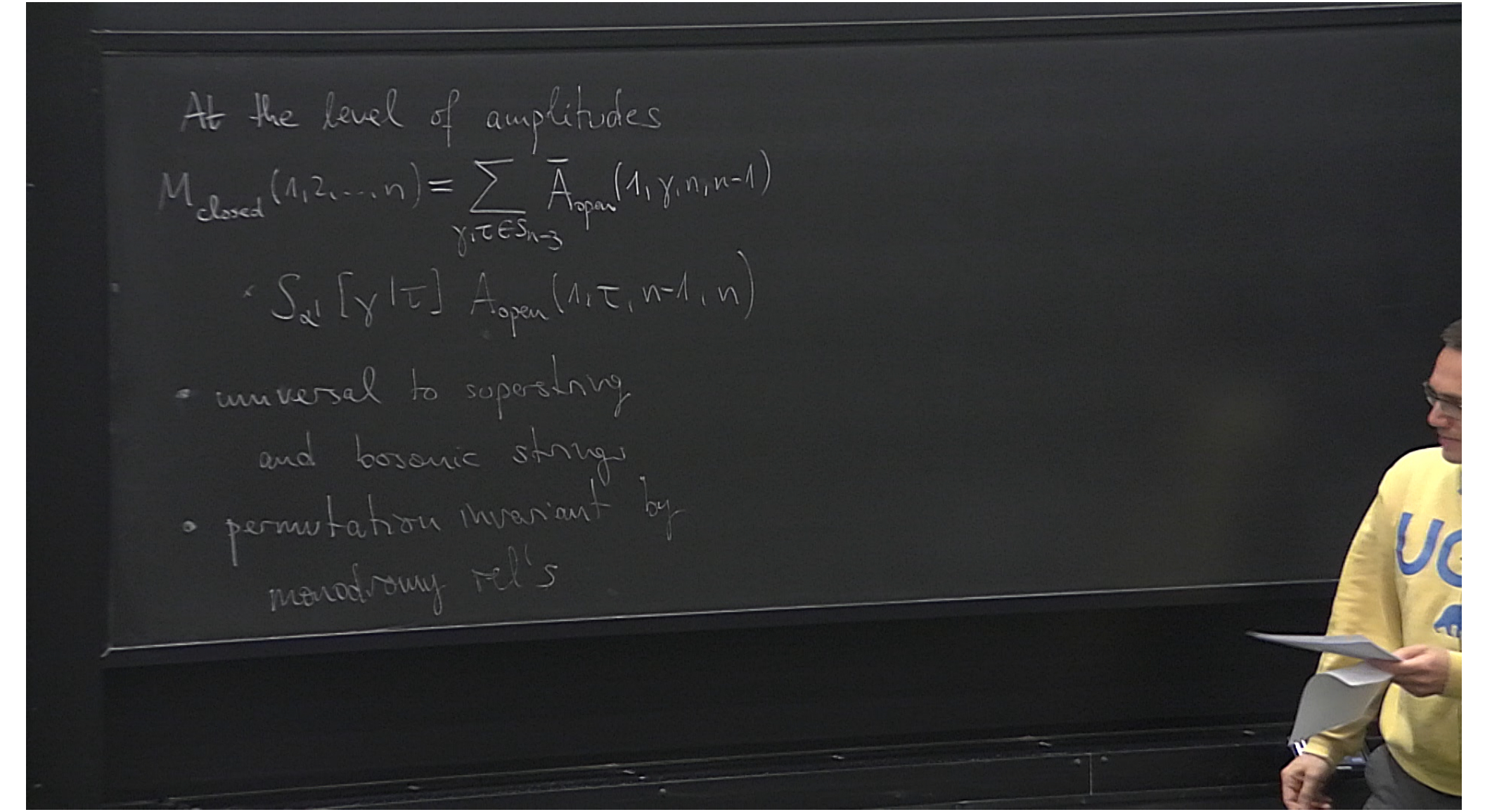
using $\alpha' \to 0$ behaviour of S_α

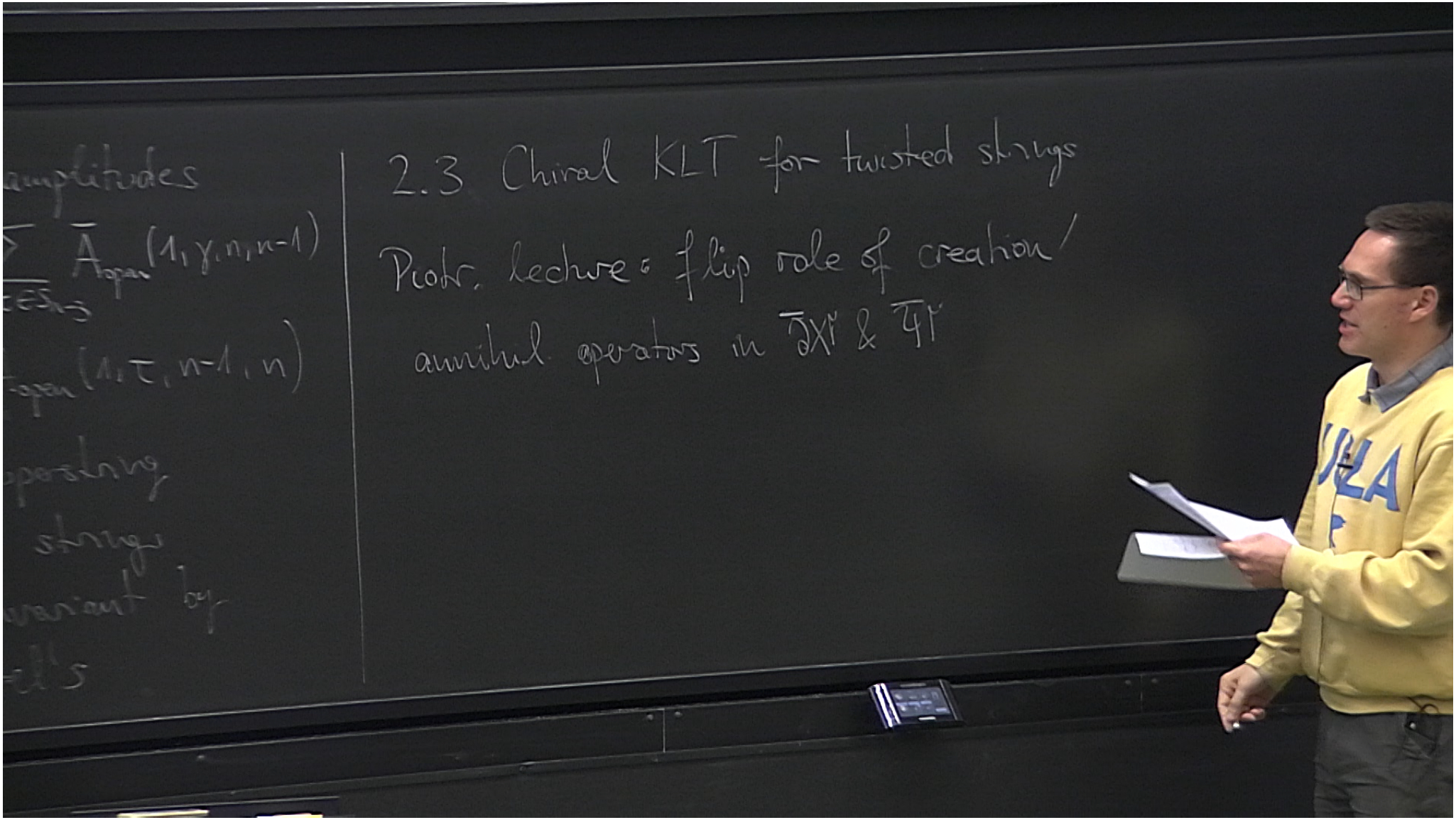

At the level of amplitudes
$M_{closed}(1,2,\dots,n) = \sum_{\gamma,\tau \in S_{n-3}} \bar{A}_{open}(1,\gamma,n,n-1)$
$\cdot S_{\alpha'}[\gamma|\tau]\, A_{open}(1,\tau,n-1,n)$

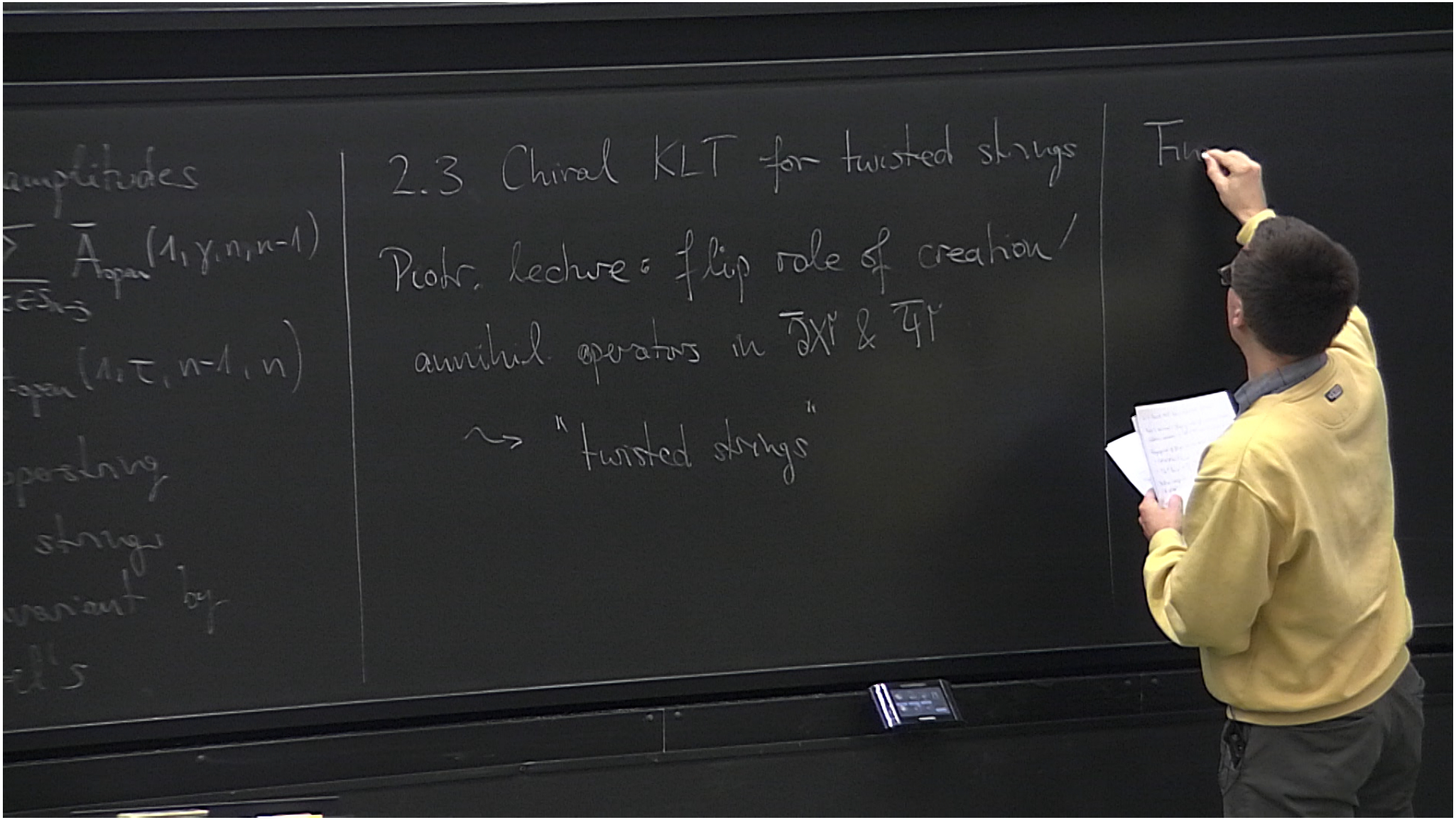
At the level of amplitudes

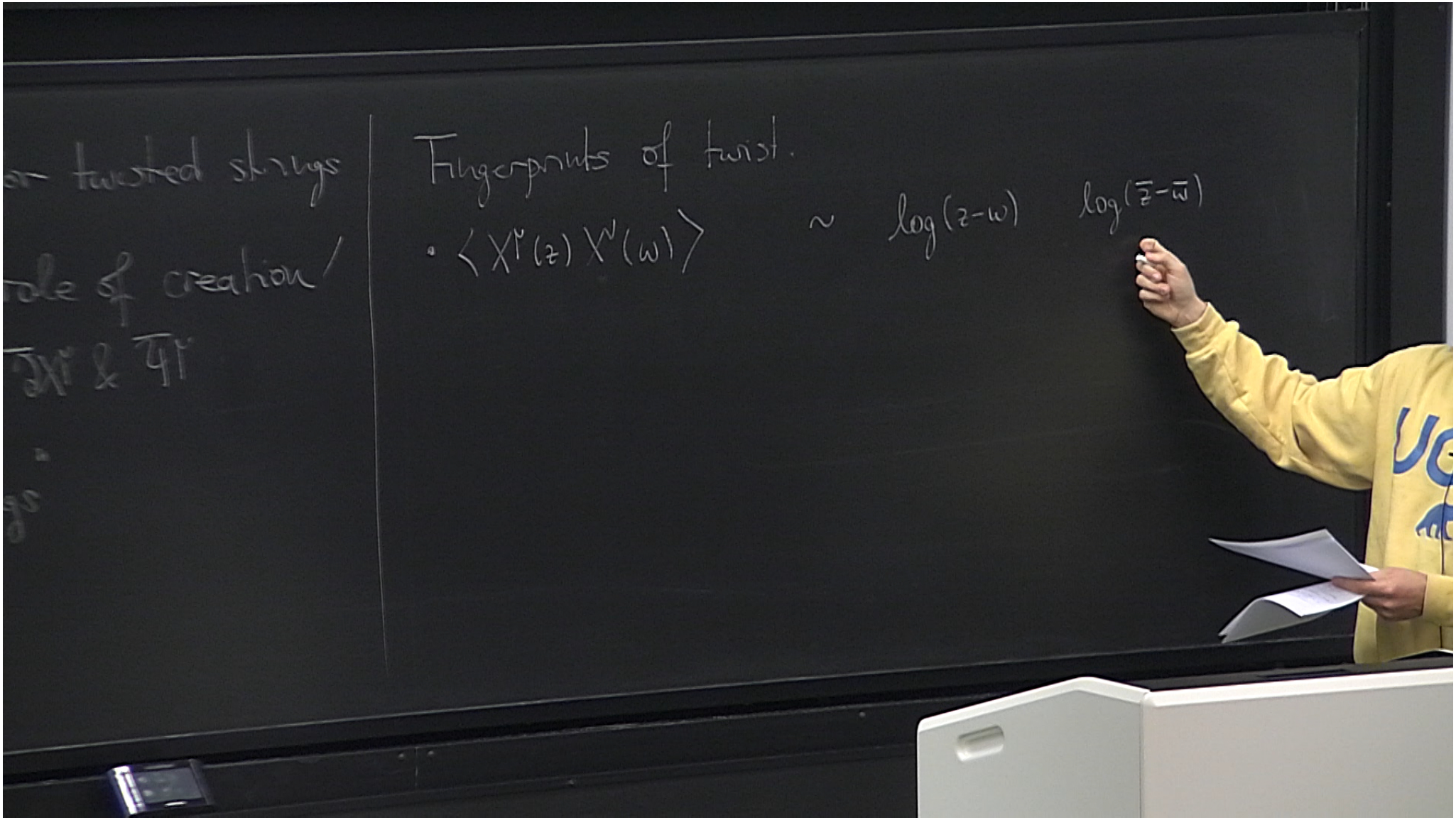
$$M_{\text{closed}}(1,2,\dots,n) = \sum_{\gamma,\tau \in S_{n-3}} \bar{A}_{\text{open}}(1,\gamma,n,n-1) \cdot S_{\alpha'}[\gamma|\tau]\, A_{\text{open}}(1,\tau,n-1,n)$$

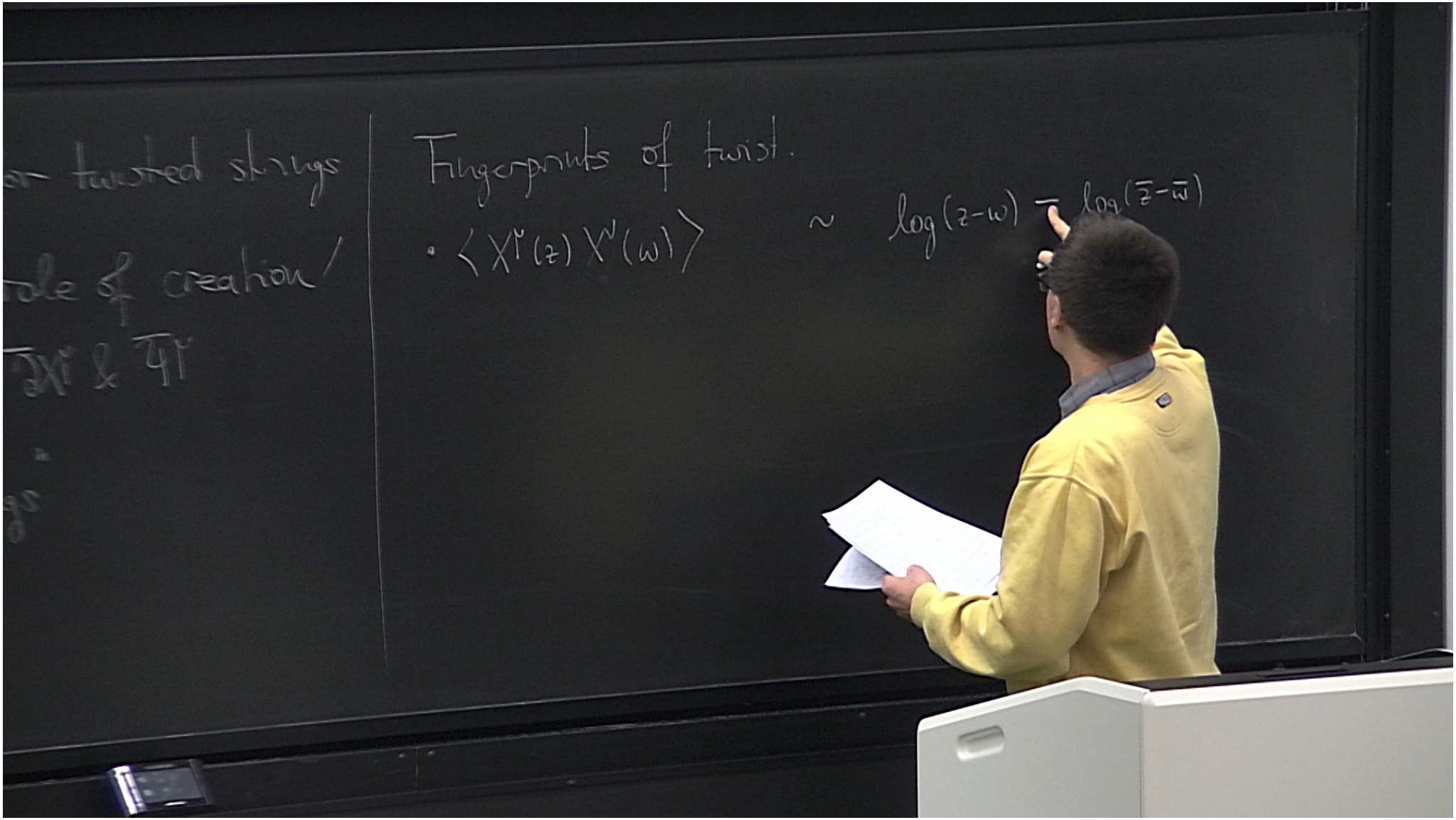
- universal to superstring and bosonic strings
- permutation invariant by monodromy rel's

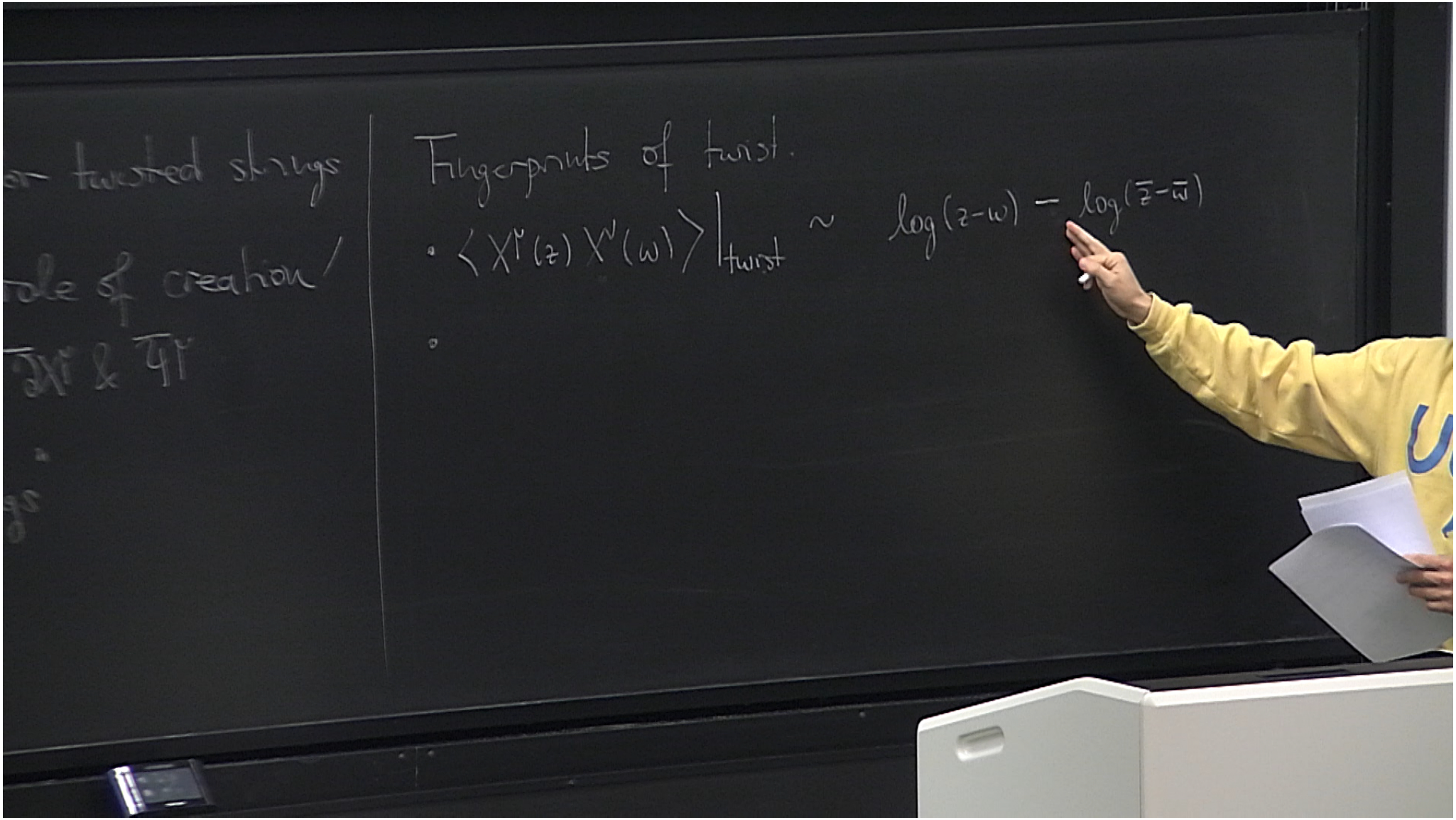


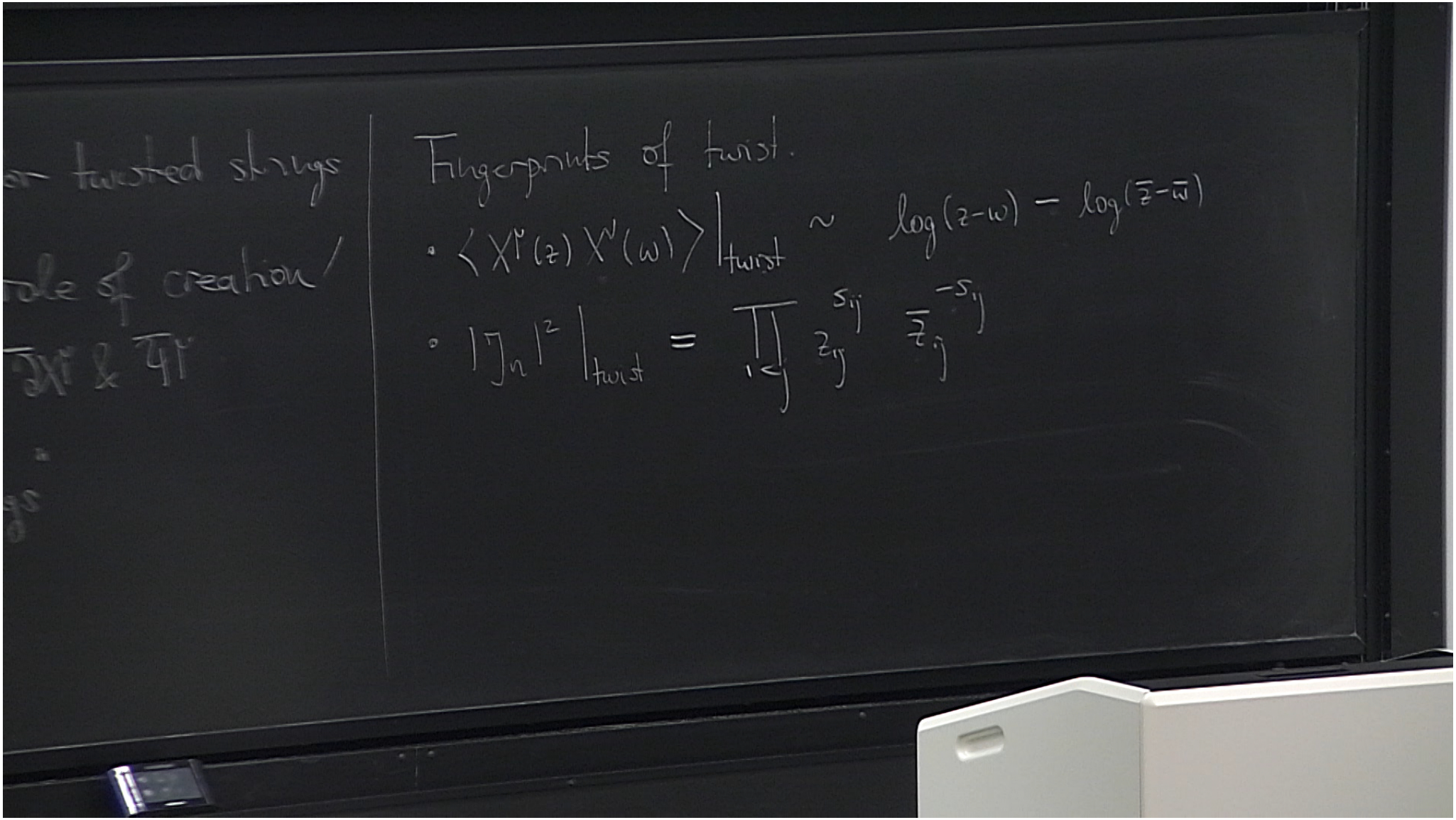

amplitudes
$\bar{A}_{open}(1,\gamma,n,n-1)$
$\bar{A}_{open}(1,\tau,n-1,n)$
2.3 Chiral KLT for twisted strings
Prof. lecture: flip role of creation/
annihil. operators in $\bar{\partial}X^\mu$ & $\bar{\psi}^\mu$
UCLA

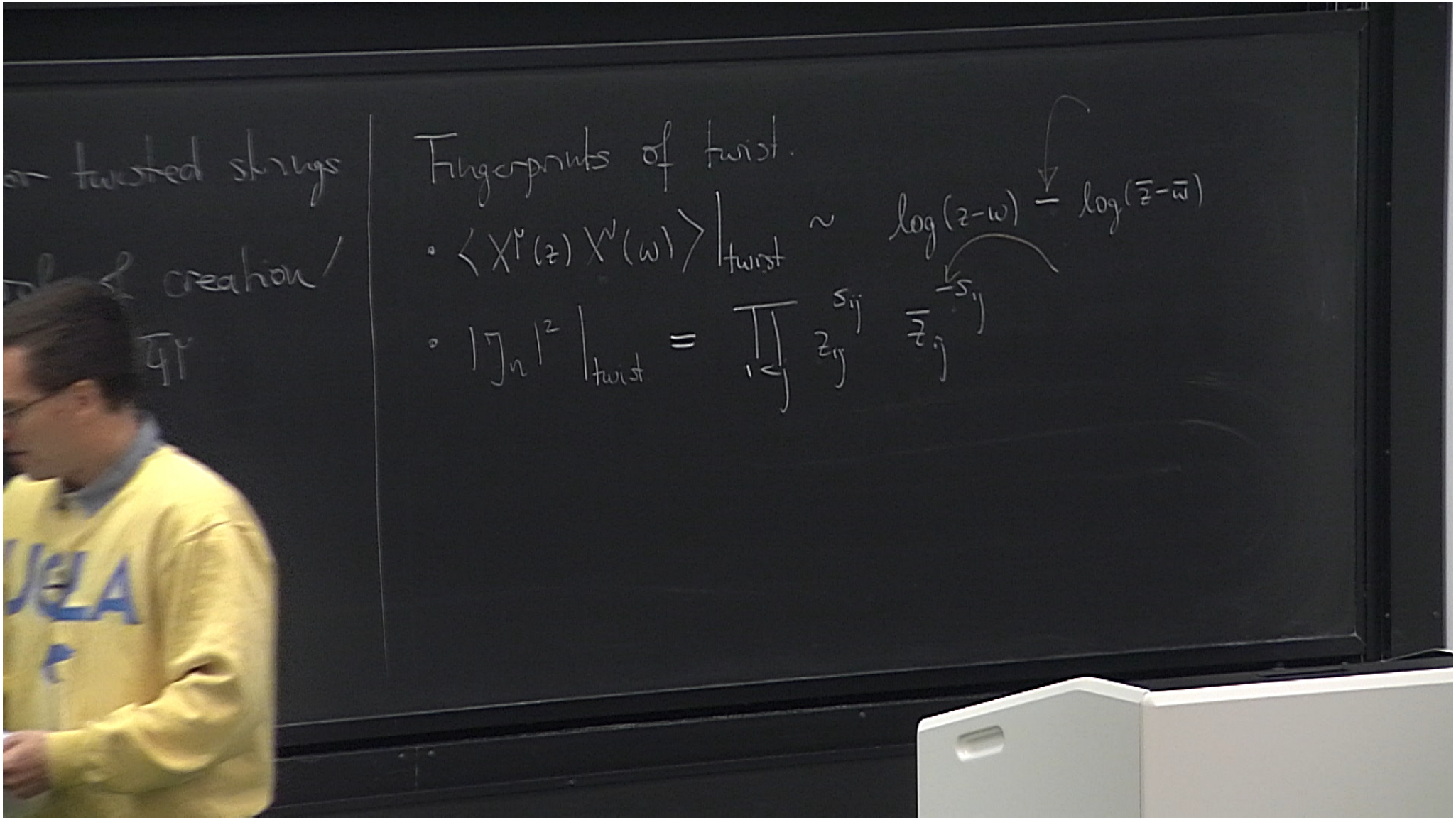

amplitudes
$\bar{A}_{open}(1,\gamma,n,n-1)$
$\bar{A}_{open}(1,\tau,n-1,n)$
2.3 Chiral KLT for twisted strings
Prof. Lechner: flip role of creation/
annihil. operators in $\overline{\partial X^\mu}$ & $\bar{\psi}^\mu$
↝ "twisted strings"

