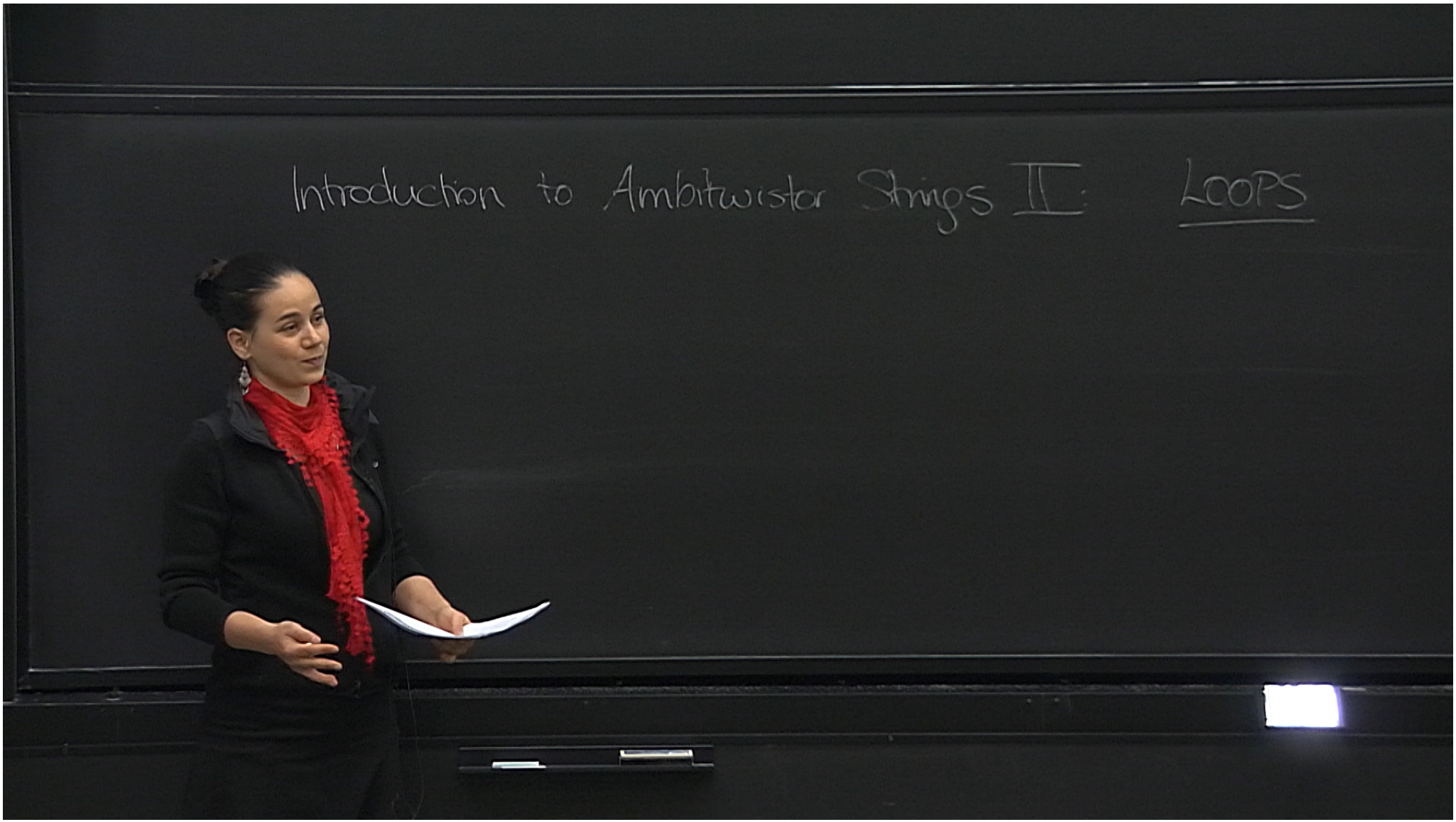
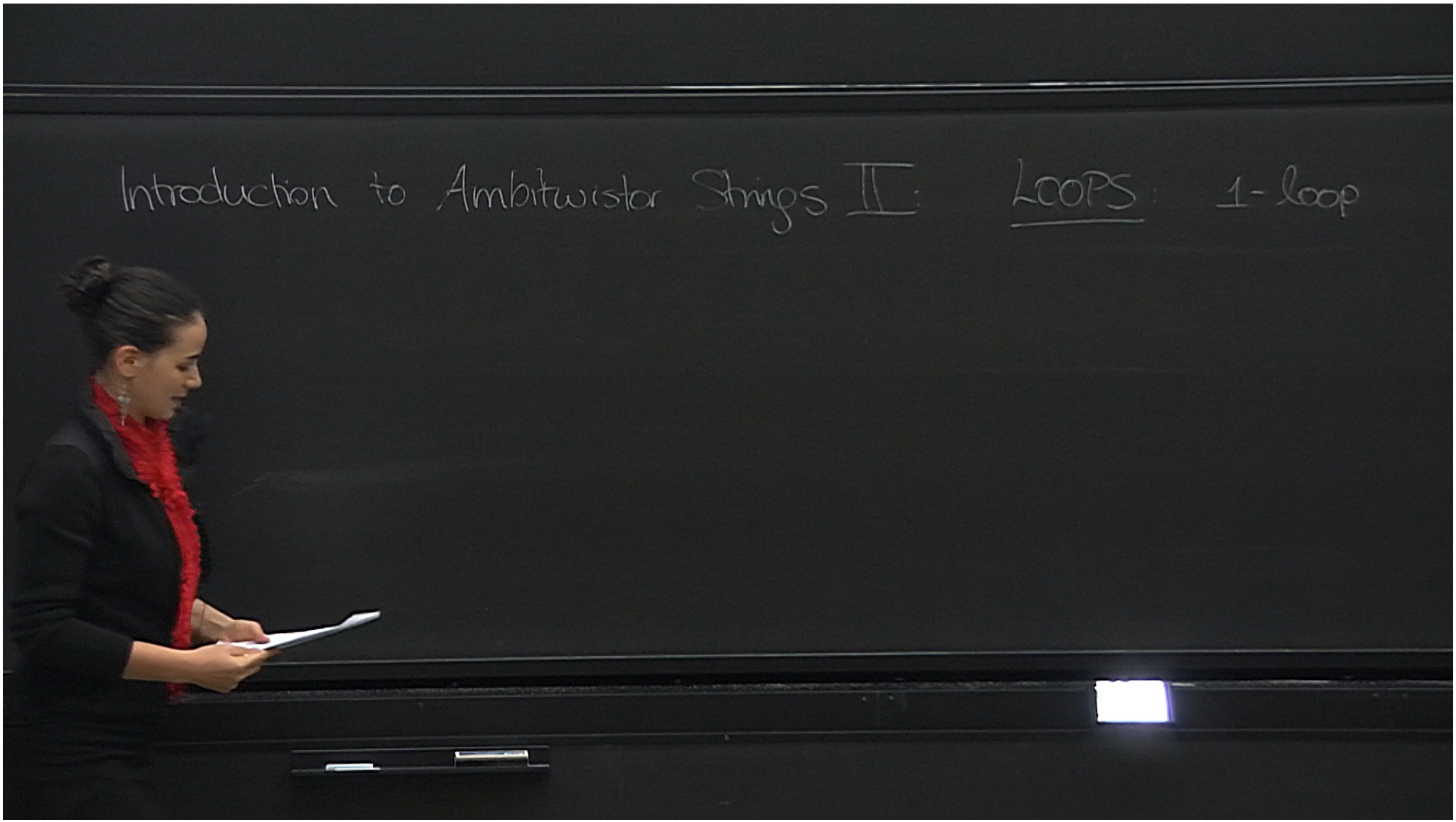
Title: PSI 2017/2018 - Scattering Amplitudes in QFT & String Theory - Lecture 8

Date: Apr 18, 2018 11:30 AM

URL: http://pirsa.org/18040042

Abstract:


Introduction to Ambitwistor Strings II: LOOPS


Introduction to Ambitwistor Strings II: LOOPS: 1-loop

# Introduction to Ambitwistor Strings II: <u>LOOPS</u>: 1-loop

$$\mathcal{M}_n = \sum_{\text{topol}} \int \frac{\mathcal{D}[X, P, \psi, \tilde{\psi} \ldots]^{e, \tilde{e}}}{\mathrm{vol}(\mathrm{Sym})}$$

Action

$$S_A = \frac{1}{2\pi}\int P\cdot\bar\partial X - \frac{\tilde e}{2}P^2 + \frac{1}{2}\psi\cdot\bar\partial\psi - \chi P\cdot\psi + \frac{1}{2}\tilde\psi\cdot\bar\partial\tilde\psi - \tilde\chi P\cdot\tilde\psi - e\left(P\cdot\partial X + \frac{1}{2}\psi\cdot\partial\psi + \frac{1}{2}\tilde\psi\cdot\partial\tilde\psi\right)$$

$X : \Sigma \to M$

$P_\mu \in \Omega^0(\Sigma, K_\Sigma)$

$\psi_\mu, \tilde\psi_\mu \in \Pi\Omega^0(\Sigma, K_\Sigma^{1/2})$

$e, \tilde e \in \Omega^{0,1}(\Sigma, T_\Sigma)$

$\chi, \tilde\chi \in \Pi\Omega^{0,1}(\Sigma, T_\Sigma^{1/2})$

Symmetries
- diffeo
- "Ambitwistor", $\delta_\alpha \tilde e = \bar\partial\alpha$
- fermionic

BRST quantization

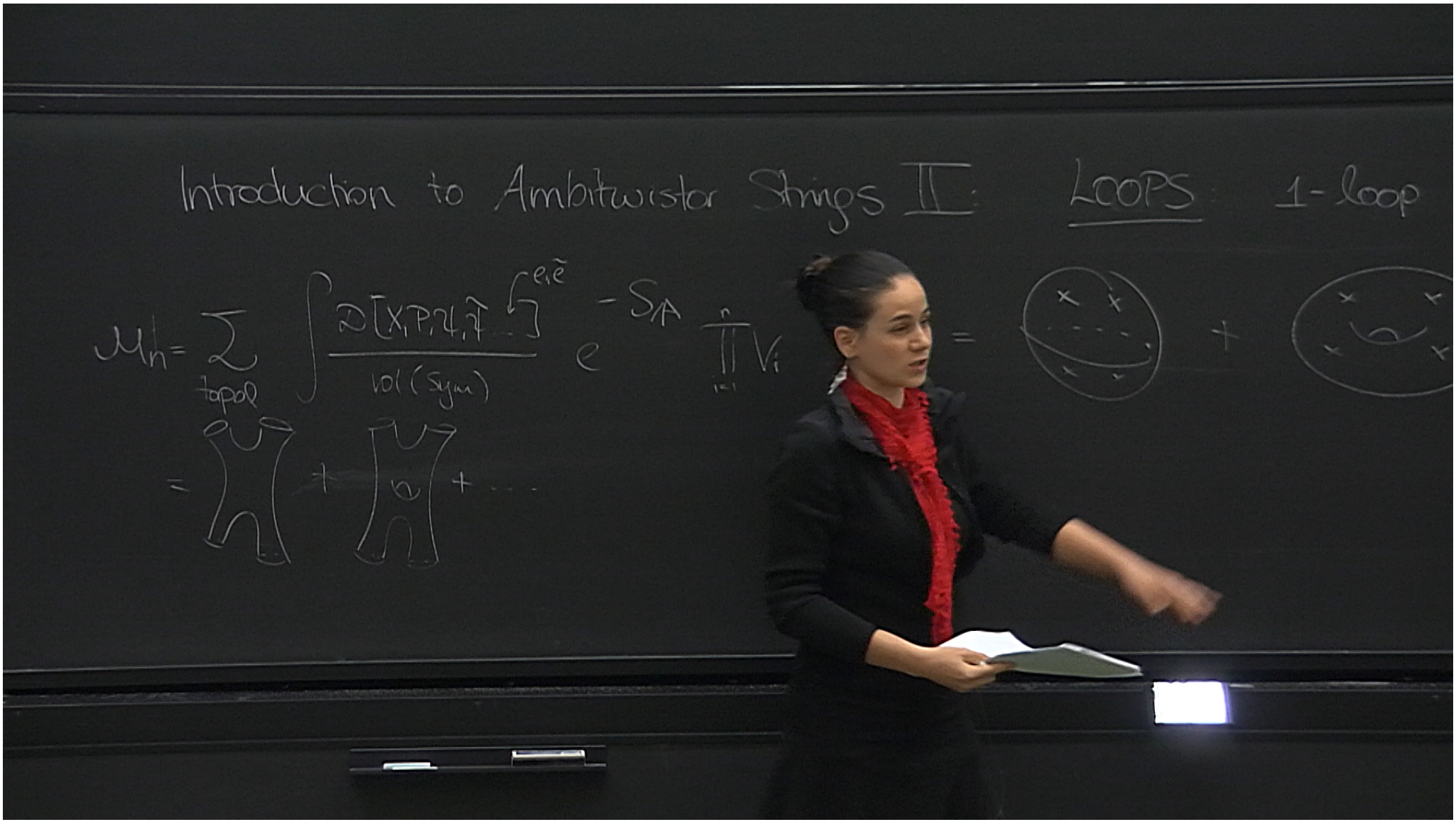
$b, \tilde b \in \Pi\Omega^0(\Sigma, K_\Sigma^2)$ $\quad \beta, \tilde\beta \in \Omega^0(\Sigma, K_\Sigma^{3/2})$

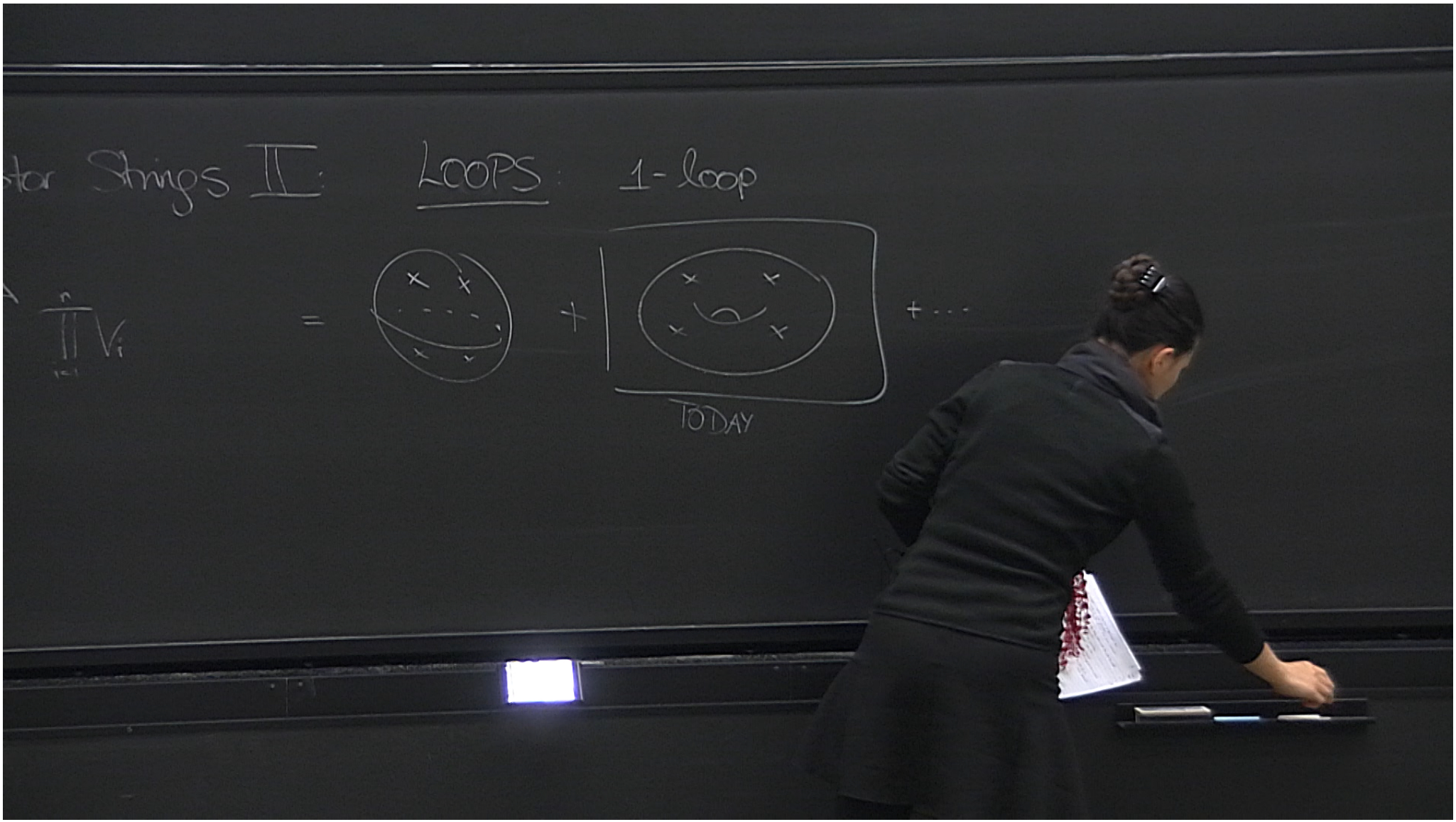
$c, \tilde c \in \Pi\Omega^0(\Sigma, T_\Sigma)$ $\quad \gamma, \tilde\gamma \in \Omega^0(\Sigma, T_\Sigma^{1/2})$

$$\tilde e_0 = \sum_{r=1}^{n-3} s_r \mu_r$$

$$S_{GF} = \frac{1}{2\pi}\int_\Sigma \{Q, \tilde b(\tilde e - \tilde e_0) + \dots\} \qquad \{Q, \tilde b\} = H$$

$$\Rightarrow S \supset \frac{1}{2\pi}\int \frac{1}{2}\tilde e P^2 + H\tilde e - \sum(s_r\mu_r H + ds_r\mu_r\tilde b)$$


Introduction to Ambitwistor Strings II: LOOPS: 1-loop
$\mathcal{M}_n = \sum_{\text{topol}} \int \frac{\mathcal{D}[X,P,\psi,\tilde{\psi}\ldots]^{e,\tilde{e}}}{\mathrm{vol}(\mathrm{Sym})} e^{-S_A} \prod_{i=1}^{n} V_i$
=
+
+ . . .
=
+


Strings II: LOOPS: 1-loop
TODAY

Action

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no $\alpha'$

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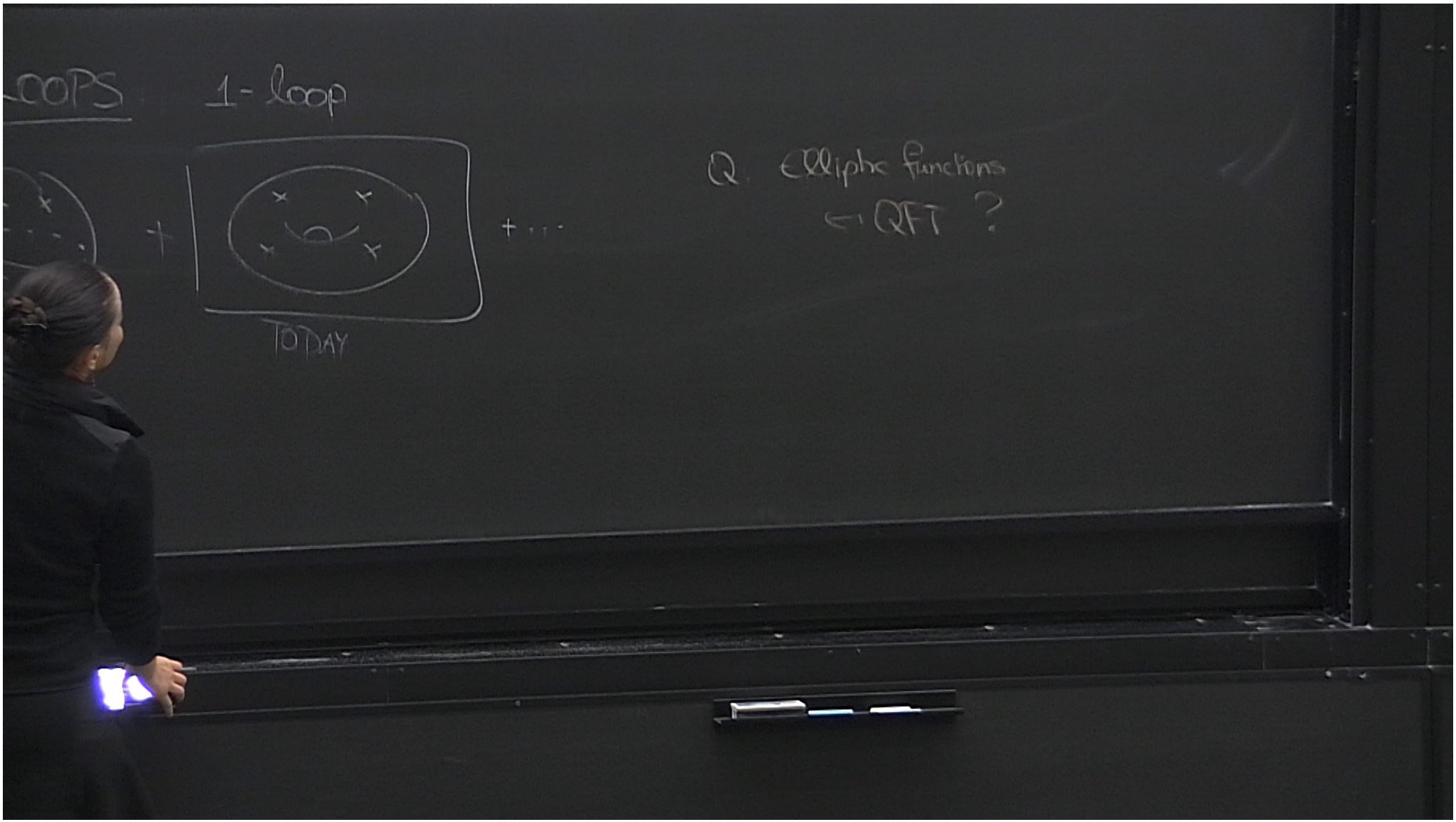
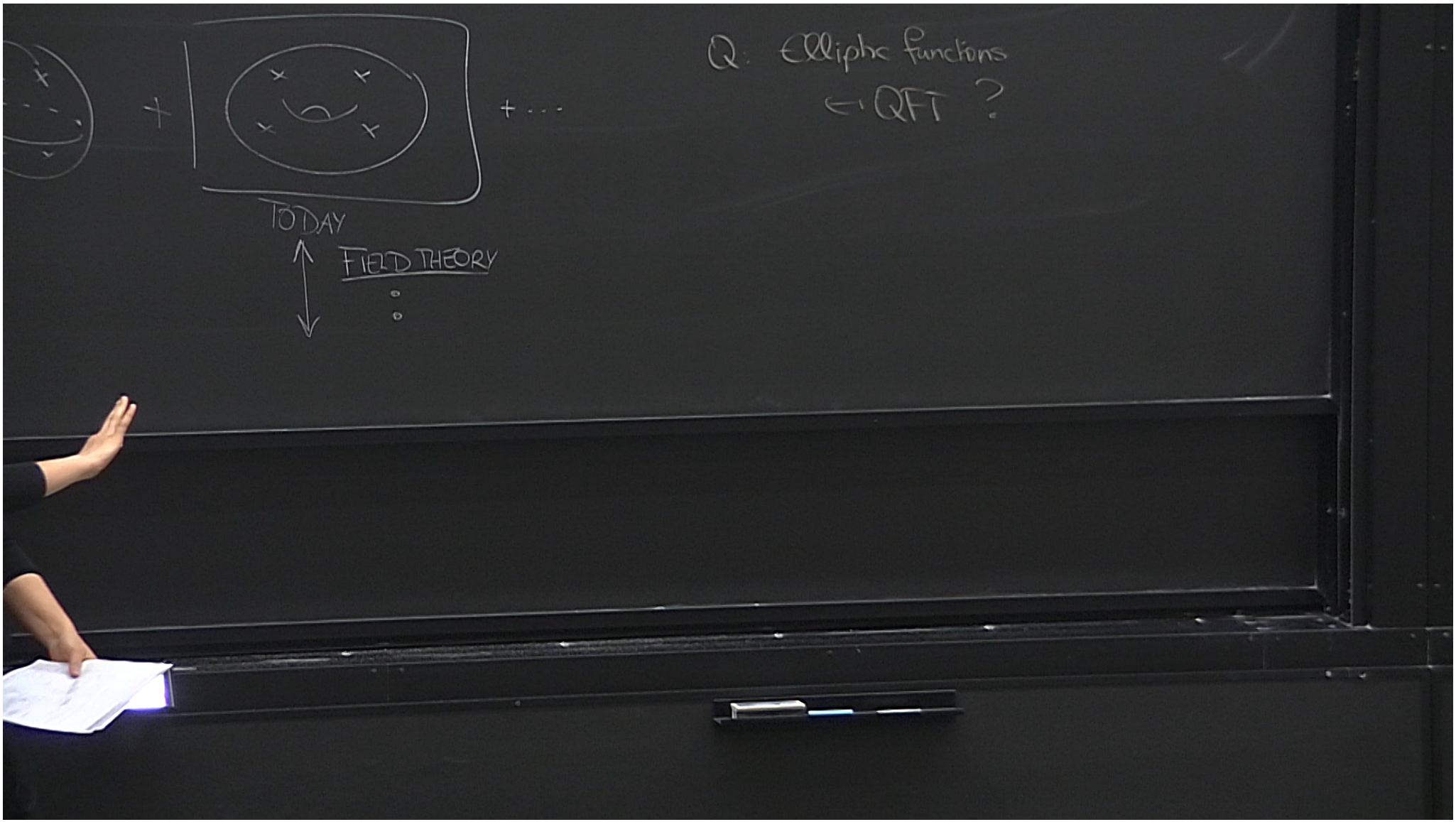
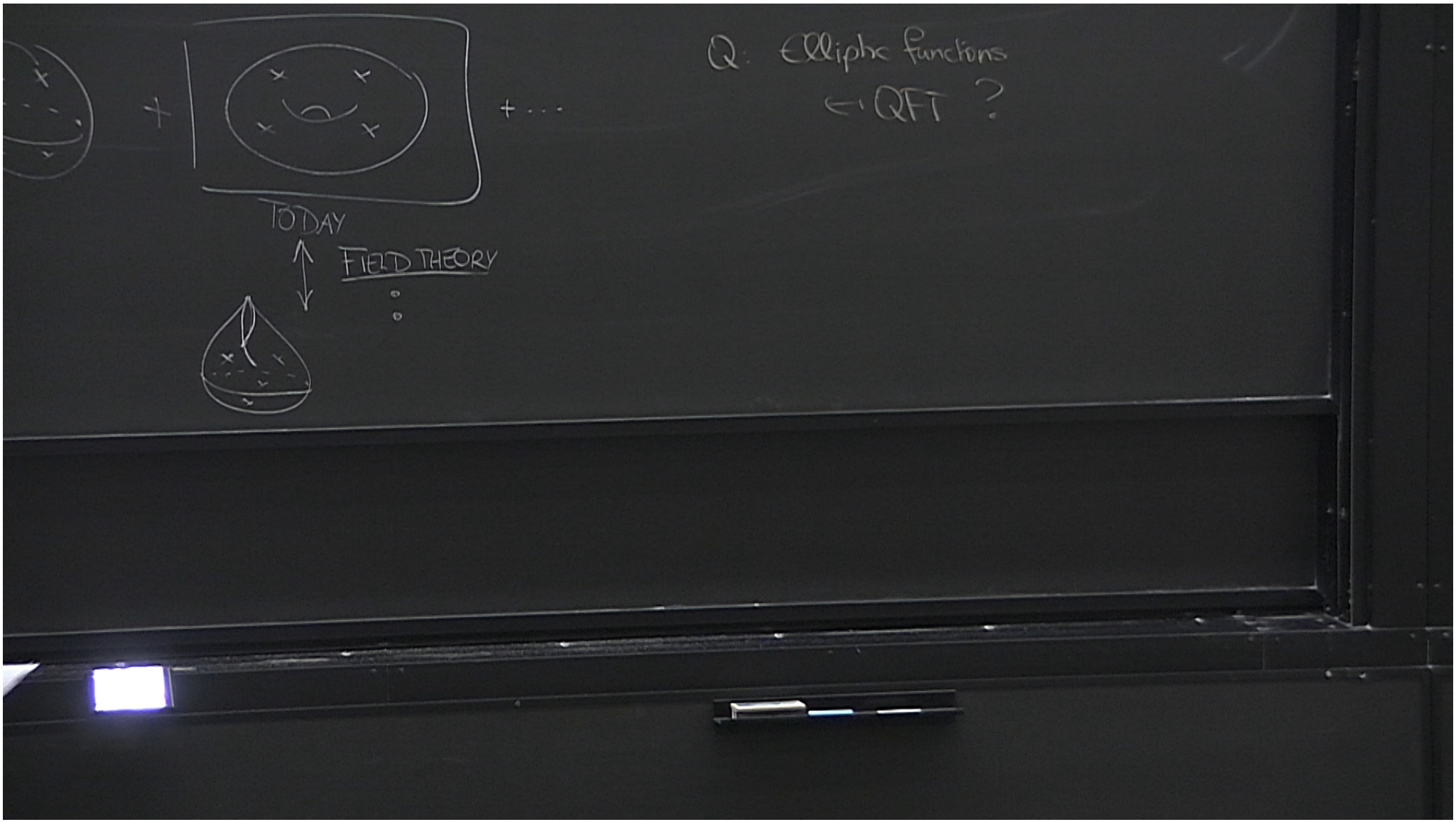

LOOPS
1-loop
+
+ . . .
TODAY
Q. Elliptic functions
⇐ QFT ?


+ ...
TODAY
FIELD THEORY
Q: Elliptic functions
⇐ QFT ?


+ ...
TODAY
FIELD THEORY
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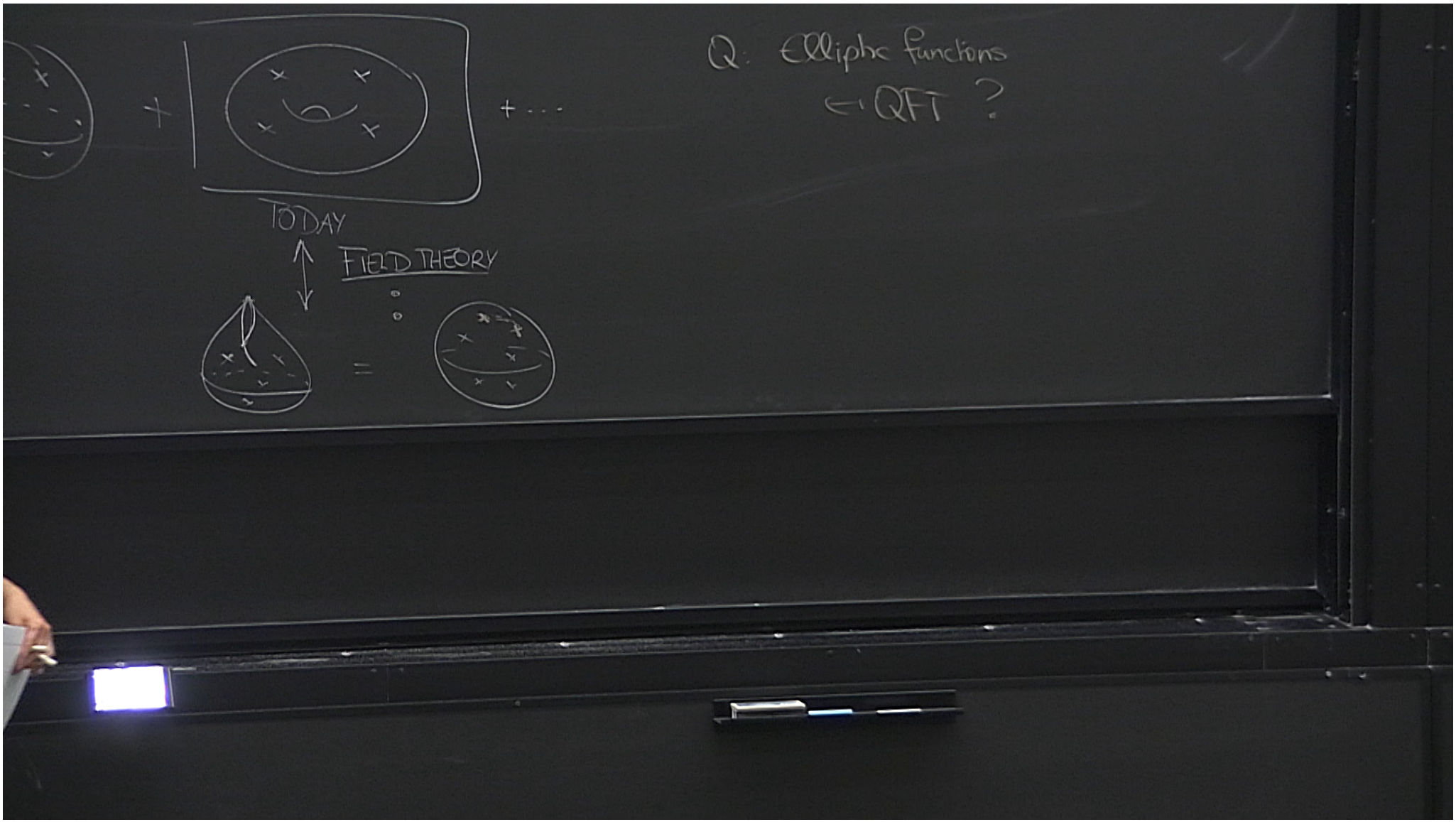
+ ...
TODAY
FIELD THEORY
=
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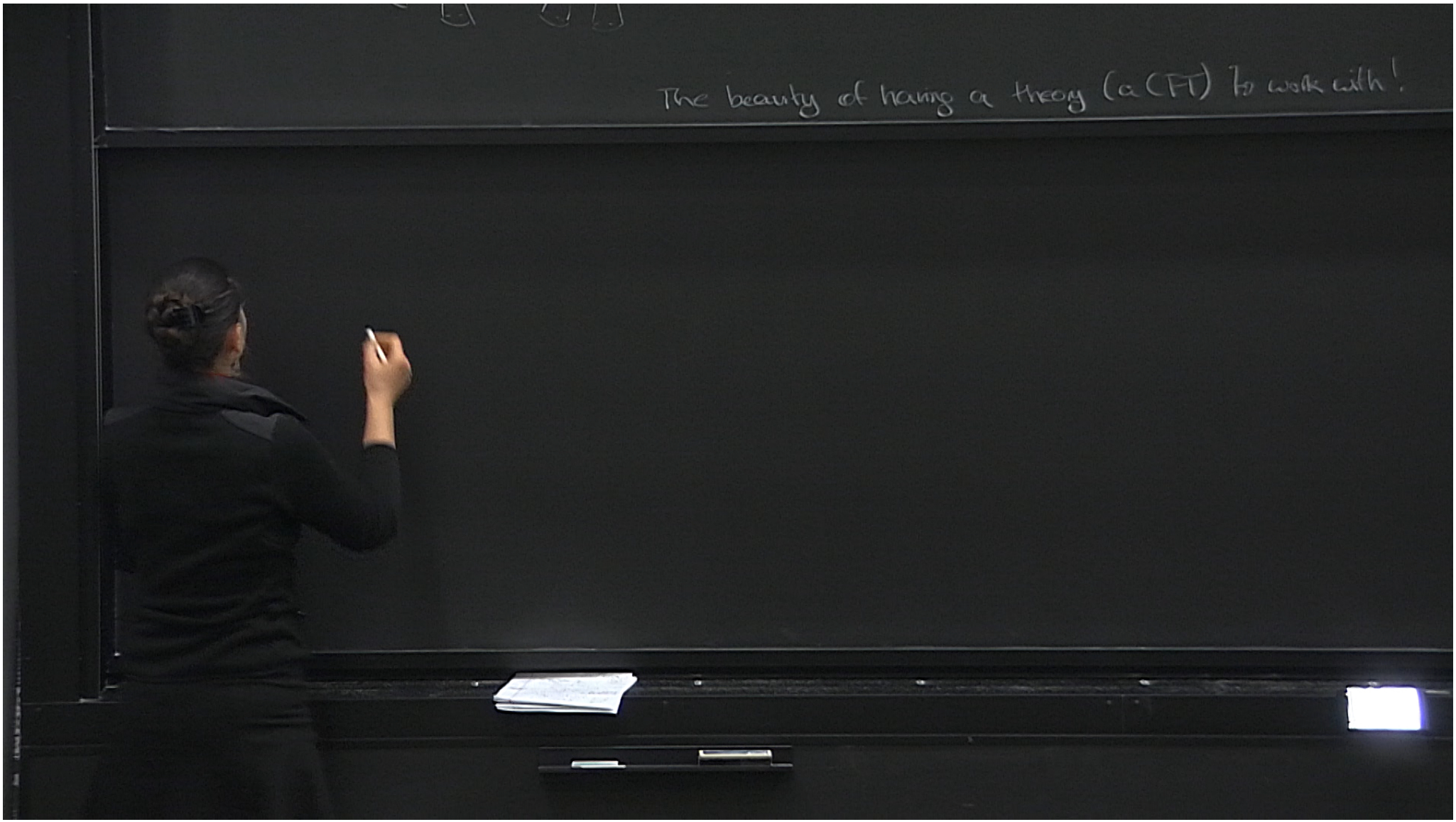
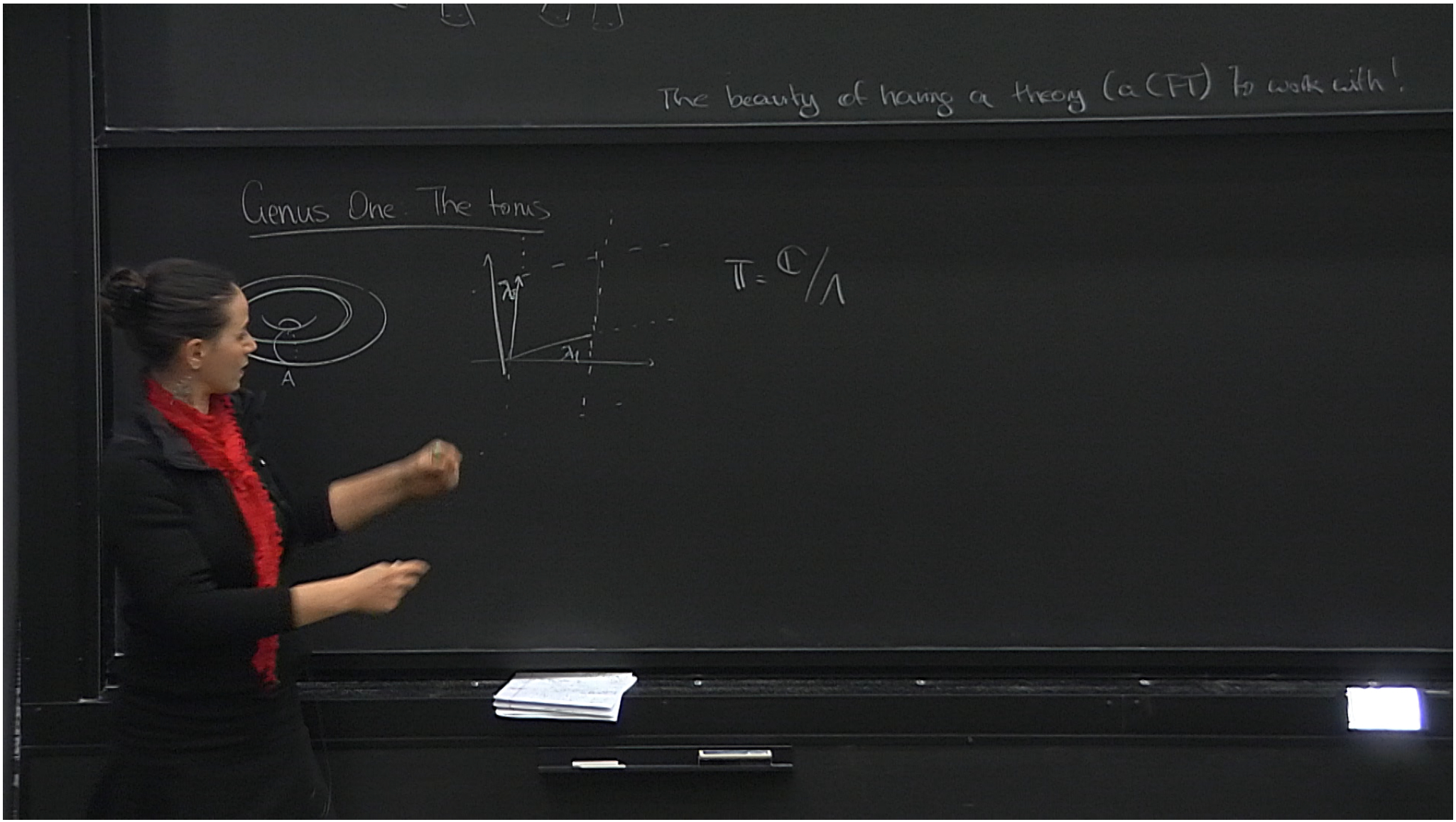

The beauty of having a theory (a CFT) to work with!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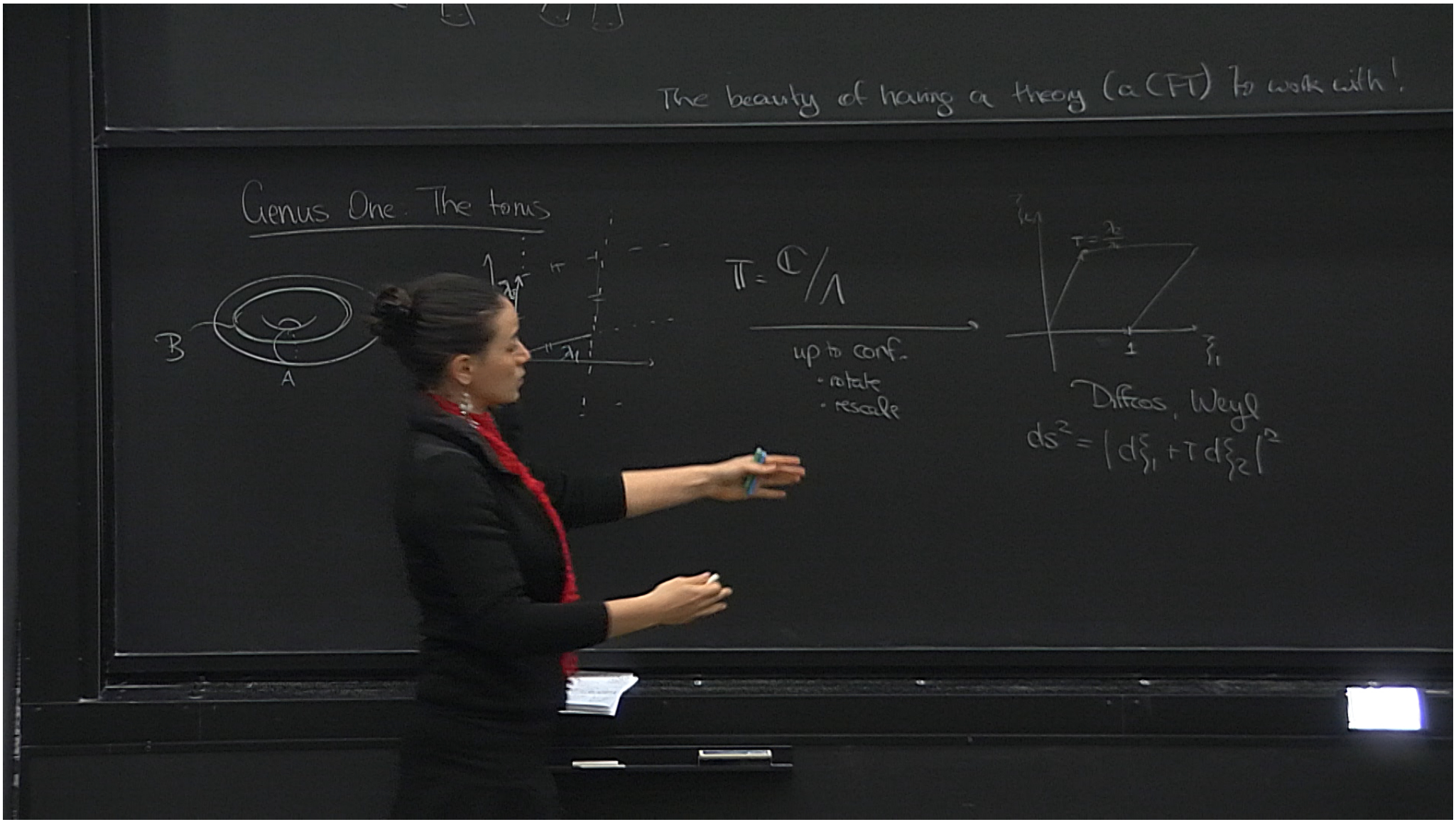
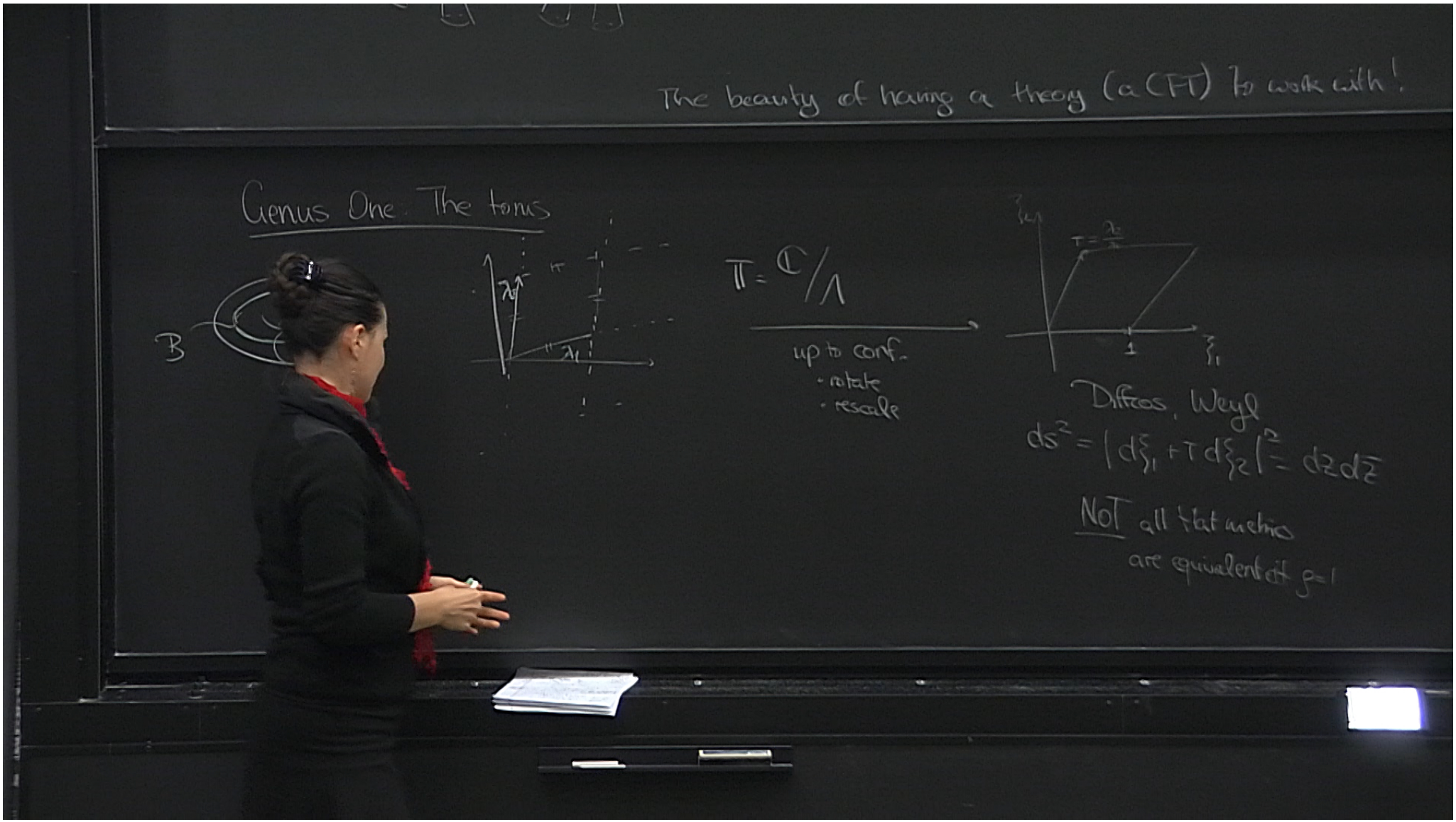
Genus One: The torus
A
$\mathbb{T} = \mathbb{C}/\Lambda$