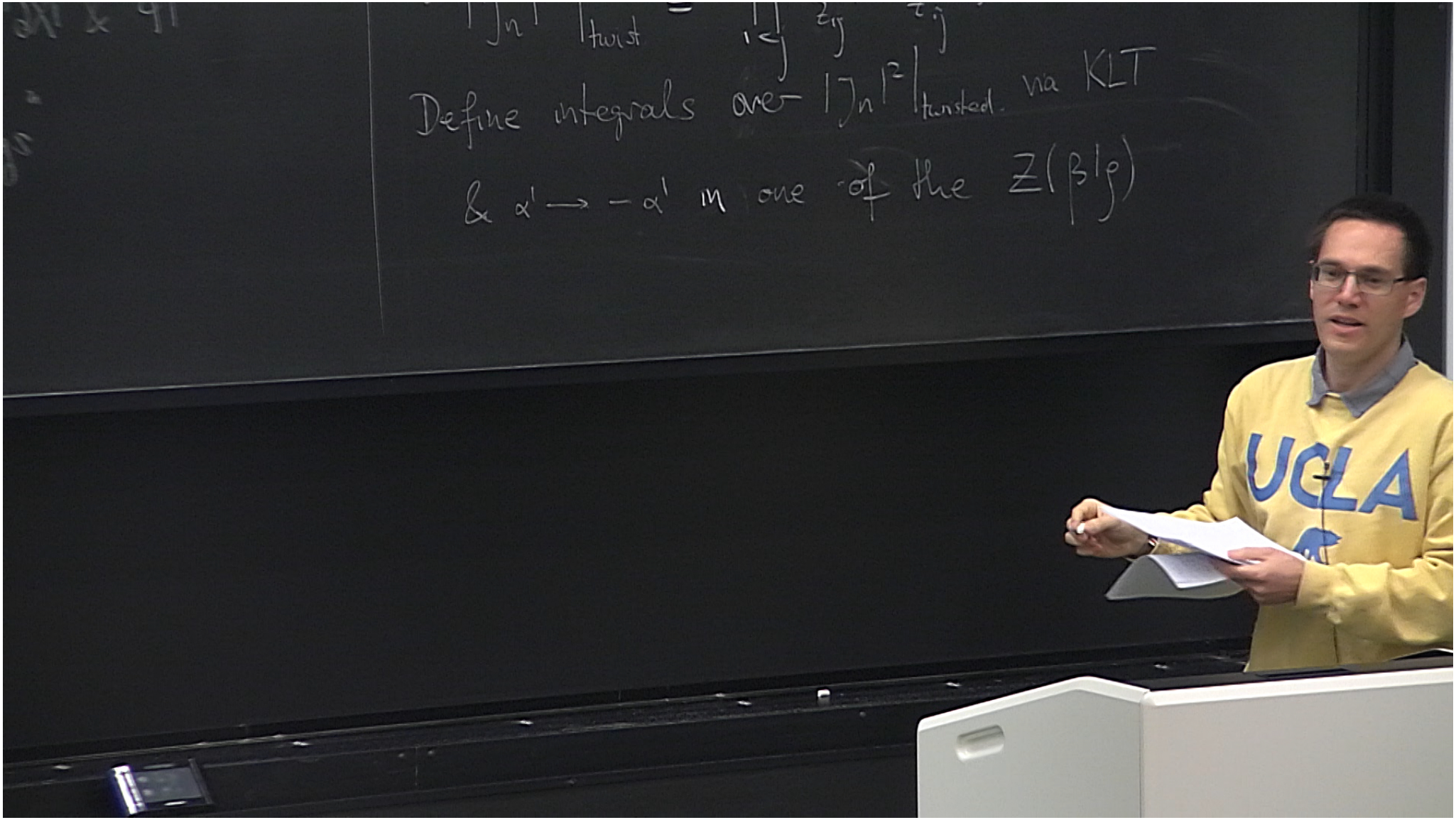

twisted strings
creation
Fingerprints of twist.
$\langle X^\mu(z) X^\nu(w) \rangle \sim \log(z-w) \quad \log(\bar{z}-\bar{w})$

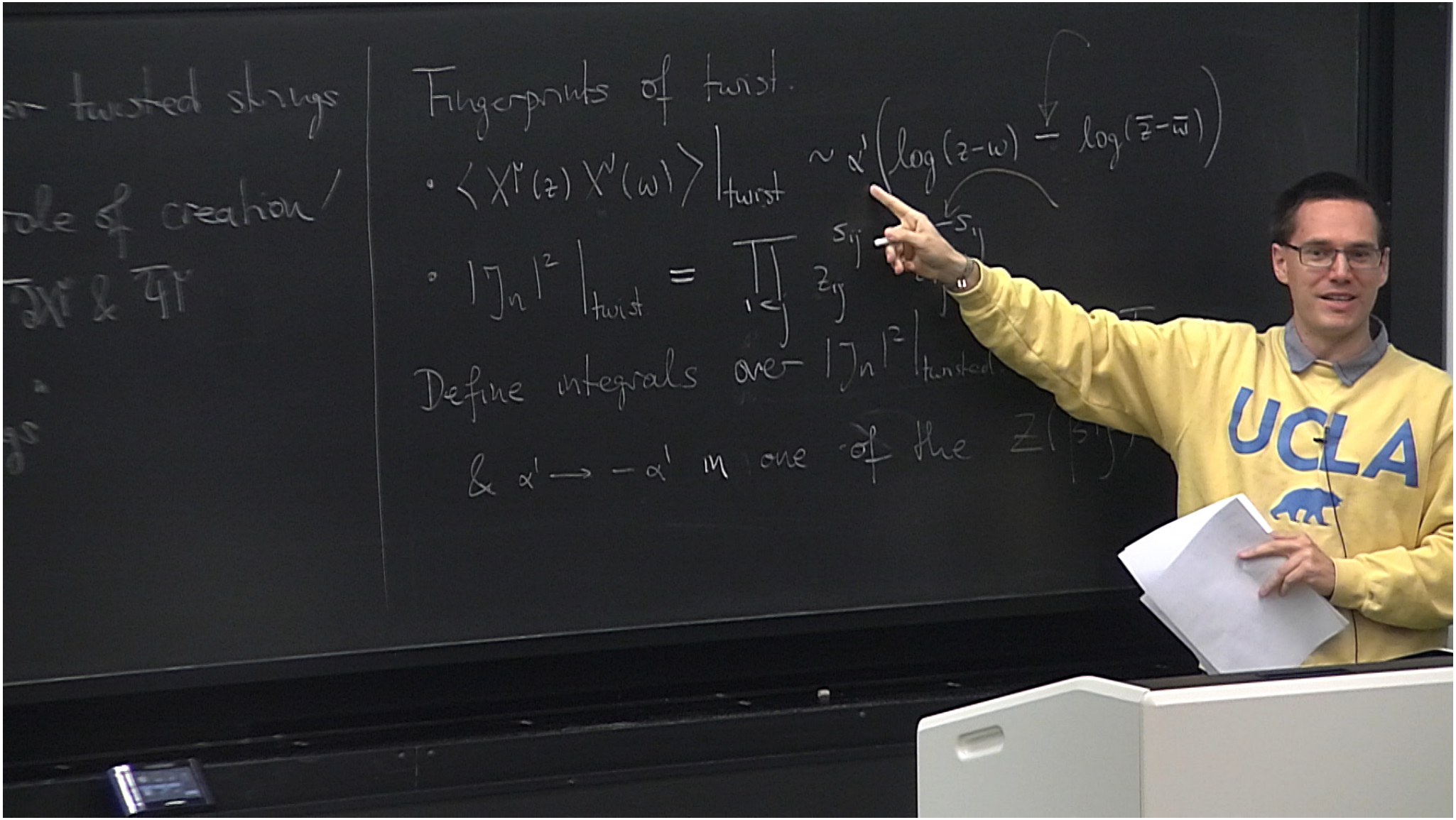

twisted strings
role of creation
Fingerprints of twist.
$\langle X^\mu(z) X^\nu(w) \rangle \sim \log(z-w) + \log(\bar{z}-\bar{w})$

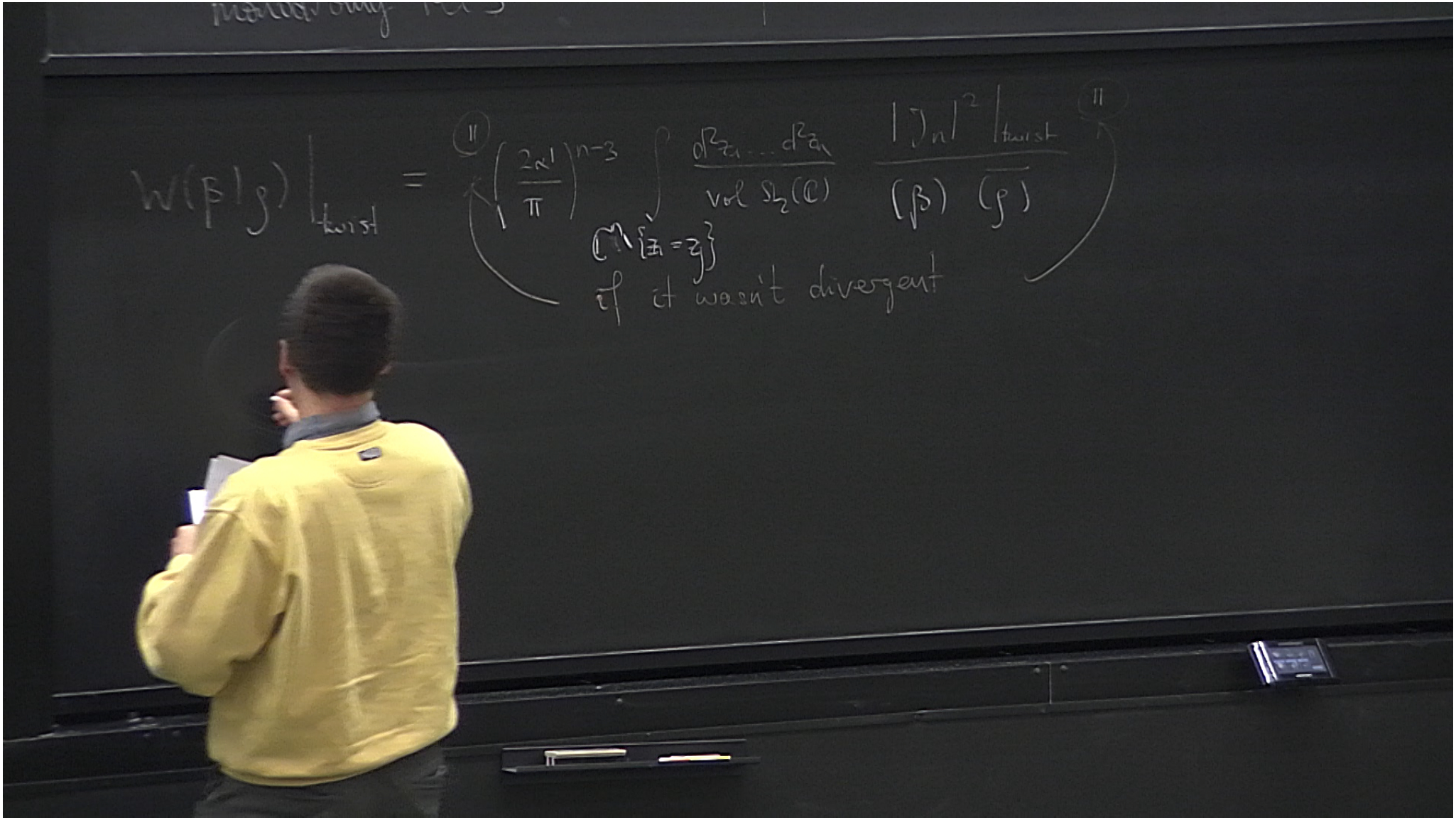

er twisted strings
role of creation/
∂X^μ & $\bar{\partial} X^\mu$
Fingerprints of twist.
$\langle X^\mu(z) X^\nu(w) \rangle|_{twist} \sim \log(z-w) - \log(\bar{z}-\bar{w})$

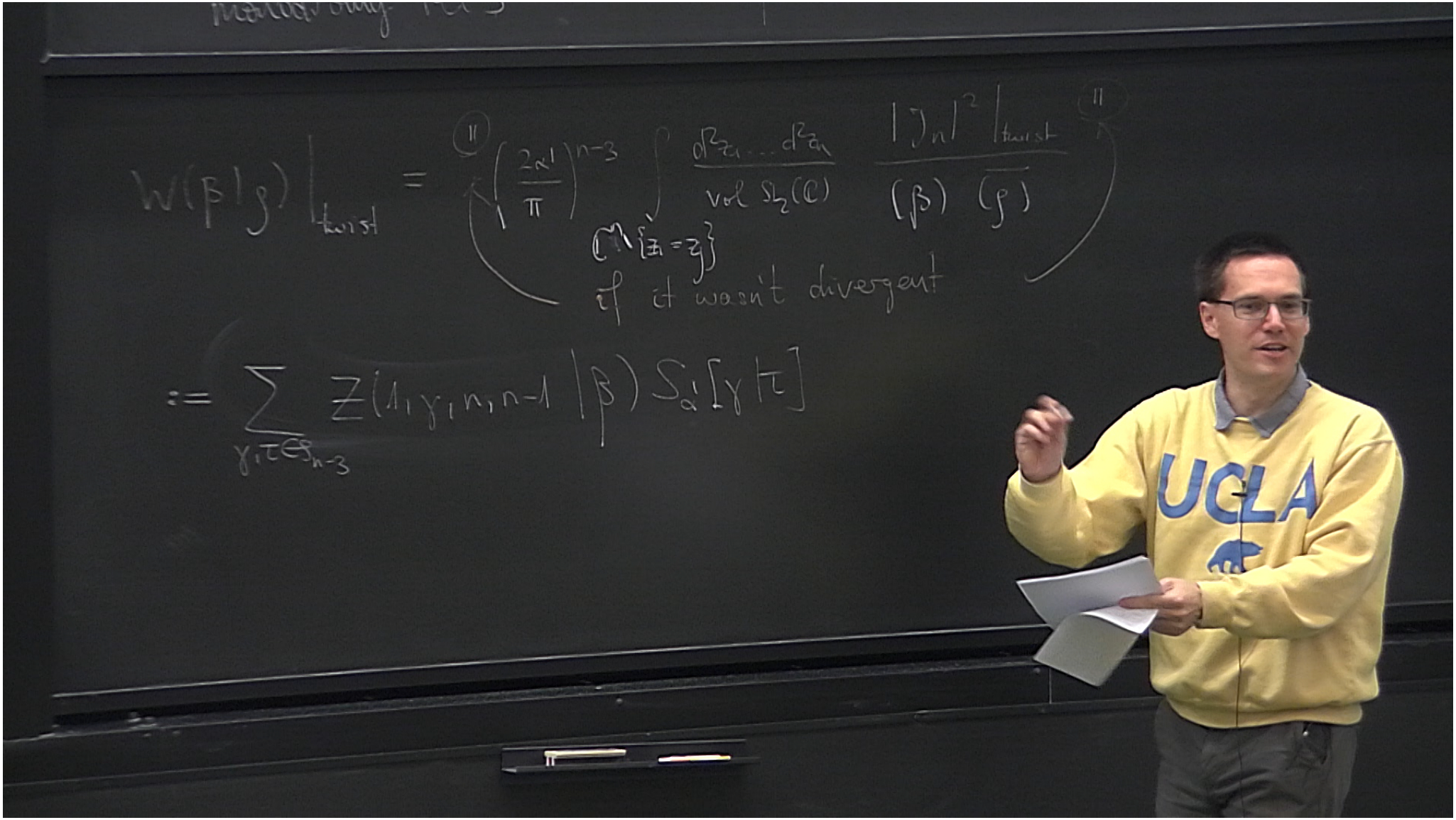

er twisted strings
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Fingerprints of twist.
$\langle X^\mu(z) X^\nu(w) \rangle|_{twist} \sim \log(z-w) - \log(\bar{z}-\bar{w})$
$|T_n|^2|_{twist} = \prod_{i<j} z_{ij}^{s_{ij}} \bar{z}_{ij}^{-s_{ij}}$


twisted strings
creation
Fingerprints of twist.
$\langle X^\mu(z) X^\nu(w) \rangle|_{twist} \sim \log(z-w) - \log(\bar{z}-\bar{w})$
$|T_n|^2|_{twist} = \prod_{i<j} z_{ij}^{s_{ij}} \bar{z}_{ij}^{-s_{ij}}$
UCLA


Define integrals over $|J_n|^2|_{twisted}$ via KLT
& $\alpha' \to -\alpha'$ in one of the $Z(\beta|\rho)$
UCLA


twisted strings
Fingerprints of twist.
$\langle X^\mu(z) X^\nu(w) \rangle|_{twist} \sim \alpha' \left(\log(z-w) + \log(\bar{z}-\bar{w}) \right)$
$|J_n|^2|_{twist} = \prod_{i<j} z_{ij}^{s_{ij}}$
Define integrals over $|J_n|^2|_{twisted}$
& $\alpha' \to -\alpha'$ in one of the
UCLA

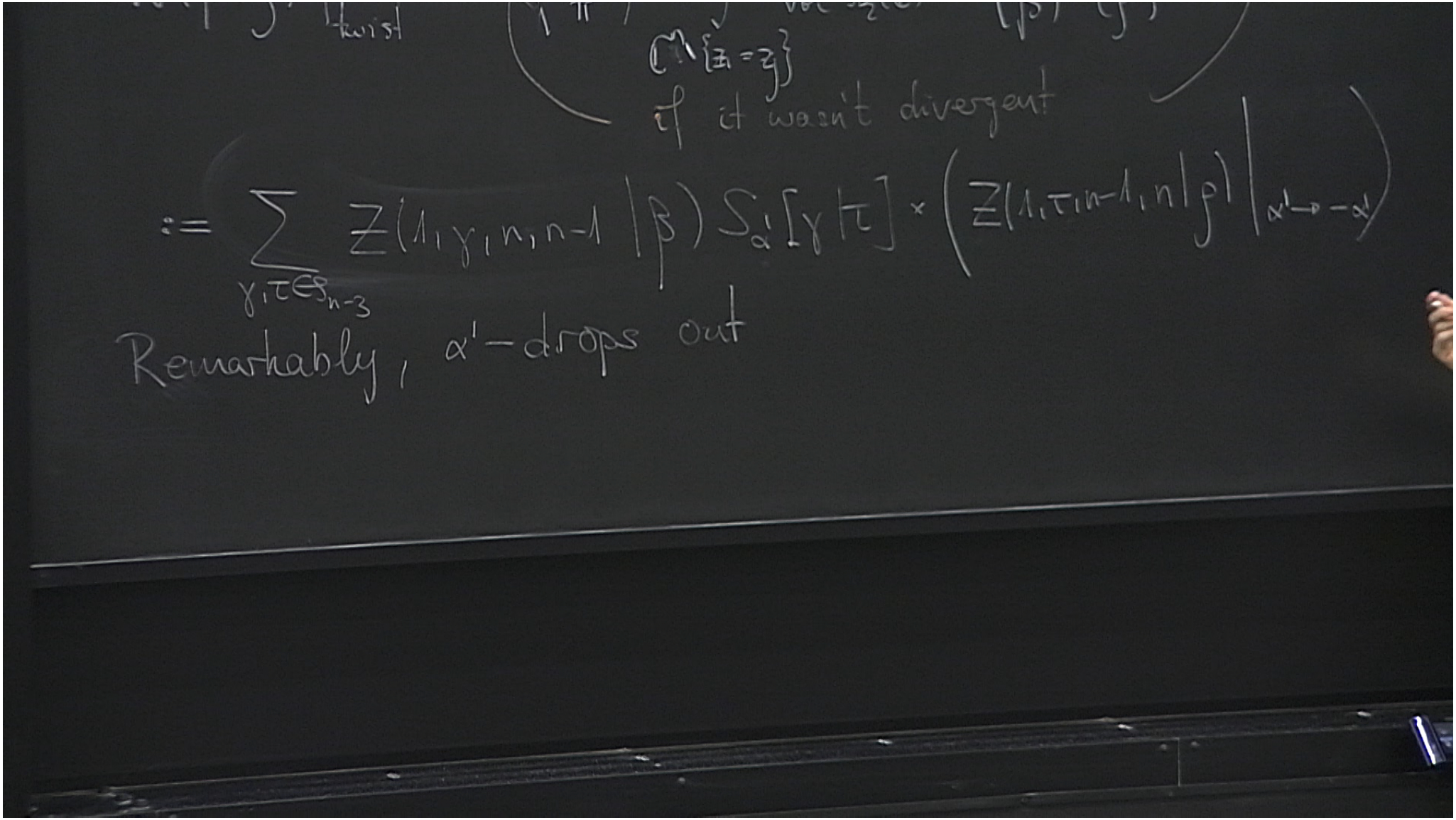

$W(\beta|\rho)\Big|_{twist} = \left(\frac{2\pi i}{\tau}\right)^{n-3} \int_{\mathcal{M}\{z_1=z\}} \frac{d^2z_1 \dots d^2z_{n-3}}{\text{vol } SL_2(\mathbb{C})} \frac{|j_n|^2\Big|_{twist}}{(\beta)\ (\rho)}$
if it wasn't divergent!

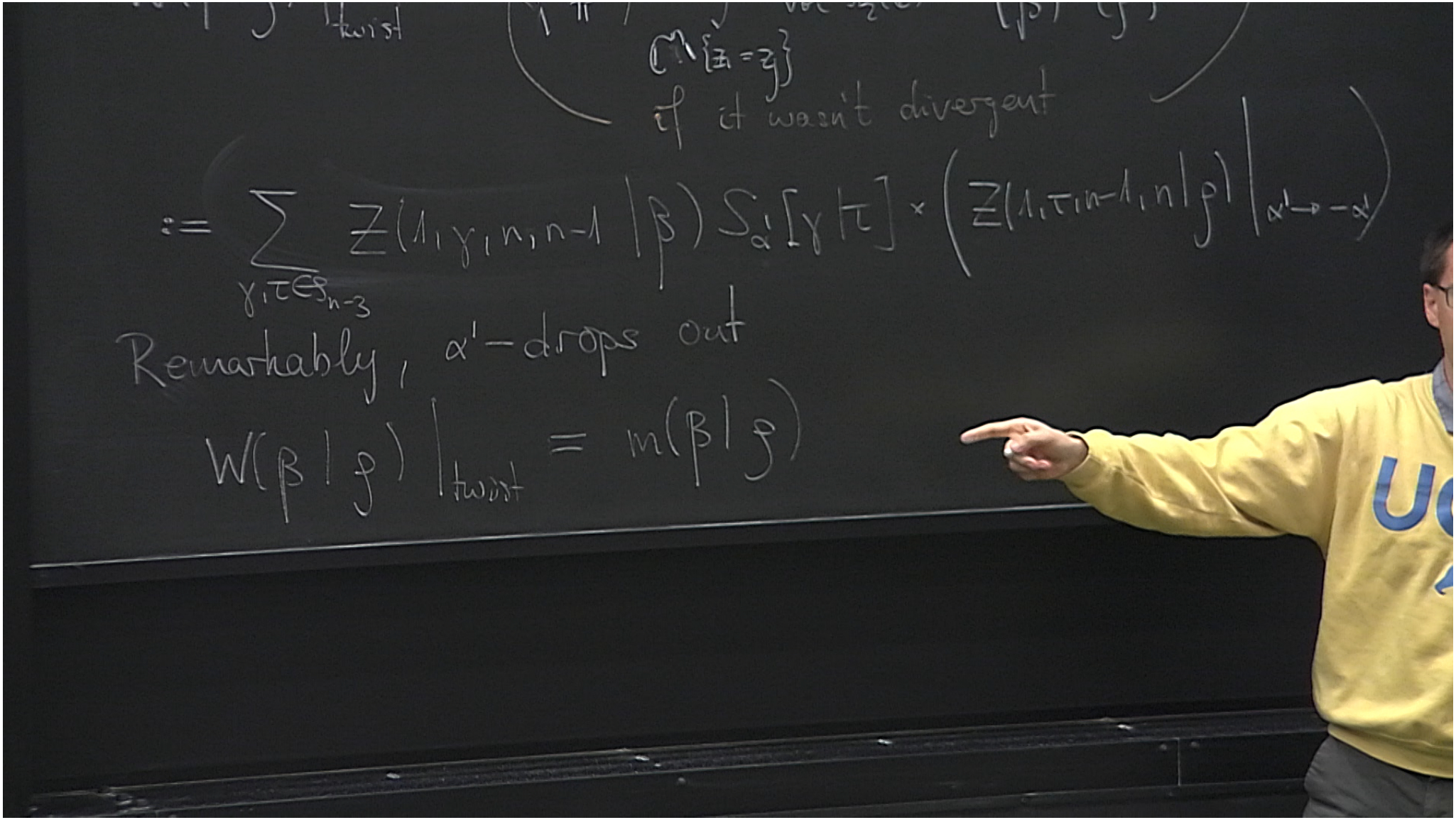

$W(\beta|\gamma)\big|_{twist} = \left(\frac{2\pi i}{\pi}\right)^{n-3} \int_{\mathcal{M}_{\{z_i=z\}}} \frac{d^2z_1 \dots d^2z_{n-3}}{vol\ SL_2(\mathbb{C})} \frac{|J_n|^2\big|_{twist}}{(\beta)\ (\gamma)}$
if it wasn't divergent!
$:= \sum_{\gamma,\tau \in \mathcal{B}_{n-3}} Z(1,\gamma,n,n-1|\beta)\, S_a^{-1}[\gamma|\tau]$
UCLA

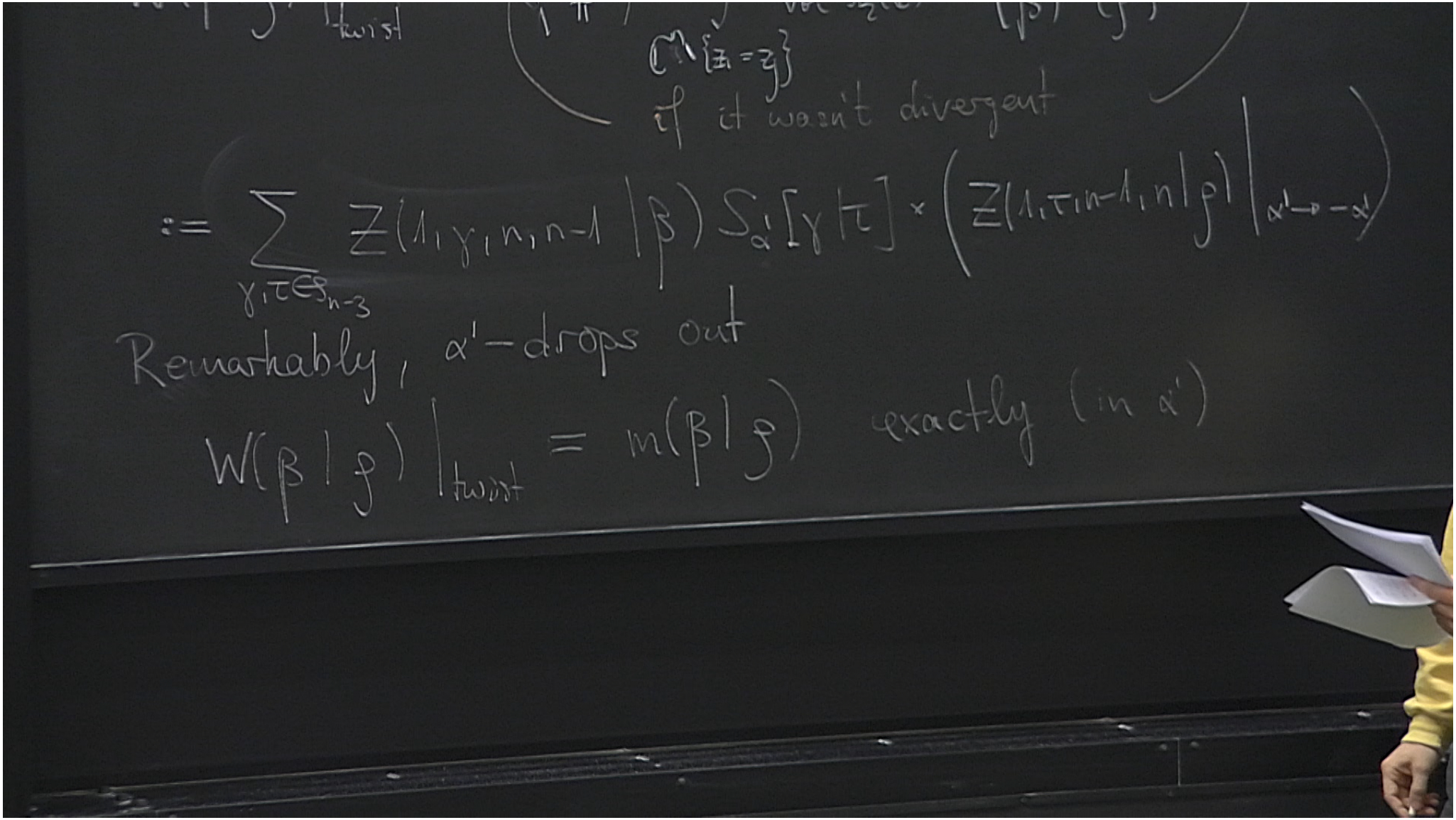
$$W(\beta|\rho)\Big|_{\text{twist}} = \left(\frac{2\alpha'}{\pi}\right)^{n-3} \int_{\mathbb{C}\setminus\{z_i=z_j\}} \frac{d^2z_4 \dots d^2z_n}{\text{vol}\, Sl_2(\mathbb{C})} \frac{|J_n|^2\Big|_{\text{twist}}}{(\beta)\ (\rho)}$$

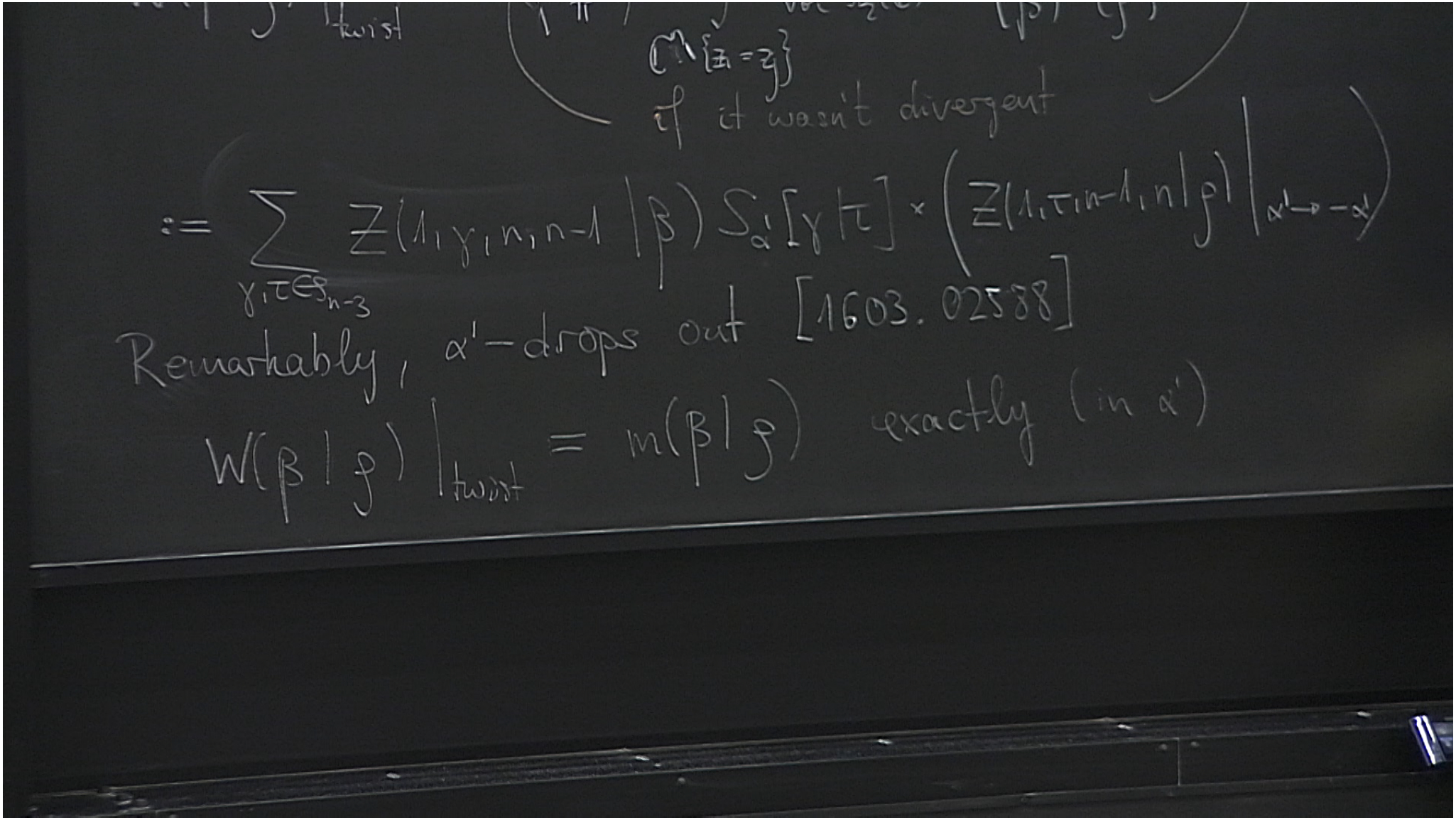
if it wasn't divergent!