The beauty of having a theory (a CFT) to work with!
Genus One. The torus
B
A
$\tau$
$\lambda_2$
$\lambda_1$
$\mathbb{T} = \mathbb{C}/\Lambda$
up to conf.
· rotate
· rescale
$\xi_2$
$\tau = \frac{\lambda_2}{\lambda_1}$
1
$\xi_1$
Diffeos, Weyl
$ds^2 = |d\xi_1 + \tau d\xi_2|^2$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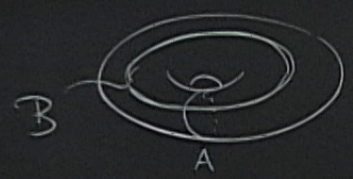
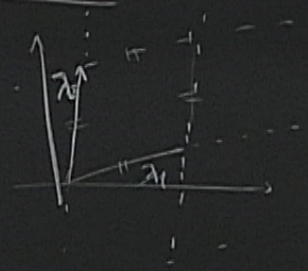
1
$\xi_1$
Diffeos, Weyl
$ds^2 = |d\xi_1 + \tau d\xi_2|^2 = dz d\bar{z}$
NOT all flat metrics
are equivalent if $g=1$

(FT) to work with!
Im τ > 0
Weyl
all flat metrics
are equivalent at g=1

The beauty of having a theory (a CFT) to work with!

## Genus One: The torus

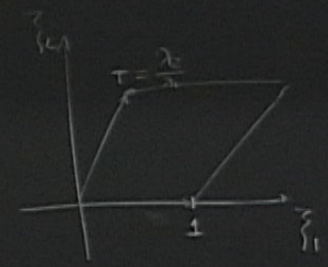



$\mathbb{T} = \mathbb{C}/\Lambda$

up to conf.
- rotate
- rescale

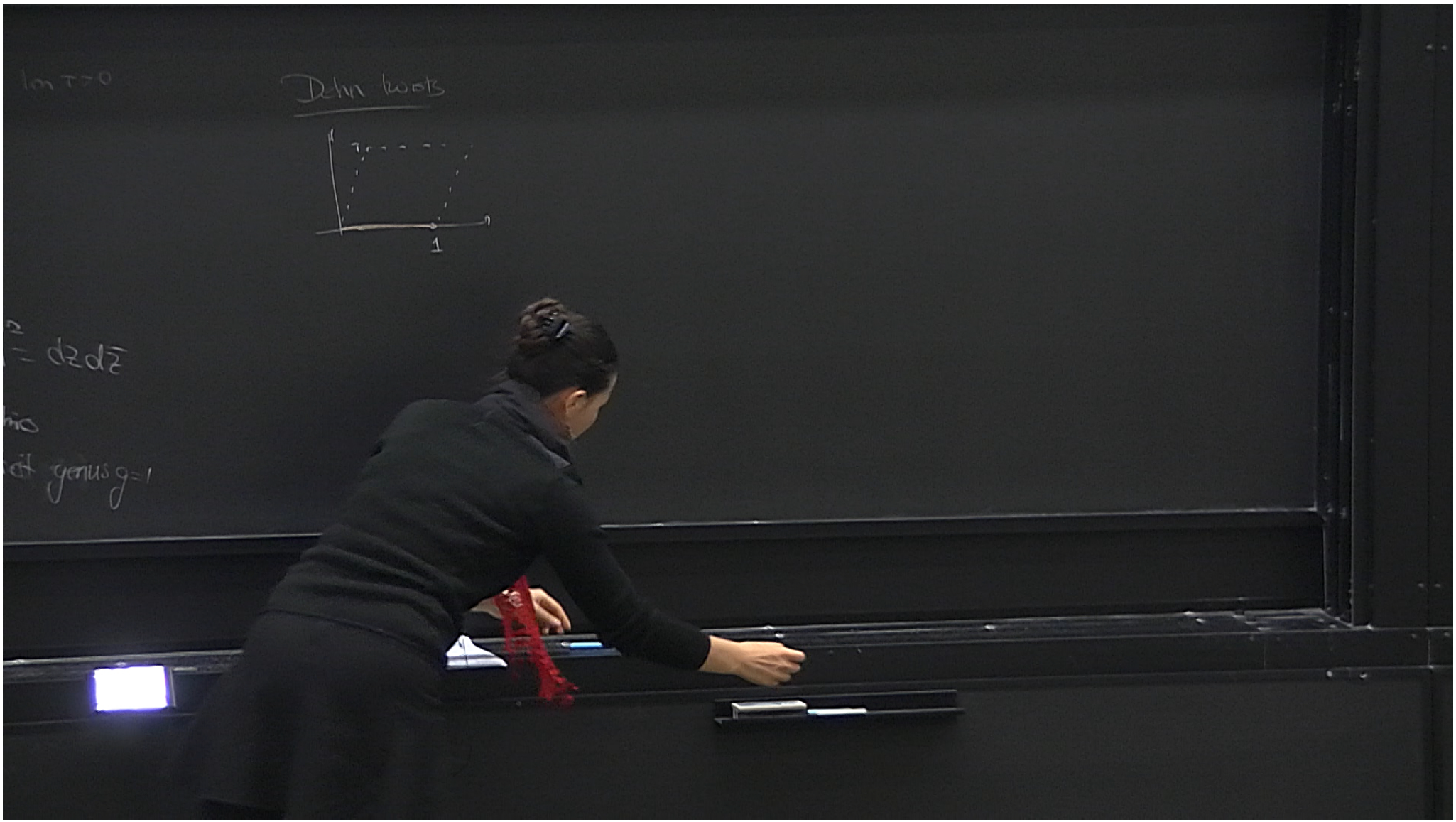
$\operatorname{Im}\tau > 0$

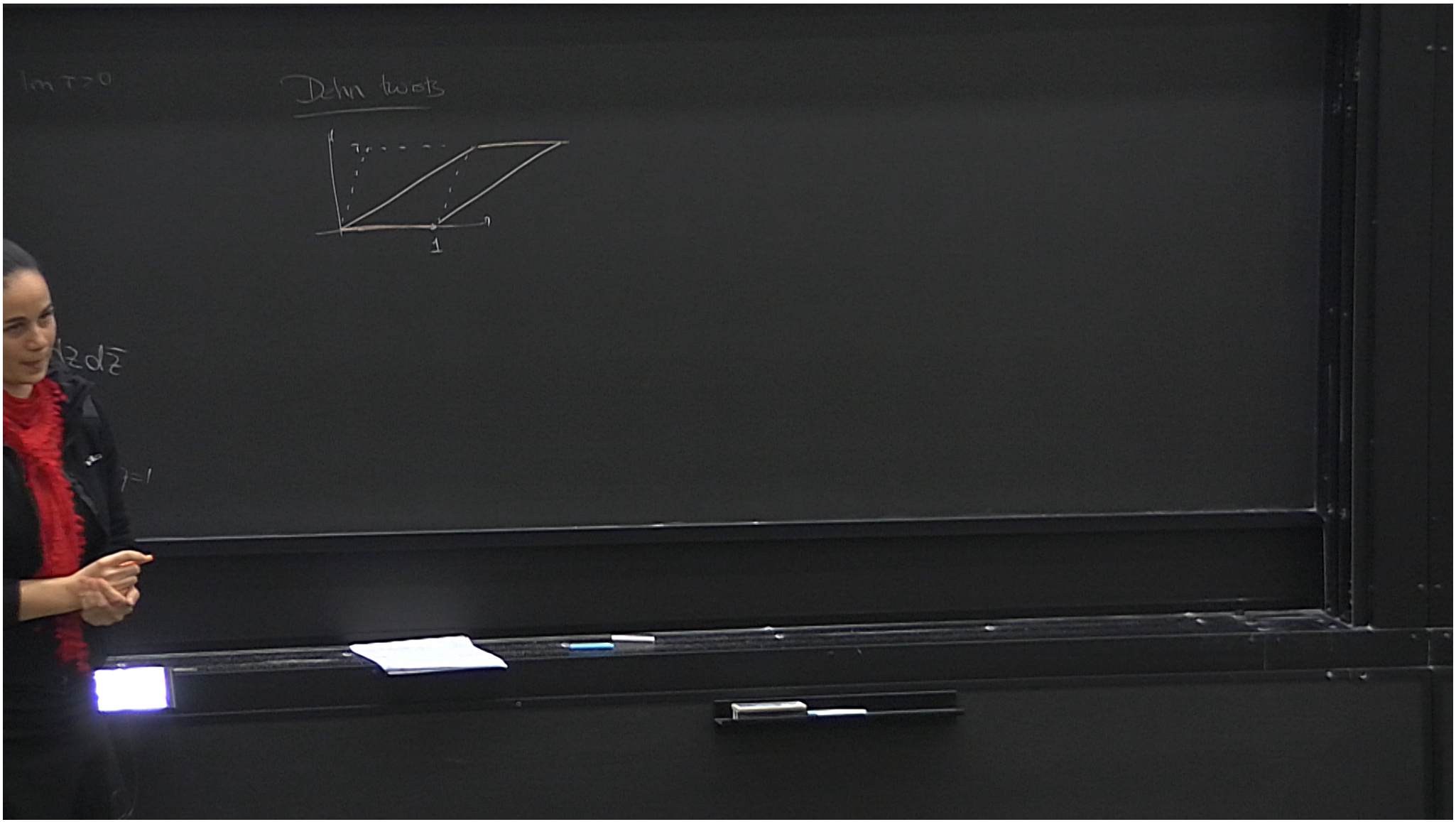


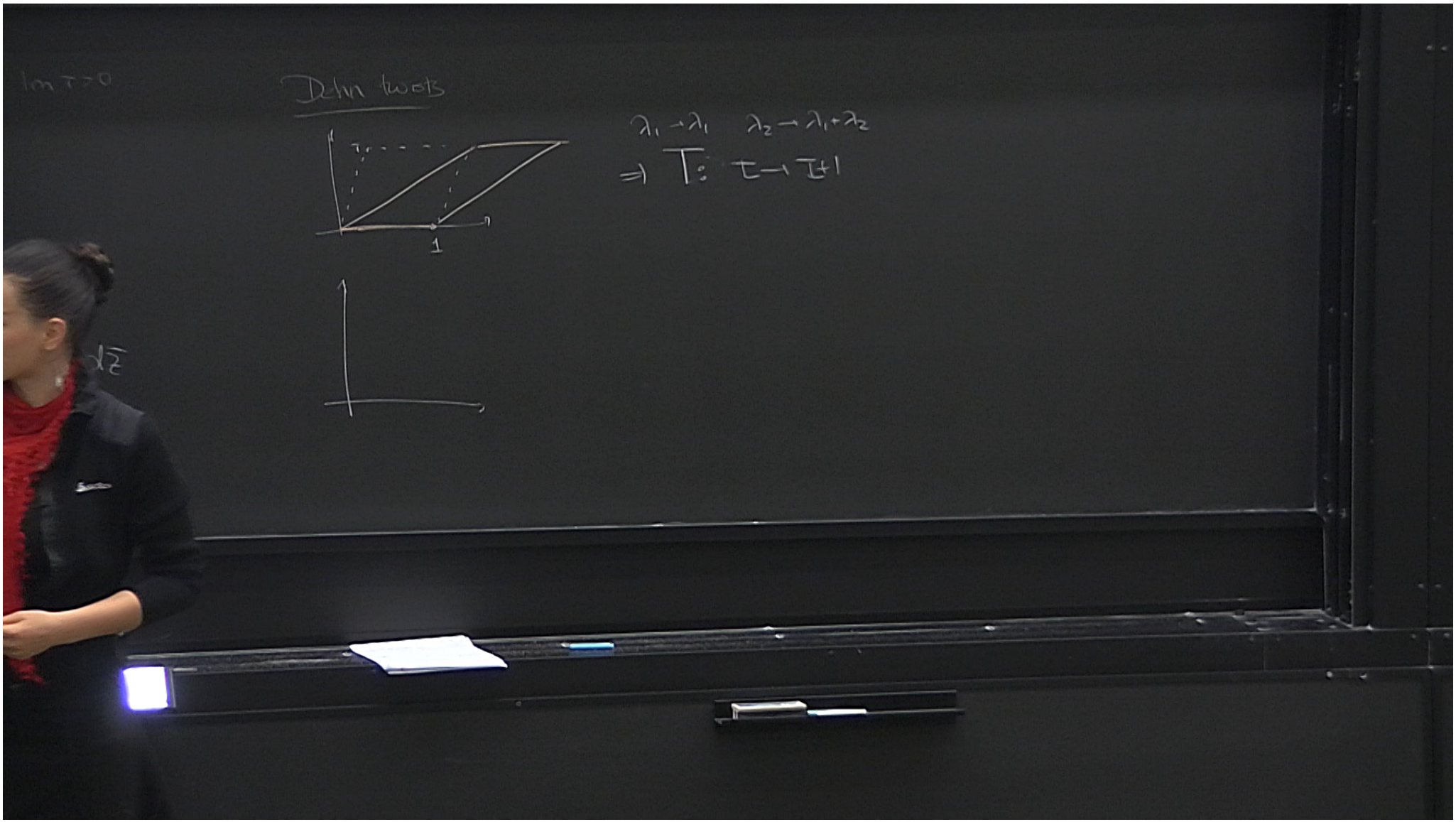

Diffeos, Weyl

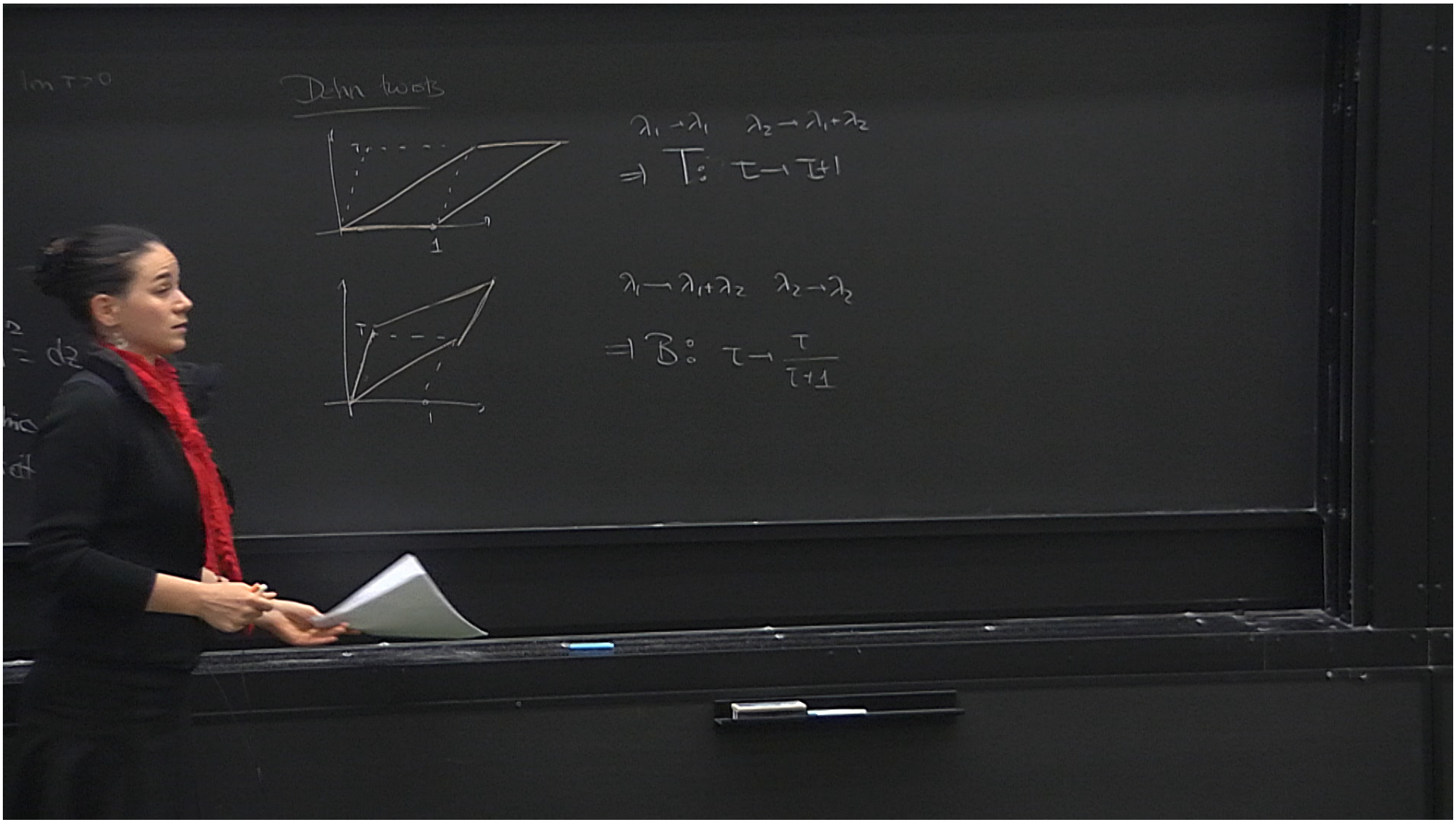
$$ds^2 = |d\xi_1 + \tau\, d\xi_2|^2 = dz\, d\bar{z}$$

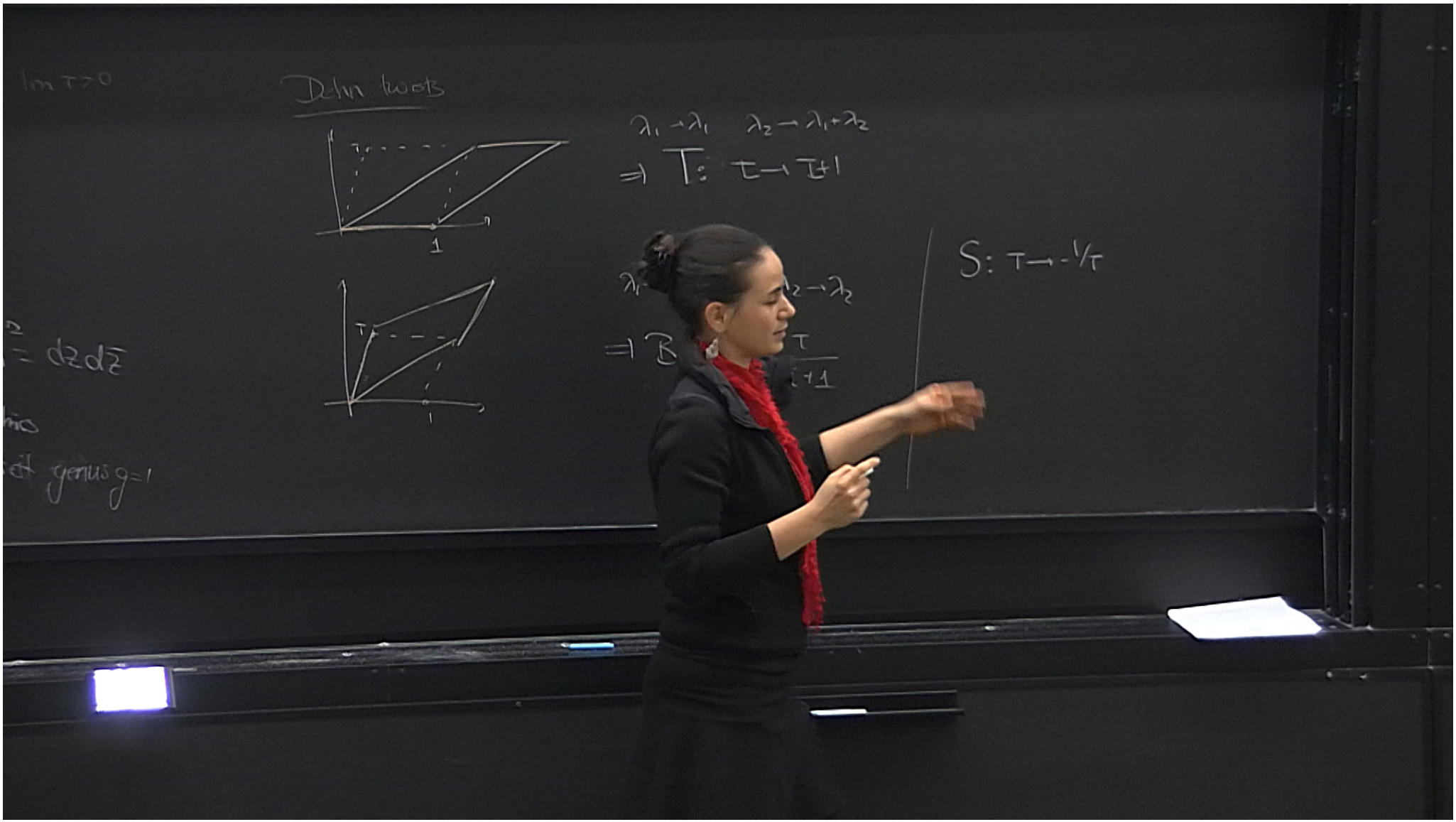
NOT all flat metrics are equivalent at genus $g=1$

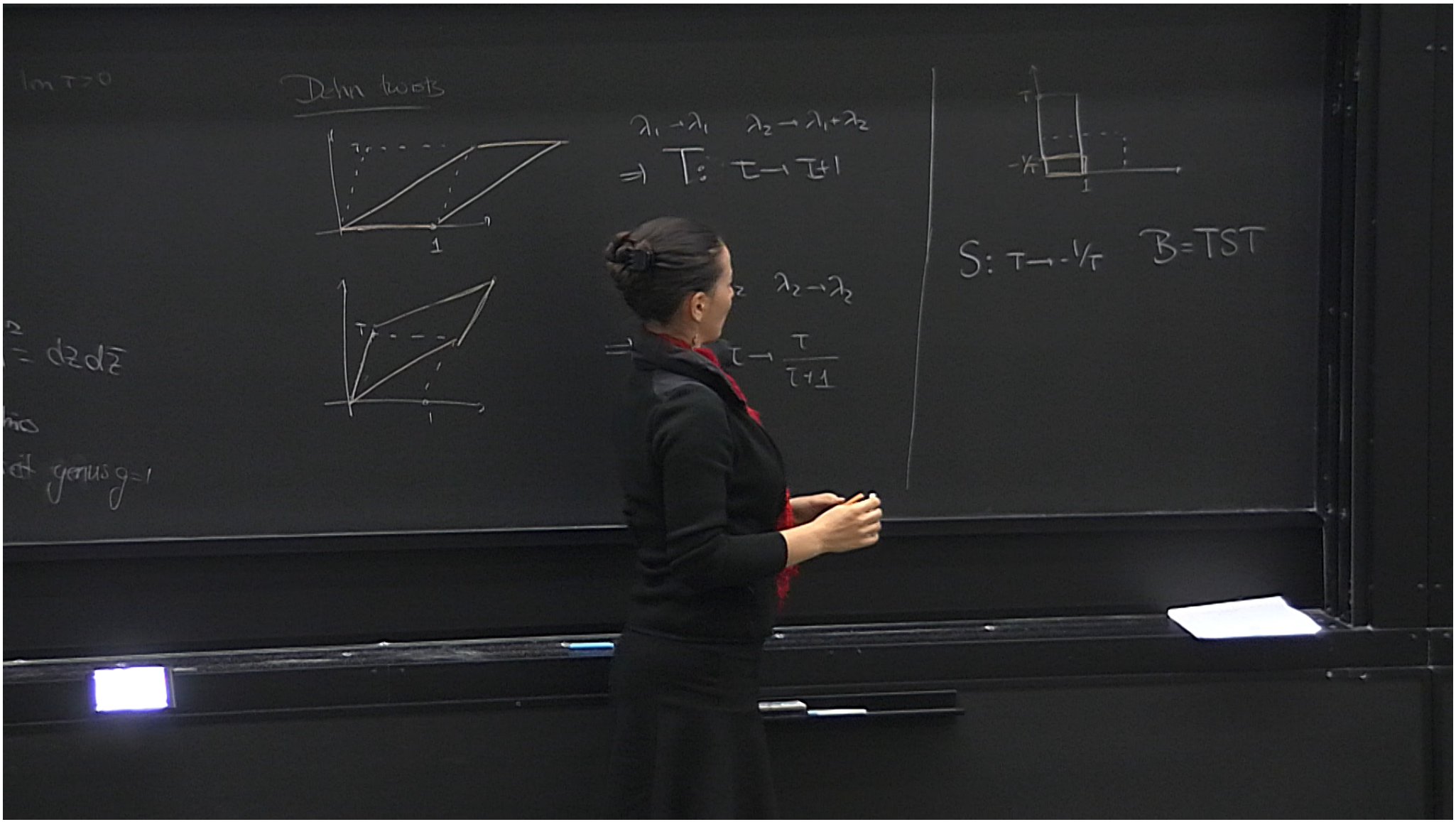

Dehn twists
1
= dzdz̄
genus g=1

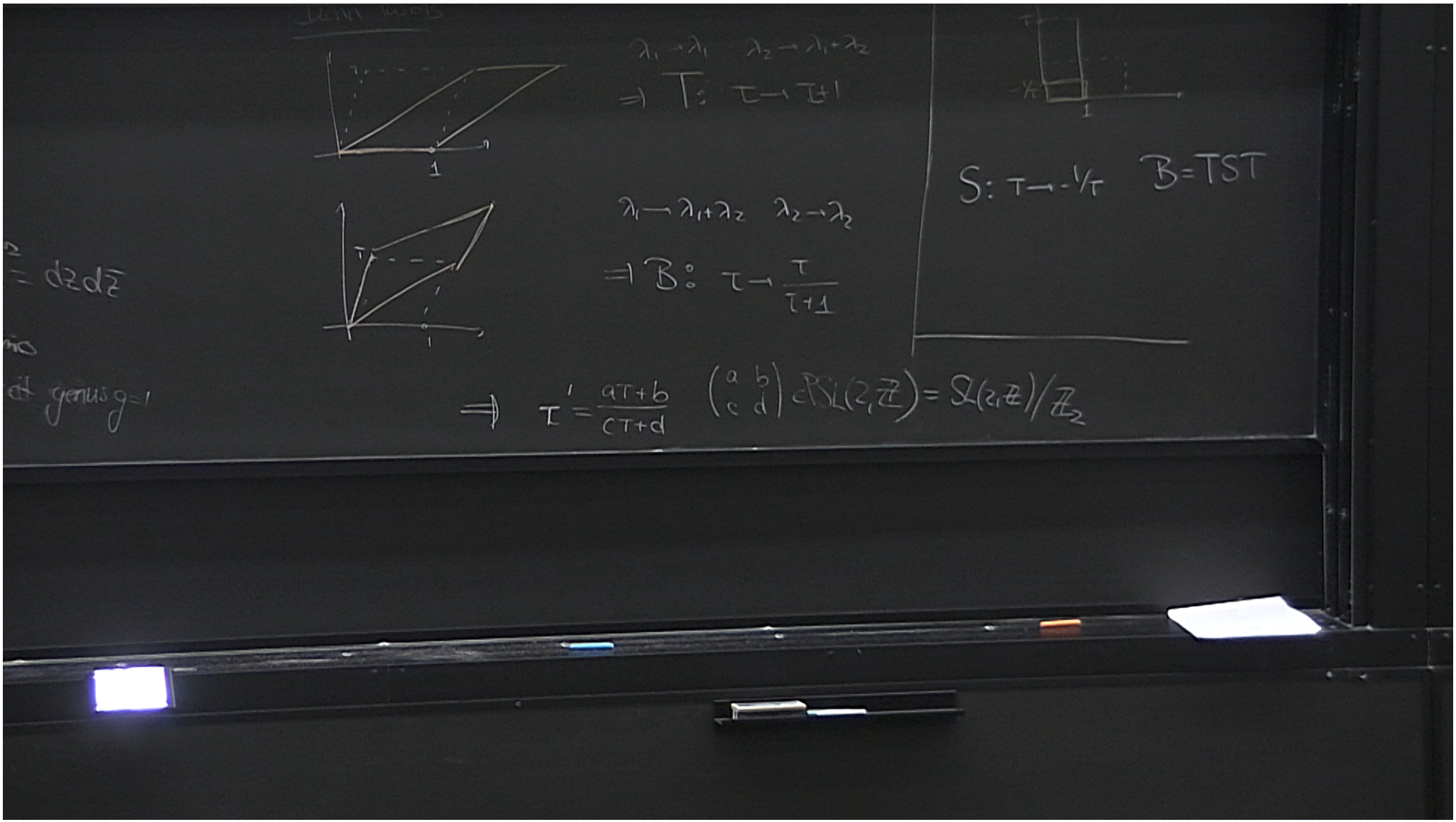

Im τ > 0
Dehn twists
1


Im τ > 0
Dehn twists
1
λ1 → λ1  λ2 → λ1 + λ2
⇒ T: τ → τ+1


Im τ > 0
Dehn twists
λ1 → λ1  λ2 → λ1 + λ2
⇒ T: τ → τ+1
λ1 → λ1 + λ2  λ2 → λ2
⇒ B: τ → τ/(τ+1)

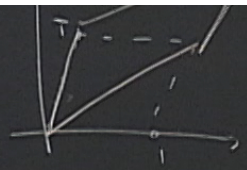

Im τ > 0
Dehn twists
λ1 → λ1  λ2 → λ1 + λ2
⇒ T: τ → τ+1
S: τ → -1/τ
genus g=1


Im τ > 0
Dehn twists
λ1 → λ1  λ2 → λ1 + λ2
⇒ T: τ → τ+1
λ2 → λ2
τ → τ/(τ+1)
genus g=1
S: τ → -1/τ  B=TST


$\lambda_1 \to \lambda_1 \quad \lambda_2 \to \lambda_1 + \lambda_2$
$\Rightarrow T: \tau \to \tau + 1$
$\lambda_1 \to \lambda_1 + \lambda_2 \quad \lambda_2 \to \lambda_2$
$\Rightarrow B: \tau \to \frac{\tau}{\tau+1}$
$S: \tau \to -1/\tau \quad B = TST$
$= dz d\bar{z}$
genus $g=1$
$\Rightarrow \tau' = \frac{a\tau+b}{c\tau+d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$

$= dz\,d\bar{z}$

genus $g=1$



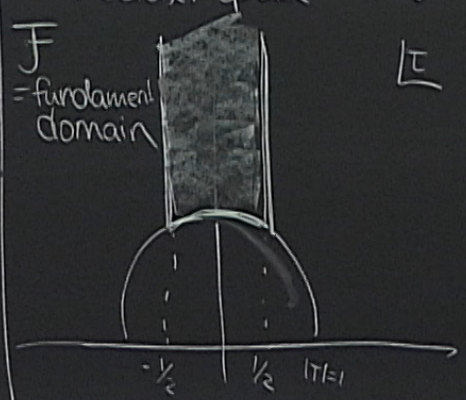

$\Rightarrow B: \tau \to \frac{\tau}{\tau+1}$

$$\Rightarrow \tau' = \frac{a\tau+b}{c\tau+d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$$

moduli space $\mathcal{M} = \frac{\{\tau \mid \operatorname{Im}\tau \geq 0\}}{PSL(2,\mathbb{Z})} = \{\tau \mid \operatorname{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}],\ |\tau| \geq 1\}$

shift by T  using S

$\mathcal{F}$ = fundamental domain



A

up to conf.

- rotate
- rescale

Diffeos, Weyl

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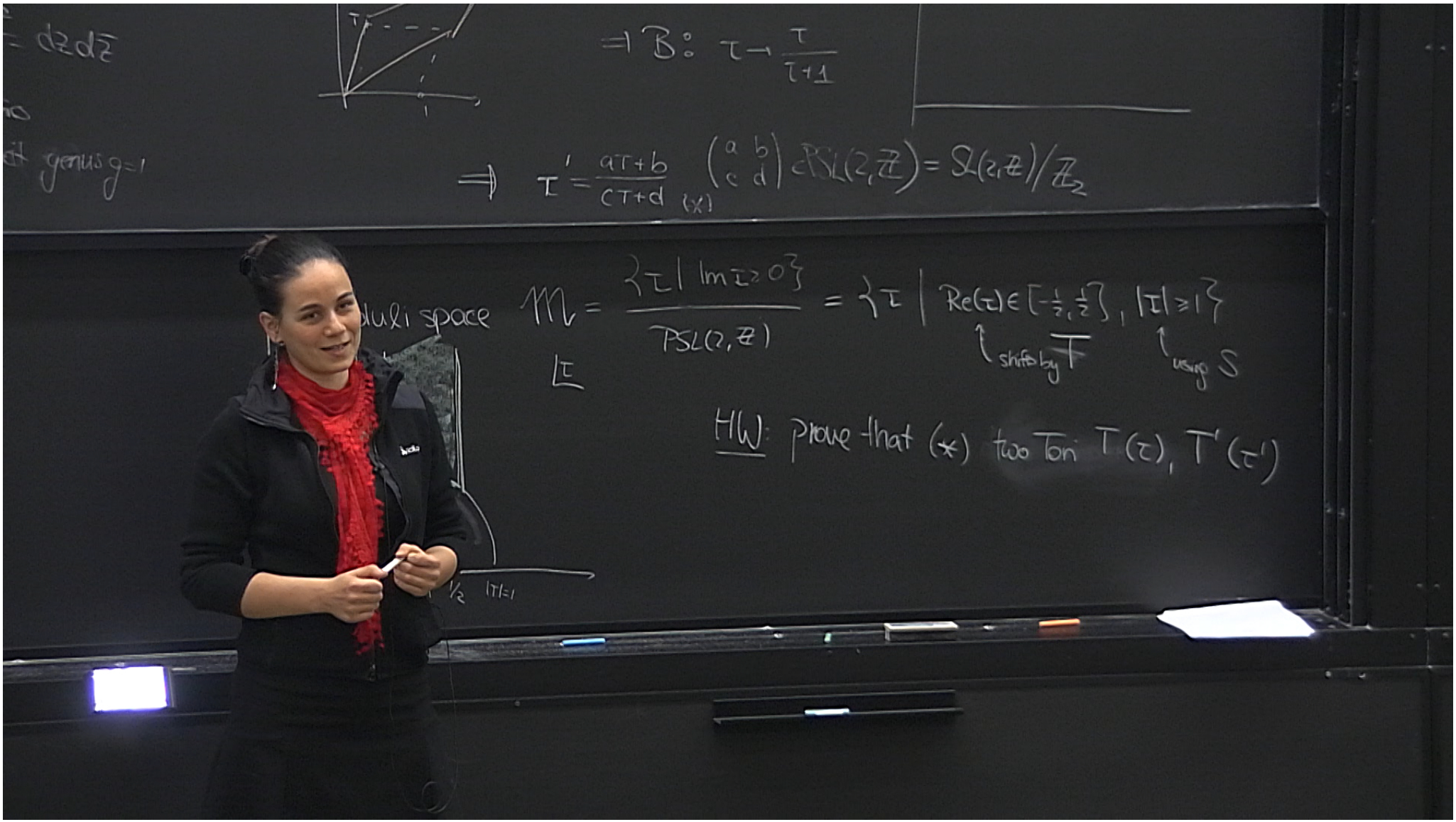
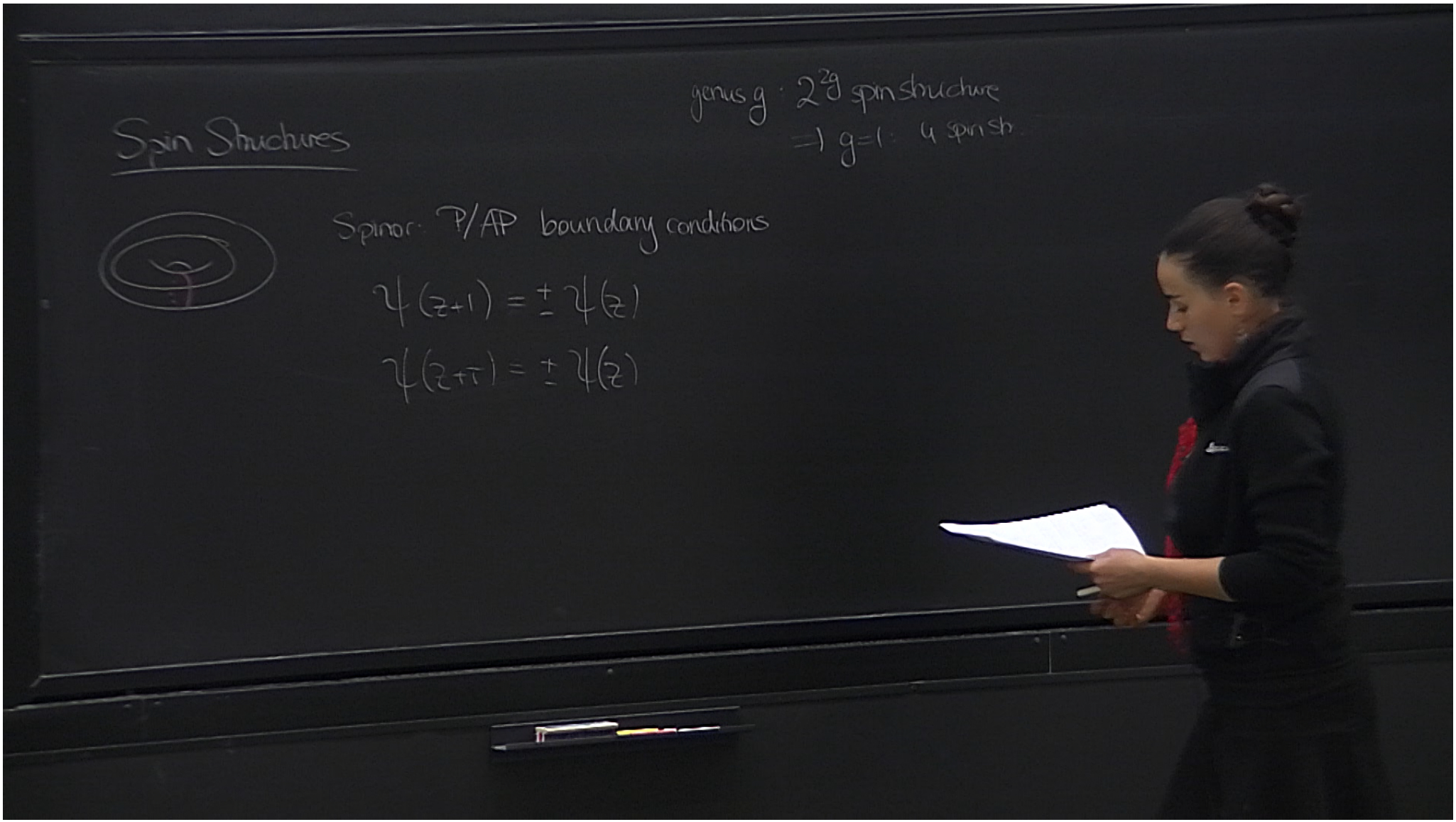
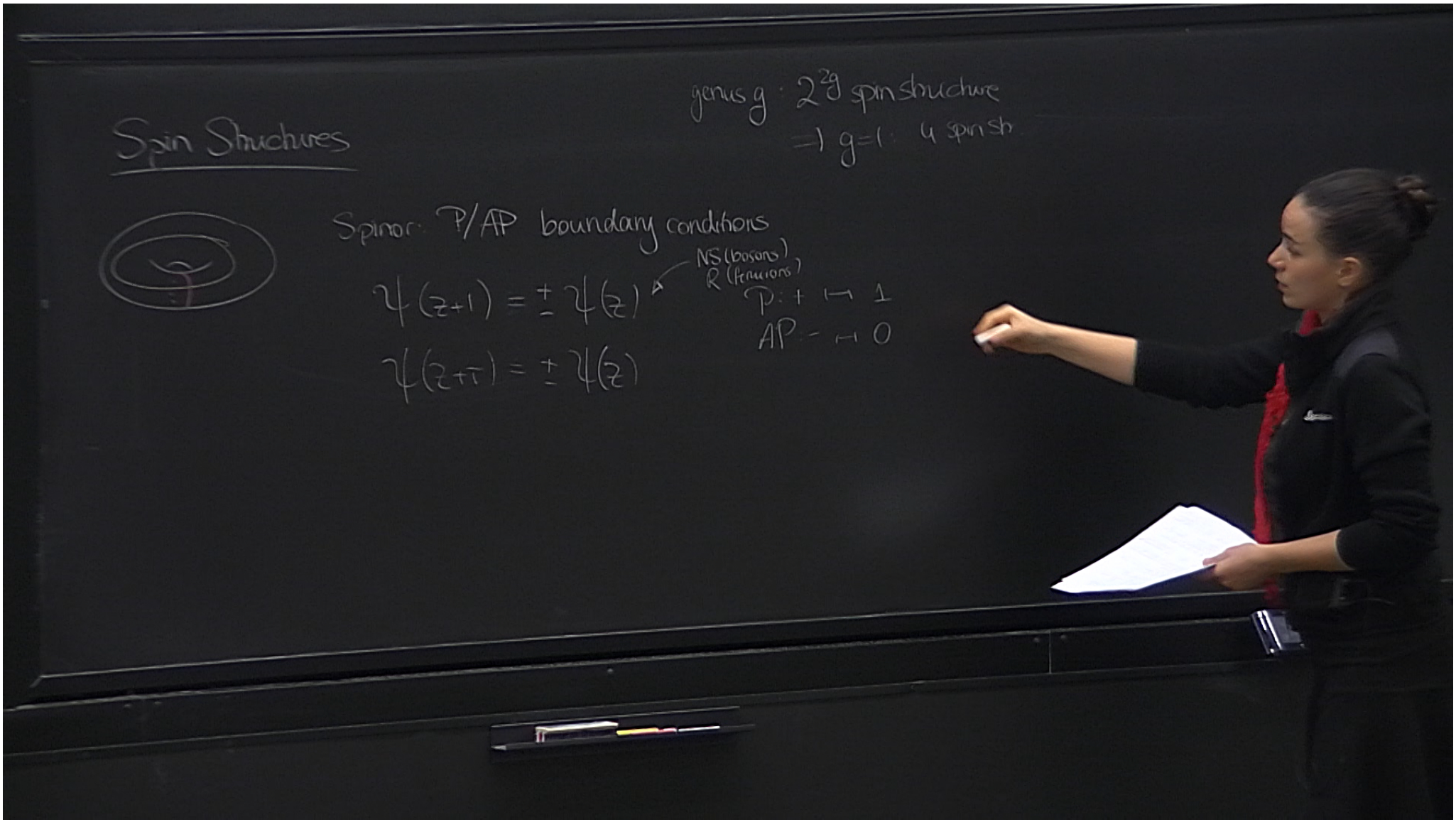
<u>NOT</u> all flat metrics are equivalent at genus $g=1$