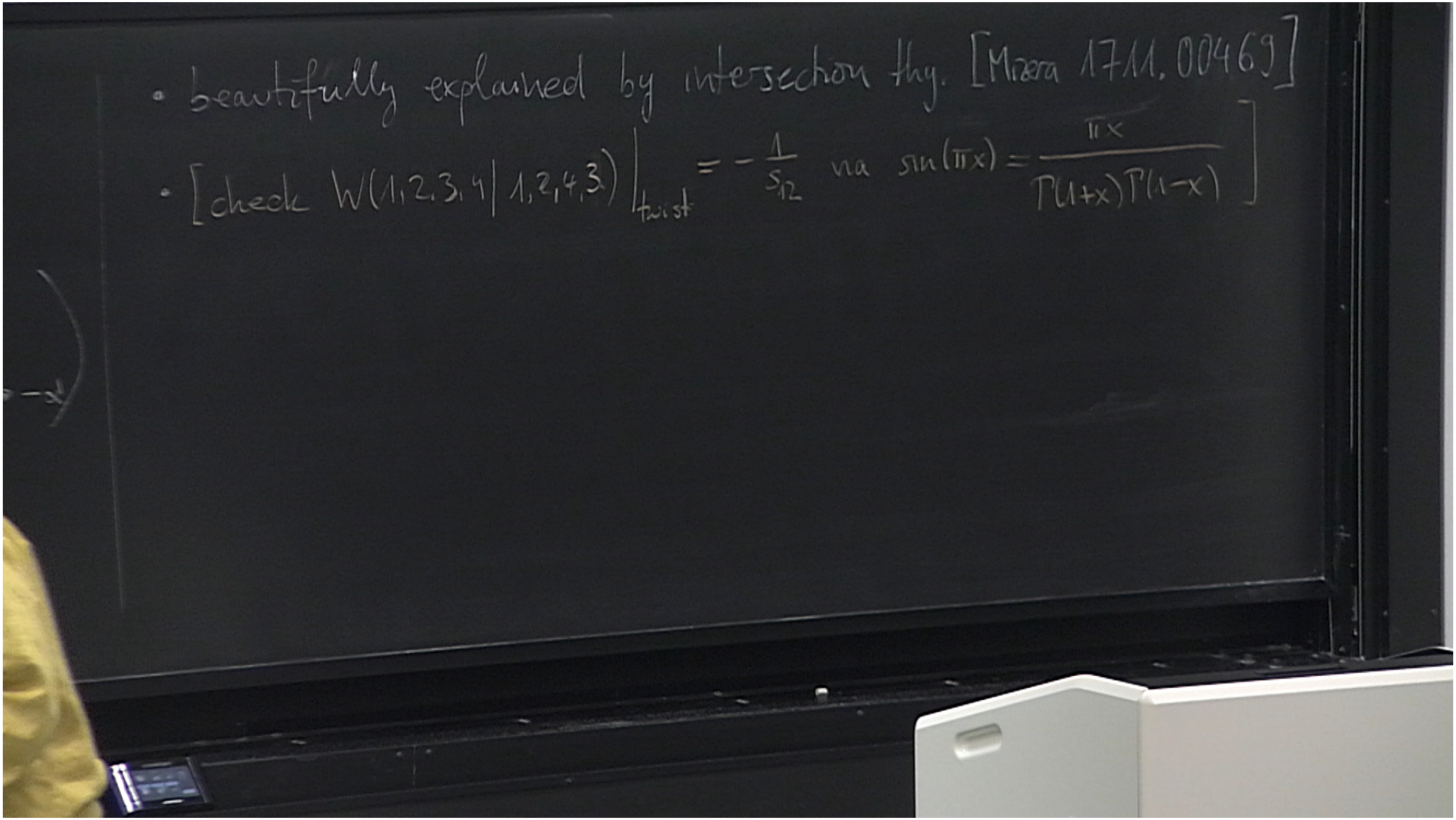
$$:= \sum_{\gamma,\tau \in S_{n-3}} Z(1,\gamma_1,\dots,n-1\,|\beta)\, S_\alpha[\gamma|\tau] \times \left(Z(1,\tau_1\dots n-1,n|\rho)\Big|_{\alpha'\to-\alpha'} \right)$$

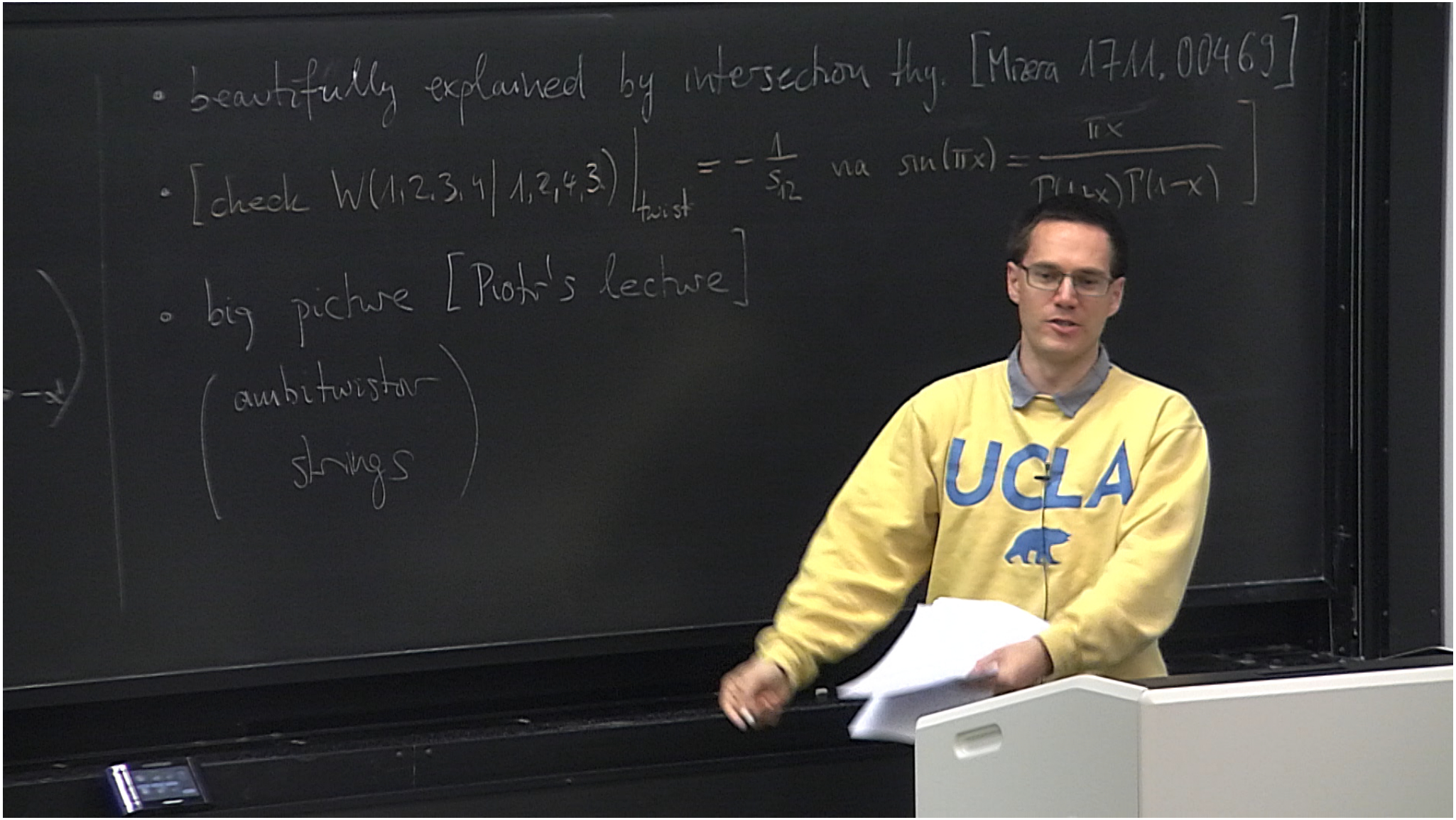

twist
$\{z_1 = z\}$
if it wasn't divergent!
$:= \sum_{\gamma,\tau \in \mathcal{S}_{n-3}} Z(1,\gamma,n,n-1\,|\beta)\, S_\alpha[\gamma|\tau] \times \left(Z(1,\tau,n-1,n|\rho)\Big|_{\alpha' \to -\alpha'} \right)$
Remarkably, α'-drops out

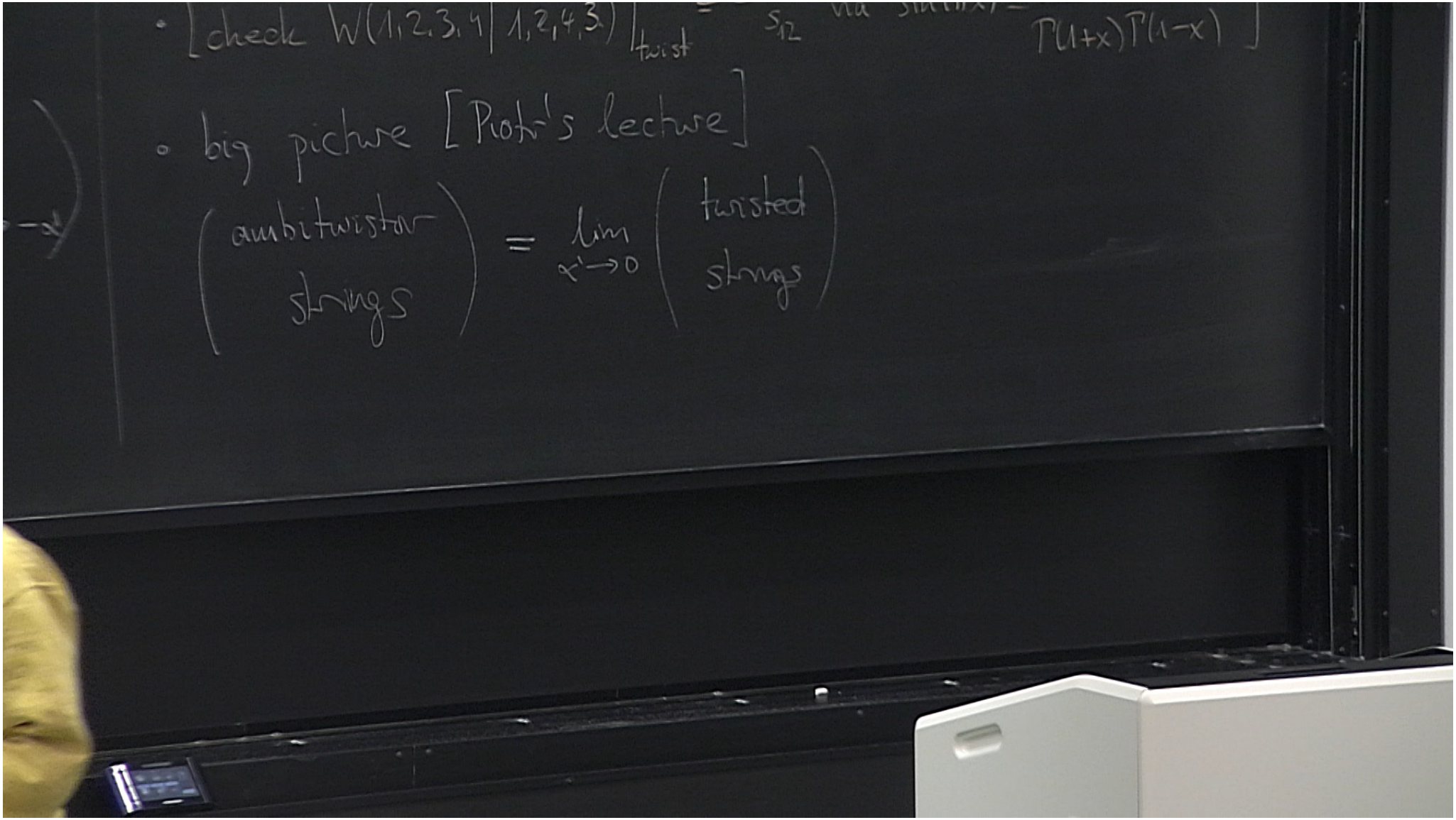

twist
$\{z_1 = z\}$
if it wasn't divergent!
$:= \sum_{\gamma_i \tau \in S_{n-3}} Z(1, \gamma_i, n, n-1 \,|\, \beta)\, S_\alpha^{-1}[\gamma|\tau] \times \left(Z(1, \tau_{in-1}, n \,|\, \rho) \Big|_{\alpha' \to -\alpha'} \right)$
Remarkably, α'-drops out
$W(\beta|\rho)\Big|_{twist} = m(\beta|\rho)$

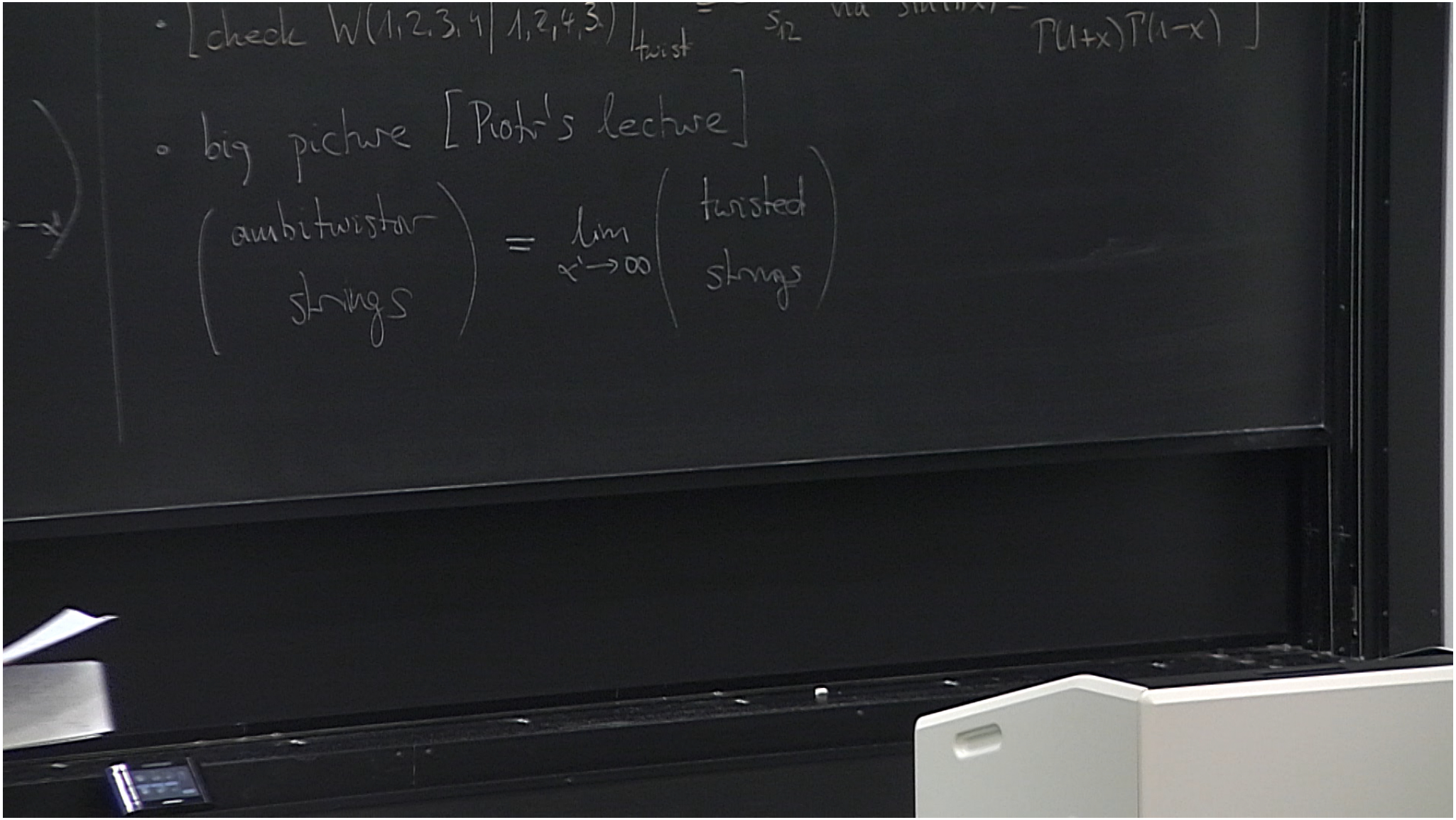

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Remarkably, α'-drops out
$W(\beta|\rho)\Big|_{twist} = m(\beta|\rho)$ exactly (in α')

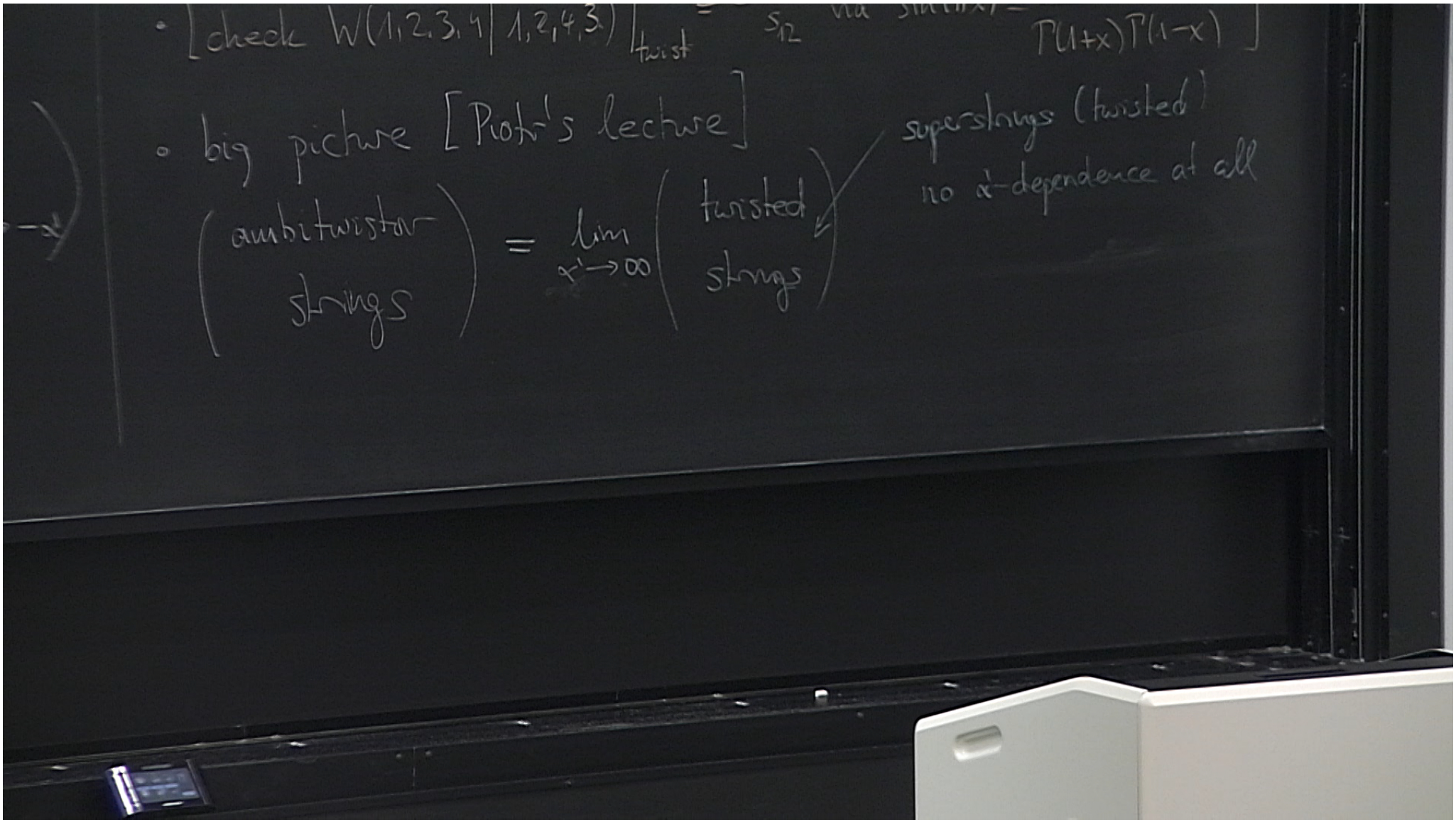

twist
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$:= \sum_{\gamma,\tau \in S_{n-3}} Z(1,\gamma,n,n-1\,|\,\beta)\, S_\alpha[\gamma\,|\,\tau] \times \left(Z(1,\tau,n-1,n\,|\,\rho)\Big|_{\alpha' \to -\alpha'} \right)$
Remarkably, α'-drops out [1603.02588]
$W(\beta\,|\,\rho)\Big|_{\text{twist}} = m(\beta\,|\,\rho)$ exactly (in α')

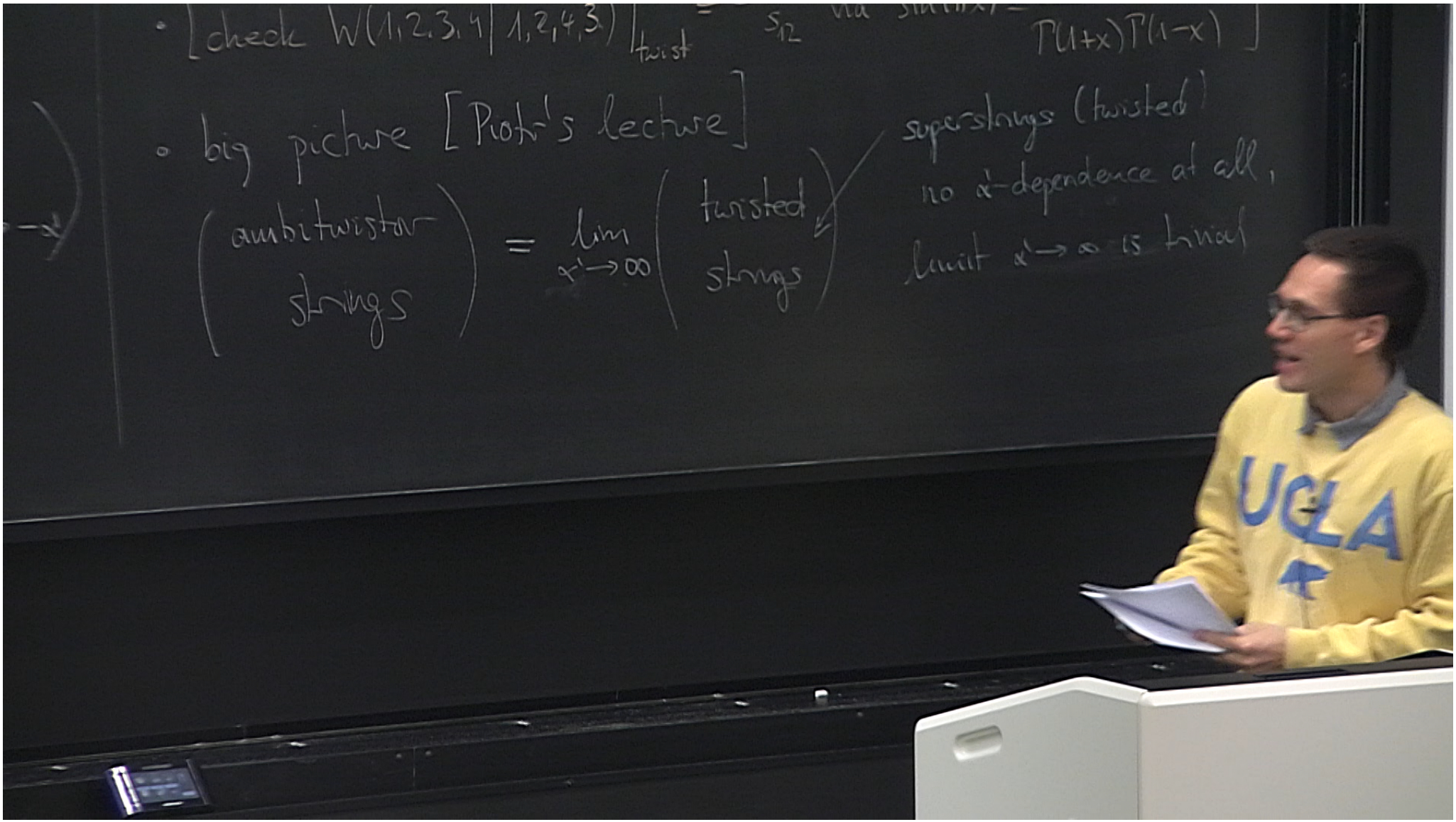

• beautifully explained by intersection thy. [Mizera 1711.00469]
• [check $W(1,2,3,4|1,2,4,3)\big|_{twist} = -\frac{1}{s_{12}}$ via $\sin(\pi x) = \frac{\pi x}{\Gamma(1+x)\Gamma(1-x)}$]

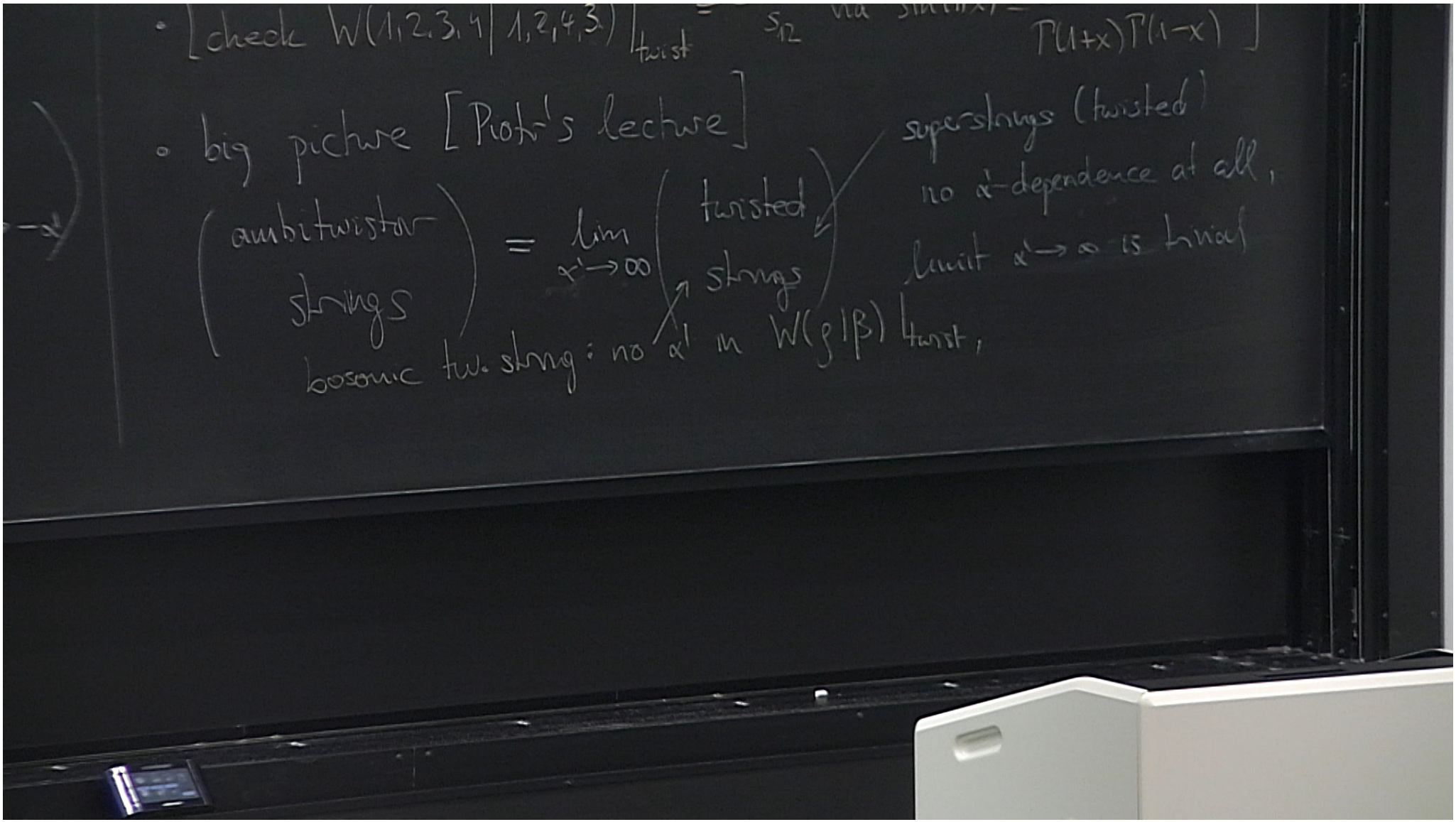

• beautifully explained by intersection thy. [Mizera 1711.00469]
• [check $W(1,2,3,4|1,2,4,3)\big|_{twist} = -\frac{1}{s_{12}}$ via $\sin(\pi x) = \frac{\pi x}{\Gamma(1+x)\Gamma(1-x)}$]
• big picture [Piotr's lecture]
(ambitwistor strings)
UCLA

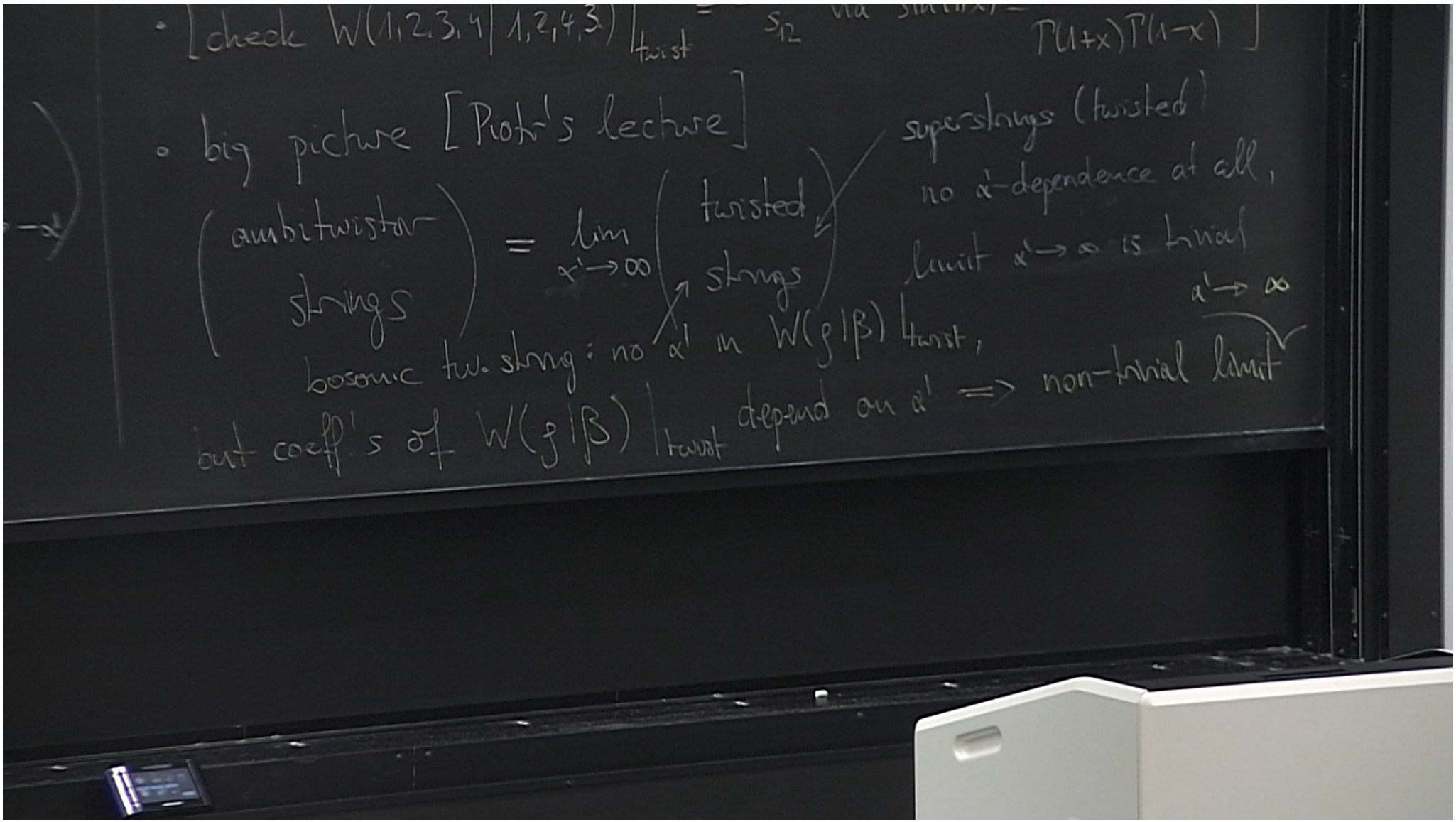

• [check $W(1,2,3,4|1,2,4,3)|_{twist}$ = s_{12} via ... = $\Gamma(1+x)\Gamma(1-x)$]
• big picture [Piotr's lecture]
(ambitwistor strings) = $\lim_{\alpha' \to 0}$ (twisted strings)