= dz dz̄
genus g=1
⇒ B: τ → τ/(τ+1)
⇒ τ' = (aτ+b)/(cτ+d) (*) (a b; c d) ∈ PSL(2,ℤ) = SL(2,ℤ)/ℤ2
duli space
M = {τ | Im τ ≥ 0} / PSL(2,ℤ) = {τ | Re(τ) ∈ [-1/2, 1/2], |τ| ≥ 1}
shifts by T
using S
HW: prove that (*) two Tori T(τ), T'(τ')
1/2
|τ|=1


Spin Structures
genus g : $2^{2g}$ spin structure
$\Rightarrow$ g=1 : 4 spin str.
Spinor: P/AP boundary conditions
$\psi(z+1) = \pm\psi(z)$
$\psi(z+\tau) = \pm\psi(z)$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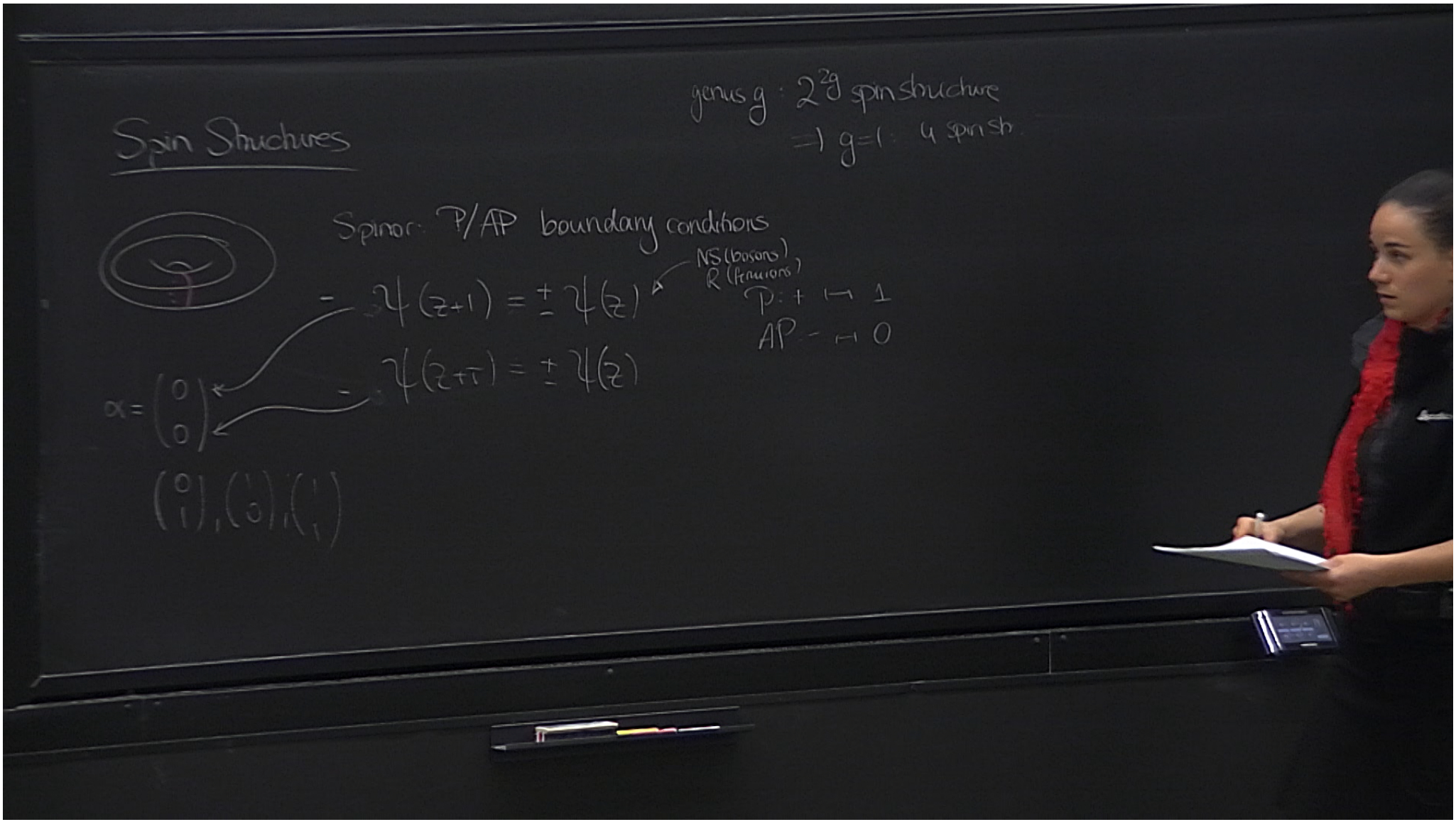
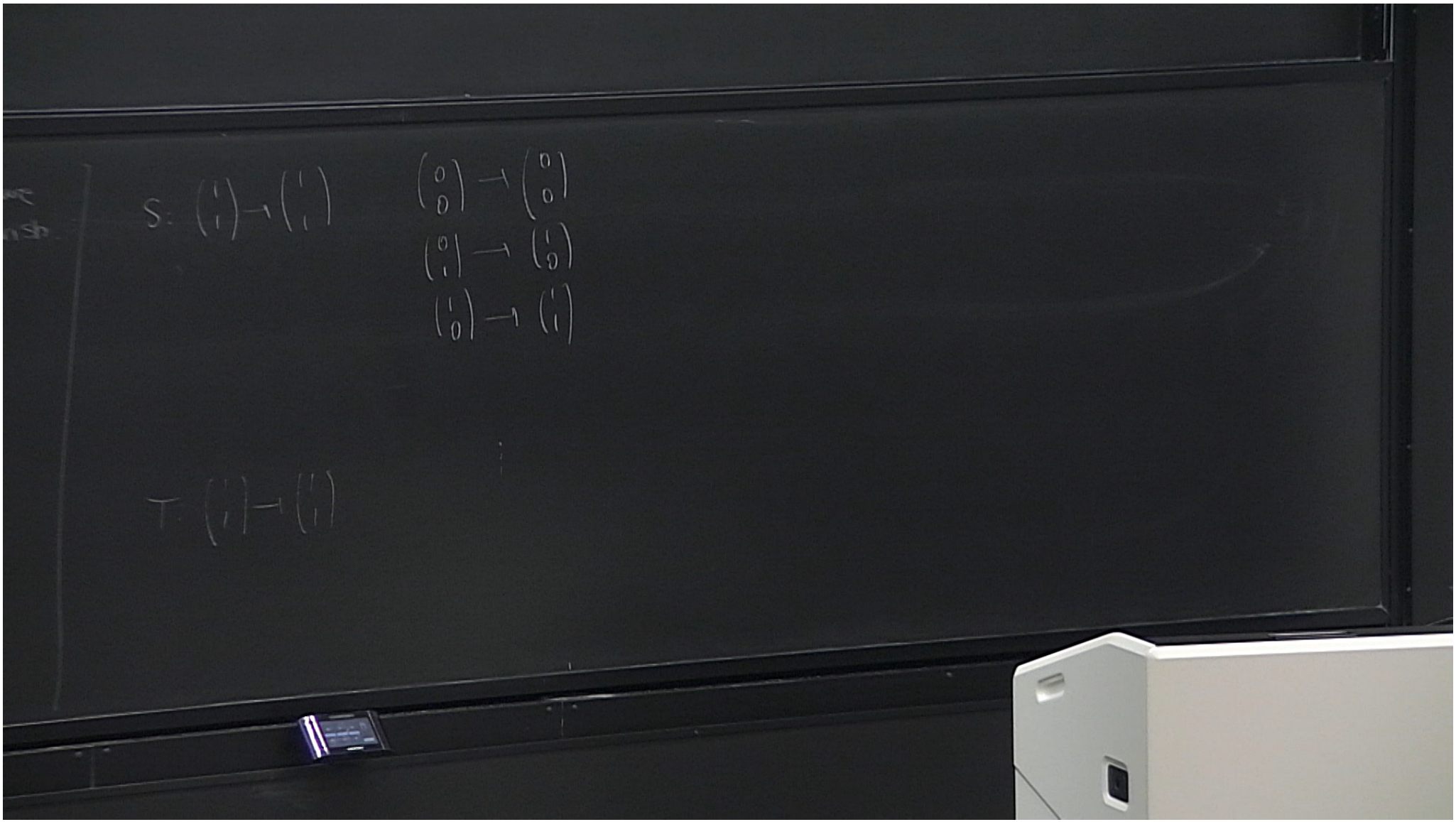
$\psi(z+1) = \pm\psi(z)$
$\psi(z+\tau) = \pm\psi(z)$
NS (bosons)
R (fermions)
P: + $\mapsto$ 1
AP: − $\mapsto$ 0


Spin Structures
genus g : 2^2g spin structure
⇒ g=1 : 4 spin str.
Spinor: P/AP boundary conditions
NS (bosons)
R (fermions)
P: + ↦ 1
AP: − ↦ 0
ψ(z+1) = ± ψ(z)
ψ(z+τ) = ± ψ(z)
α = (0 0)
(0 1), (1 0), (1 1)


S: $\begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$
$\begin{pmatrix}0\\0\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}$
$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$
$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$
T: $\begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$

$S: \begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$

$\begin{pmatrix}0\\0\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$

modular invariance

$\Rightarrow$ Sum over spin structures

$T: \begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$

$S: \begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$

odd spin struct

# zero modes = 1

$$\begin{pmatrix}0\\0\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$$

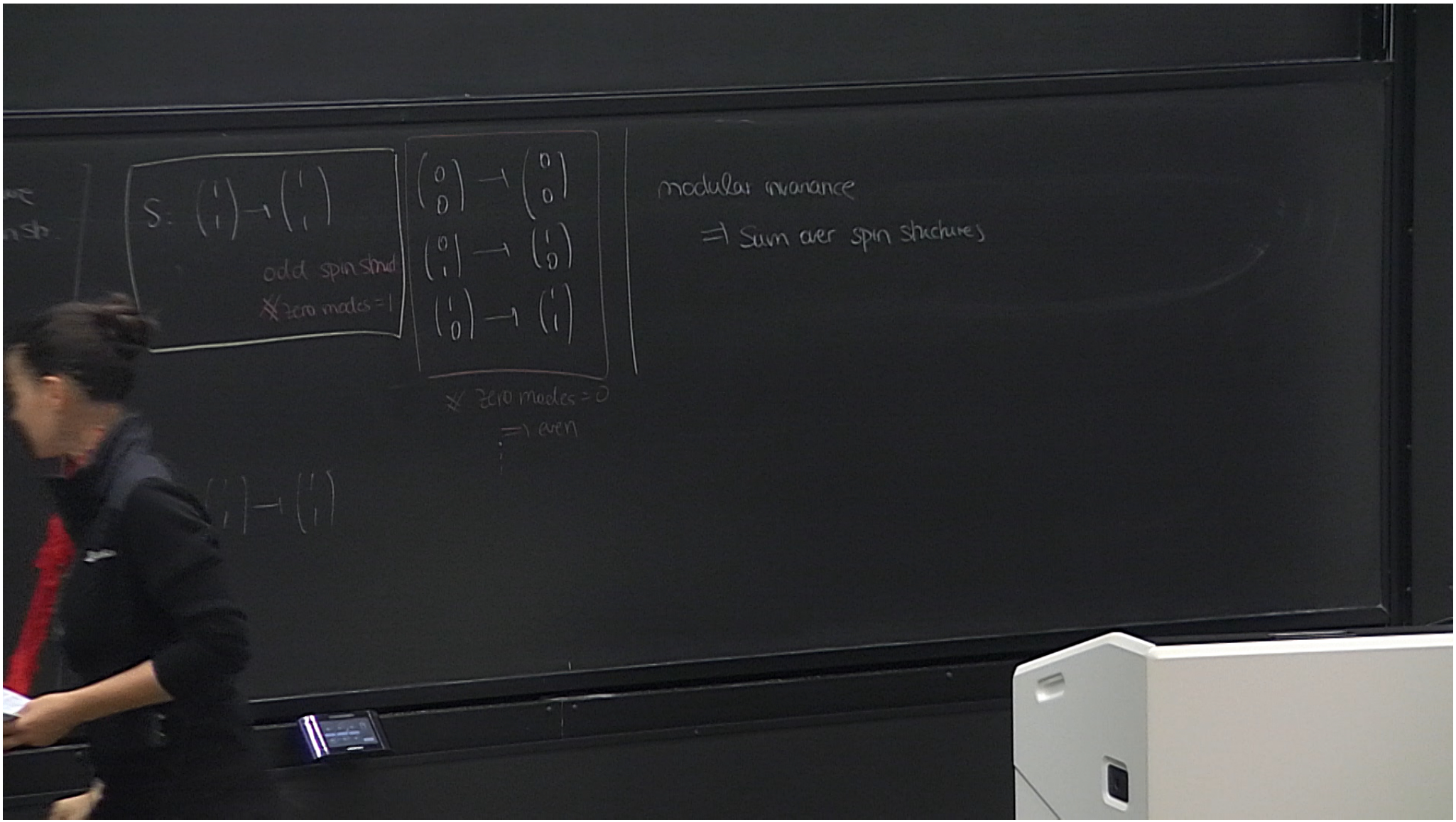
# zero modes = 0

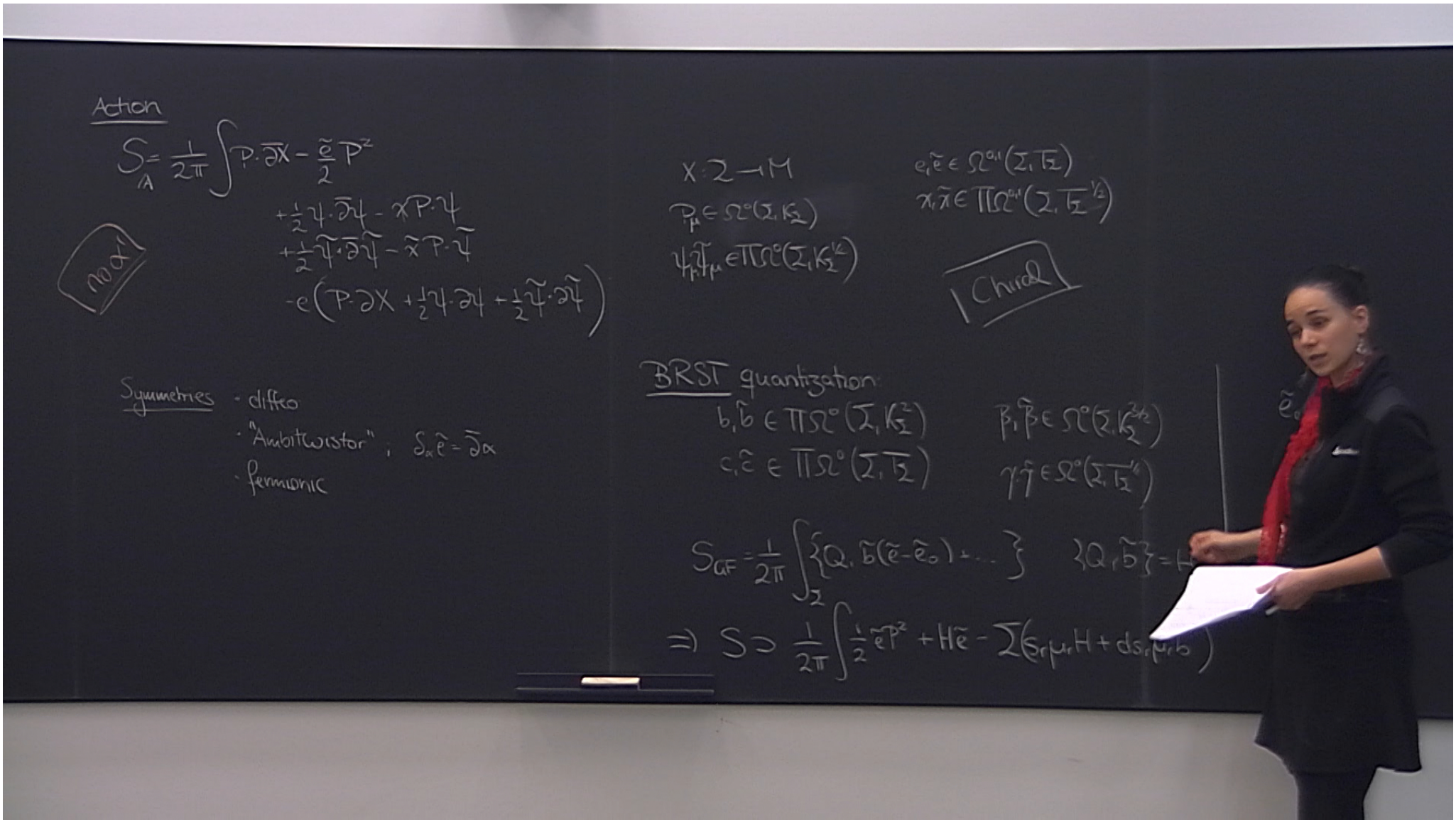
$\Rightarrow$ even

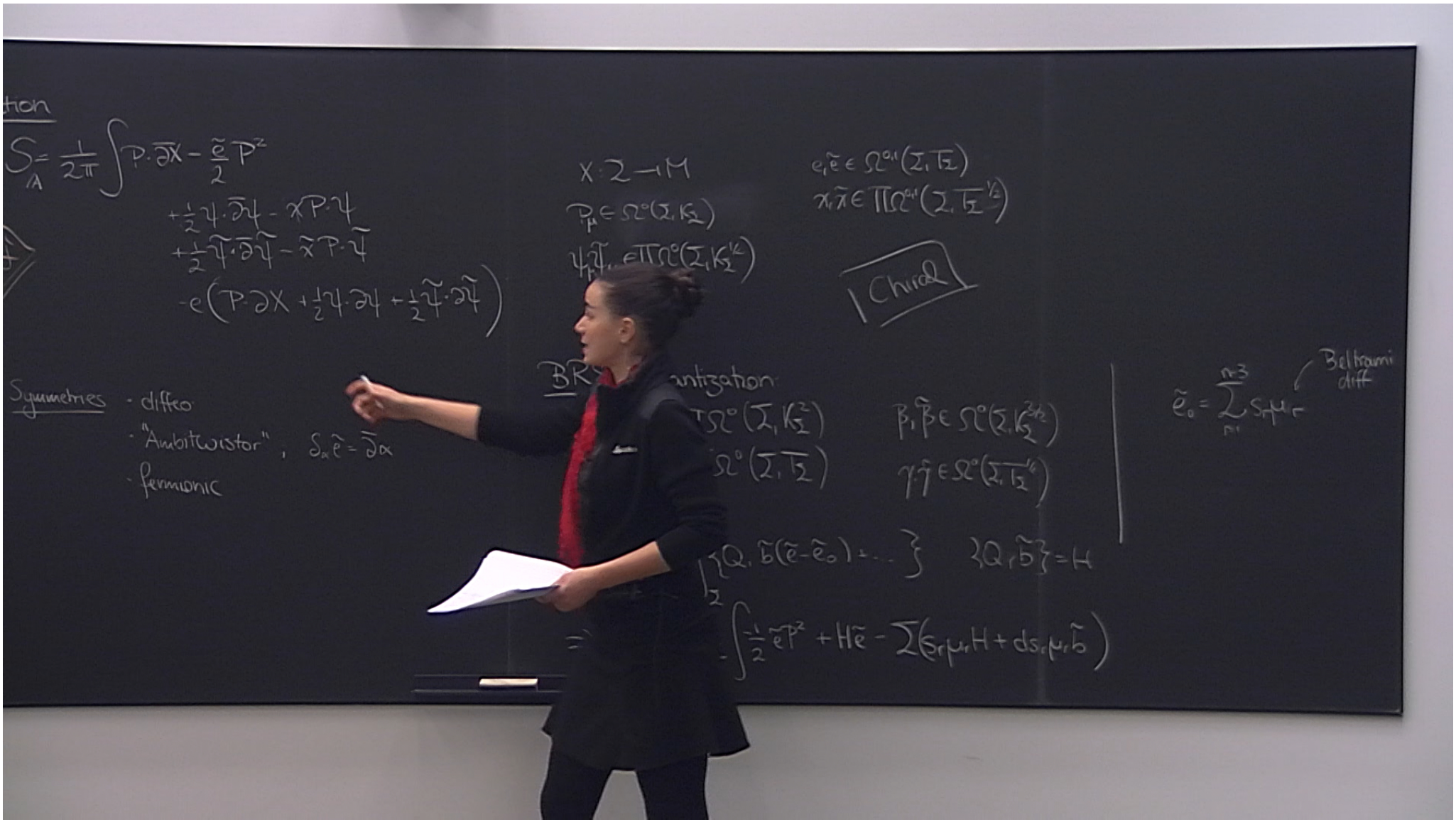
$\begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}$