• [check $W(1,2,3,4|1,2,4,3)|_{twist}$ = s_{12} via ... = $\frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma(1+x)\Gamma(1-x)}$]
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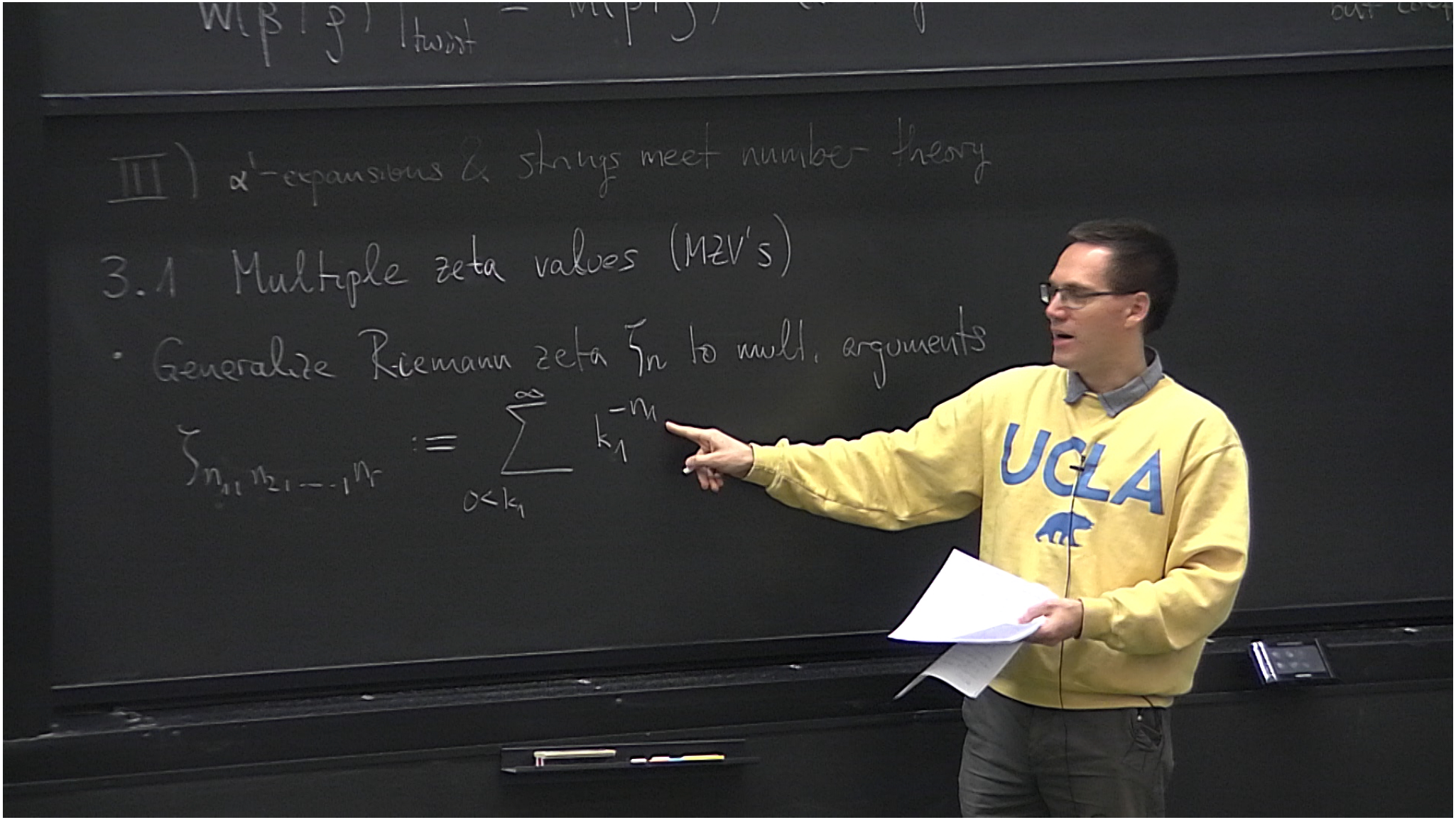

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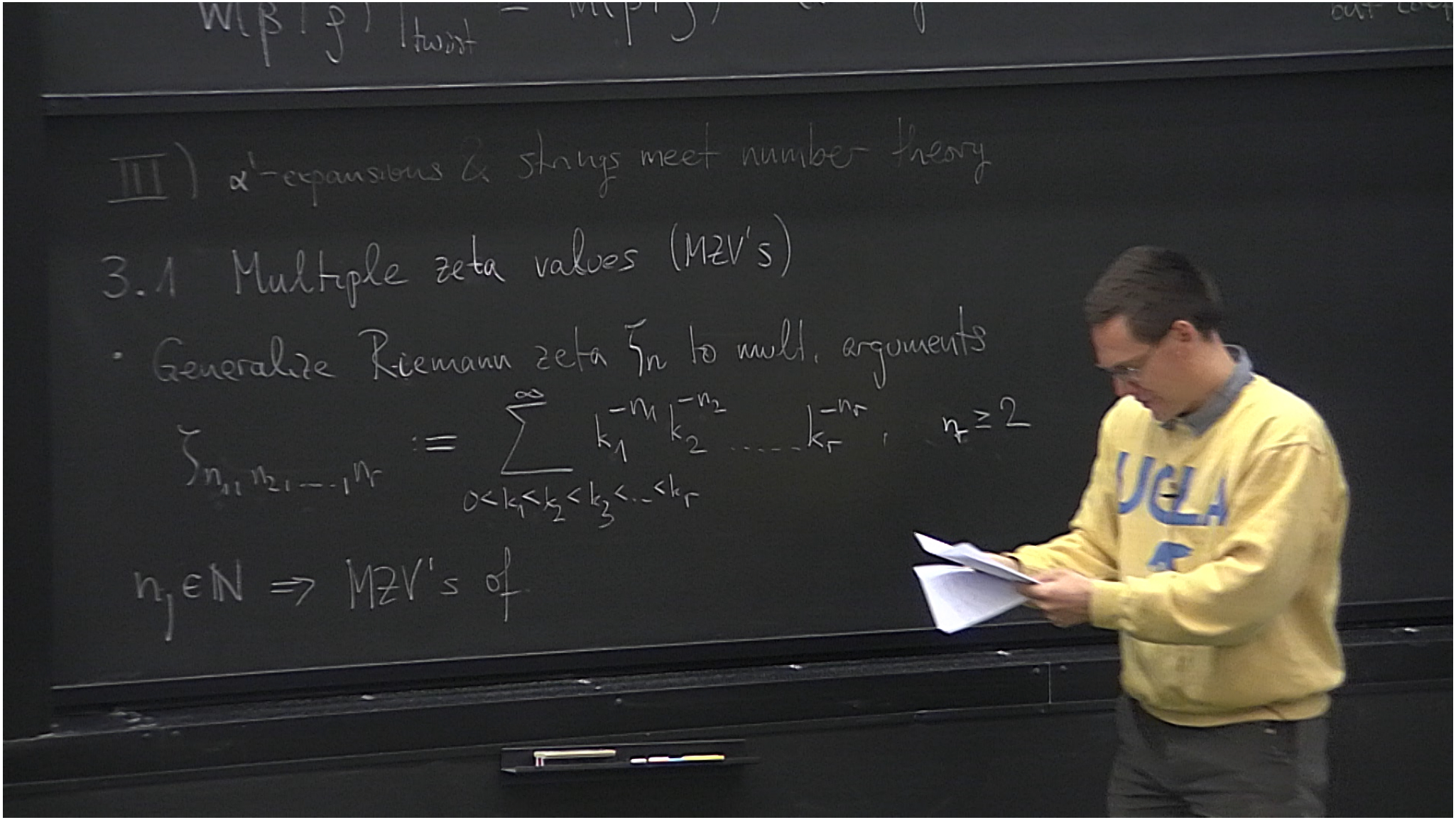

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bosonic tw. string: no α' in $W(g|\beta)|_{twist}$,
$\alpha'\to\infty$
out coeff's of $W(g|\beta)|_{twist}$ depend on α' ⇒ non-trivial limit

III) α'-expansions & strings meet number-theory

3.1 Multiple zeta values (MZV's)


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$\zeta_{n_1,n_2,\ldots,n_r} := \sum_{0<k_1<\cdots<k_r<\infty} k_1^{-n_1} \cdots k_r^{-n_r}$
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$n_i \in \mathbb{N} \Rightarrow$ MZV's of
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$W(\beta, g)|_{twist} = M(p_i, \ldots)$

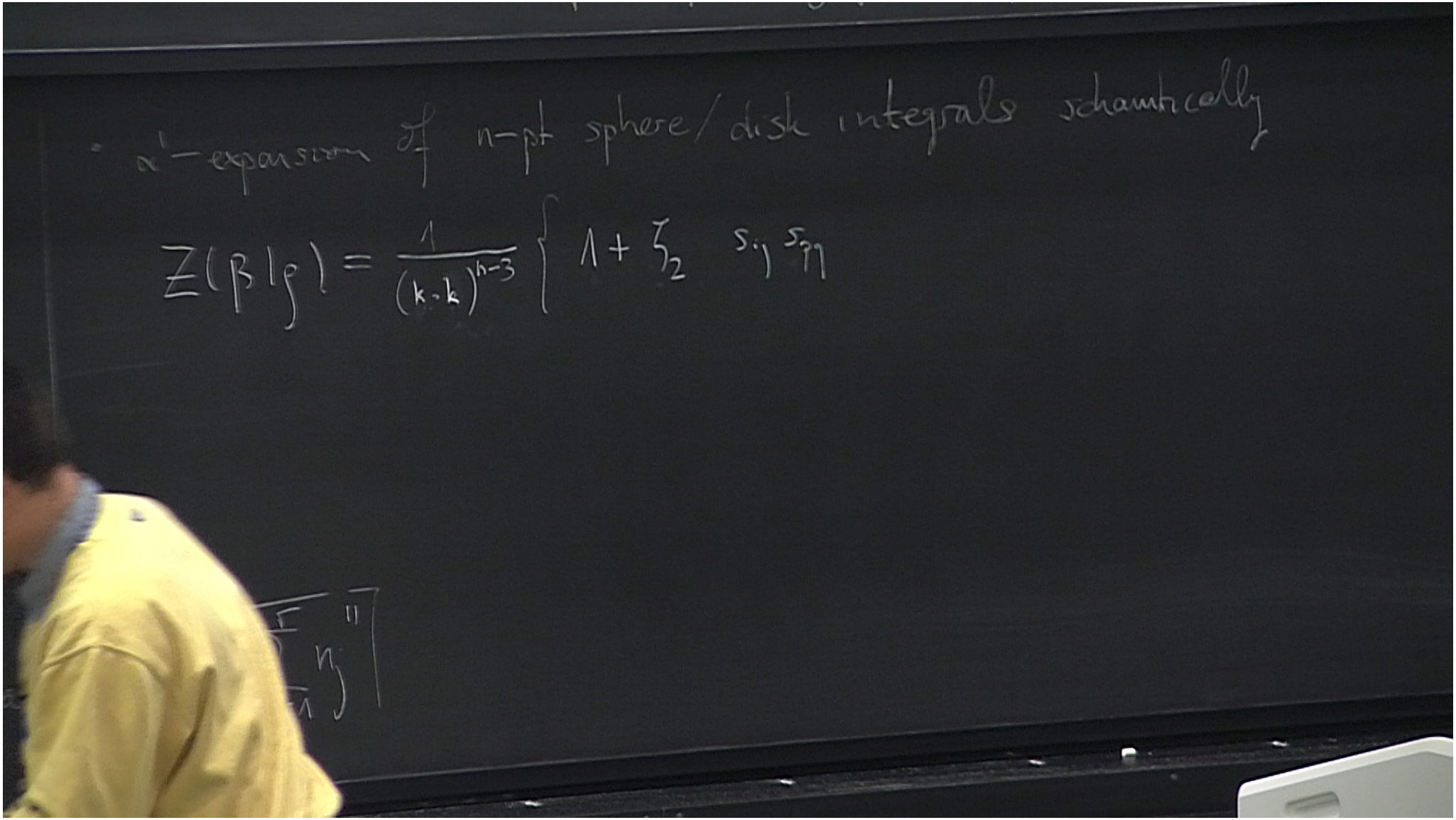
III) α'-expansions & strings meet number theory

3.1 Multiple zeta values (MZV's)

- Generalize Riemann zeta ζ_n to mult. arguments

$$\zeta_{n_1, n_2, \ldots, n_r} := \sum_{0<k_1<k_2<k_3<\ldots<k_r} k_1^{-n_1} k_2^{-n_2} \cdots k_r^{-n_r}, \quad n_i \geq 2$$

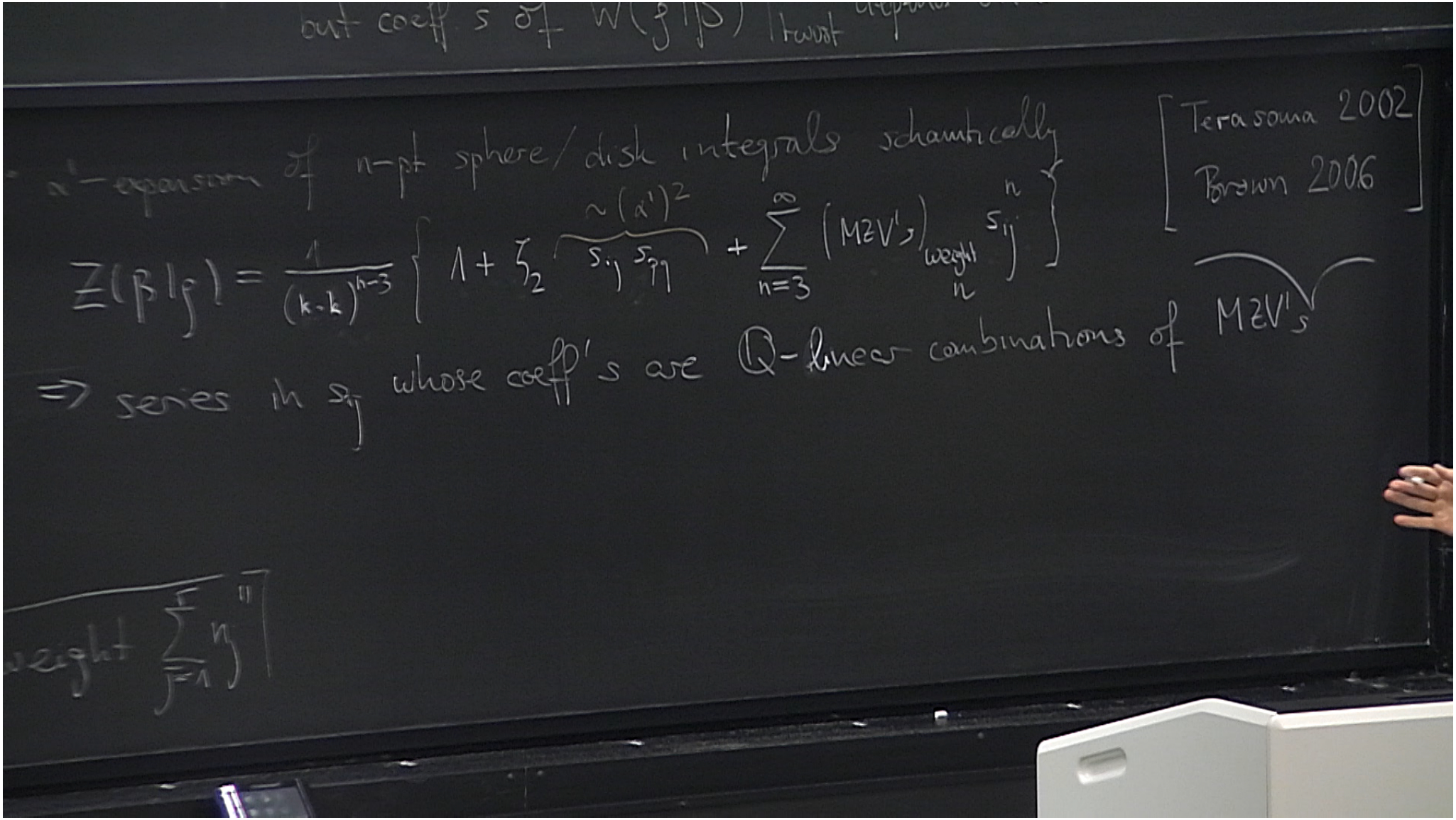
$n_i \in \mathbb{N} \Rightarrow$ MZV's of "depth r" and "(transcendental) weight $\sum_{j=1}^{r} n_j$"

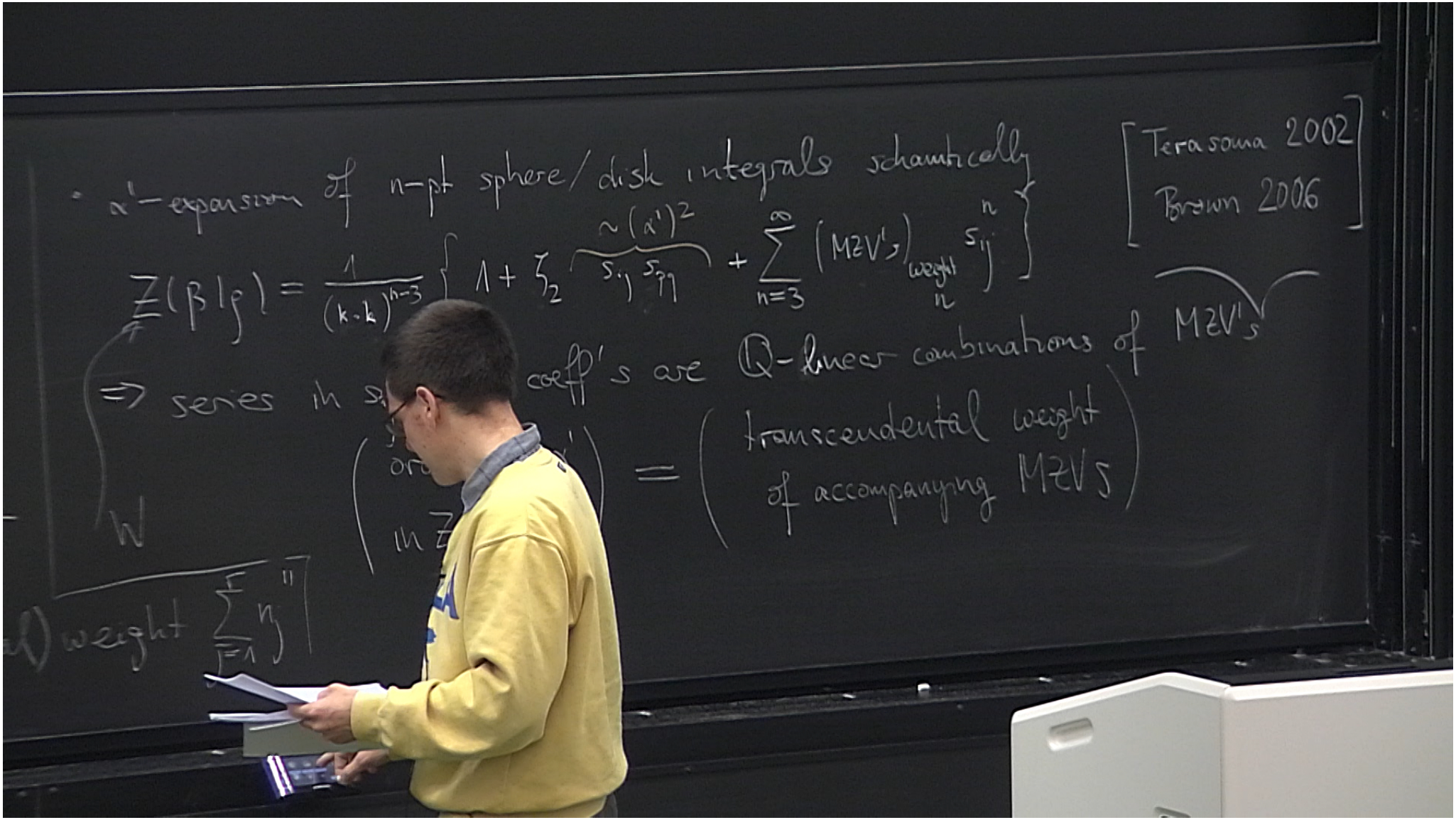

α'-expansion of n-pt sphere/disk integrals schematically
$Z(\beta|\rho) = \frac{1}{(k\cdot k)^{n-3}} \left\{ 1 + \zeta_2 \; s_{ij} s_{jj} \right.$

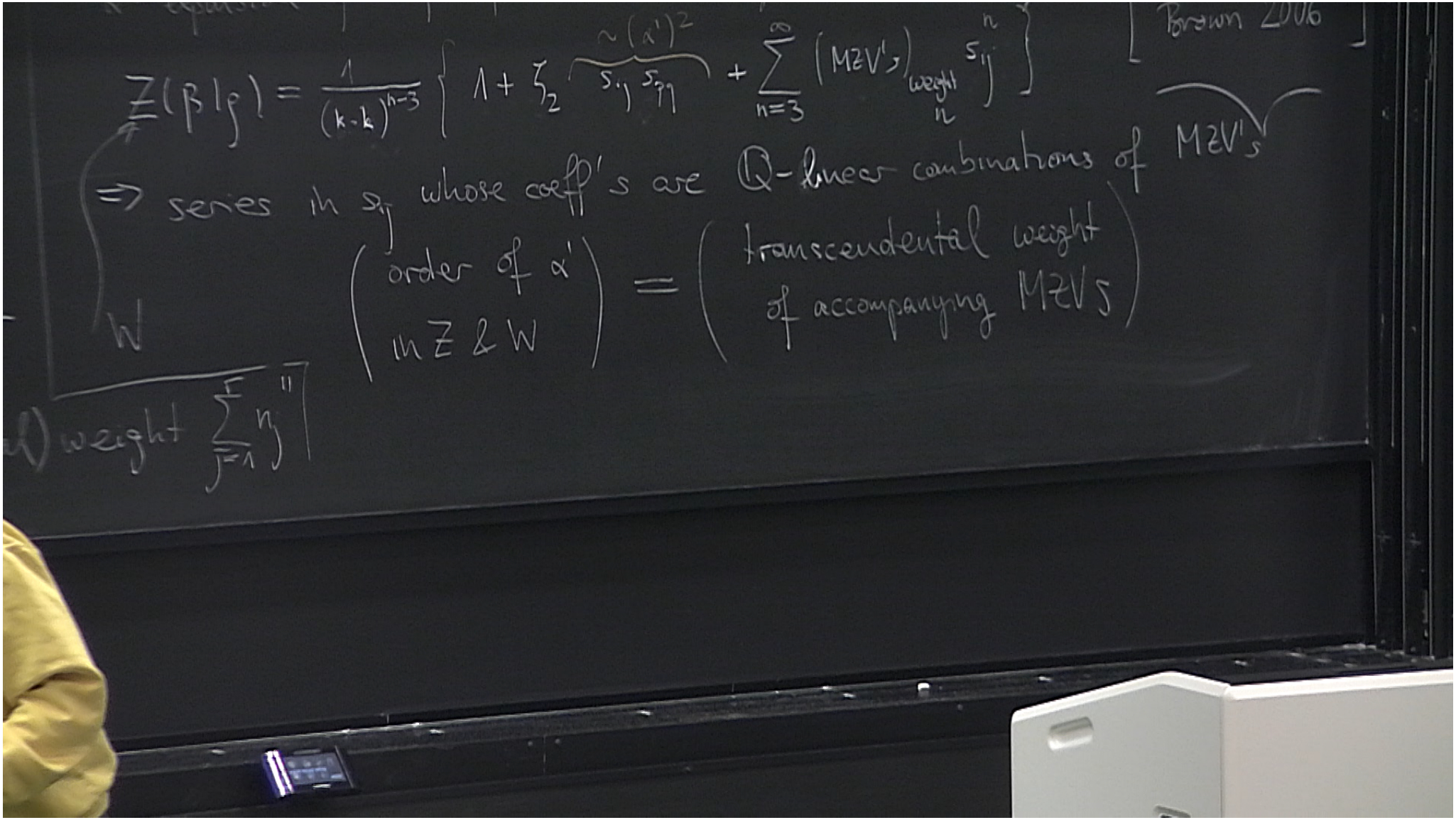
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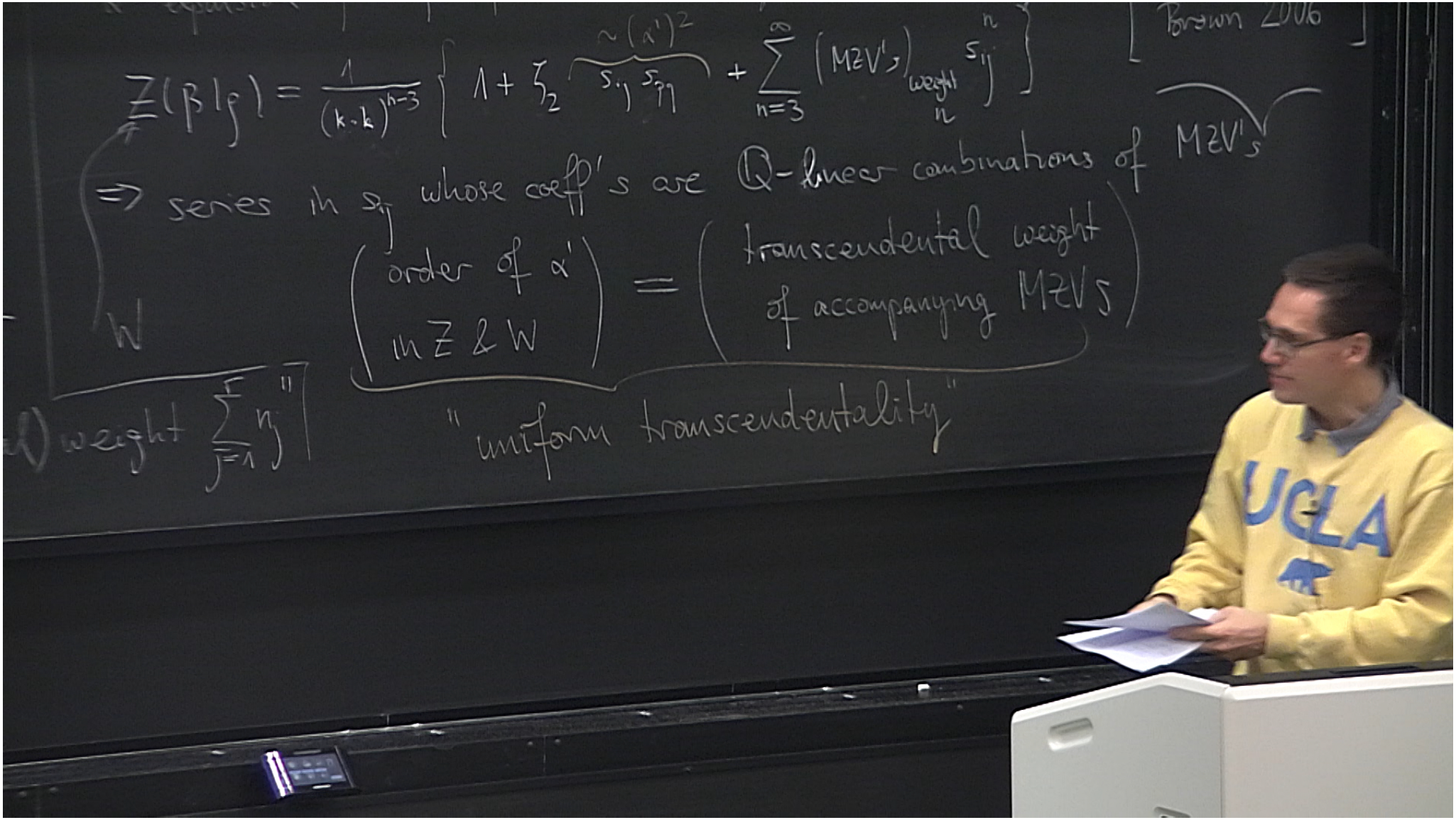
$$Z(\beta|\rho) = \frac{1}{(k\cdot k)^{n-3}} \left\{ 1 + \zeta_2 \underbrace{\sim (\alpha')^2}_{s_{ij}\, s_{jj}} + \sum_{n=3}^{\infty} (\mathrm{MZV's})_{\substack{\text{weight} \\ n}} s_{ij}^{n} \right\}$$

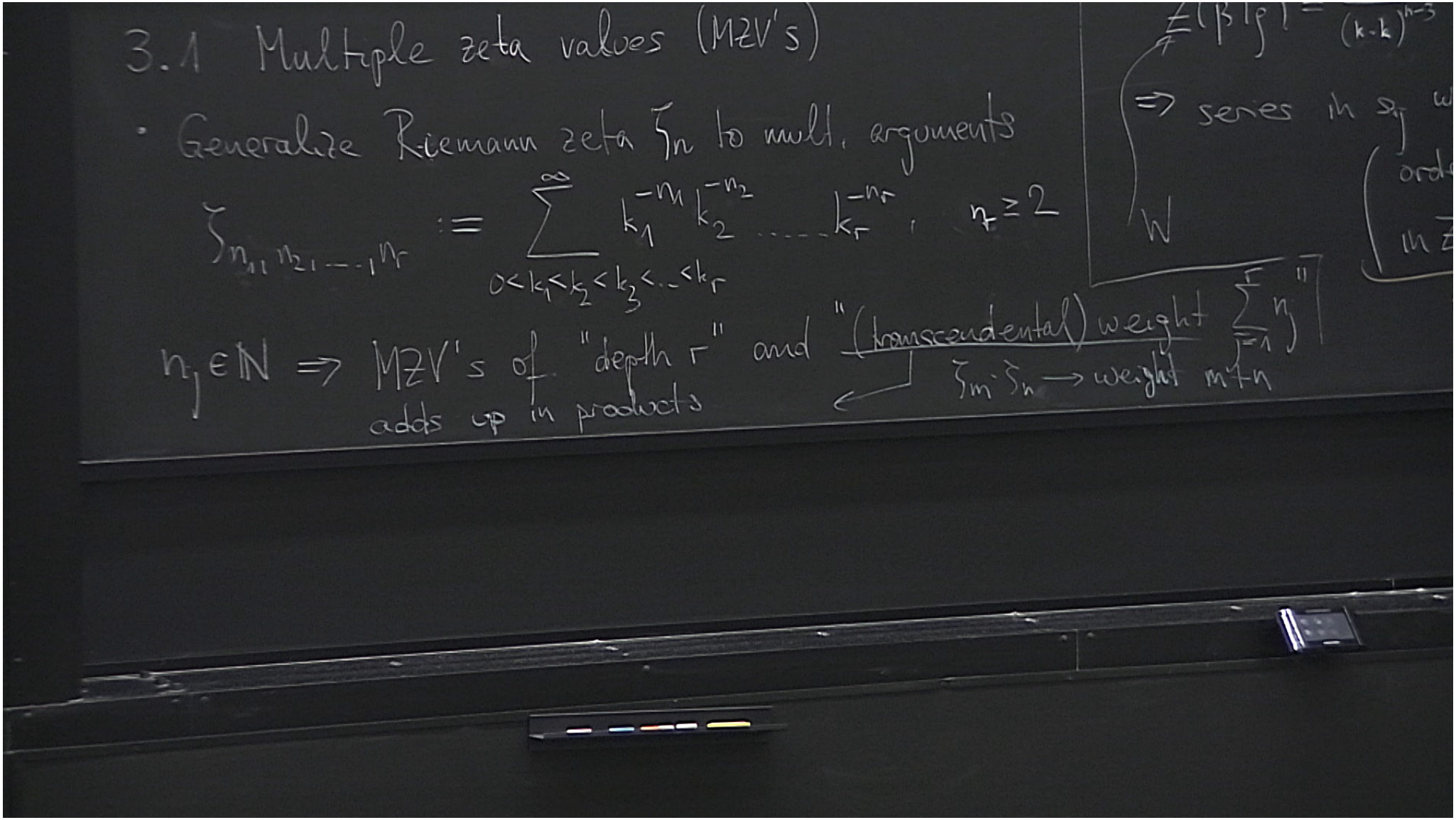
$$\text{weight} \sum_{j=1}^{r} n_j \overset{!!}{=} 7$$

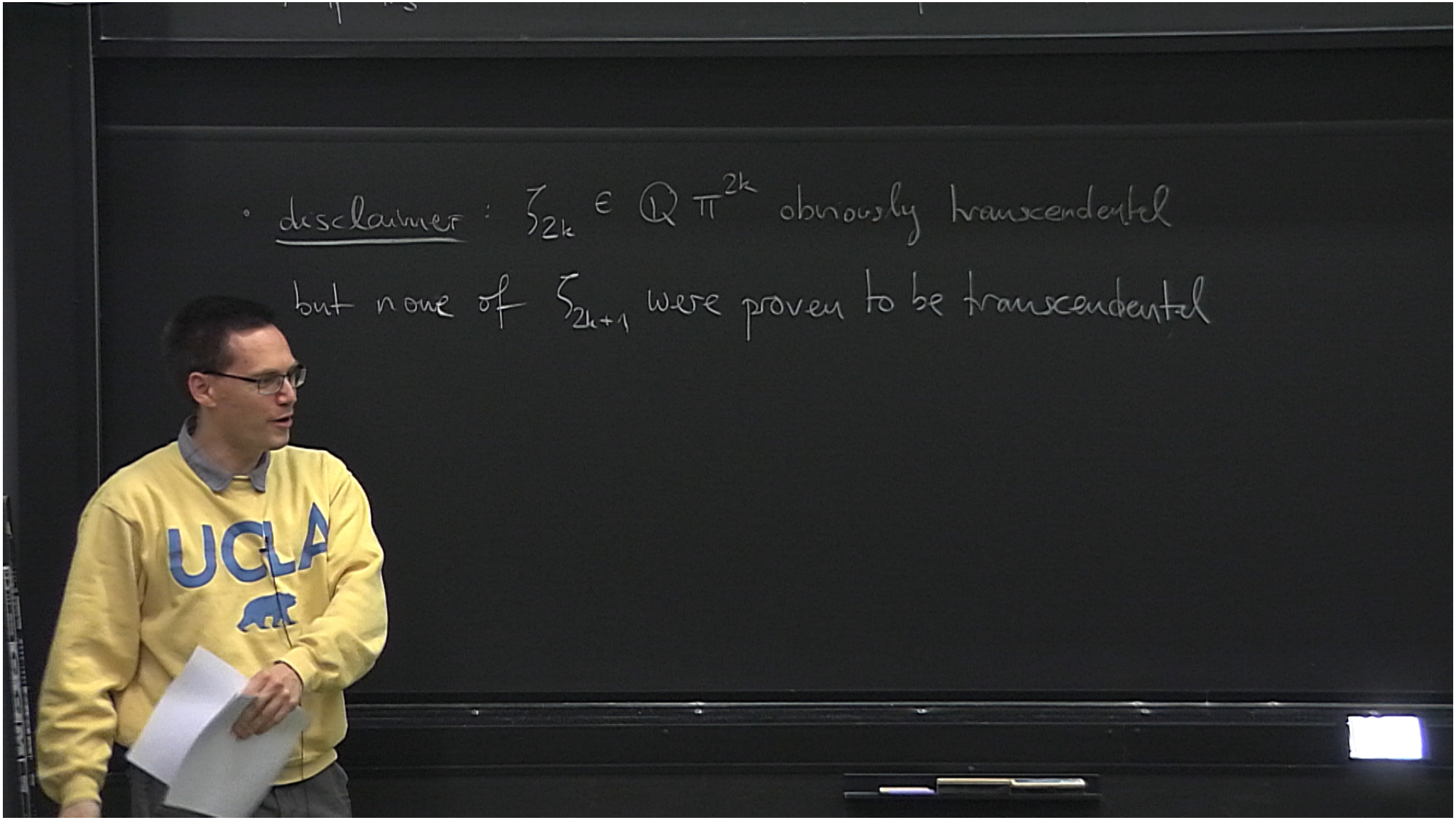

out coeff's of W(β|β)
α'-expansion of n-pt sphere/disk integrals schematically
[Terasoma 2002
Brown 2006]
$Z(\beta|\beta) = \frac{1}{(k-k)^{n-3}} \left\{ 1 + \zeta_2 \overbrace{s_{ij}\, s_{ij}}^{\sim(\alpha')^2} + \sum_{n=3}^{\infty} (\mathrm{MZV's})_{\substack{\text{weight} \\ n}} s_{ij}^{n} \right\}$
MZV's
⇒ series in s_{ij} whose coeff's are $\mathbb{Q}$-linear combinations of
weight $\sum_{j=1}^{r} n_j$

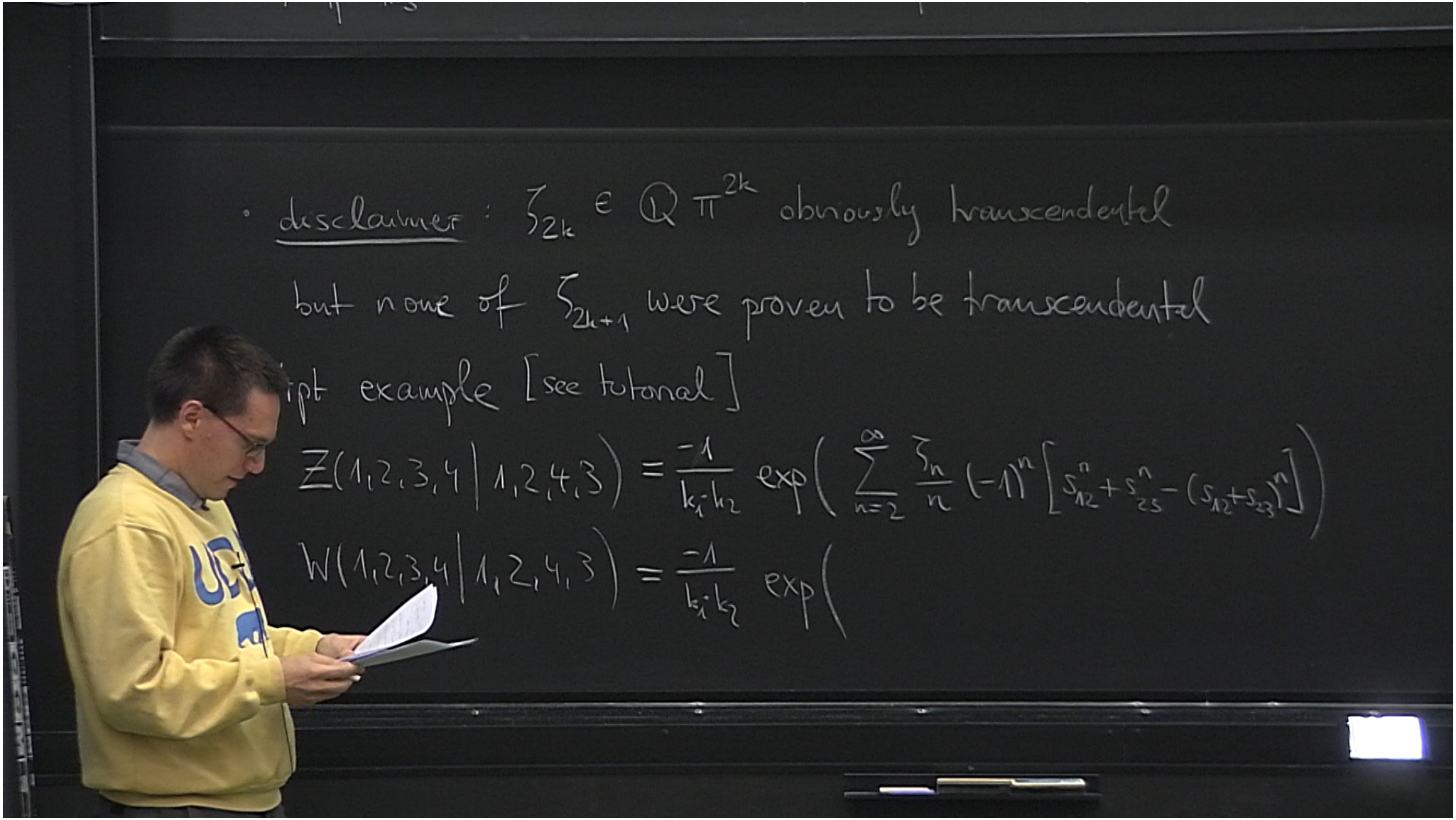

α'-expansion of n-pt sphere/disk integrals schematically
[Terasoma 2002
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Z(β|ρ) = 1/(k-k)^{n-3} { 1 + ζ_2 s_ij s_jj + Σ_{n=3}^{∞} (MZV's)_{weight n} s_ij^n }
~(α')^2
⇒ series in s
coeff's are Q-linear combinations of MZV's
(transcendental weight of accompanying MZVs)
weight Σ_{j=1}^{r} n_j


$Z(\beta|_\Gamma) = \frac{1}{(k-k)^{n-3}} \left\{ 1 + \xi_2 \overbrace{s_{ij} s_{ij}}^{\sim (\alpha')^2} + \sum_{n=3}^{\infty} (\text{MZV's})_{\text{weight } n} s_{ij}^n \right\}$
Brown 2006
$\Rightarrow$ series in s_{ij} whose coeff's are $\mathbb{Q}$-linear combinations of MZV's
W
$\left(\begin{array}{l}\text{order of } \alpha' \\ \text{in } Z \text{ \& } W\end{array}\right) = \left(\begin{array}{l}\text{transcendental weight} \\ \text{of accompanying MZVs}\end{array}\right)$
weight $\sum_{j=1}^{F} n_j$


$Z(\beta|_\Gamma) = \frac{1}{(k\cdot k)^{n-3}} \left\{ 1 + \zeta_2 \overbrace{s_{ij} s_{jj}}^{\sim(\alpha')^2} + \sum_{n=3}^{\infty} (\text{MZV's})_{\text{weight } n} s_{ij}^{n} \right\}$
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"uniform transcendentality"
W
weight $\sum_{j=1}^{r} n_j$
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• Generalize Riemann zeta ζ_n to mult. arguments
$\zeta_{n_1,n_2,\dots,n_r} := \sum_{0<k_1<k_2<k_3<\dots<k_r} k_1^{-n_1} k_2^{-n_2} \dots k_r^{-n_r}$, $n_i \geq 2$
$n_i \in \mathbb{N} \Rightarrow$ MZV's of "depth r" and "(transcendental) weight $\sum_{i=1}^{r} n_i$"
adds up in products
$\zeta_m \cdot \zeta_n \rightarrow$ weight $m+n$
$\Rightarrow$ series in s_j


• disclaimer : $\zeta_{2k} \in \mathbb{Q}\,\pi^{2k}$ obviously transcendental
but none of ζ_{2k+1} were proven to be transcendental
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ipt example [see tutorial]
$Z(1,2,3,4|1,2,4,3) = \frac{-1}{k_1 k_2} \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} (-1)^n \left[s_{12}^n + s_{23}^n - (s_{12}+s_{23})^n \right] \right)$
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- 4pt example [see tutorial]

confirms uniform transcendentality

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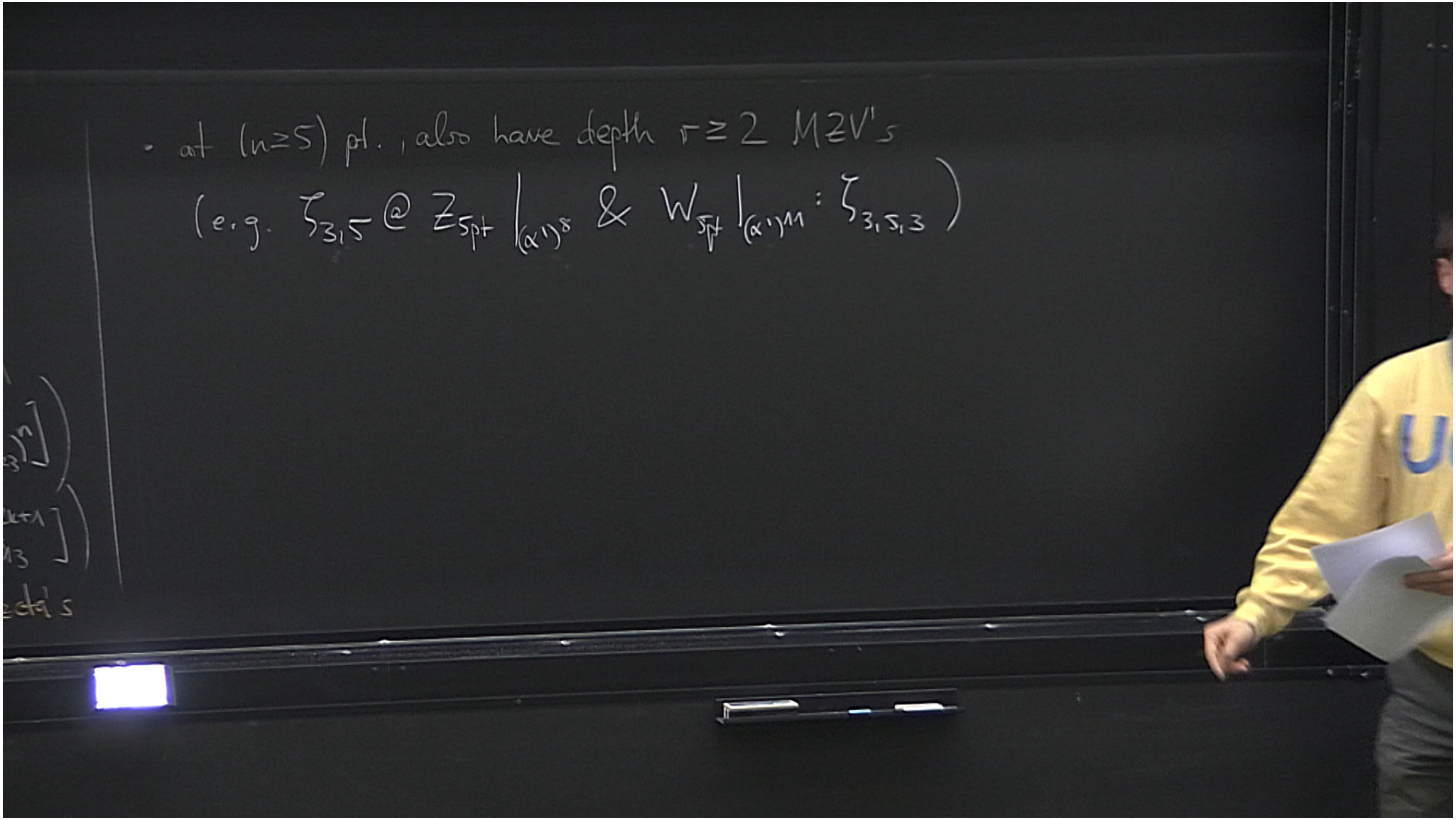
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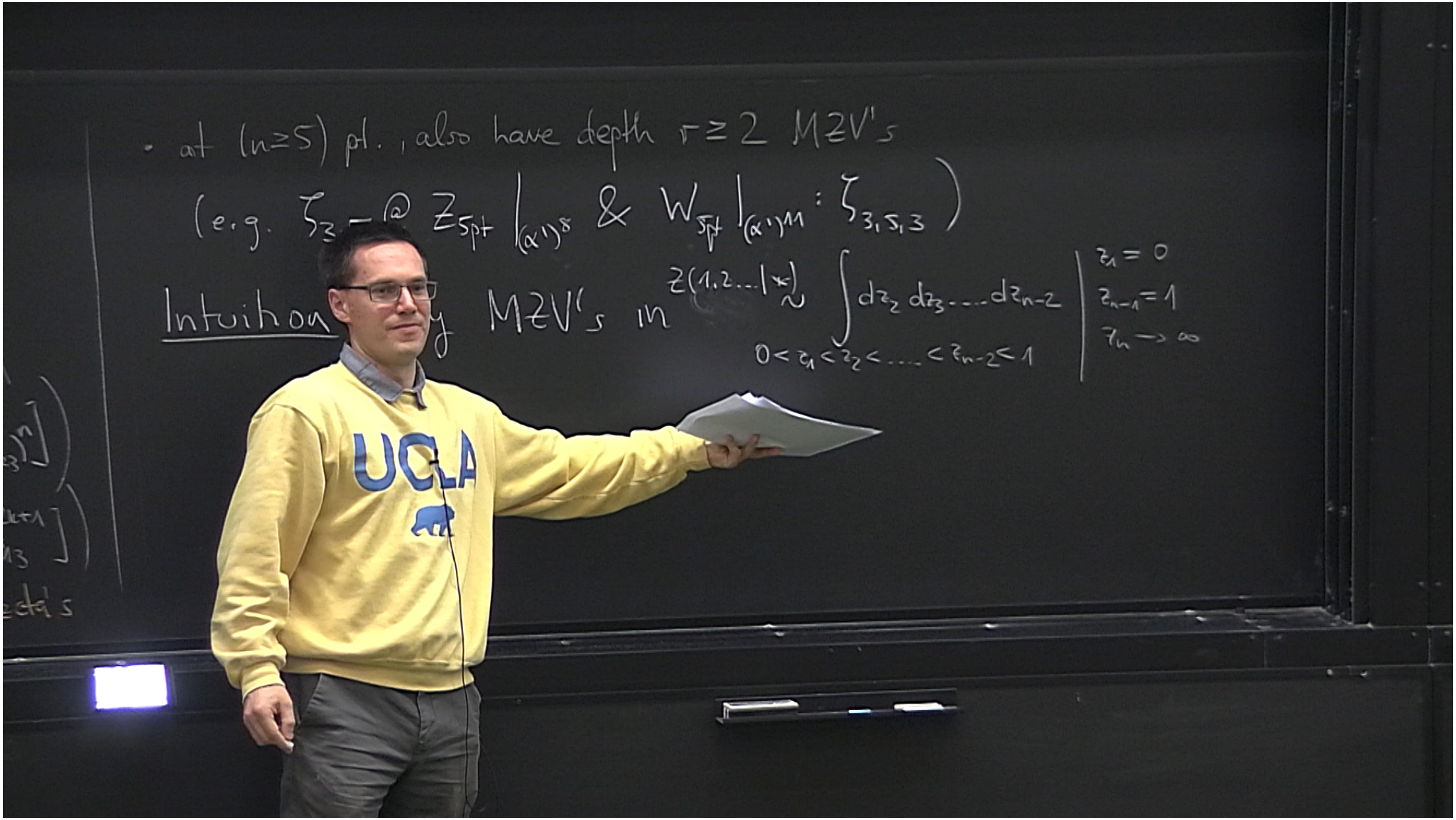
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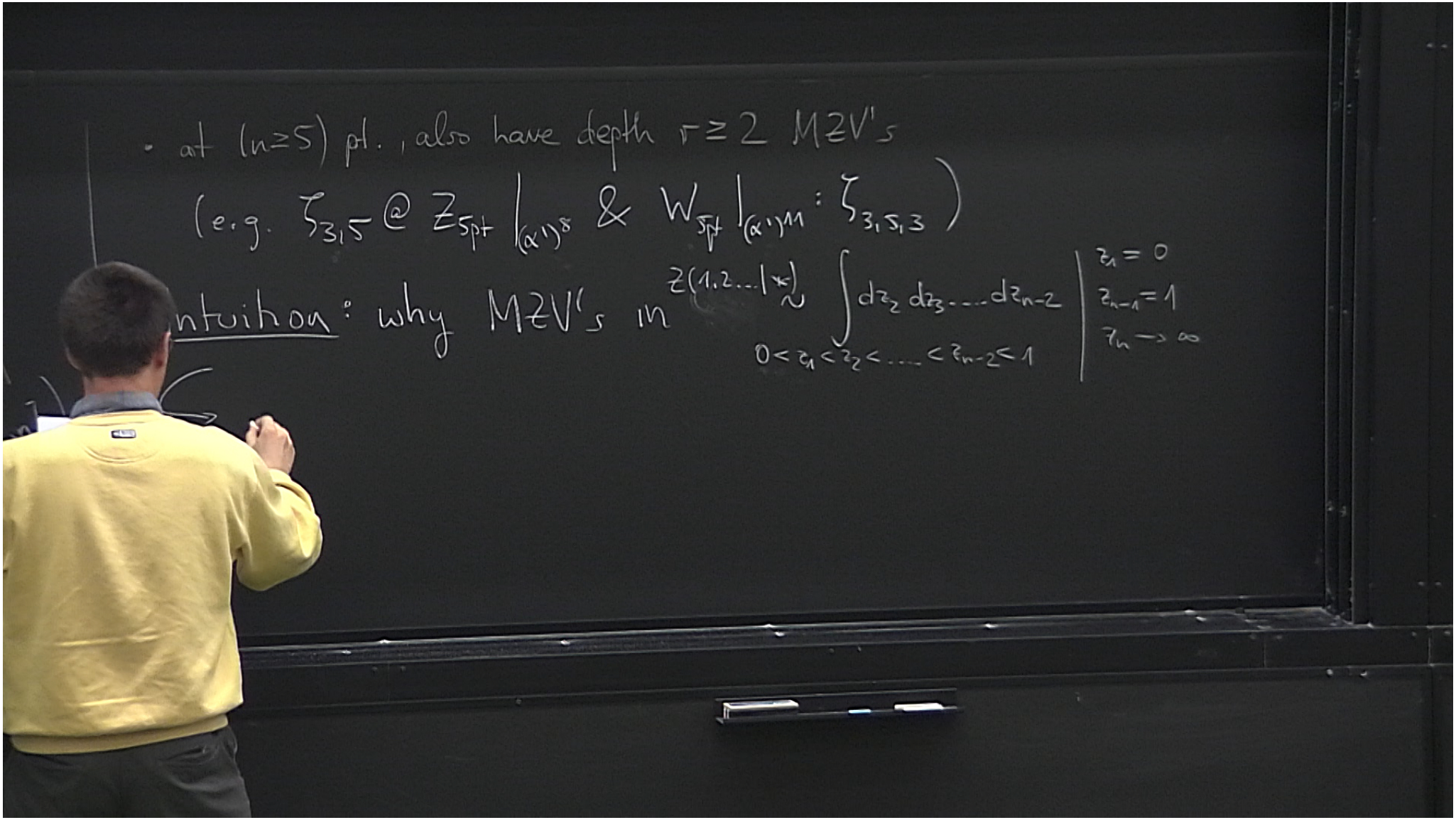
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no even zeta's


• at (n≥5) pt., also have depth r≥2 MZV's
(e.g. $\zeta_{3,5}$ @ $Z_{5pt}|_{(\alpha')^8}$ & $W_{5pt}|_{(\alpha')^{11}}$: $\zeta_{3,5,3}$)


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Intuition
MZV's in
$z(1,2\ldots|*) \sim \int dz_2\, dz_3 \ldots dz_{n-2}$
$0<z_1<z_2<\ldots<z_{n-2}<1$
$z_1 = 0$
$z_{n-1} = 1$
$z_n \to \infty$
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Intuition: why MZV's in $z(1,2,\ldots|*)$
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- at $(n \geq 5)$ pt., also have depth $r \geq 2$ MZV's

  (e.g. $\zeta_{3,5} \subset Z_{5pt}\big|_{(\alpha')^8}$ & $W_{5pt}\big|_{(\alpha')^{11}} : \zeta_{3,5,3}$)

Intuition: why MZV's in $Z(1,2,\ldots|\ast)$ $\sim \int_{0<z_1<z_2<\ldots<z_{n-2}<1} dz_2\, dz_3 \ldots dz_{n-2}$ $\Big|$ $z_1 = 0$, $z_{n-1} = 1$, $z_n \to \infty$

↪ can map MZV to iterated int's $\quad \omega_0(z) = \frac{dz}{z}$, $\omega_1(z) = \frac{dz}{1-z}$

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↪ can map MZV to iterated int's

$\omega_0(z) = \frac{dz}{z}$; $\omega_1(z) = \frac{dz}{1-z}$

$\zeta(a_1, a_2, \ldots, a_k) := \int_{0<z_1<z_2<\ldots<z_k<1} \omega_{a_1}(z_1)\, \omega_{a_2}(z_2) \cdots \omega_{a_k}(z_k)$

- at ($n \geq 5$) pt., also have depth $r \geq 2$ MZV's

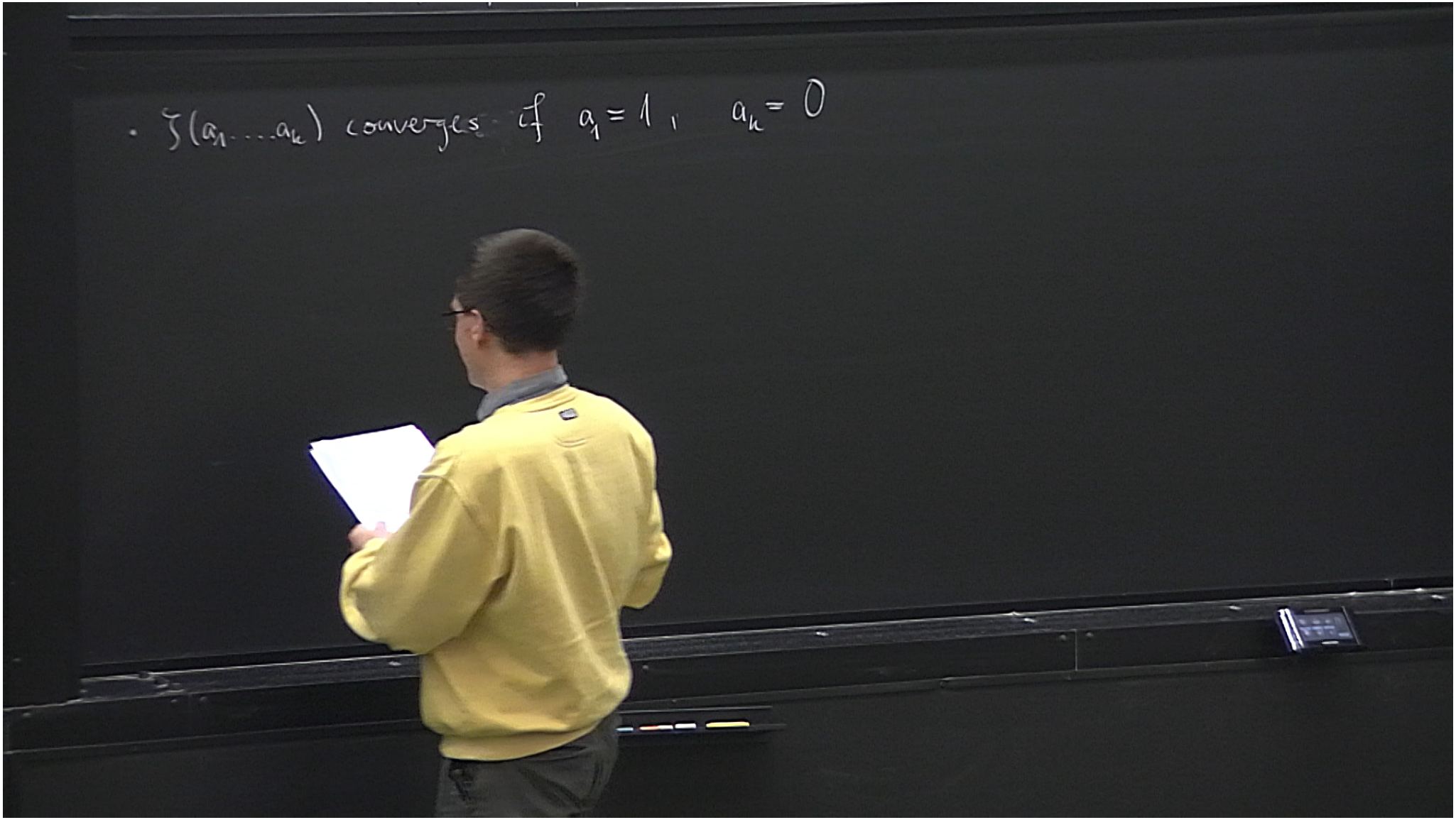
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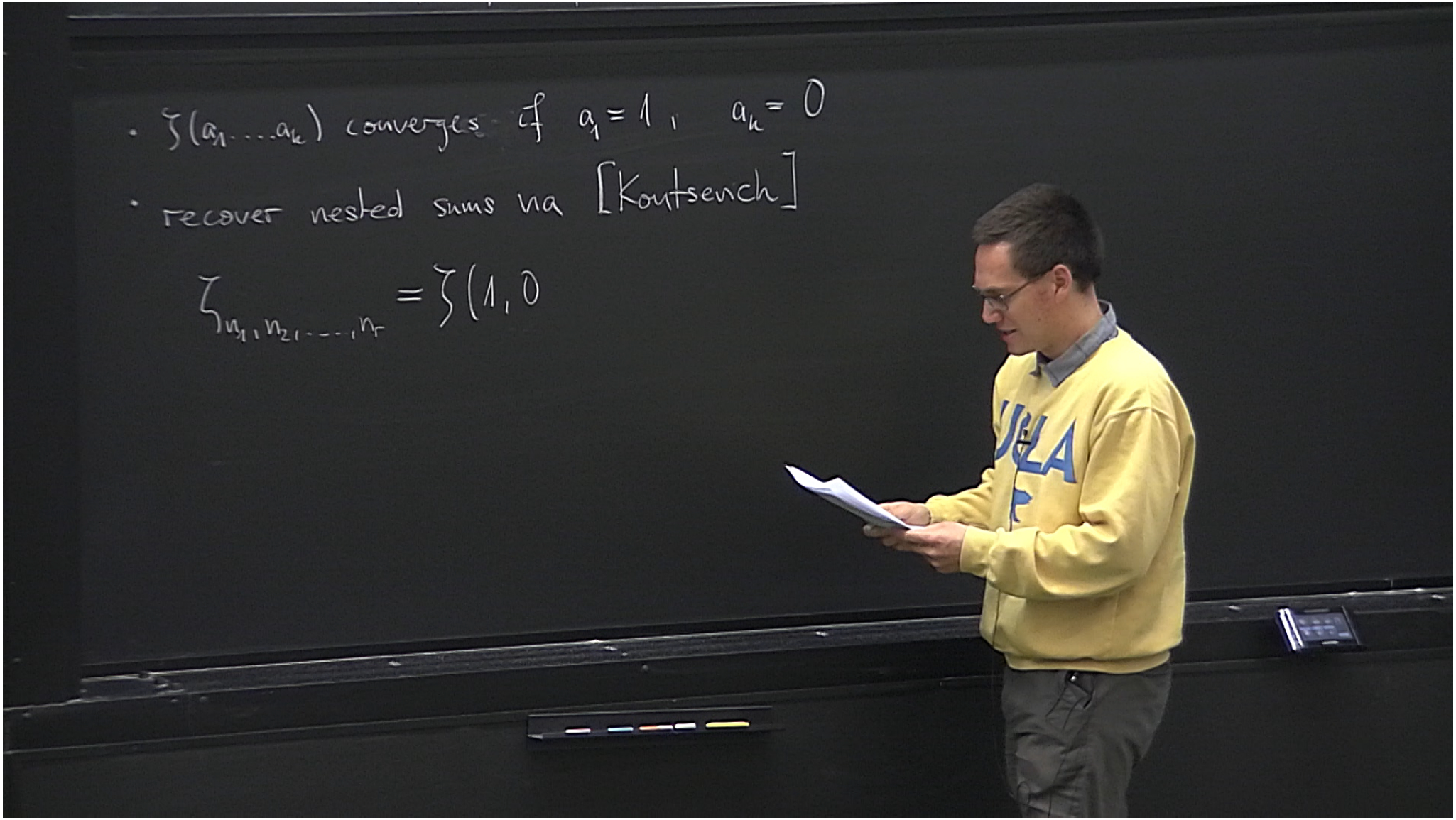
Intuition: why MZV's in $Z(1,2,\ldots,n|*) \sim \int_{0<z_1<z_2<\ldots<z_{n-2}<1} dz_2\, dz_3 \ldots dz_{n-2} \Big|_{z_1=0,\ z_{n-1}=1,\ z_n \to \infty}$