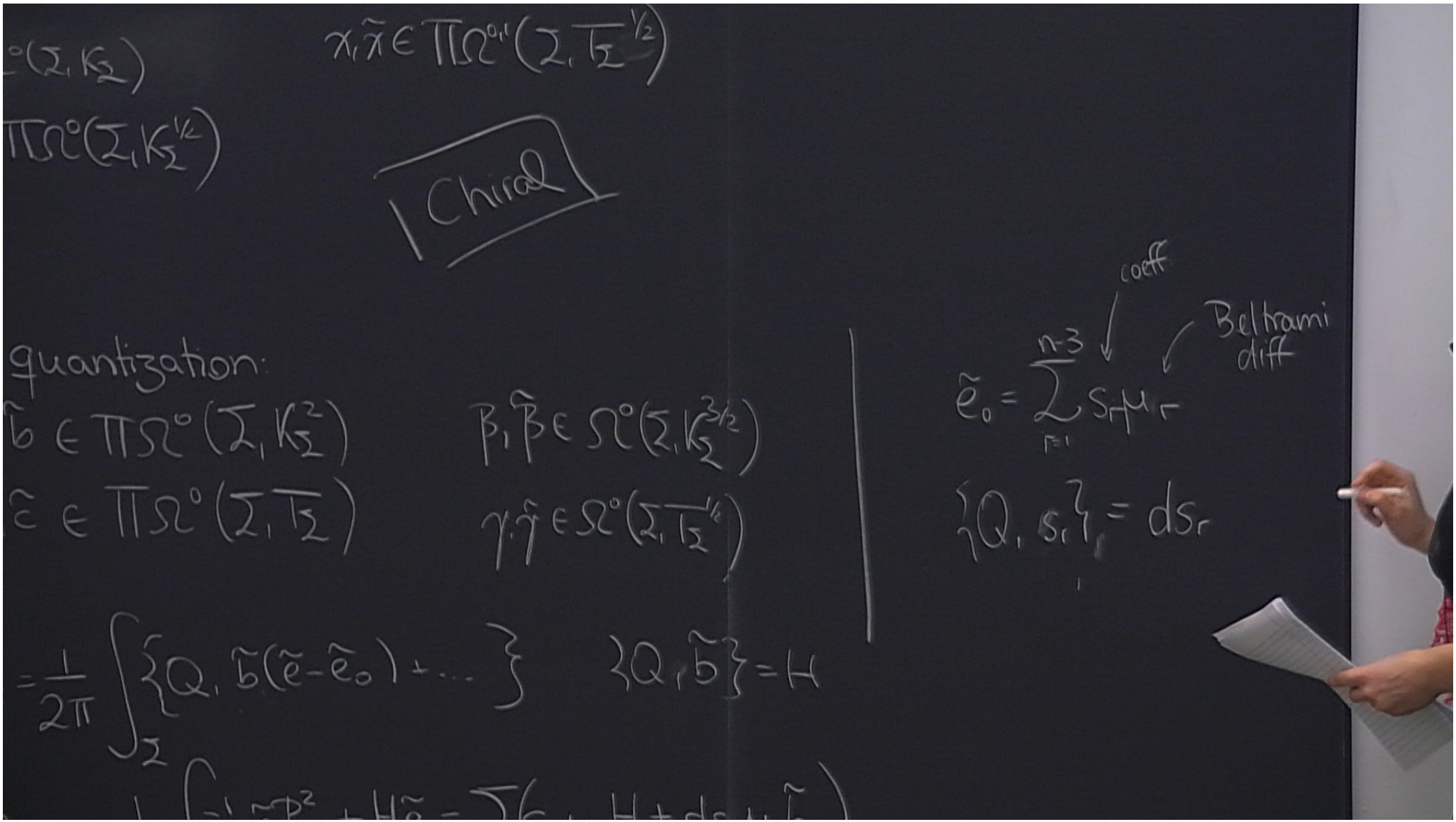
modular invariance

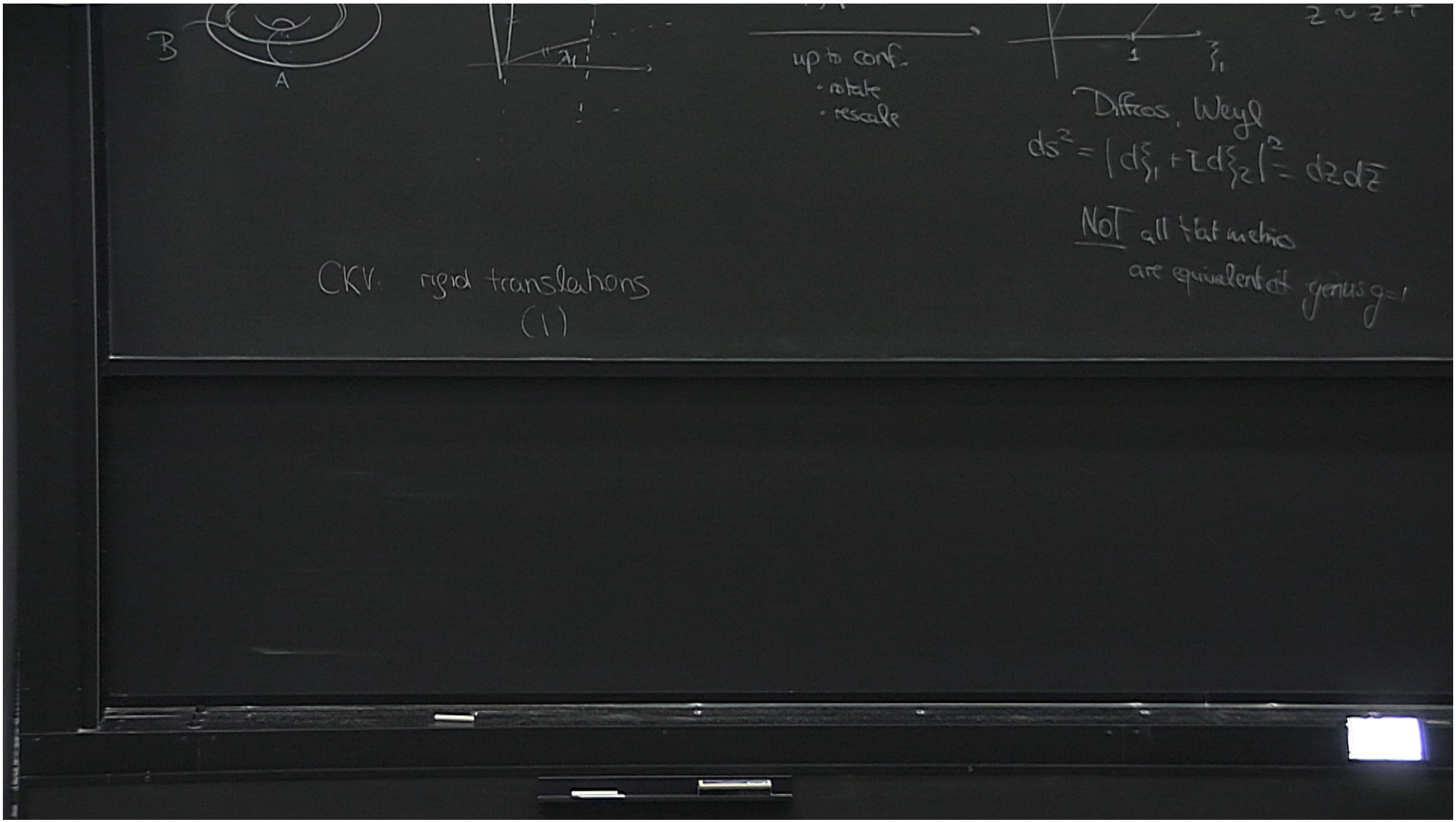
$\Rightarrow$ Sum over spin structures


S: (1 1) → (1 1)
odd spin struct
# zero modes = 1
(0 0) → (0 0)
(0 1) → (1 0)
(1 0) → (1 1)
# zero modes = 0
⇒ even
modular invariance
⇒ Sum over spin structures

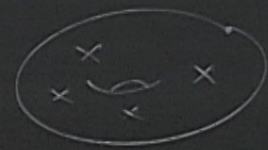

Action
$S_A = \frac{1}{2\pi}\int P\cdot\bar\partial X - \frac{\tilde e}{2}P^2$
$+\frac{1}{2}\psi\cdot\bar\partial\psi - \chi P\cdot\psi$
$+\frac{1}{2}\tilde\psi\cdot\bar\partial\tilde\psi - \tilde\chi P\cdot\tilde\psi$
$-e\left(P\cdot\partial X + \frac{1}{2}\psi\cdot\partial\psi + \frac{1}{2}\tilde\psi\cdot\partial\tilde\psi\right)$
no $\alpha'$
$X:\Sigma\to M$
$P_\mu\in\Omega^0(\Sigma,K_\Sigma)$
$\psi_\mu,\tilde\psi_\mu\in\Pi\Omega^0(\Sigma,K_\Sigma^{1/2})$
$e,\tilde e\in\Omega^{0,1}(\Sigma,T_\Sigma)$
$\chi,\tilde\chi\in\Pi\Omega^{0,1}(\Sigma,T_\Sigma^{1/2})$
Chiral
Symmetries
· diffeo
· "Ambitwistor", $\delta_\alpha\tilde e=\bar\partial\alpha$
· fermionic
BRST quantization:
$b,\tilde b\in\Pi\Omega^0(\Sigma,K_\Sigma^2)$
$c,\tilde c\in\Pi\Omega^0(\Sigma,T_\Sigma)$
$\beta,\tilde\beta\in\Omega^0(\Sigma,K_\Sigma^{3/2})$
$\gamma,\tilde\gamma\in\Omega^0(\Sigma,T_\Sigma^{1/2})$
$S_{GF}=\frac{1}{2\pi}\int_\Sigma\{Q,\tilde b(\tilde e-\tilde e_0)+\dots\}$
$\{Q,\tilde b\}=H$
$\Rightarrow S\supset\frac{1}{2\pi}\int\frac{1}{2}\tilde eP^2+H\tilde e-\Sigma(s_r\mu_r H+ds_r\mu_r b)$


Chiral
Symmetries
· diffeo
· "Ambitwistor", $\delta_\alpha \tilde{e} = \bar{\partial}\alpha$
· fermionic
Beltrami diff

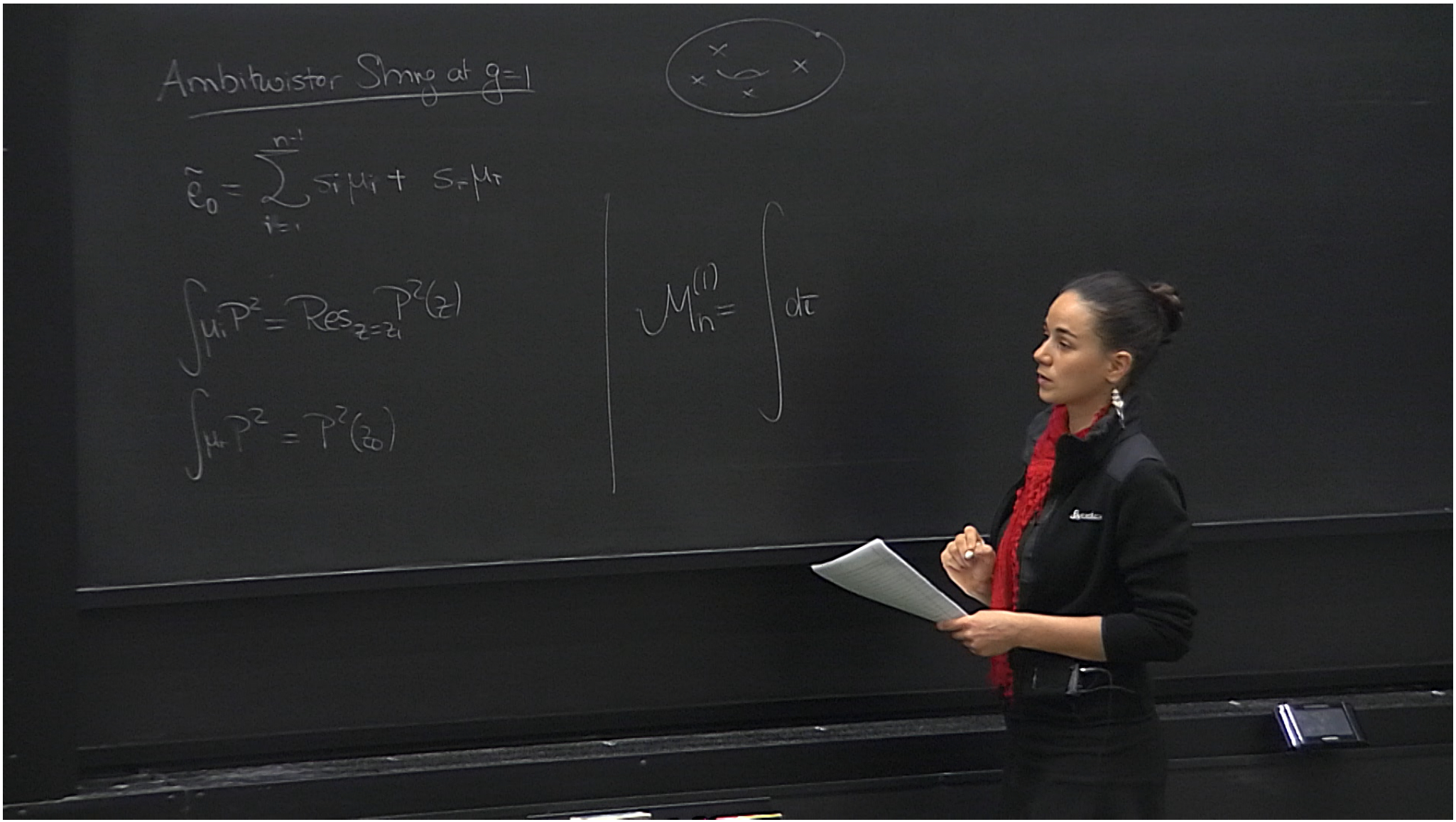

$x, \tilde{x} \in \Pi\Omega^{0,1}(\Sigma, T_\Sigma^{1/2})$
$\Pi\Omega^0(\Sigma, K_\Sigma^{1/2})$
Chiral
quantization:
$\tilde{b} \in \Pi\Omega^0(\Sigma, K_\Sigma^2)$
$\beta, \tilde{\beta} \in \Omega^0(\Sigma, K_\Sigma^{3/2})$
$\tilde{c} \in \Pi\Omega^0(\Sigma, T_\Sigma)$
$\gamma, \tilde{\gamma} \in \Omega^0(\Sigma, T_\Sigma^{1/2})$
$= \frac{1}{2\pi}\int_\Sigma \{Q, \tilde{b}(\tilde{e} - \tilde{e}_0) + \dots\}$
$\{Q, \tilde{b}\} = H$
coeff
Beltrami diff
$\tilde{e}_0 = \sum_{r=1}^{n-3} s_r \mu_r$
$\{Q, s_r\} = ds_r$

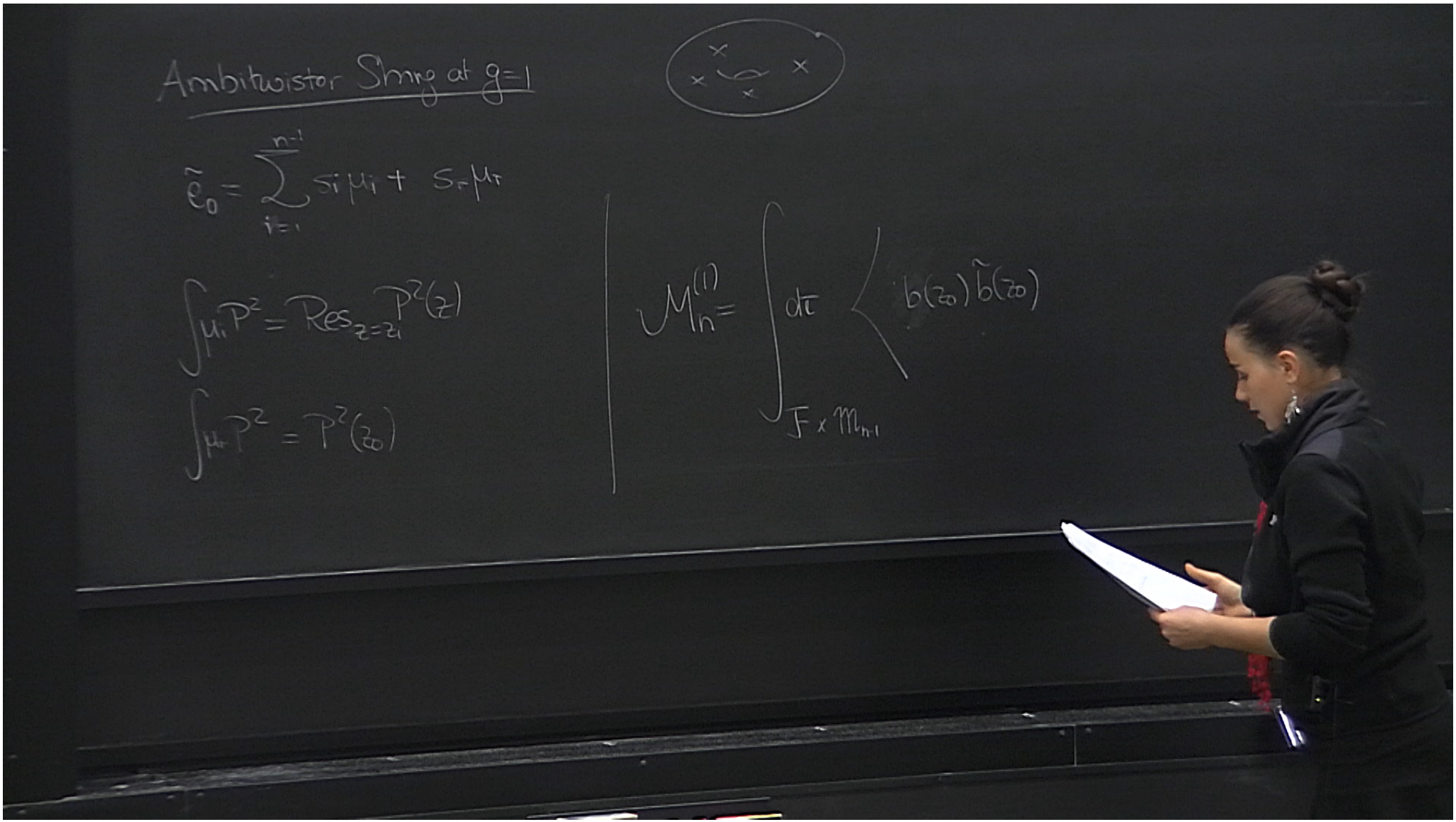

B
A
$\lambda_1$
up to conf.
· rotate
· rescale
1
$\xi_1$
$z \sim z + \tau$
Diffeos, Weyl
$ds^2 = |d\xi_1 + \tau d\xi_2|^2 = dz\,d\bar{z}$
NOT all flat metrics
are equivalent at genus $g=1$
CKV: rigid translations
(1)

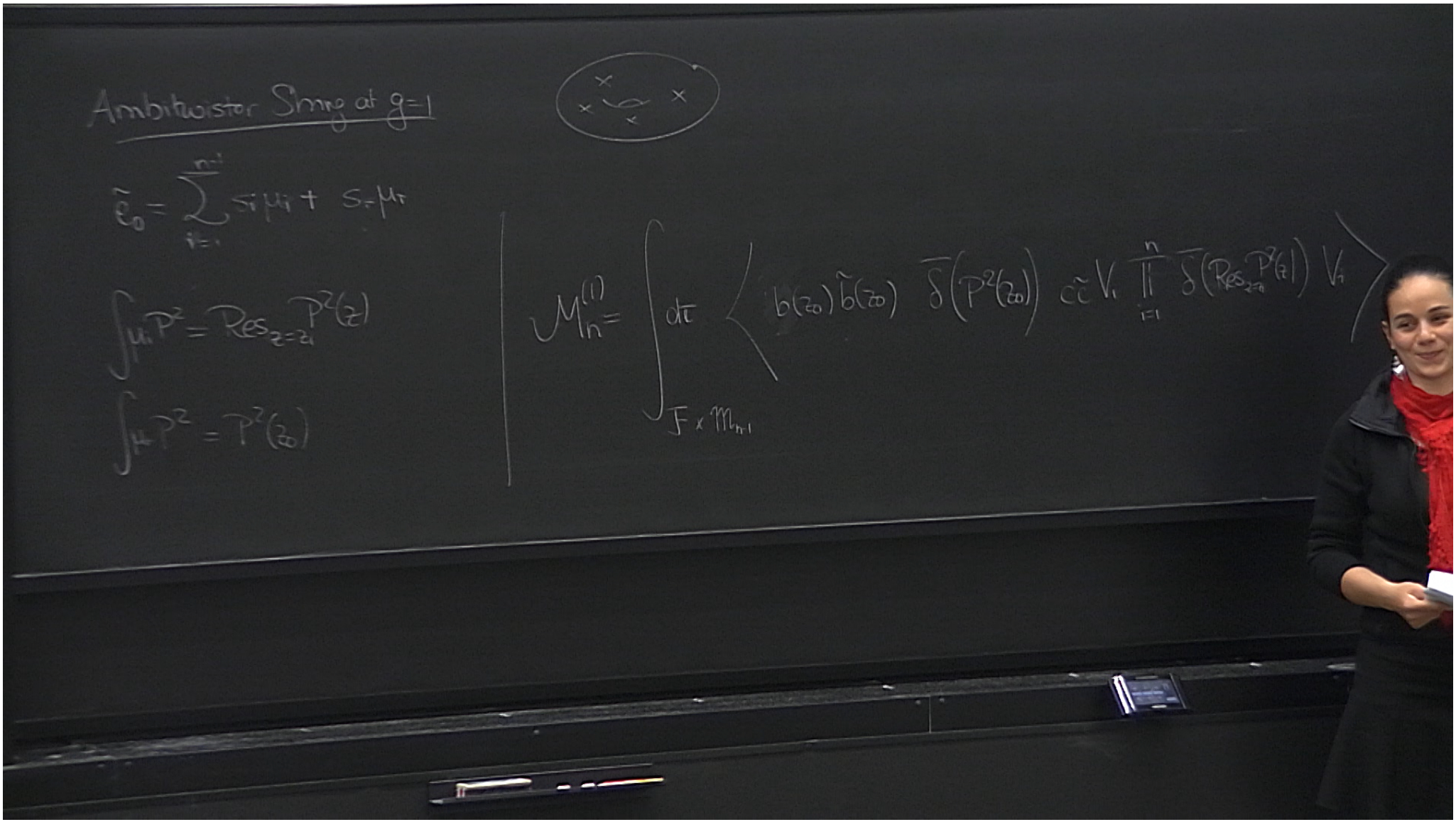
## Ambitwistor String at $g=1$



$$\tilde{\ell}_0 = \sum_{i=1}^{n-1} s_i \mu_i + t$$


Ambitwistor String at g=1
$\tilde{\ell}_0 = \sum_{i=1}^{n-1} s_i \mu_i + s_r \mu_r$
$\int \mu_i P^2 = \mathrm{Res}_{z=z_i} P^2(z)$
$\int \mu_r P^2 = P^2(z_0)$
$\mathcal{M}_n^{(1)} = \int d\tau$


Ambitwistor String at g=1
$\tilde{\ell}_0 = \sum_{i=1}^{n-1} s_i \mu_i + s_r \mu_r$
$\int \mu_i P^2 = \mathrm{Res}_{z=z_i} P^2(z)$
$\int \mu_r P^2 = P^2(z_0)$
$\mathcal{M}_n^{(1)} = \int_{\mathcal{F} \times \mathcal{M}_{n-1}} d\tau \langle b(z_0) \tilde{b}(z_0)$


Ambitwistor String at g=1
$\tilde{\mathcal{E}}_0 = \sum_{i=1}^{n-1} s_i \mu_i + s_r \mu_r$
$\int \mu_i P^2 = \mathrm{Res}_{z=z_i} P^2(z)$
$\int \mu_r P^2 = P^2(z_0)$
$\mathcal{M}_n^{(1)} = \int_{\mathcal{F} \times \mathcal{M}_{n-1}} d\tau \left\langle b(z_0)\tilde{b}(z_0)\, \delta(P^2(z_0))\, c\tilde{c}\, V_1 \prod_{i=1}^{n} \delta(\mathrm{Res}_{z=z_i} P^2(z))\, V_i \right\rangle$

# Vertex operators

$$\mathcal{U}_1 = c\tilde{c}\,\delta(\gamma)\,\delta(\tilde{\gamma})\,\epsilon_i\psi\,\tilde{\epsilon}_i\cdot\tilde{\psi}\,e^{ik_i\cdot X}$$