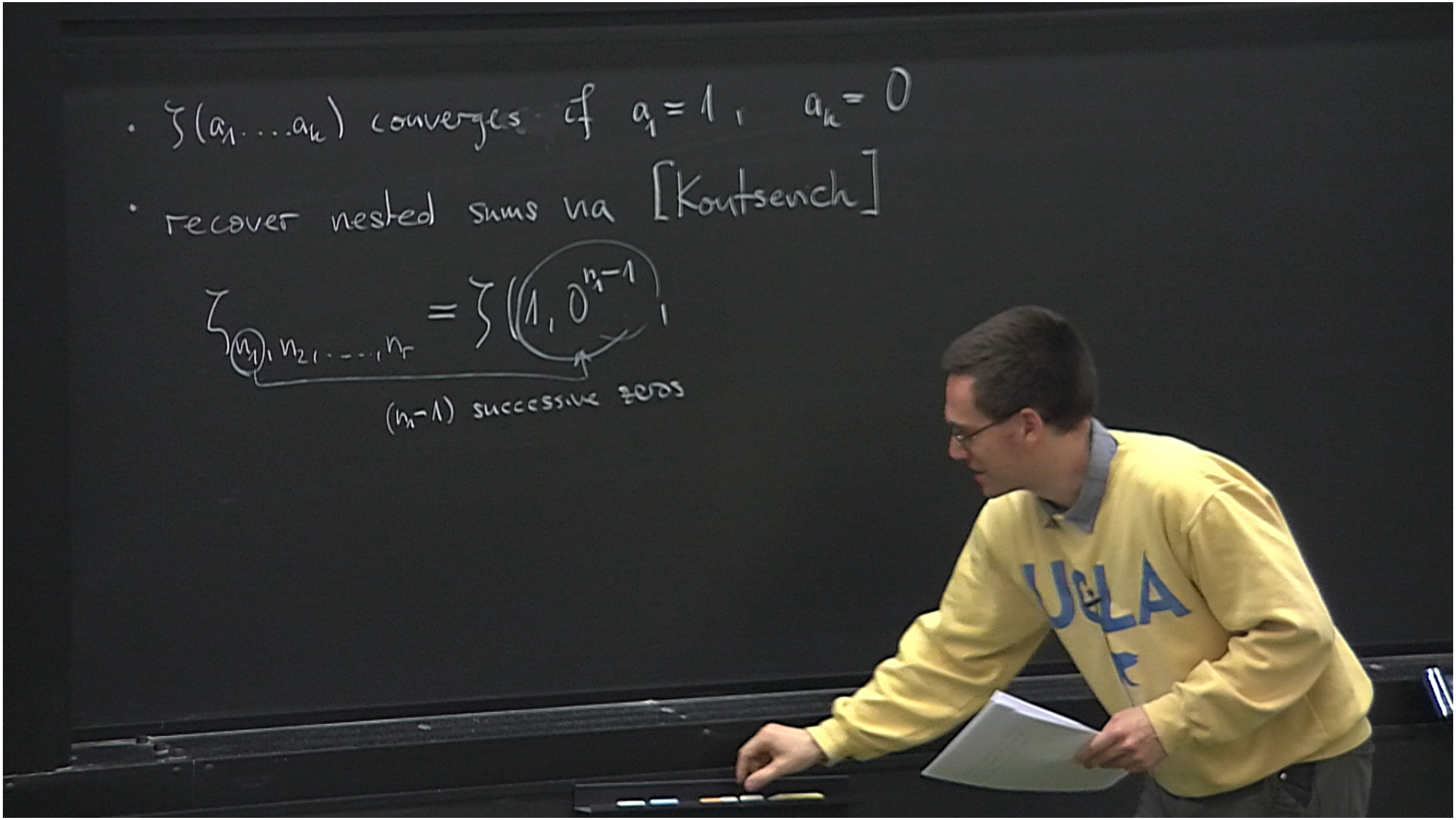
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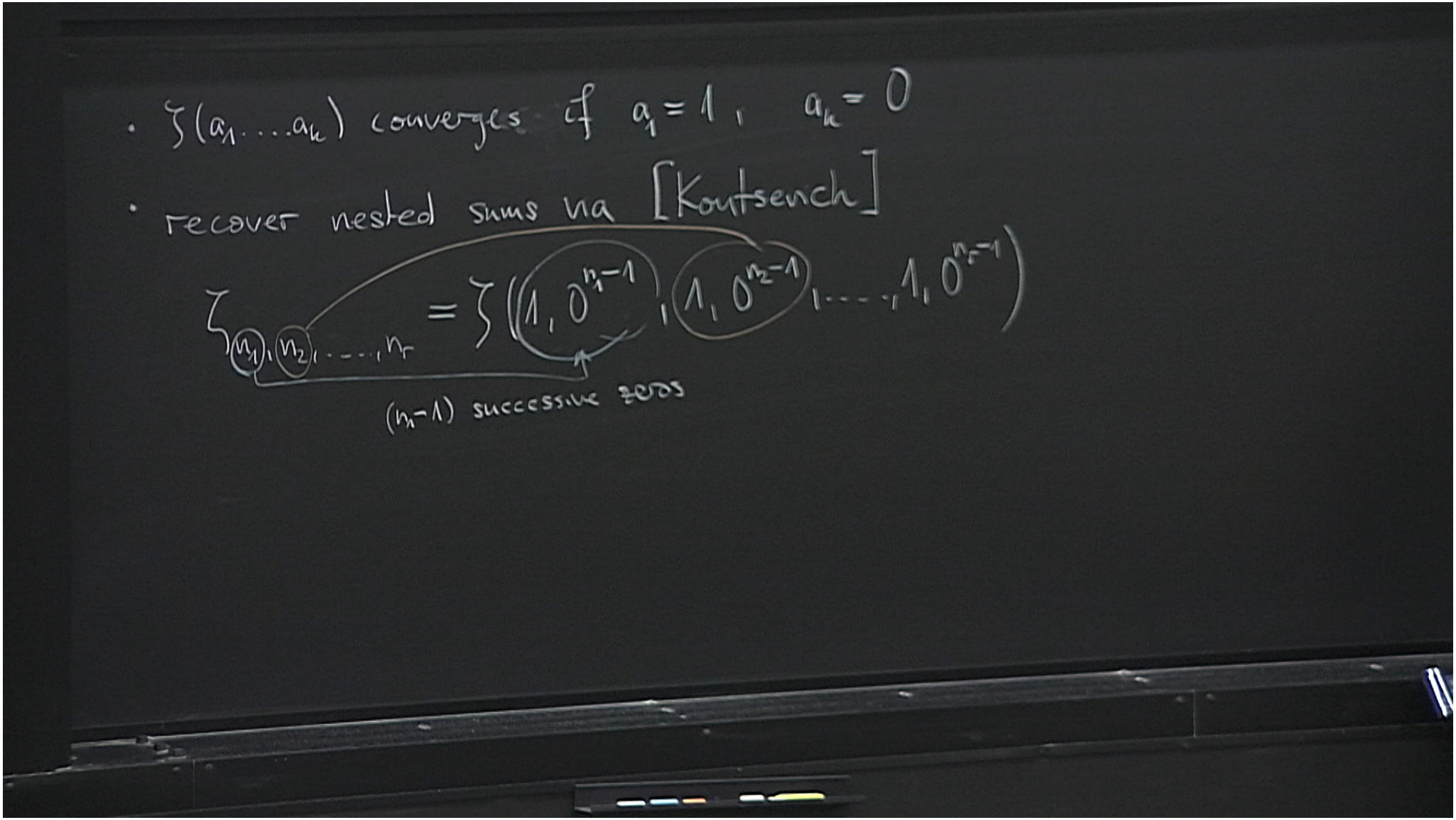
$$\omega_0(z) = \frac{dz}{z} \; ; \; \omega_1(z) = \frac{dz}{1-z}$$

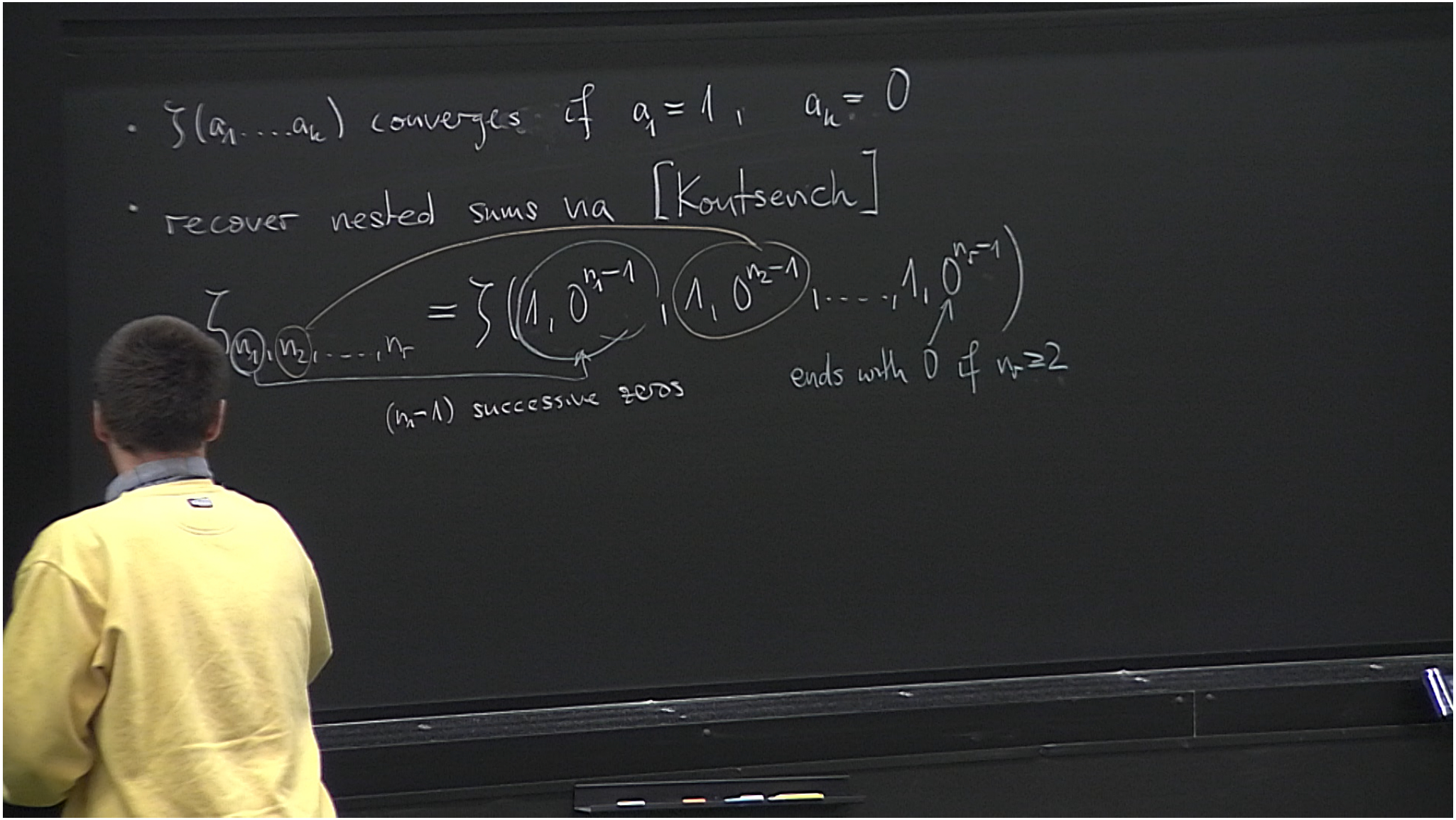
$$\zeta(a_1, a_2, \ldots, a_k) := \int_{0<z_1<z_2<\ldots<z_k<1} \omega_{a_1}(z_1)\, \omega_{a_2}(z_2) \cdots \omega_{a_k}(z_k)$$


· ζ(a1 ... ak) converges if a1 = 1, ak = 0


• $\zeta(a_1 \ldots a_k)$ converges if $a_1 = 1$, $a_k = 0$
• recover nested sums via [Koutsenich]
$\zeta_{n_1, n_2, \ldots, n_r} = \zeta(1, 0$
UCLA


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$\zeta_{n_1, n_2, \ldots, n_r} = \zeta(\{1, 0^{n_1-1}\}, 1)$
$(n_1 - 1)$ successive zeros
UCLA


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$\zeta_{n_1, n_2, \ldots, n_r} = \zeta(1, 0^{n_1-1}, 1, 0^{n_2-1}, \ldots, 1, 0^{n_r-1})$
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(n_1-1) successive zeros
ends with 0 if $n_r\geq 2$

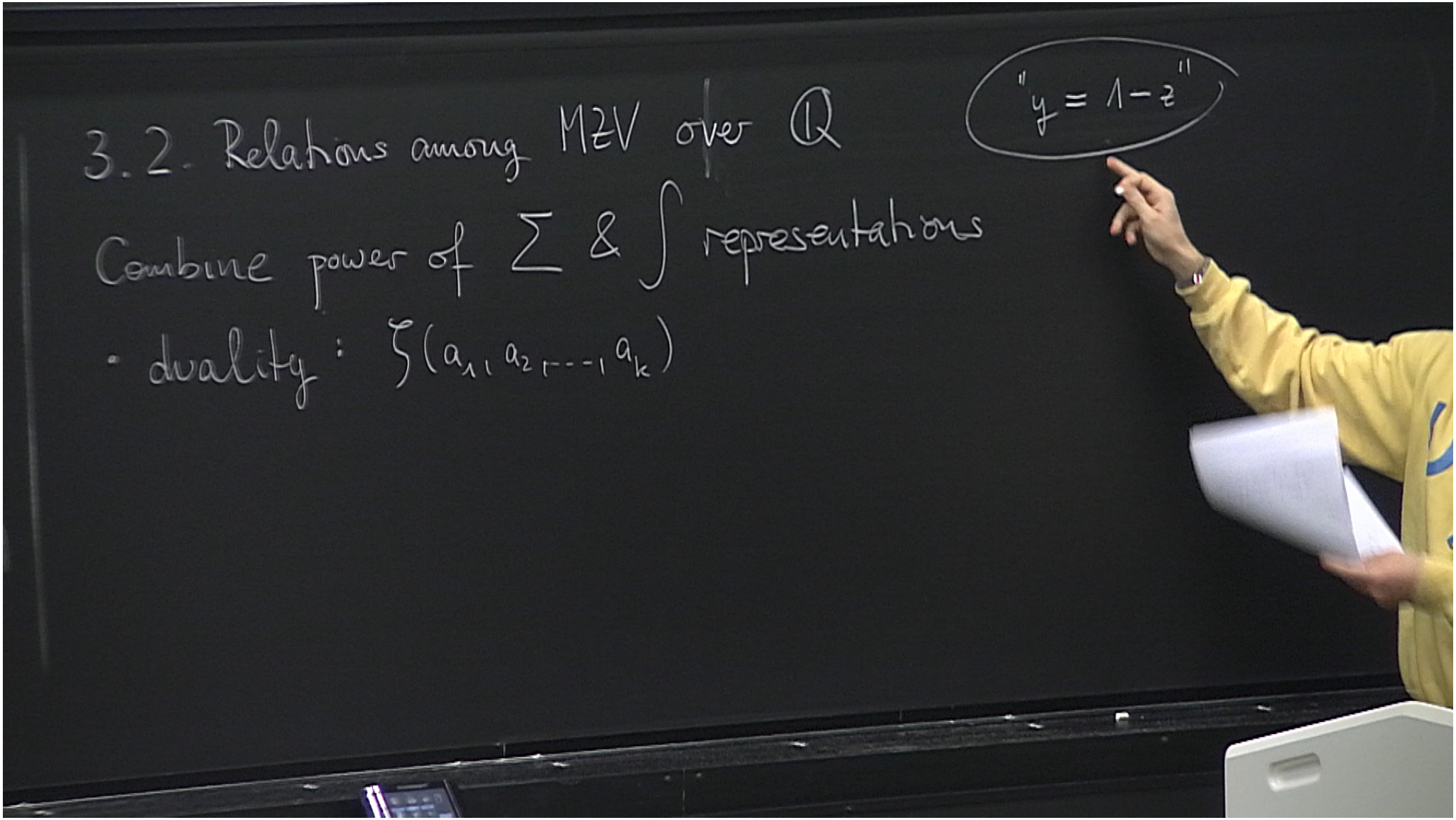
- $\zeta(a_1, \ldots, a_k)$ converges if $a_1 \neq 1$, $a_k \neq 0$
- recover nested sums via [Koutsenich]

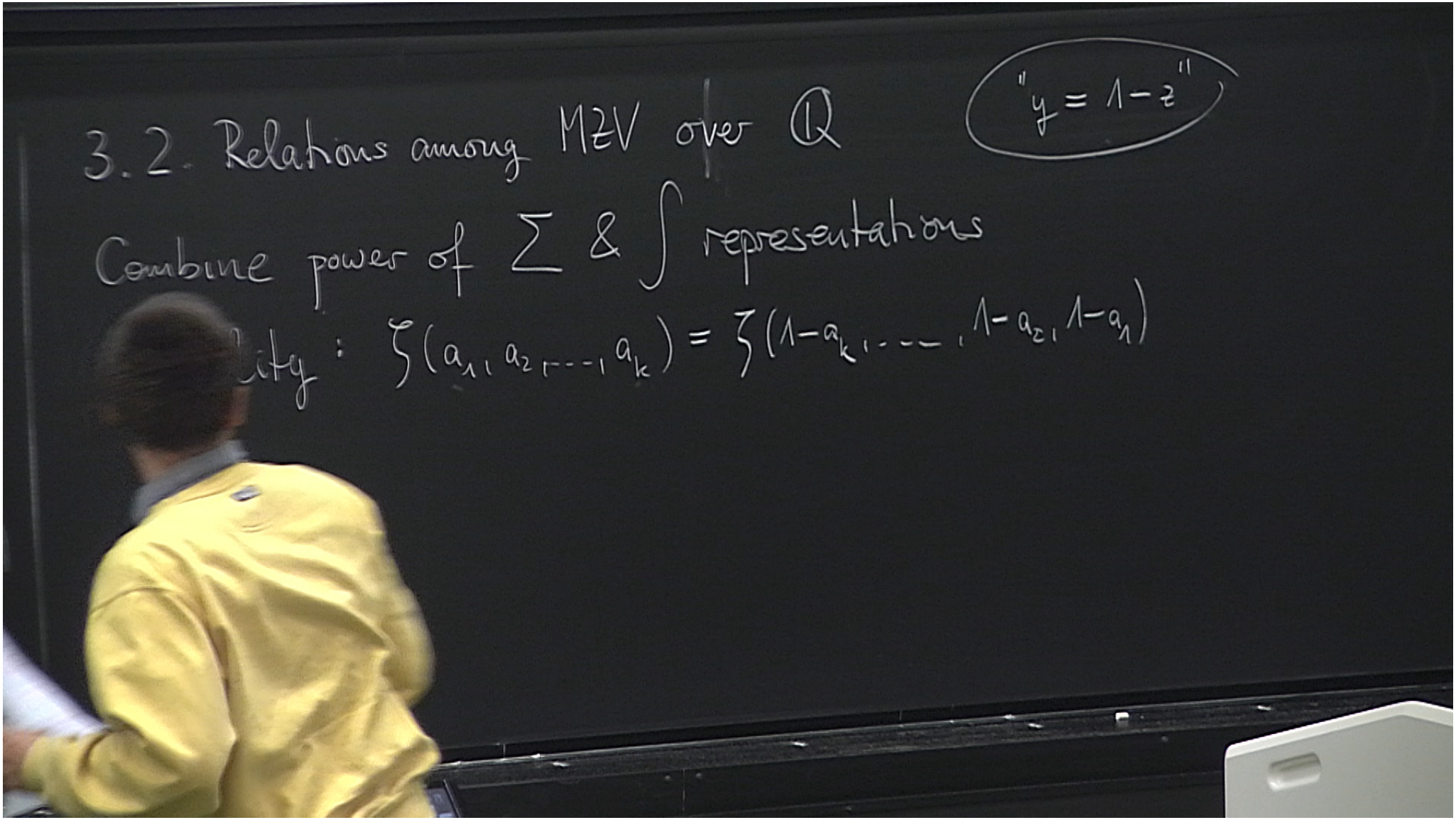
$$\zeta(\underbrace{n_1}, \underbrace{n_2}, \ldots, n_r) = \zeta(\underbrace{1, 0^{n_1-1}}, \underbrace{1, 0^{n_2-1}}, \ldots, 1, 0^{n_r-1})$$

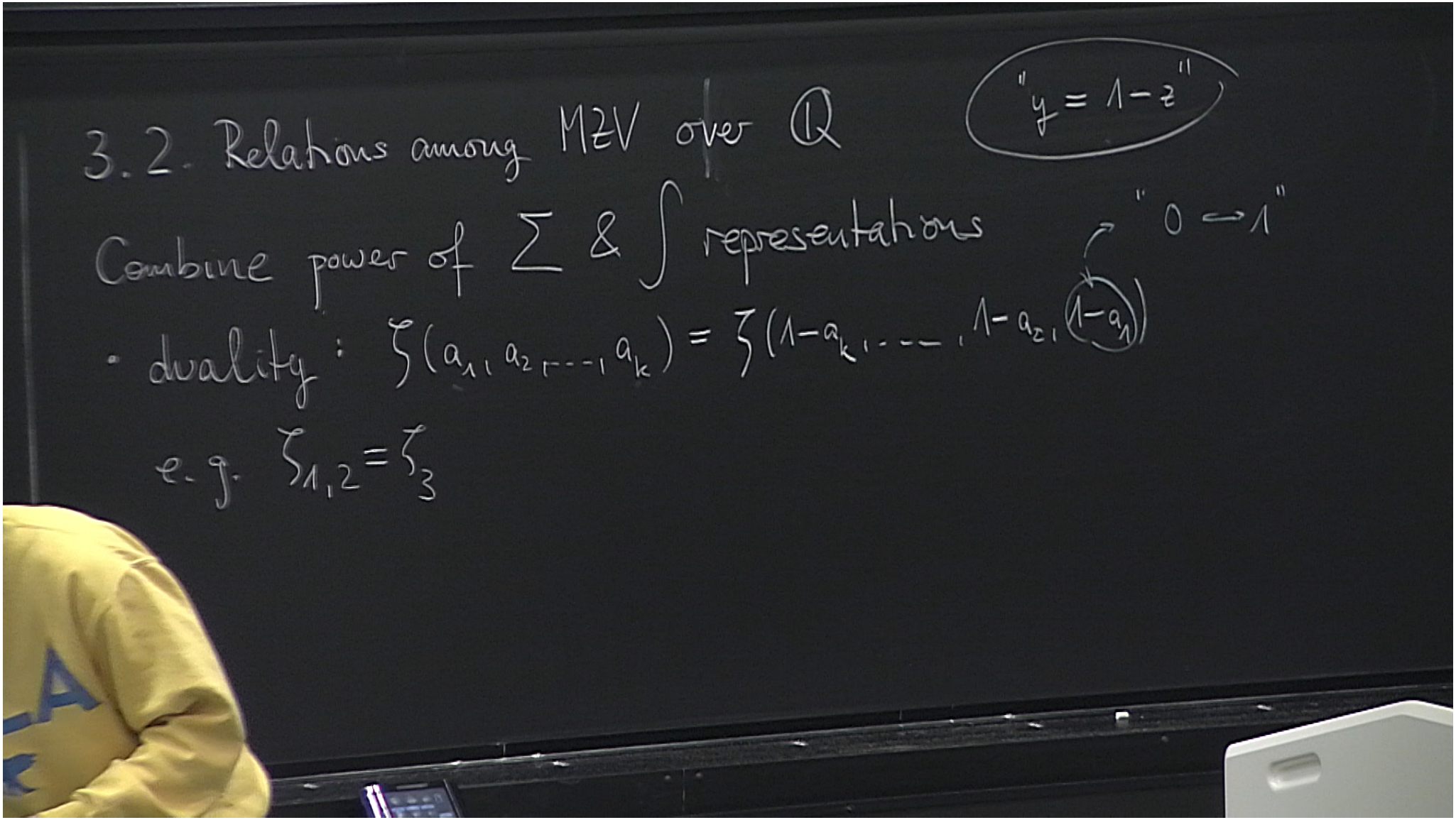
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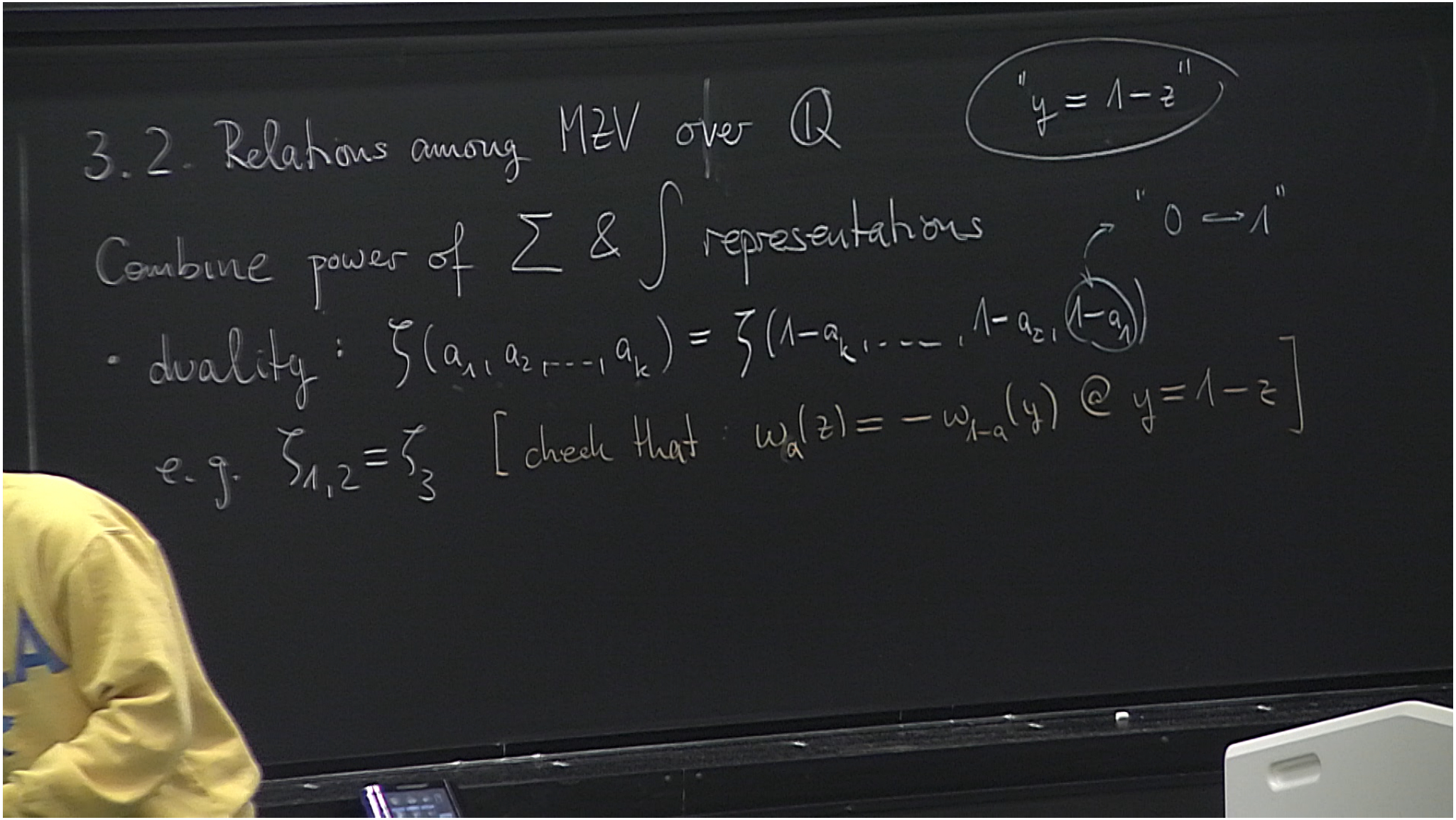
ends with 0 if $n_r \geq 2$

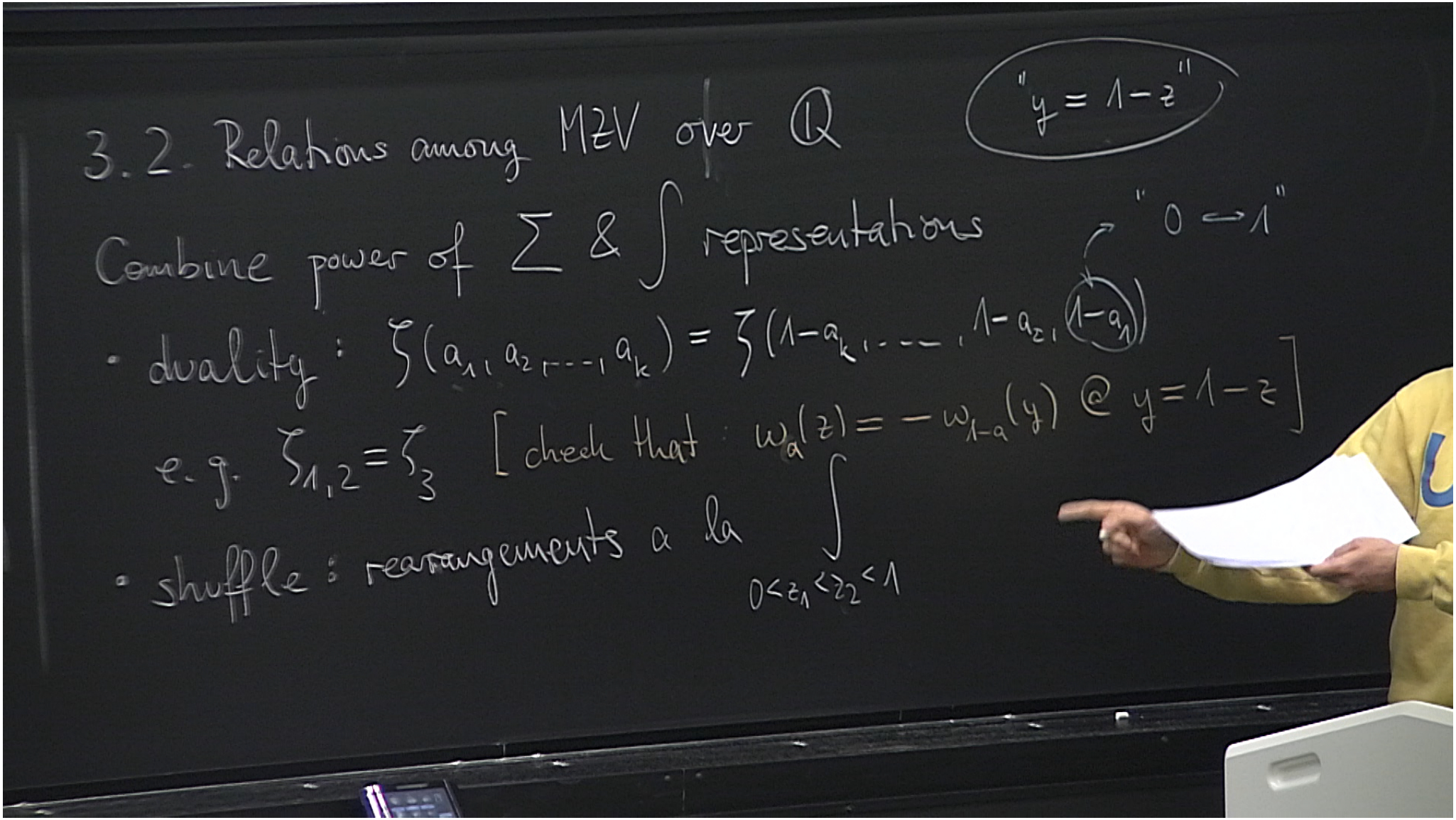
- depth $r \leftrightarrow \#(w_1)$, weight $\sum_{j=1}^{r} n_j \leftrightarrow \#(\text{integrations})$


3.2. Relations among MZV over ℚ
"y = 1 − z"
Combine power of Σ & ∫ representations
• duality : ζ(a1, a2, ..., ak)


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ity : ζ(a_1, a_2, ..., a_k) = ζ(1−a_k, ..., 1−a_2, 1−a_1)


3.2. Relations among MZV over ℚ
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"0 ↔ 1"
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• shuffle: rearrangements a la ∫
0 < z_1 < z_2 < 1

3.2. Relations among MZV over $\mathbb{Q}$

"$y = 1 - z$"

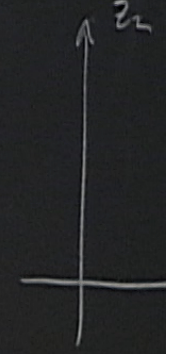
Combine power of $\sum$ & $\int$ representations

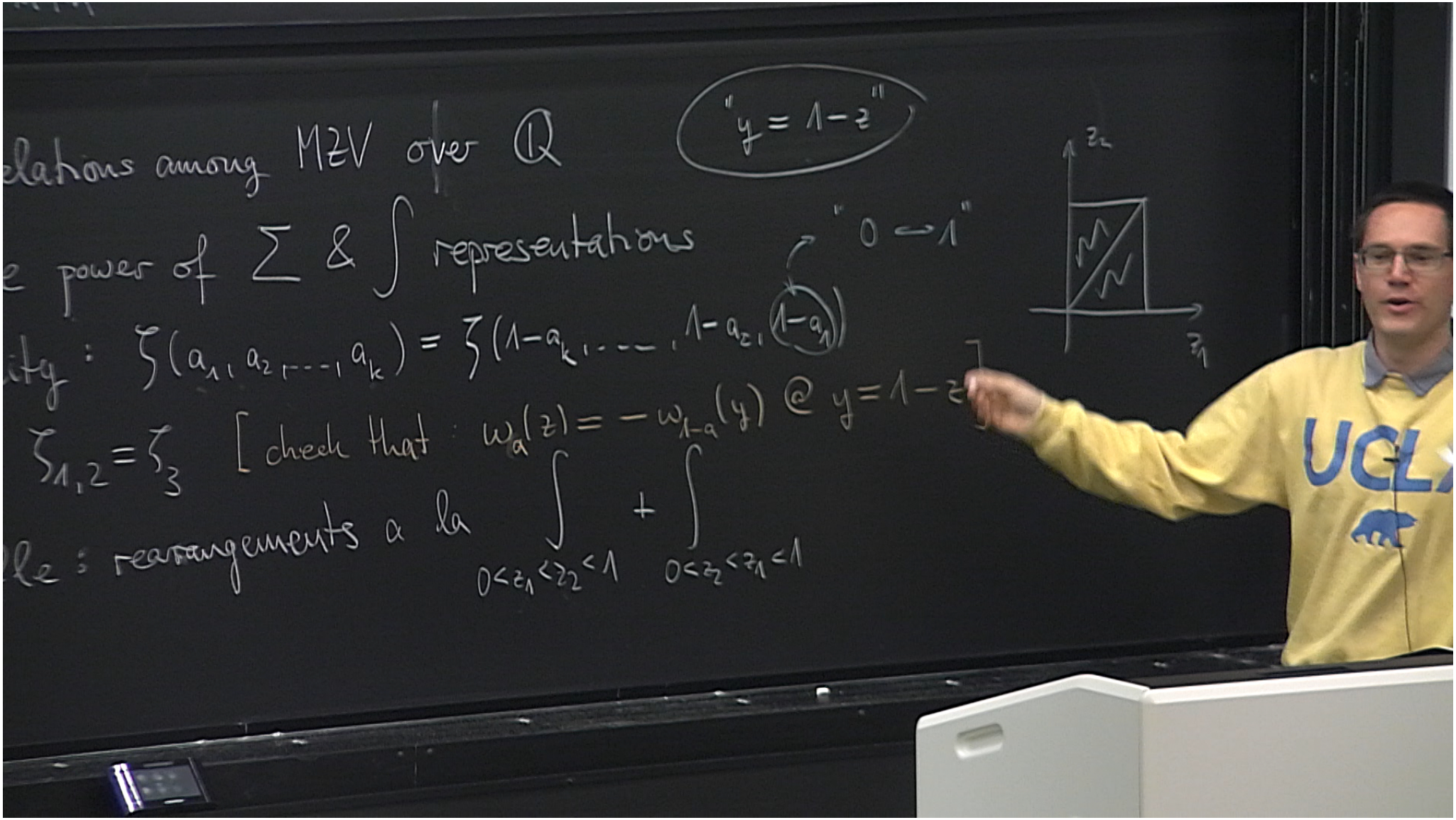
"$0 \leftrightarrow 1$"