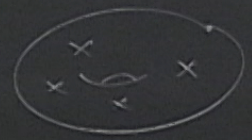
$$k_i^2 = \epsilon_i\cdot k_i = \tilde{\epsilon}_i\cdot k_i = 0$$

$$\epsilon_i^{\mu\nu} = \epsilon_i^{(\mu}\epsilon_i^{\nu)}$$

$$V_1 = \left(\epsilon_i\cdot P(\sigma_i) + \epsilon_i\cdot\psi\, k_i\cdot\psi\right)\left(\tilde{\epsilon}_i\cdot P(\sigma_i) + \tilde{\epsilon}_i\cdot\tilde{\psi}\, k_i\,\tilde{\psi}\right)e^{ik_i X}$$

$$V_1 = \int_I d\sigma_i\,\bar{\delta}\left(k_i P(\sigma_i)\right)V_1$$

## Ambitwistor String at $g=1$

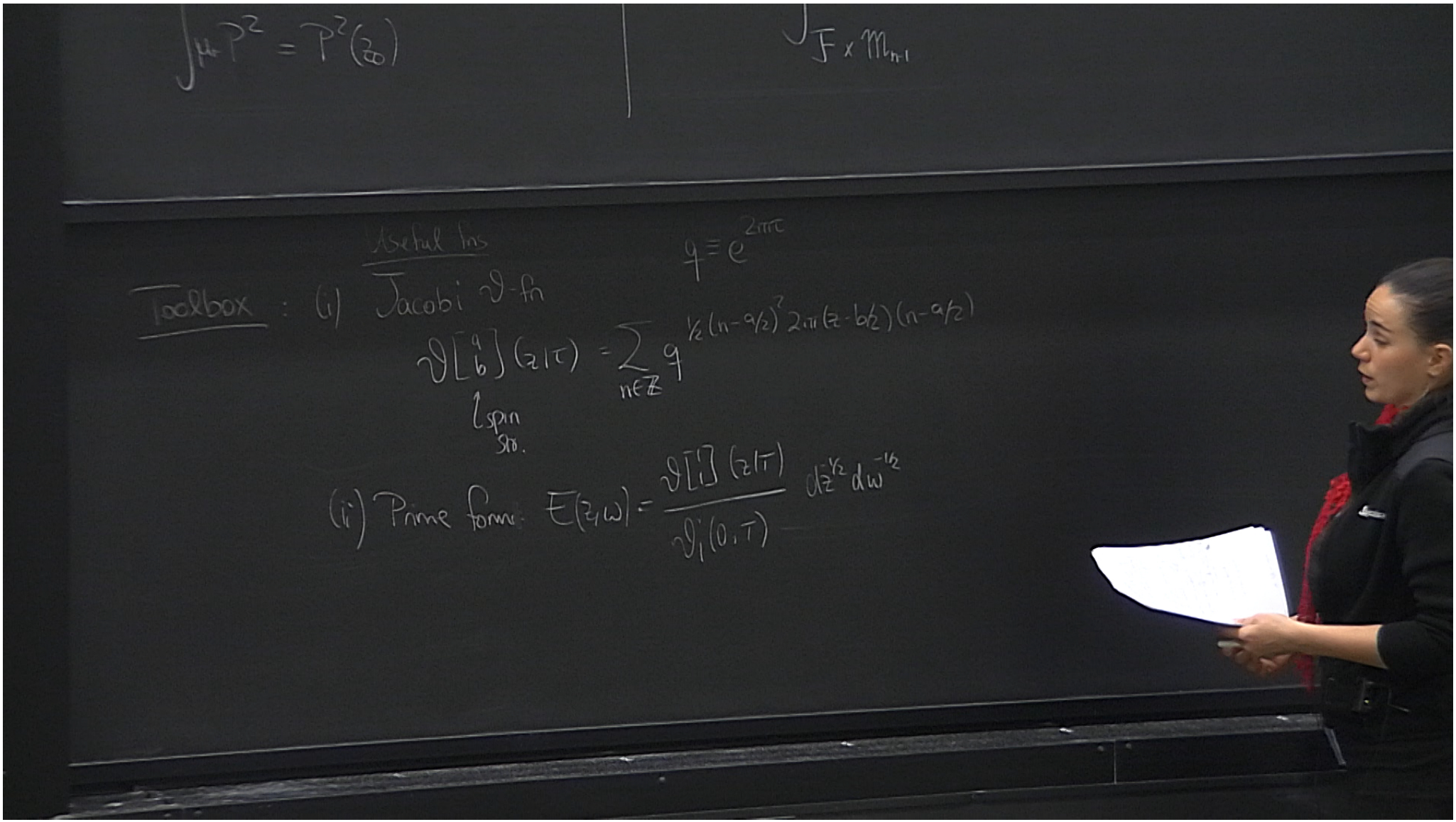


$$\tilde{\mathcal{E}}_0 = \sum_{i=1}^{n-1} s_i \mu_i + s_r \mu_r$$

$$\int \mu_i P^2 = \mathrm{Res}_{z=z_i} P^2(z)$$

$$\int \mu_r P^2 = P^2(z_0)$$

$$\mathcal{M}_n^{(1)} = \int_{\mathcal{F} \times \mathcal{M}_{n-1}} d\tau \left\langle b(z_0)\tilde{b}(z_0)\, \bar{\delta}\big(P^2(z_0)\big)\, c\tilde{c}\, V_n \int \underbrace{\prod_{i=1}^{n-1} \bar{\delta}\big(\mathrm{Res}_{z=z_i} P^2(z)\big)}_{\mu_i} V_i \right\rangle$$


$\int \mu_+ P^2 = P^2(z_0)$
$\int_{F \times M_{n-1}}$
Useful fns
$q \equiv e^{2\pi i \tau}$
Toolbox : (i) Jacobi $\vartheta$-fn
$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-a/2)^2 2\pi i (z-b/2)(n-a/2)}$
spin str.
(ii) Prime form: $E(z,w) = \frac{\vartheta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z|\tau)}{\vartheta_1'(0,\tau)} dz^{-1/2} dw^{-1/2}$

$$\int_{\mu_+} P^2 = P^2(z_0)$$

$$\int_{F \times \mathcal{M}_{n,1}}$$

Useful fns

$$q \equiv e^{2\pi i \tau}$$

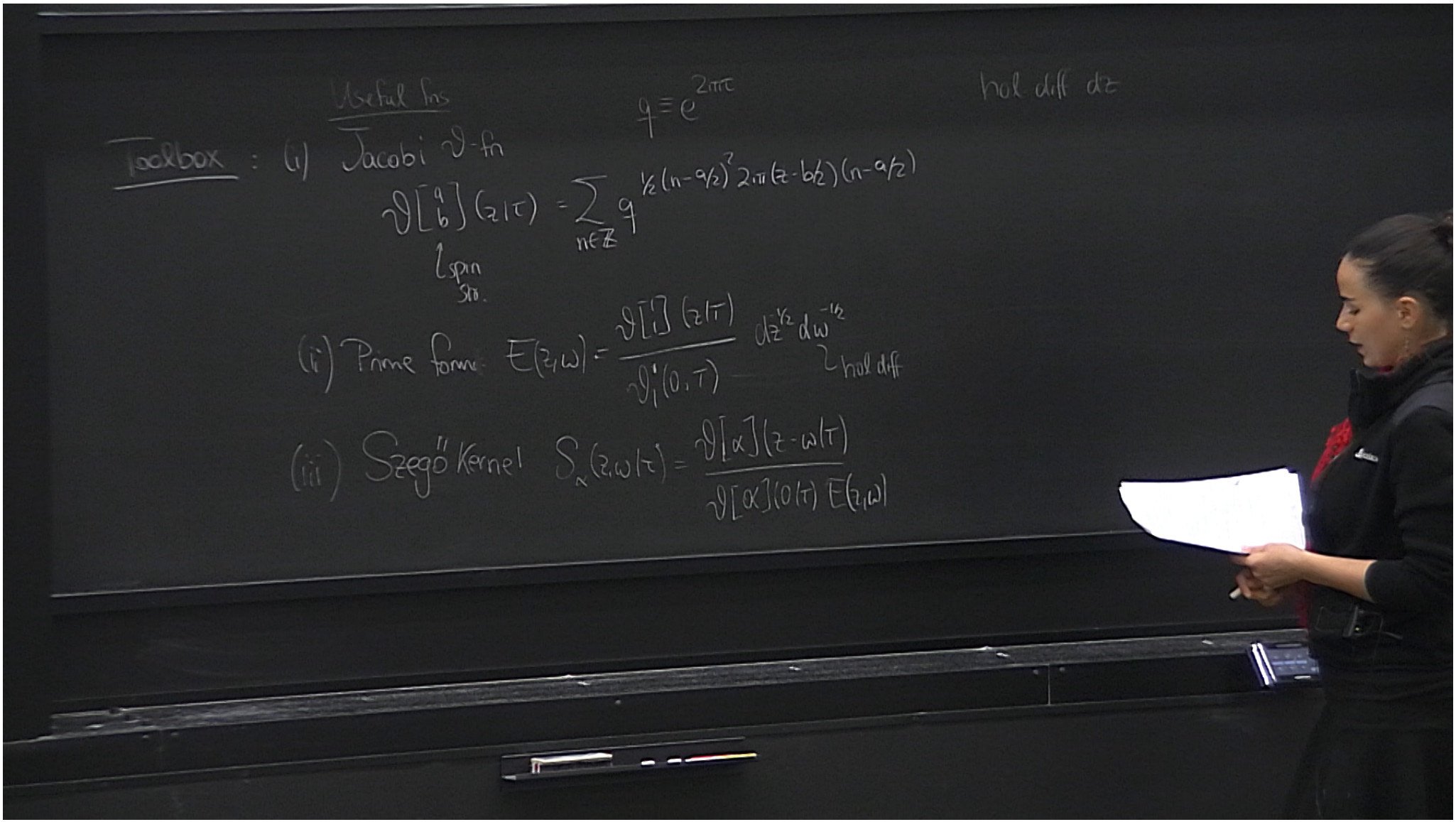
Toolbox : (i) Jacobi $\vartheta$-fn

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-a/2)^2} e^{2\pi i (z-b/2)(n-a/2)}$$

spin str.

(ii) Prime form: $E(z,w) = \dfrac{\vartheta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z|\tau)}{\vartheta_1'(0,\tau)} dz^{-1/2} dw^{-1/2}$

hol diff


Useful fns
$q \equiv e^{2\pi i \tau}$
hol diff $dz$
Toolbox : (i) Jacobi $\vartheta$-fn
$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a/2)^2} 2\pi i (z-b/2)(n-a/2)$
spin str.
(ii) Prime form $E(z,w) = \frac{\vartheta[1](z|\tau)}{\vartheta_1'(0,\tau)} dz^{1/2} dw^{-1/2}$
hol diff
(iii) Szegő Kernel $S_\alpha(z,w|\tau) = \frac{\vartheta[\alpha](z-w|\tau)}{\vartheta[\alpha](0|\tau)\, E(z,w)}$

Useful fns

$q \equiv e^{2\pi i \tau}$

hol diff $dz$

Toolbox : (i) Jacobi $\vartheta$-fn

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a/2)^2} e^{2\pi i (z-b/2)(n-a/2)}$$

spin str.

Asymptotics

(ii) Prime form $E(z,w) = \dfrac{\vartheta[1](z|\tau)}{\vartheta_1'(0,\tau)} dz^{1/2} dw^{-1/2}$ $\sim (z-w)\, dz^{-1/2} dw^{-1/2}$

hol diff

(iii) Szegő Kernel $S_\alpha(z,w|\tau) = \dfrac{\vartheta[\alpha](z-w|\tau)}{\vartheta[\alpha](0|\tau)\, E(z,w)}$ $\sim \dfrac{1}{z-w}$

Useful fns

$q \equiv e^{2\pi i \tau}$

hol diff $dz$

Toolbox : (i) Jacobi $\vartheta$-fn

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a/2)^2} e^{2\pi i (z-b/2)(n-a/2)}$$

spin str.

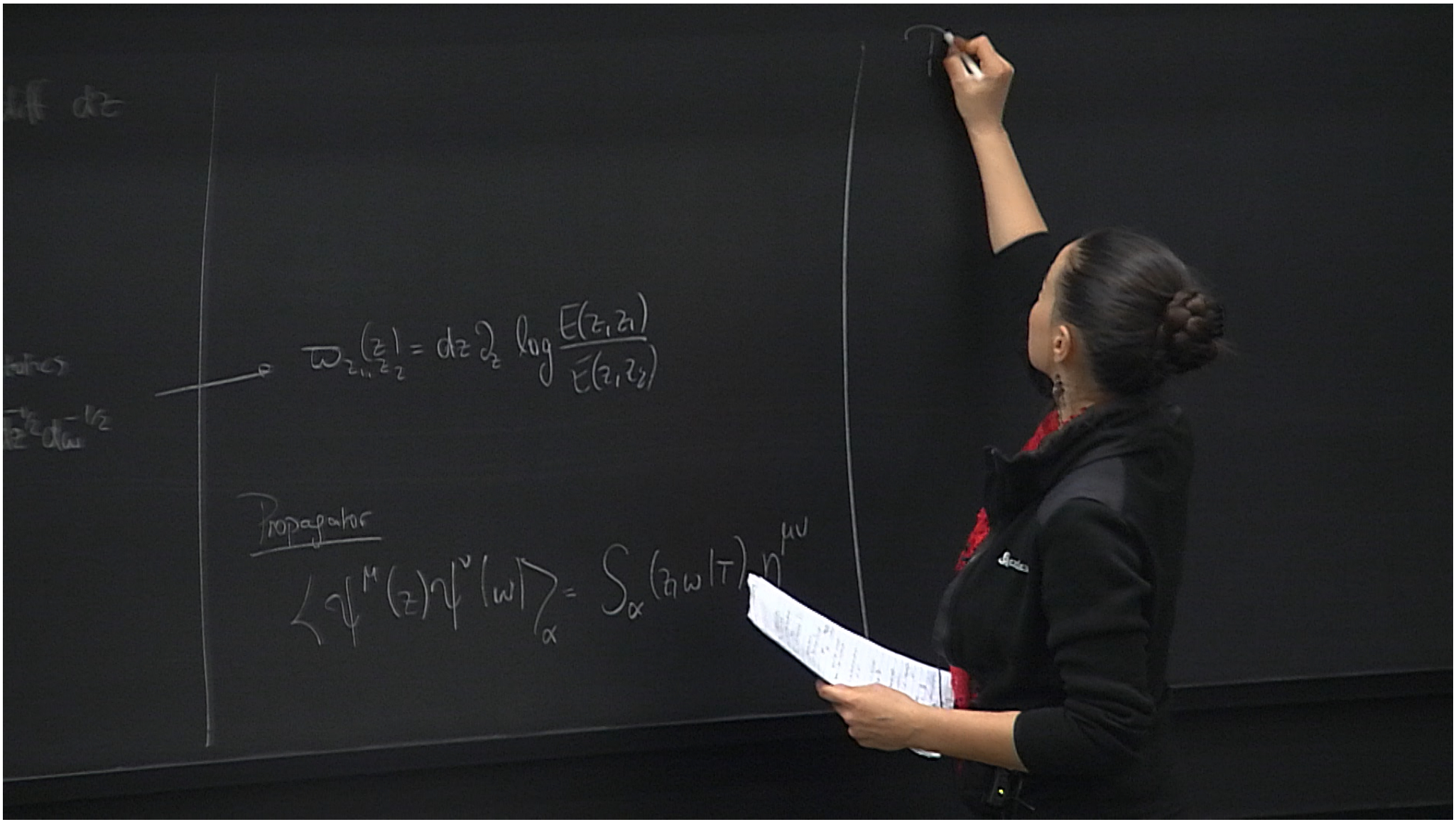
Asymptotics

(ii) Prime form: $E(z,w) = \dfrac{\vartheta[1](z|\tau)}{\vartheta_1'(0,\tau)} dz^{1/2} dw^{-1/2}$ $\sim (z-w)\, dz^{-1/2} dw^{-1/2}$

hol diff

(iii) Szegő Kernel $S_\alpha(z,w|\tau) = \dfrac{\vartheta[\alpha](z-w|\tau)}{\vartheta[\alpha](0|\tau)\, E(z,w)}$ $\sim \dfrac{1}{z-w}$

$\langle \psi^\mu(z)\psi^\nu(w)\rangle_\alpha = S_\alpha(z,w|\tau)\,\eta^{\mu\nu}$


$\omega_{z_1,z_2}(z) = dz\,\partial_z \log \frac{E(z,z_1)}{E(z,z_2)}$
Propagator
$\langle \psi^\mu(z)\psi^\nu(w)\rangle_\alpha = S_\alpha(z,w|\tau)\,\eta^{\mu\nu}$

## Vertex operators

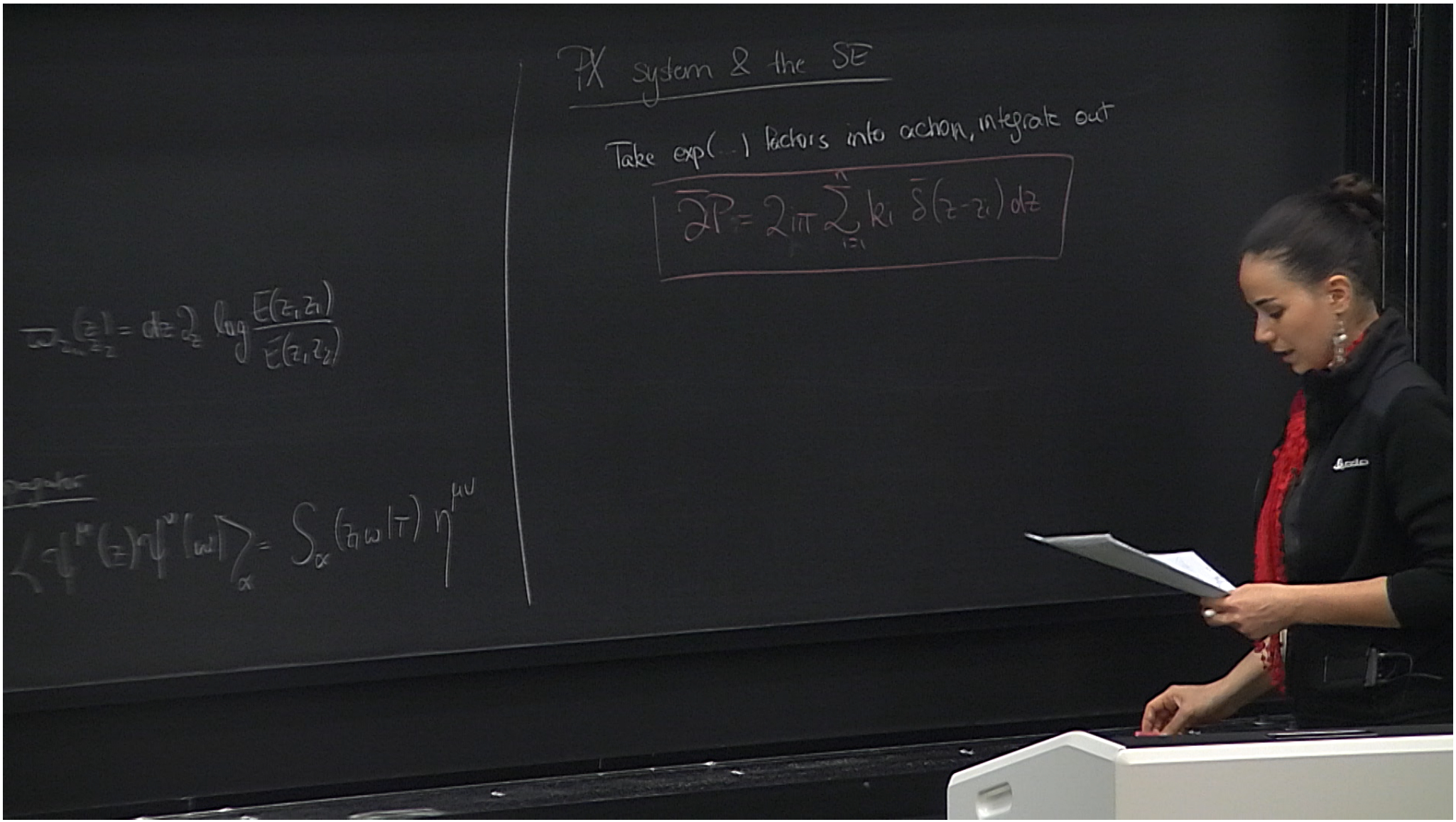
$$\mathcal{U}_1 = c\tilde{c}\,\delta(\gamma)\,\delta(\tilde{\gamma})\,\epsilon_i\cdot\psi\,\tilde{\epsilon}_i\cdot\tilde{\psi}\,e^{ik_i\cdot X}$$

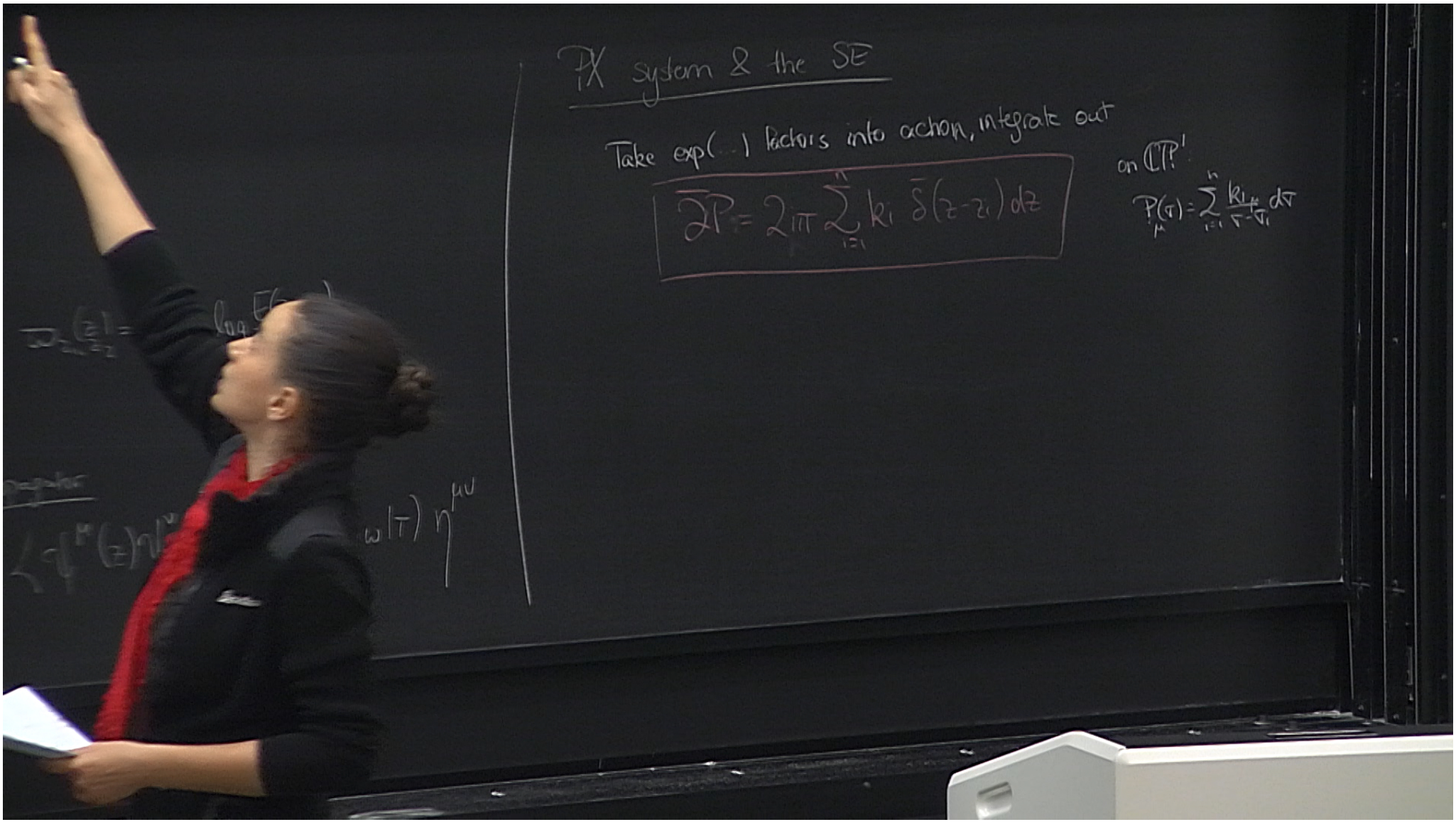
$$k_i^2 = \epsilon_i\cdot k_i = \tilde{\epsilon}_i\cdot k_i = 0$$

$$\epsilon_i^{\mu\nu} = \epsilon_i^{(\mu}\epsilon_i^{\nu)}$$

$$V_1 = \left(\epsilon_i\cdot P(\sigma_i) + \epsilon_i\cdot\psi\, k_i\cdot\psi\right)\left(\tilde{\epsilon}_i\cdot P(\sigma_i) + \tilde{\epsilon}_i\cdot\tilde{\psi}\, k_i\cdot\tilde{\psi}\right)e^{ik_i\cdot X}$$

$$\mathcal{V}_1 = \int_\Sigma d\sigma\,\bar{\delta}\left(k_i\cdot P(\sigma_i)\right)V_1$$