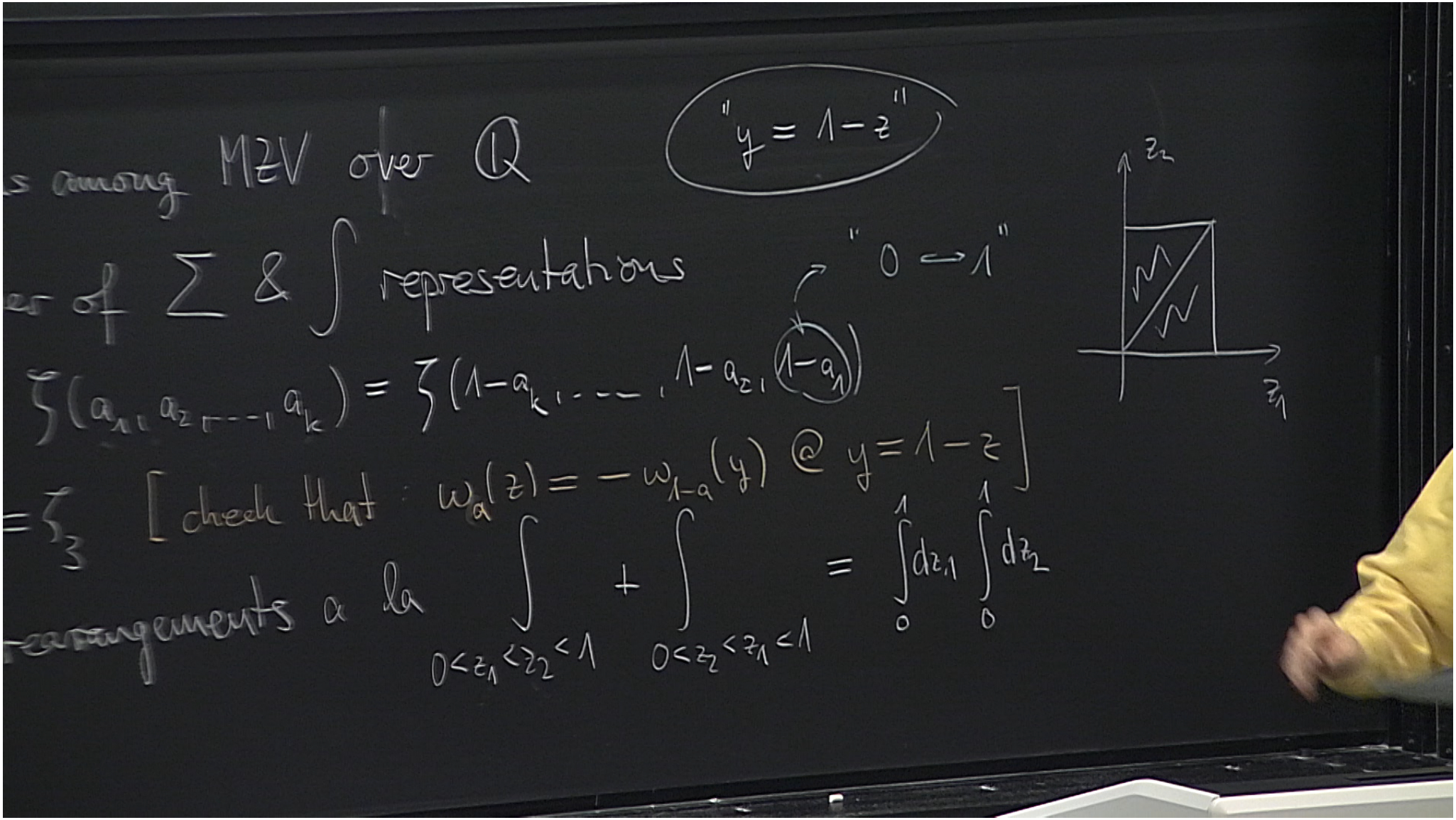
- duality: $\zeta(a_1, a_2, \ldots, a_k) = \zeta(1-a_k, \ldots, 1-a_2, 1-a_1)$

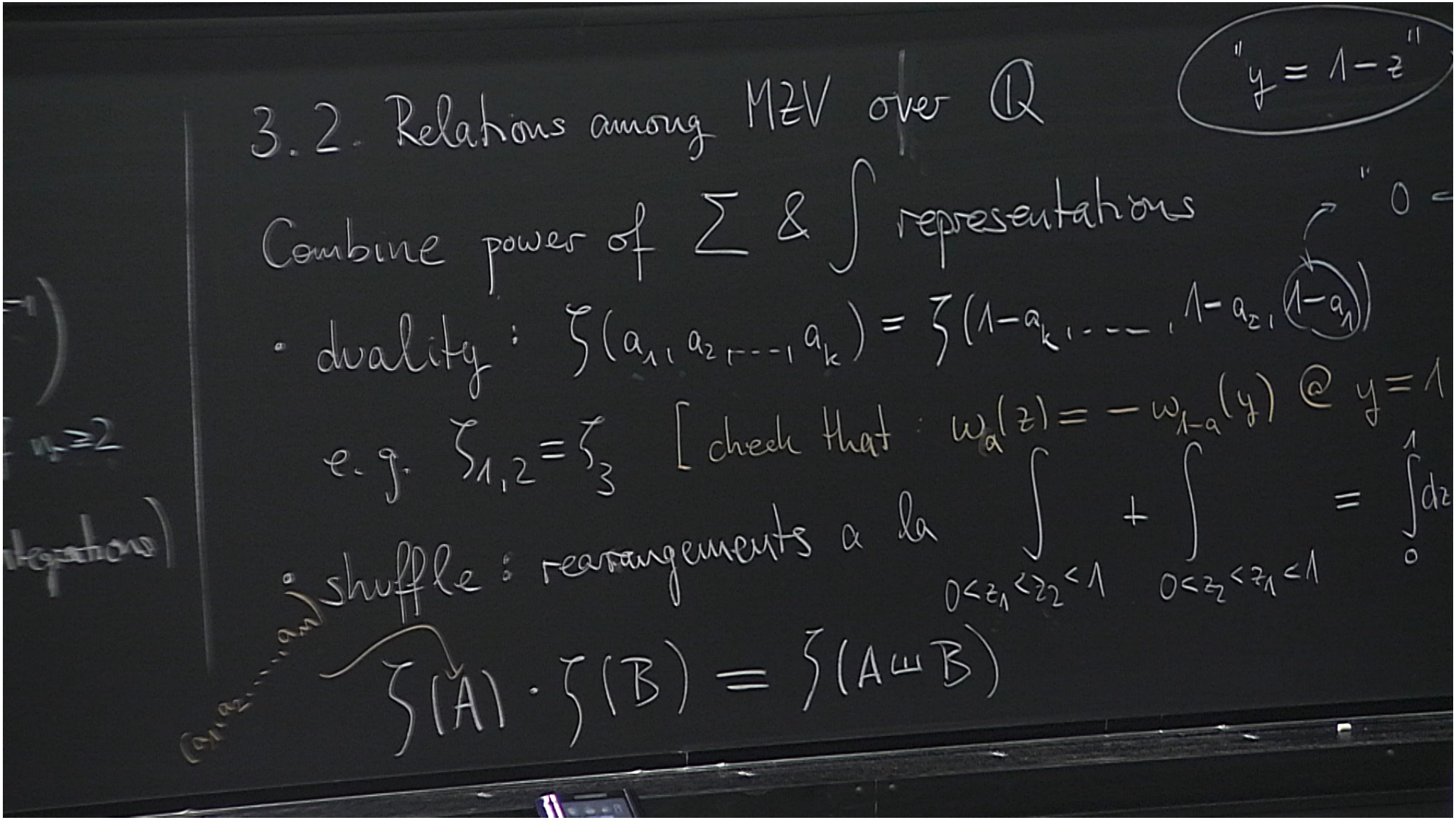
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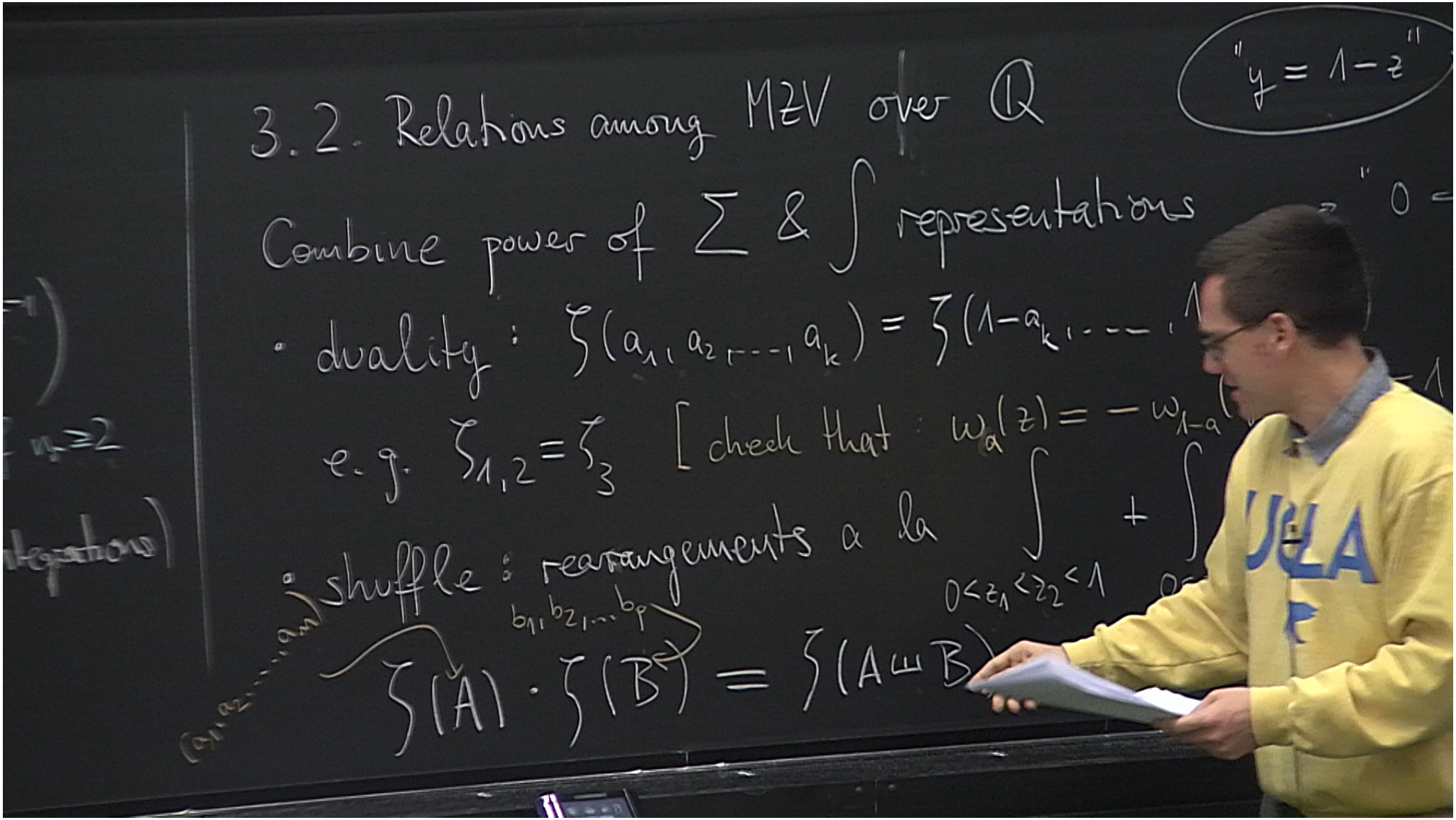
- shuffle: rearrangements a la $\int_{0<z_1<z_2<1} + \int_{0<z_2<z_1<1}$

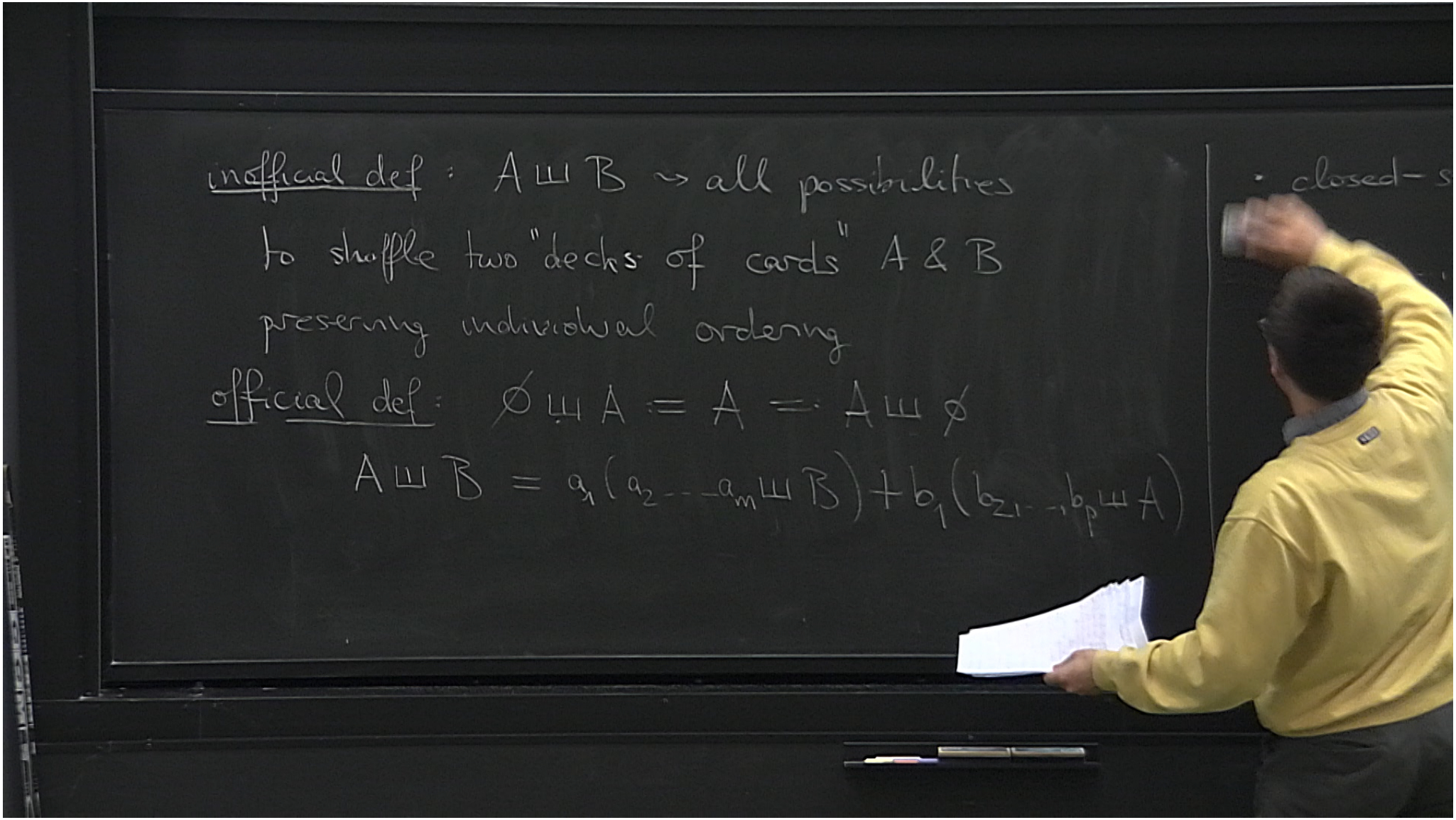



elations among MZV over ℚ
"y = 1 − z"
e power of Σ & ∫ representations
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ity : ζ(a1, a2, ..., ak) = ζ(1−ak, ..., 1−a2, 1−a1)
ζ1,2 = ζ3 [check that ωa(z) = −ω1−a(y) @ y = 1−z
le : rearrangements a la ∫ + ∫
0<z1<z2<1 0<z2<z1<1
z2
z1
UCLA


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∫_{0<z_1<z_2<1} + ∫_{0<z_2<z_1<1} = ∫_0^1 dz_1 ∫_0^1 dz_2
z_2
z_1


"y = 1 − z"
3.2. Relations among MZV over ℚ
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[check that: ωa(z) = −ω1−a(y) @ y = 1
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0<z1<z2<1
0<z2<z1<1
ζ(A) · ζ(B) = ζ(A ⧢ B)


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• shuffle : rearrangements a la ∫ + ∫
0<z1<z2<1
ζ(A) · ζ(B) = ζ(A ⧢ B)


unoffical def : A ⧢ B → all possibilities
to shuffle two "decks of cards" A & B
preserving individual ordering
official def: ∅ ⧢ A := A =: A ⧢ ∅
A ⧢ B = a1(a2 - - -am ⧢ B) + b1(b2 - - -, bp ⧢ A)
• closed-s

assume: $\zeta(P+Q) = \zeta(P) + \zeta(Q)$

formal sum of ordered sets

4pt,

$$\mathcal{A}(\beta|\rho) = -\frac{\sin(\pi s_{12})}{2\pi\alpha'} Z(1,2,3,4|\beta)\, Z(1,2,4,3|\rho)$$

assume: $\zeta(P+Q) = \zeta(P) + \zeta(Q)$

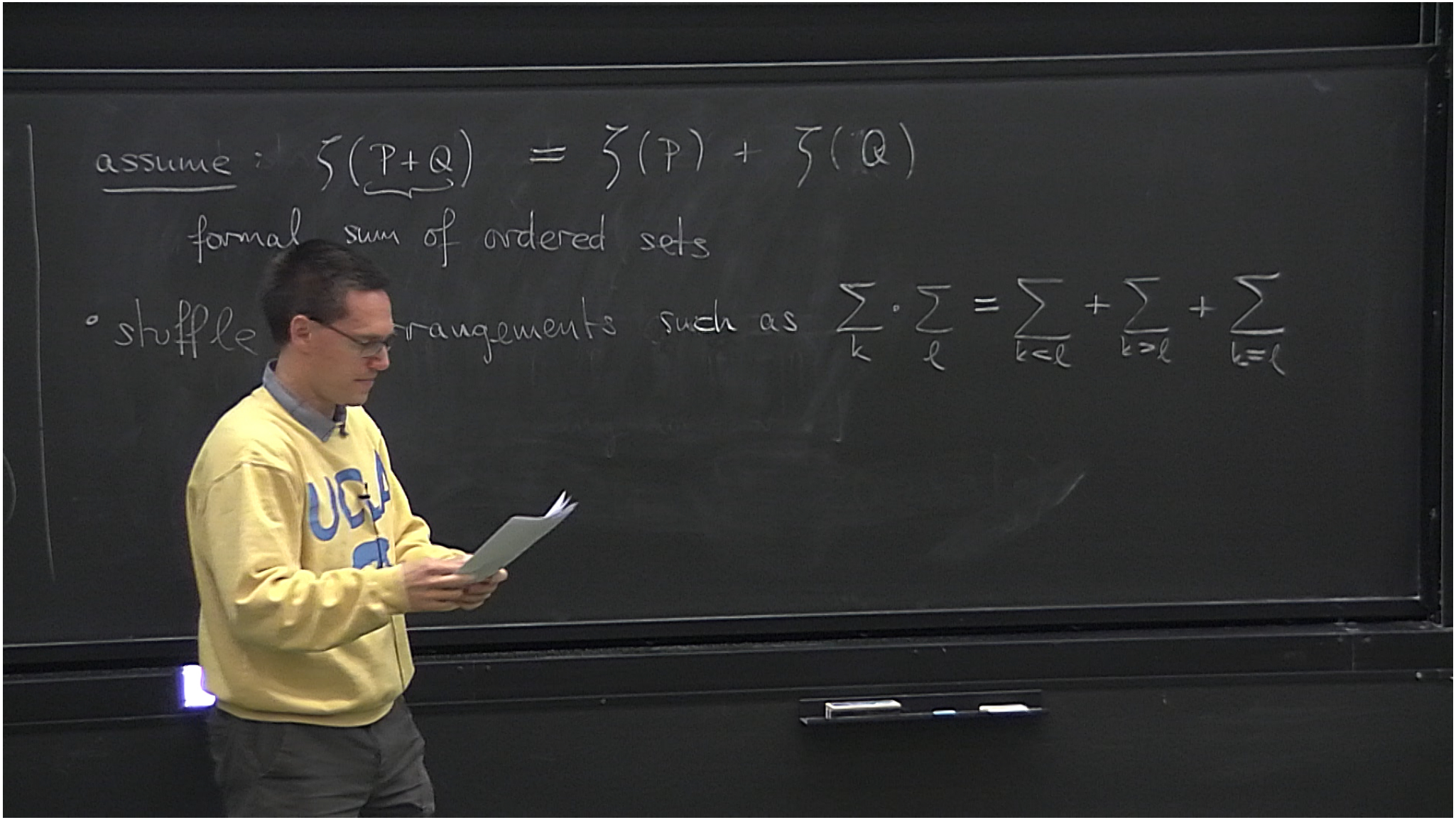
formal sum of ordered sets

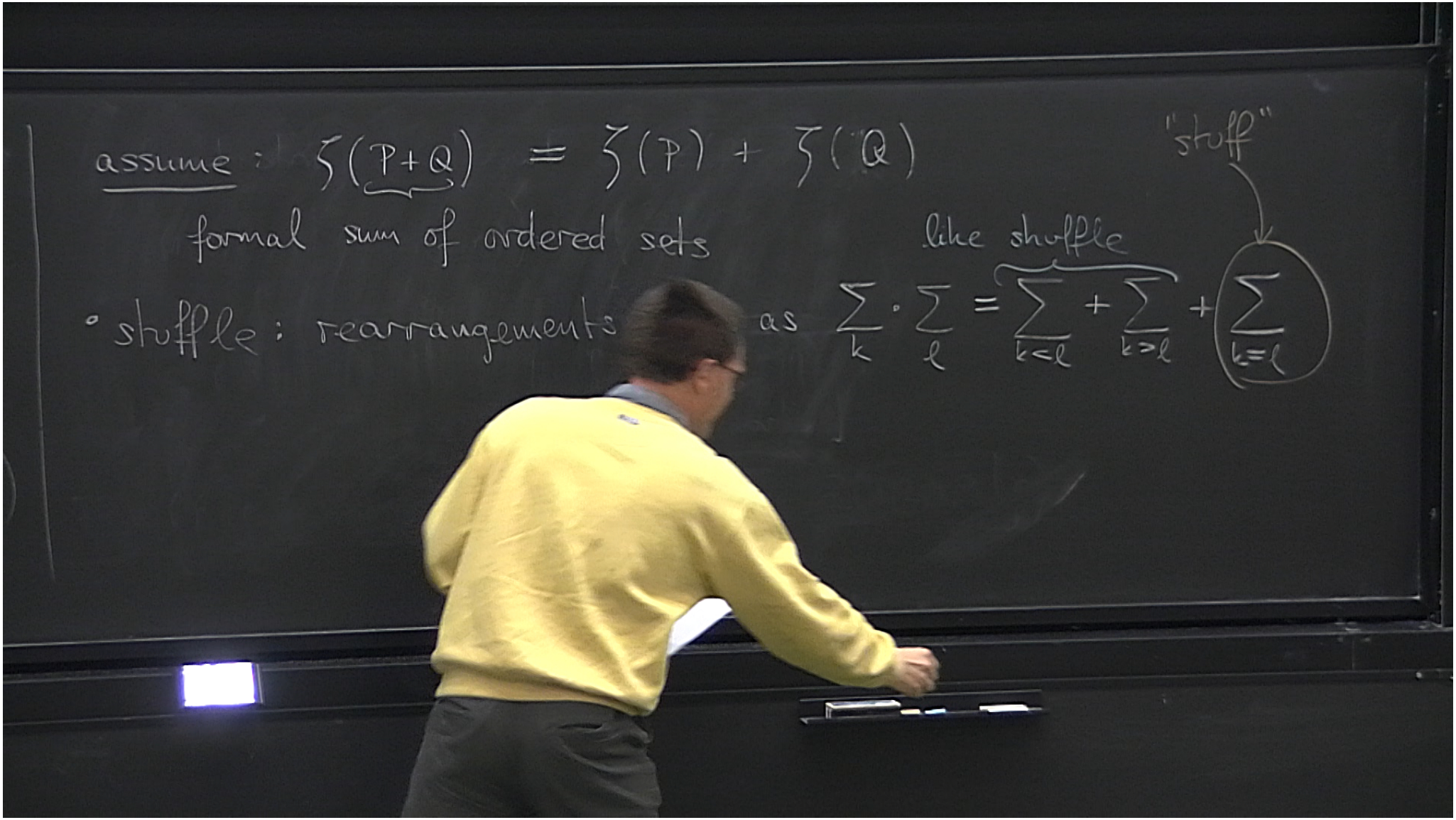
- stuffle

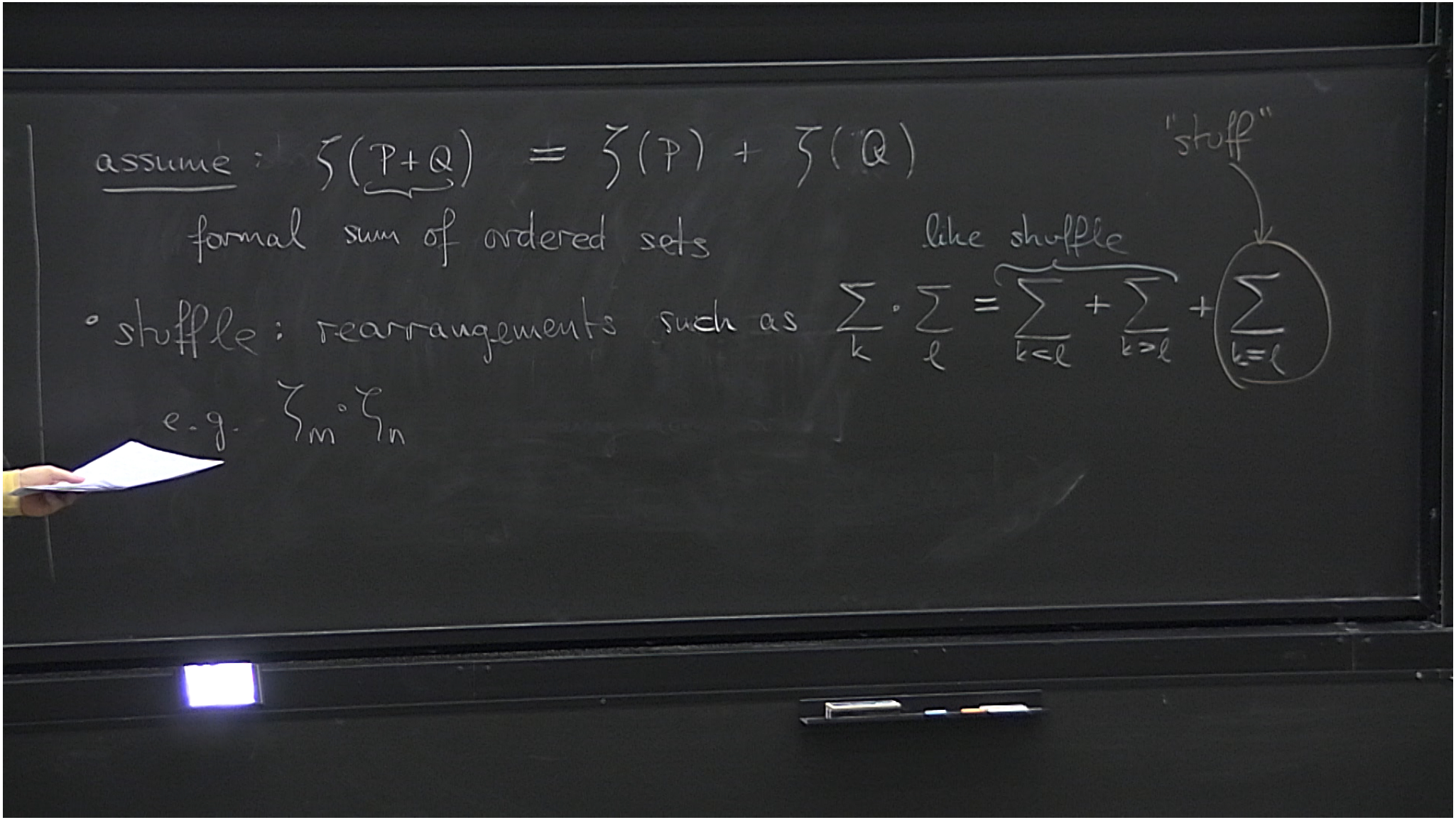
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formal sum of ordered sets

- stuffle: rearrangements such as $\sum_k \cdot \sum_\ell$


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formal sum of ordered sets
"stuff"
like shuffle
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e.g. $\zeta_m \cdot \zeta_n$

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formal sum of ordered sets

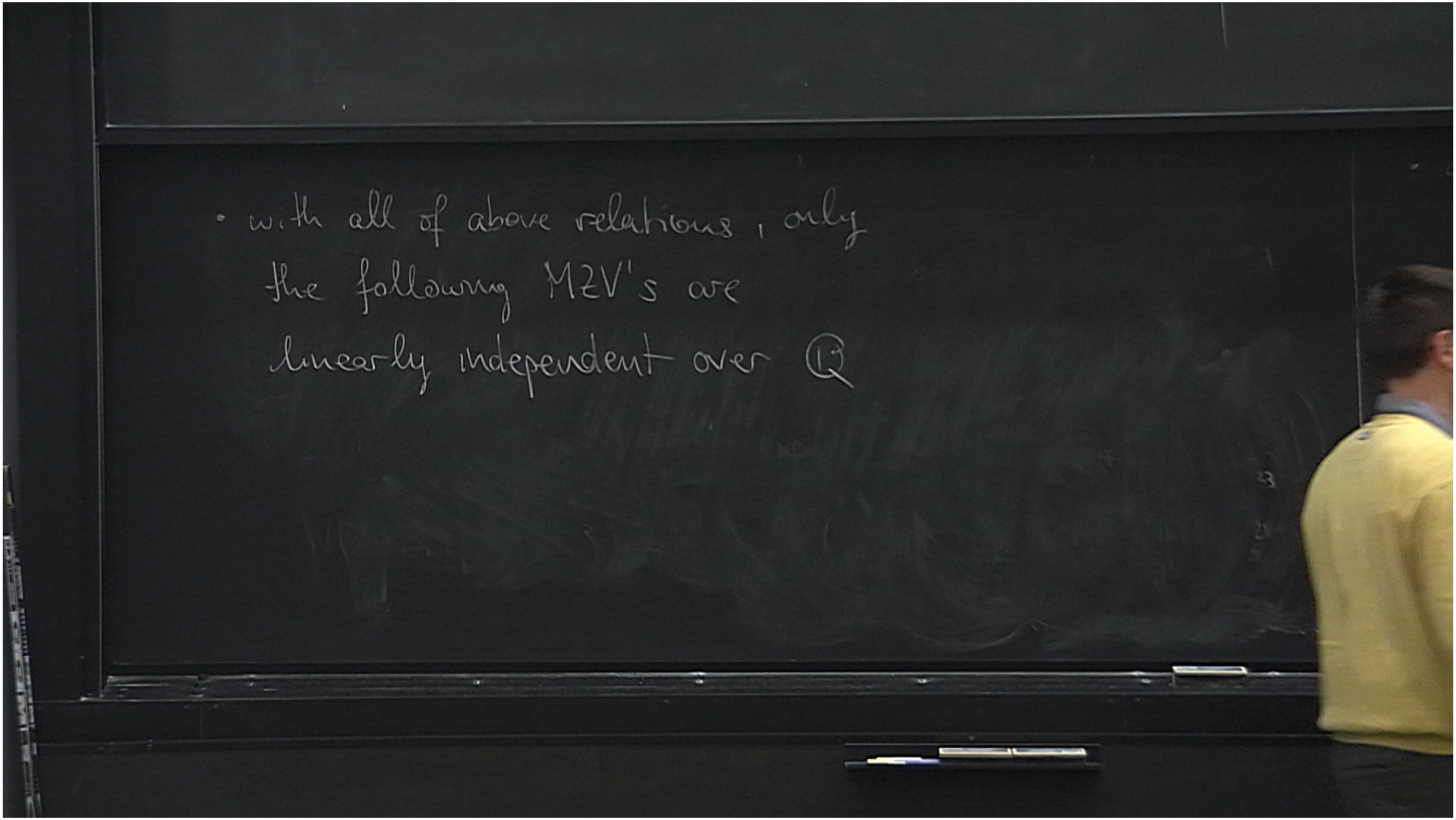
"stuff"

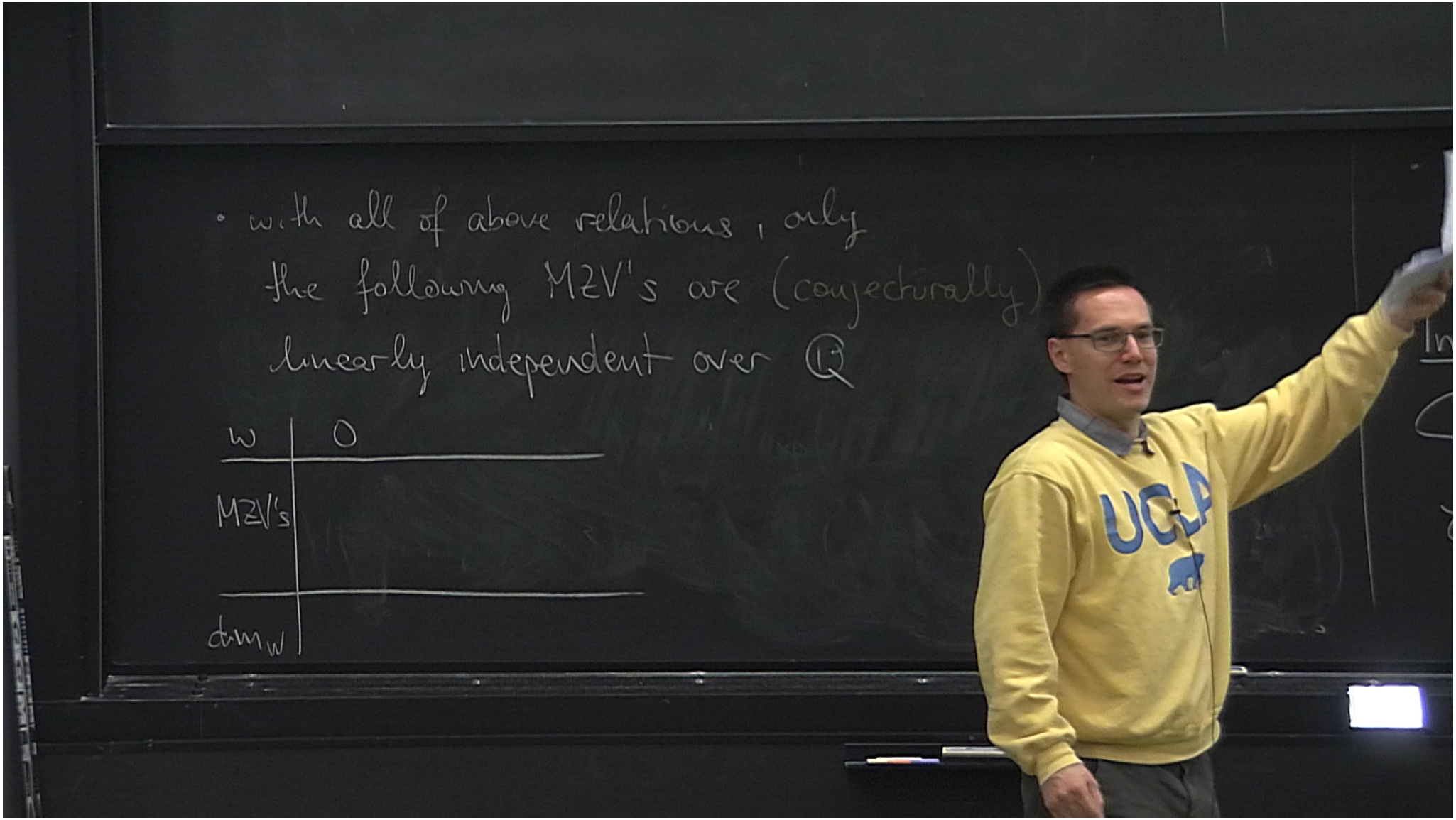
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e.g. $\zeta_m \cdot \zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}$

[check that]


• with all of above relations, only
the following MZV's are
linearly independent over ℚ


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the following MZV's are (conjecturally)
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w | 0
MZV's
dim_w

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[check that]

- all of duality, shuffle & stuffle preserve the weight

- at $(n=5)$ pt., also have depth $r \geq 2$ MZV's

(e.g. $\zeta_{3,5}$ @ $Z_{5pt}|_{(\alpha')^8}$ & $W_{5pt}|_{(\alpha')^{11}}$: $\zeta_{3,5}$

Intuition: why MZV's in $\int_{0<z_1<z_2<\cdots<z_n<1} dz_2 \, a$

↳ can map MZV to iterated int's

$\zeta(a_1, a_2, \ldots, a_k) := \int_{0<z_1<z_2<\cdots<z_k<1} \omega_{a_1}(z_1)\, \omega_{a_2}(z_2) \cdots$

$z_{n-1} = 1$

$z_n \to \infty$

$(z) = \frac{dz}{1-z}$

e.g. $\zeta_m \cdot \zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}$

[check that]

- all of duality, shuffle & stuffle preserve the weight!

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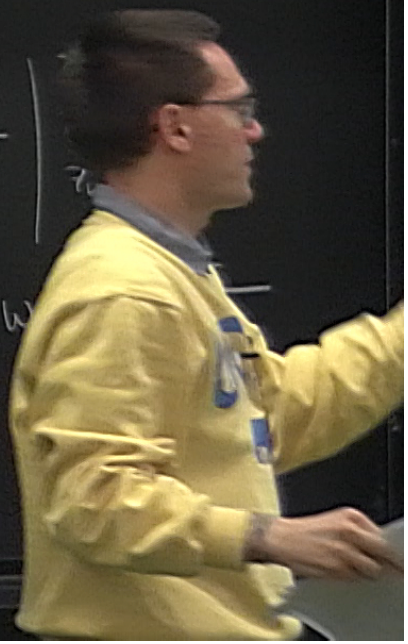
(e.g. $\zeta_{3,5}$ @ $Z_{5pt}|_{(\alpha')^8}$ & $W_{5pt}|_{(\alpha')^{11}}$: $\zeta_{3,5,3}$)

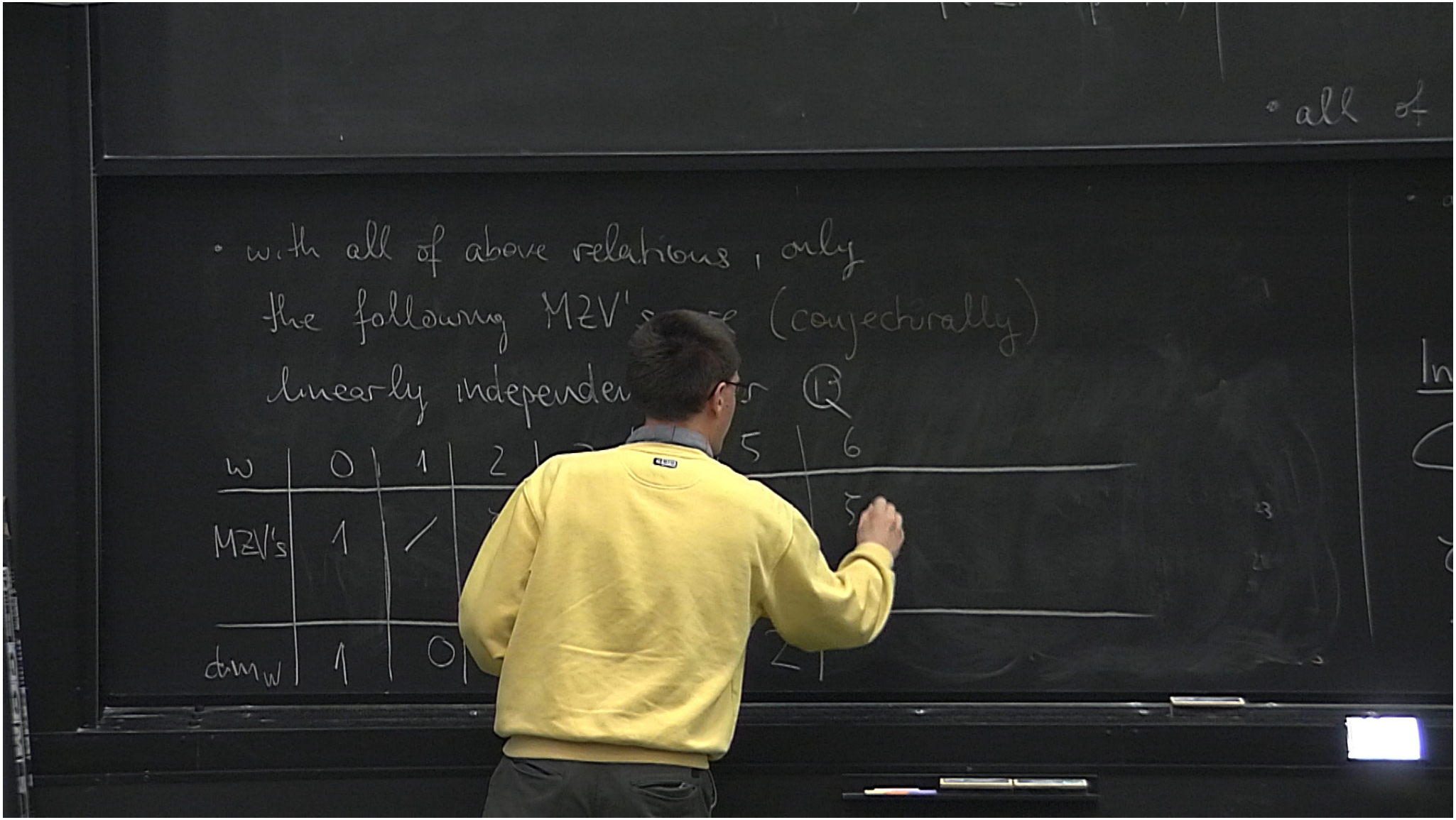
Intuition: why MZV's in $z(1,2,\ldots,1,*)$ $\sim \int_{0<z_1<z_2<\ldots<z_{n-2}<1} dz_2\, dz_3 \ldots dz_{n-2}$

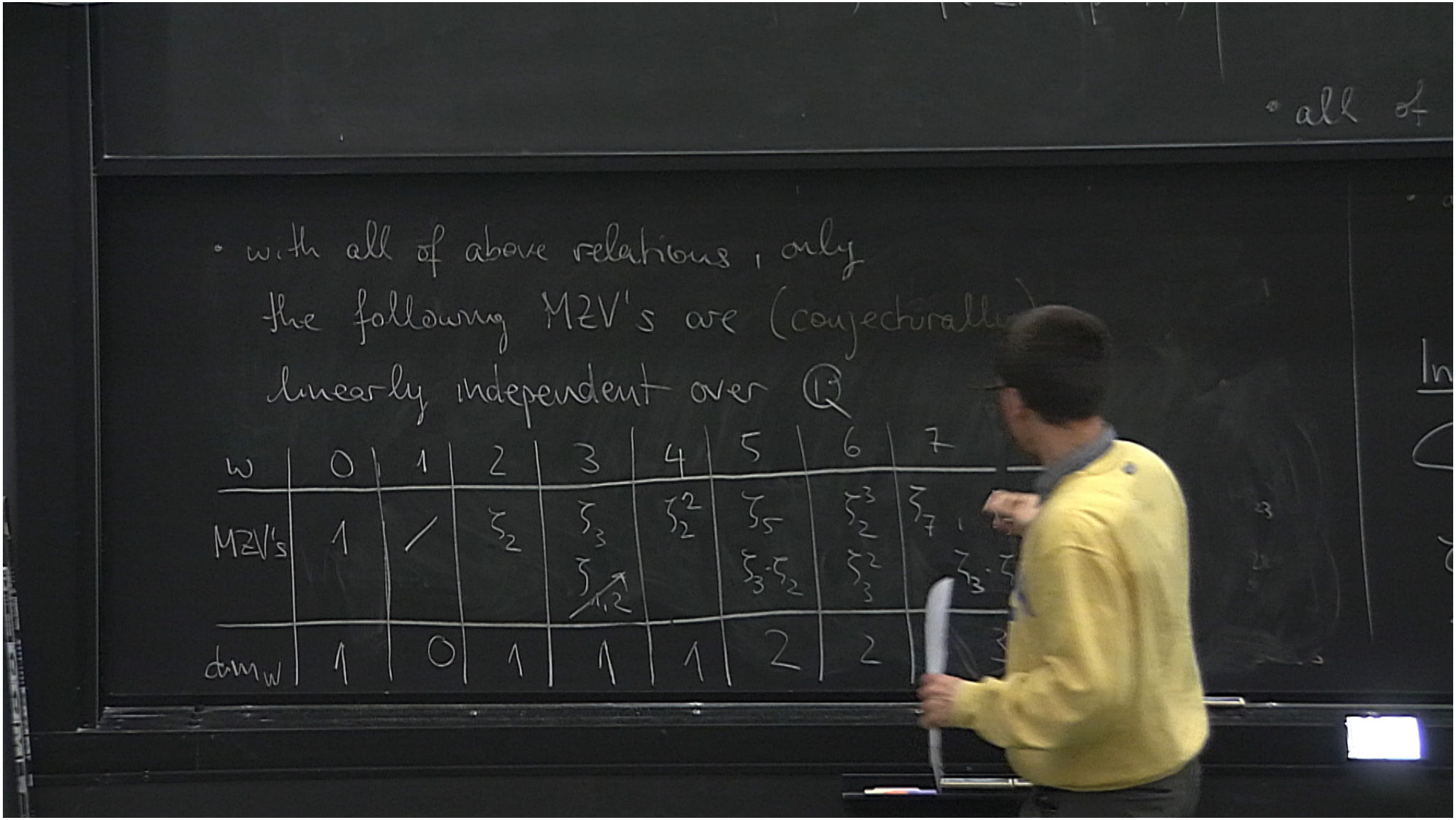
↪ can map MZV to iterated int's

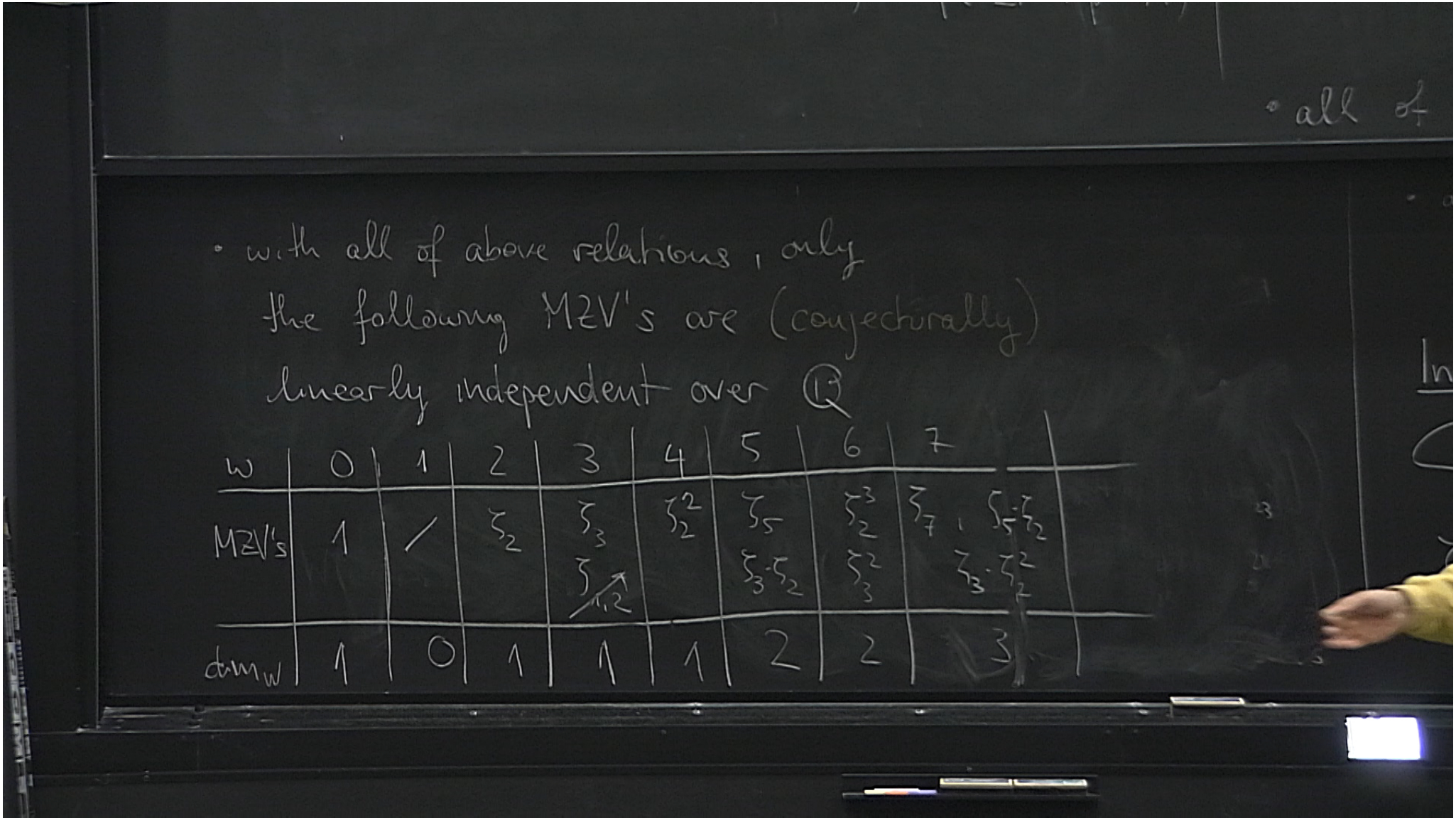
$\omega_0(z) = \frac{dz}{z}$; ω

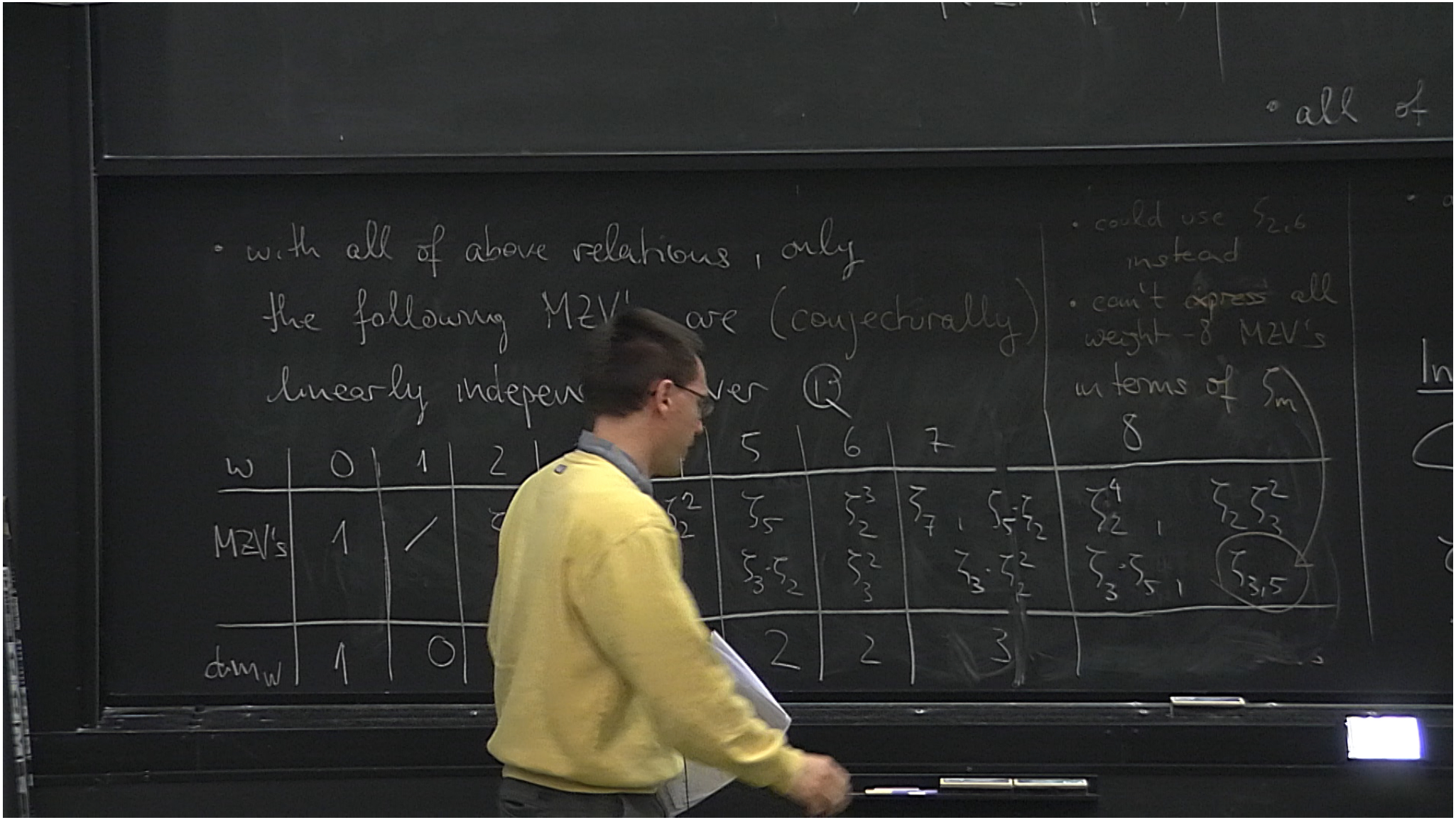
$\zeta(a_1, a_2, \ldots, a_k) := \int_{0<z_1<z_2<\ldots<z_k<1} \omega_{a_1}(z_1)\, \omega_{a_2}(z_2) \cdots \omega_{a_k}(z_k)$

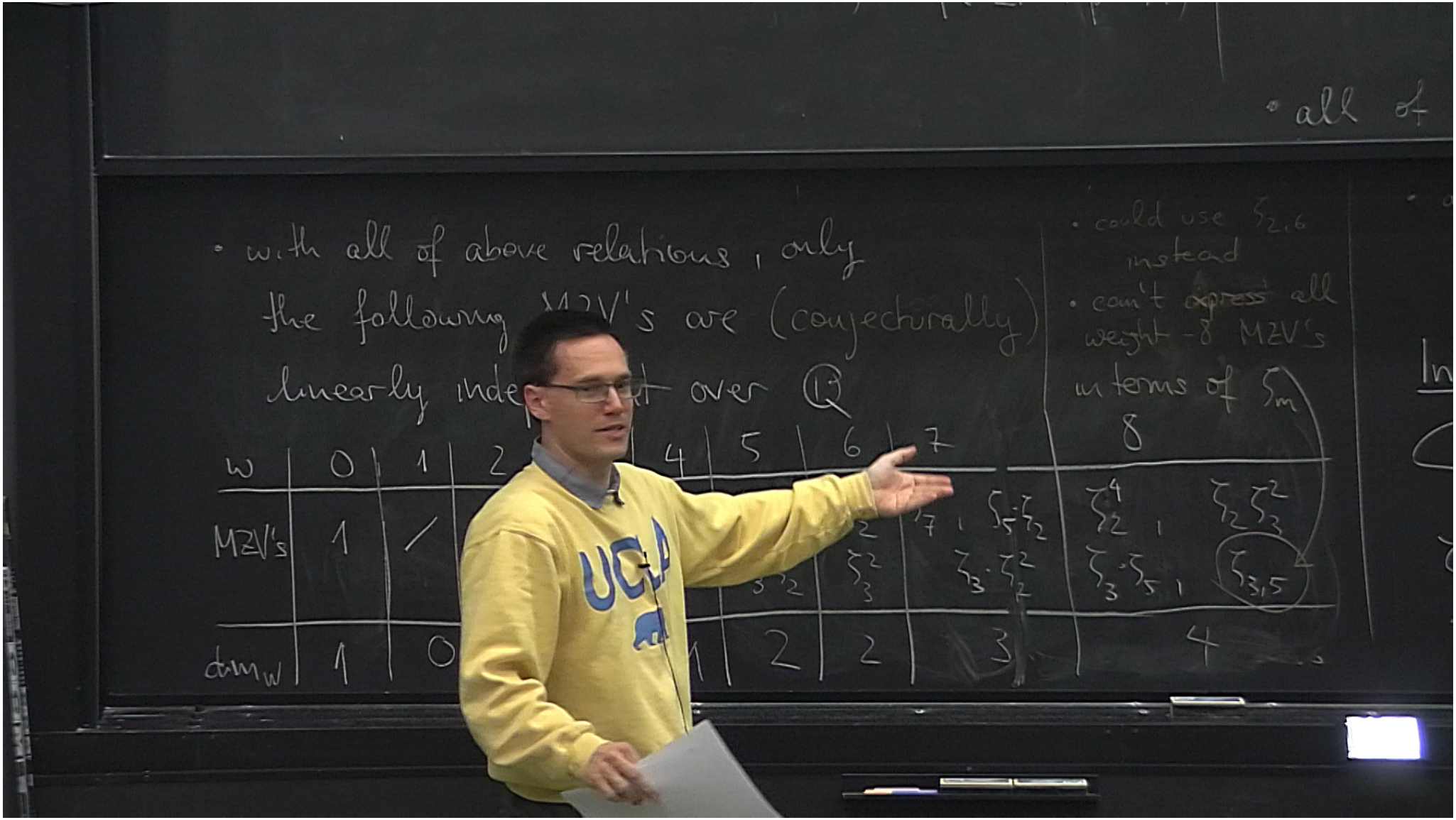


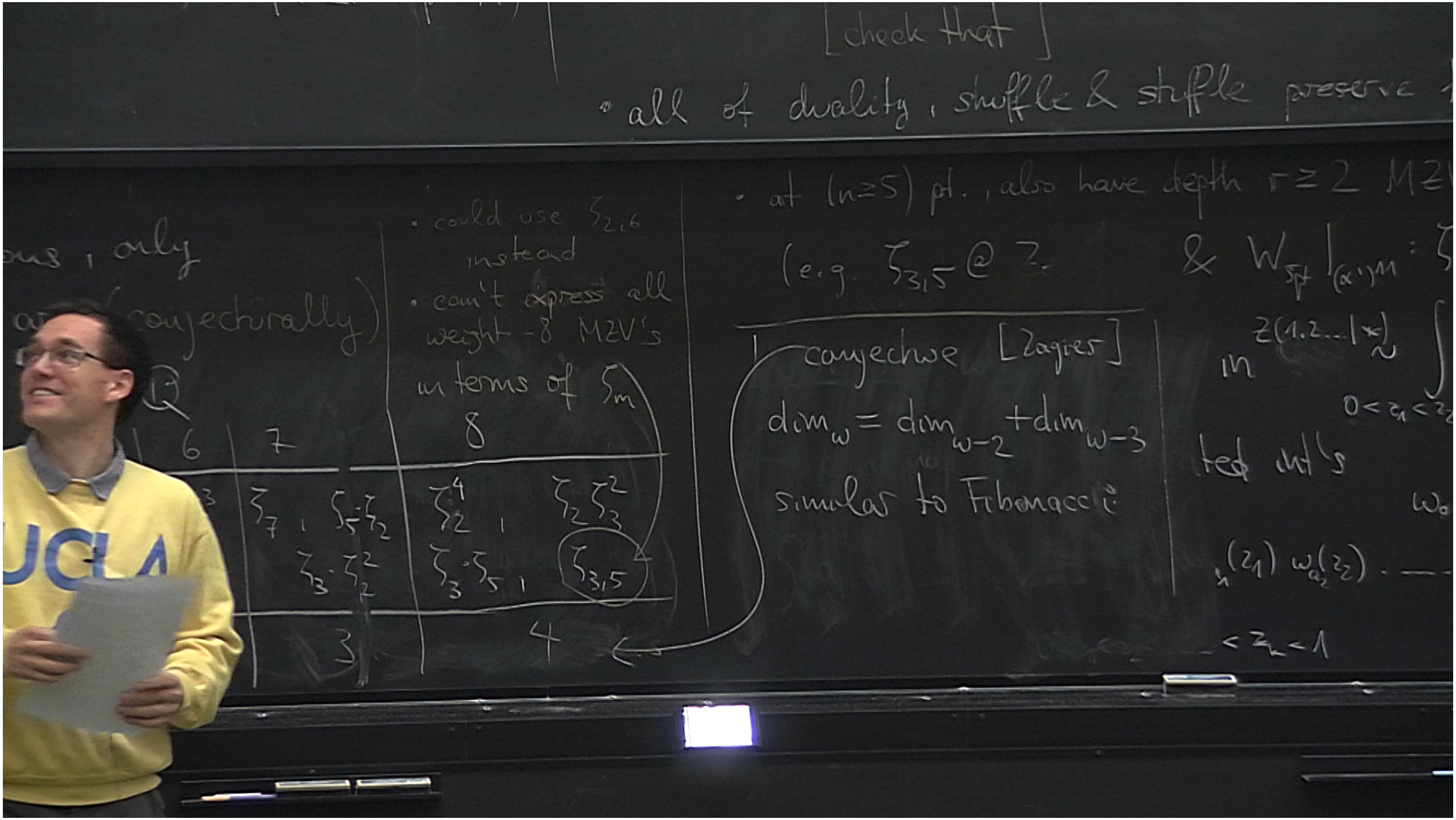

• all of
• with all of above relations, only
the following MZV's
linearly independe
w	0	1	2	5	6
MZV's	1	/			
dim_w	1	0			


• all of
• with all of above relations, only
the following MZV's are (conjecturally)
linearly independent over ℚ
w | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7
MZV's | 1 | / | ζ_2 | ζ_3 | ζ_2^2 | ζ_5 | ζ_2^3 | ζ_7,
| | | | $\zeta_{1,2}$ | | $\zeta_3 \cdot \zeta_2$ | ζ_3^2 | $\zeta_3 \cdot$
$\dim_w$ | 1 | 0 | 1 | 1 | 1 | 2 | 2 |
In


• all of
• with all of above relations, only
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w | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7
MZV's | 1 | / | ζ_2 | ζ_3 | ζ_2^2 | ζ_5 | ζ_2^3 | ζ_7, $\zeta_5\cdot\zeta_2$
| | | | $\zeta_{1,2}$ | | $\zeta_3\cdot\zeta_2$ | ζ_3^2 | $\zeta_3\cdot\zeta_2^2$
$\dim_w$ | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3


• all of
• with all of above relations, only
the following MZV's are (conjecturally)
linearly indepen ver ℚ
• could use ζ2,6 instead
• can't express all weight-8 MZV's in terms of ζm
8
w | 0 | 1 | 2 | 5 | 6 | 7
MZV's | 1 | / | ζ5 | ζ2^3 | ζ7, ζ5·ζ2 | ζ2^4, ζ2^2·ζ3
ζ3·ζ2 | ζ3^2 | ζ3·ζ2^2 | ζ3·ζ5, ζ3,5
dim_w | 1 | 0 | 2 | 2 | 3


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linearly
over ℚ
• could use ζ2,6 instead
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w | 0 | 1 | 2 | 4 | 5 | 6 | 7 | 8
MZV's | 1 | / | ζ2 | ζ3 | ζ7, ζ5·ζ2, ζ3·ζ2² | ζ2⁴, ζ2ζ3, ζ3·ζ5, ζ3,5
dim_w | 1 | 0 | 1 | 2 | 2 | 3 | 4


[check that
• all of duality, shuffle & stuffle preserve
• could use $\zeta_{2,6}$ instead
• can't express all weight-8 MZV's in terms of ζ_m
8
6
7
ζ_7, $\zeta_5\zeta_2$
$\zeta_3\cdot\zeta_2^2$
ζ_2^4, $\zeta_2\zeta_3^2$
$\zeta_3\cdot\zeta_5$, $\zeta_{3,5}$
3
4
• at $(n\geq 5)$ pl., also have depth $r\geq 2$ MZ
(e.g. $\zeta_{3,5}$ @ 7
conjecture [Zagier]
$\dim_w = \dim_{w-2} + \dim_{w-3}$
similar to Fibonacci
& $W_{5,7}|_{(\alpha)_M}$
$0<z_1<z_2$
$0<z_k<1$