PX system & the SE
Take exp( ) factors into action, integrate out
$\bar{\partial} P = 2i\pi \sum_{i=1}^{n} k_i \, \bar{\delta}(z - z_i) \, dz$
$\omega_{z_1 z_2}(z) = dz \, \partial_z \log \frac{E(z, z_1)}{E(z, z_2)}$
$\langle \psi^\mu(z) \psi^\nu(w) \rangle_\alpha = S_\alpha(z, w \mid \tau) \, \eta^{\mu\nu}$


PX system & the SE
Take exp( ) factors into action, integrate out
$\bar{\partial} P = 2i\pi \sum_{i=1}^{n} k_i \, \bar{\delta}(z - z_i) \, dz$
on $\mathbb{CP}^1$
$P_\mu(\sigma) = \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i} d\sigma$

$\log \frac{E(z_1, z_1)}{E(z_1, z_2)}$

$S_\alpha(z, w | \tau)\, \eta^{\mu\nu}$

## PX system & the SE

Take exp(...) factors into action, integrate out

$$\boxed{\bar{\partial} P = 2\pi \sum_{i=1}^{n} k_i\, \bar{\delta}(z - z_i)\, dz}$$

on $\mathbb{CP}^1$:

$$P_\mu(\sigma) = \sum_{i=1}^{n} \frac{k_{i\mu}}{\sigma - \sigma_i}\, d\sigma$$

on torus: $P =$

Useful fns

$q \equiv e^{2\pi i \tau}$

$\boxed{\text{hol diff } dz}$

Toolbox : (i) Jacobi $\vartheta$-fn

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a/2)^2} e^{2\pi i (z-b/2)(n-a/2)}$$

spin str.

(ii) Prime form: $E(z,w) = \dfrac{\vartheta[1](z|\tau)}{\vartheta_1'(0,\tau)} dz^{-1/2} dw^{-1/2}$ hol diff

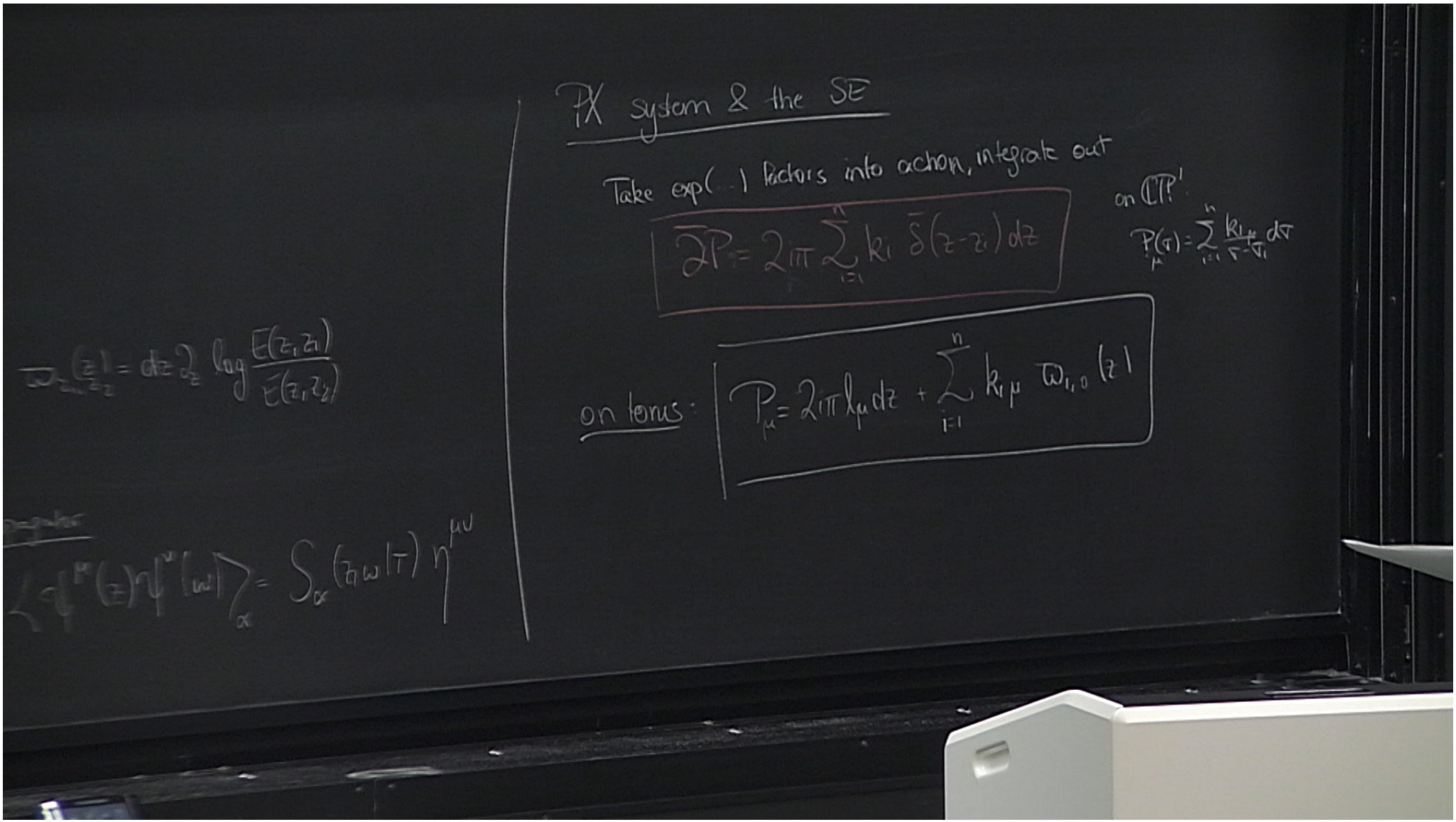
Asymptotics $\sim (z-w)\, dz^{-1/2} dw^{-1/2}$

(iii) Szegő Kernel $S_\alpha(z,w|\tau) = \dfrac{\vartheta[\alpha](z-w|\tau)}{\vartheta[\alpha](0|\tau)\, E(z,w)} \sim \dfrac{1}{z-w}$

$\omega_{z_1 w_1 z_2} = dz\, \partial_z \log \dfrac{E(z_1 \ldots}{E(z \ldots}$

Propagator

$\langle \psi^\mu(z) \psi^\nu(w) \rangle_\alpha = \ldots$


PX system & the SE
Take exp(...) factors into action, integrate out
$\bar{\partial} P = 2i\pi \sum_{i=1}^{n} k_i \bar{\delta}(z - z_i)\, dz$
on $\mathbb{CP}^1$:
$P_\mu(\sigma) = \sum_{i=1}^{n} \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma$
on torus:
$P_\mu = 2i\pi l_\mu dz + \sum_{i=1}^{n} k_{i\mu}\, \varpi_{1,0}(z)$
$\varpi_{z_1 z_2}(z) = dz\, \partial_z \log \frac{E(z, z_1)}{E(z, z_2)}$
$\langle \psi^\mu(z) \psi^\nu(w) \rangle_\alpha = S_\alpha(z, w | \tau)\, \eta^{\mu\nu}$

# PX system & the SE

Take exp( ) factors into action, integrate out

$$\boxed{\bar{\partial} P = 2i\pi \sum_{i=1}^{n} k_i \, \bar{\delta}(z - z_i)\, dz}$$

on $\mathbb{CP}^1$:

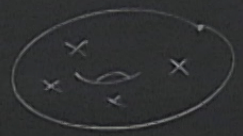
$$P_\mu(\sigma) = \sum_{i=1}^{n} \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma$$

on torus:

$$\boxed{P_\mu = 2i\pi \underbrace{l_\mu}_{\sim \text{ loop momentum}} dz + \sum_{i=1}^{n} k_{i\mu}\, \omega_{1,0}(z)}$$

$\eta^{\mu\nu}$

## Ambitwistor String at $g=1$

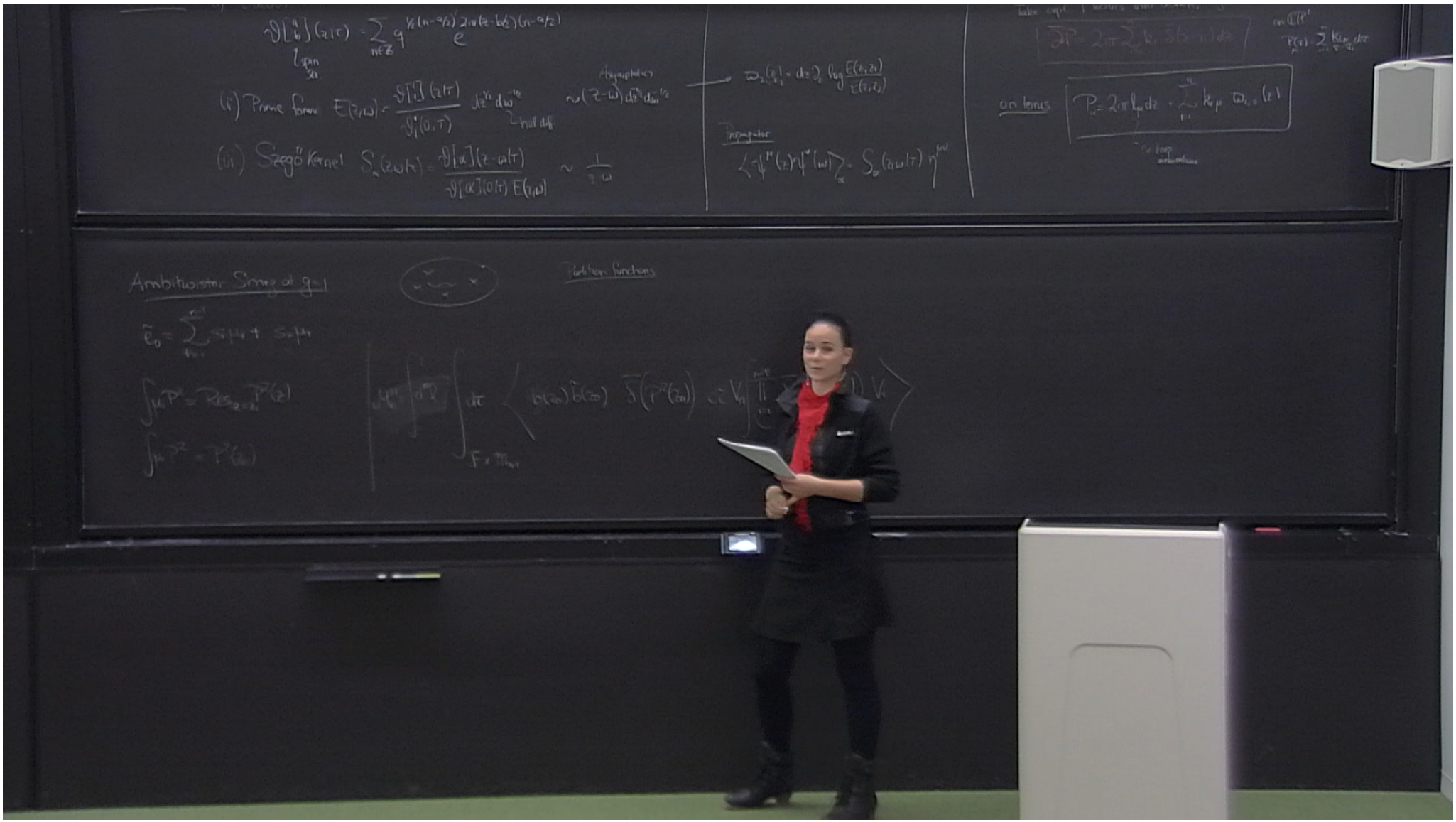


$$\tilde{\ell}_0 = \sum_{i=1}^{n-1} s_i \mu_i + s_T \mu_T$$

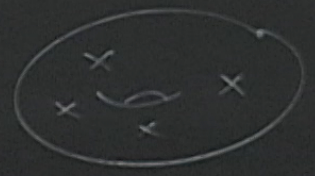
$$\int \mu_i P^2 = \mathrm{Res}_{z=z_i} P^2(z)$$

$$\int \mu_T P^2 = P^2(z_0)$$

$$\mathcal{M}_n^{(1)} = \int d^{10}\ell \int_{\mathcal{F}\times\mathcal{M}_{n-1}} d\tau \left\langle b(z_0)\tilde{b}(z_0)\, \bar{\delta}\big(P^2(z_0)\big)\, c\tilde{c}\, V_n \underbrace{\int \prod_{i=1}^{n-1} \bar{\delta}\big(k_i\cdot P(z_i)\big)}_{\mu_i} V_i \right\rangle$$


Ambitwistor Strings at g=1
Partition functions
on torus

## Ambitwistor String at $g=1$



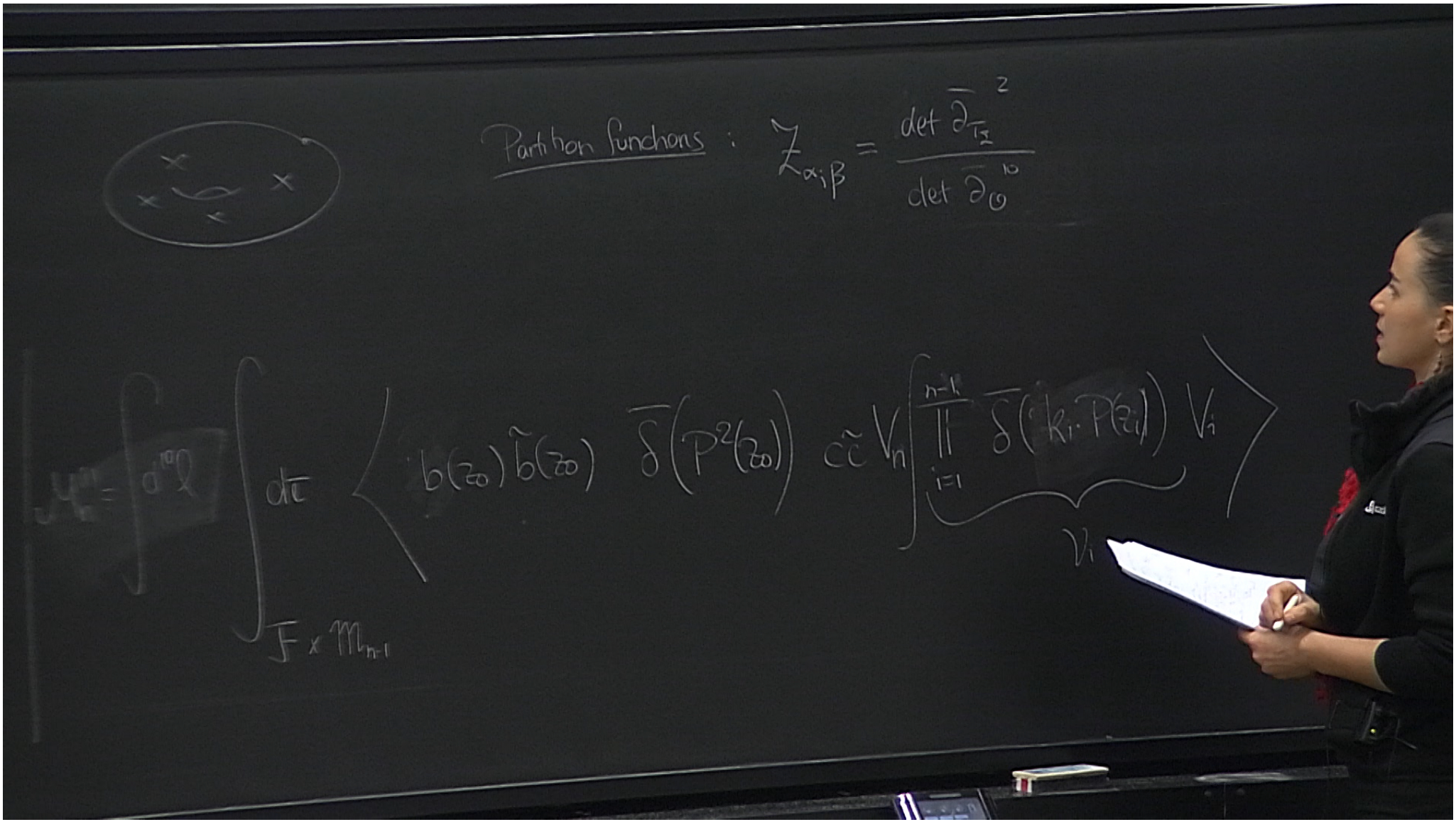
Partition functions

$$\tilde{\mathcal{E}}_0 = \sum_{i=1}^{n-1} s_i \mu_i + s_- \mu_+$$

$$\int \mu_i P^2 = \mathrm{Res}_{z=z_i} P^2(z)$$

$$\int \mu_+ P^2 = P^2(z_0)$$

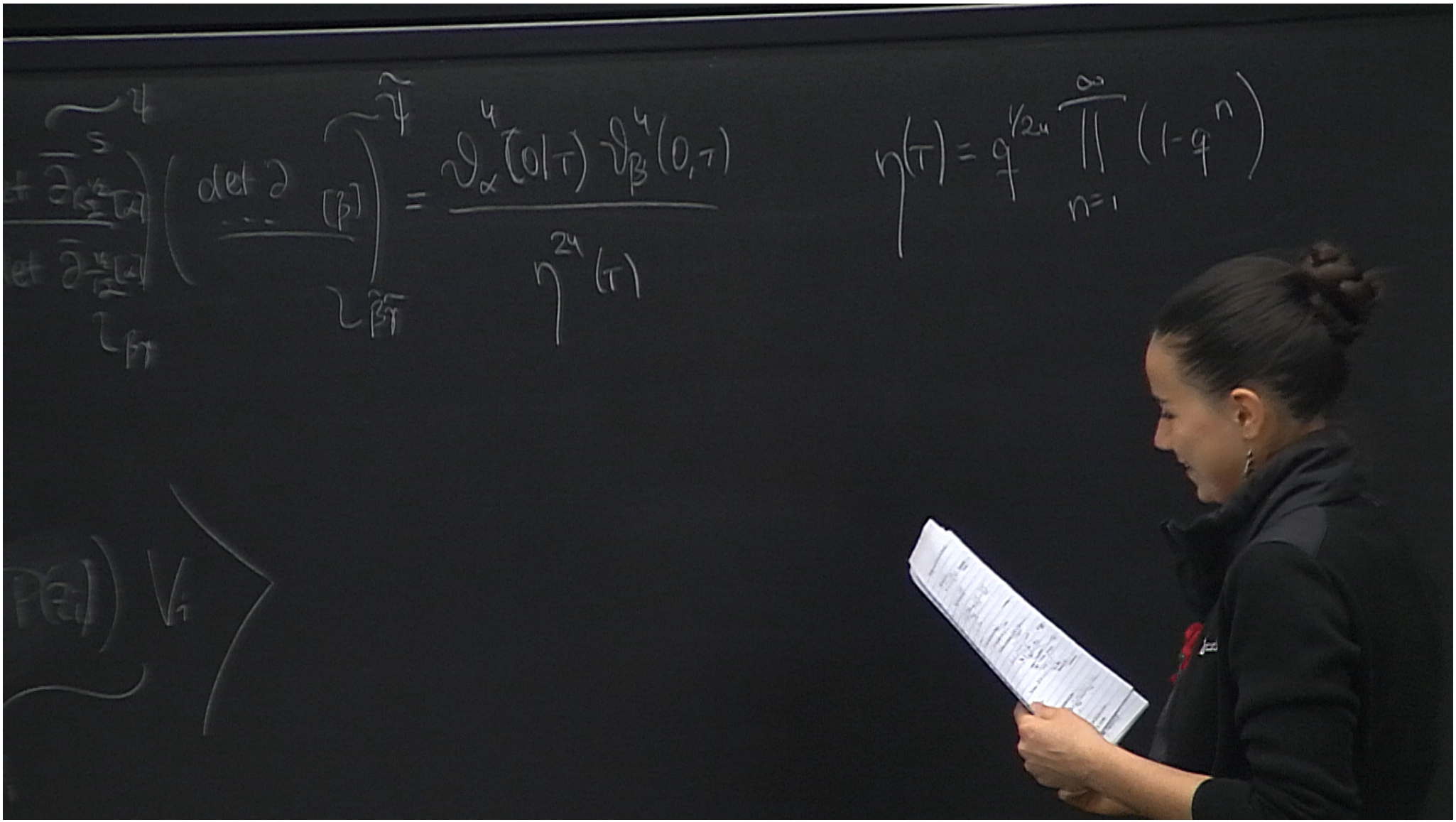
$$\mathcal{M}_n^{(1)} = \int d^{10}\ell \int_{\mathcal{F}\times\mathcal{M}_{n-1}} d\tau \left\langle b(z_0)\tilde{b}(z_0)\,\bar{\delta}\left(P^2(z_0)\right) c\tilde{c}\, V_n \int \right.$$

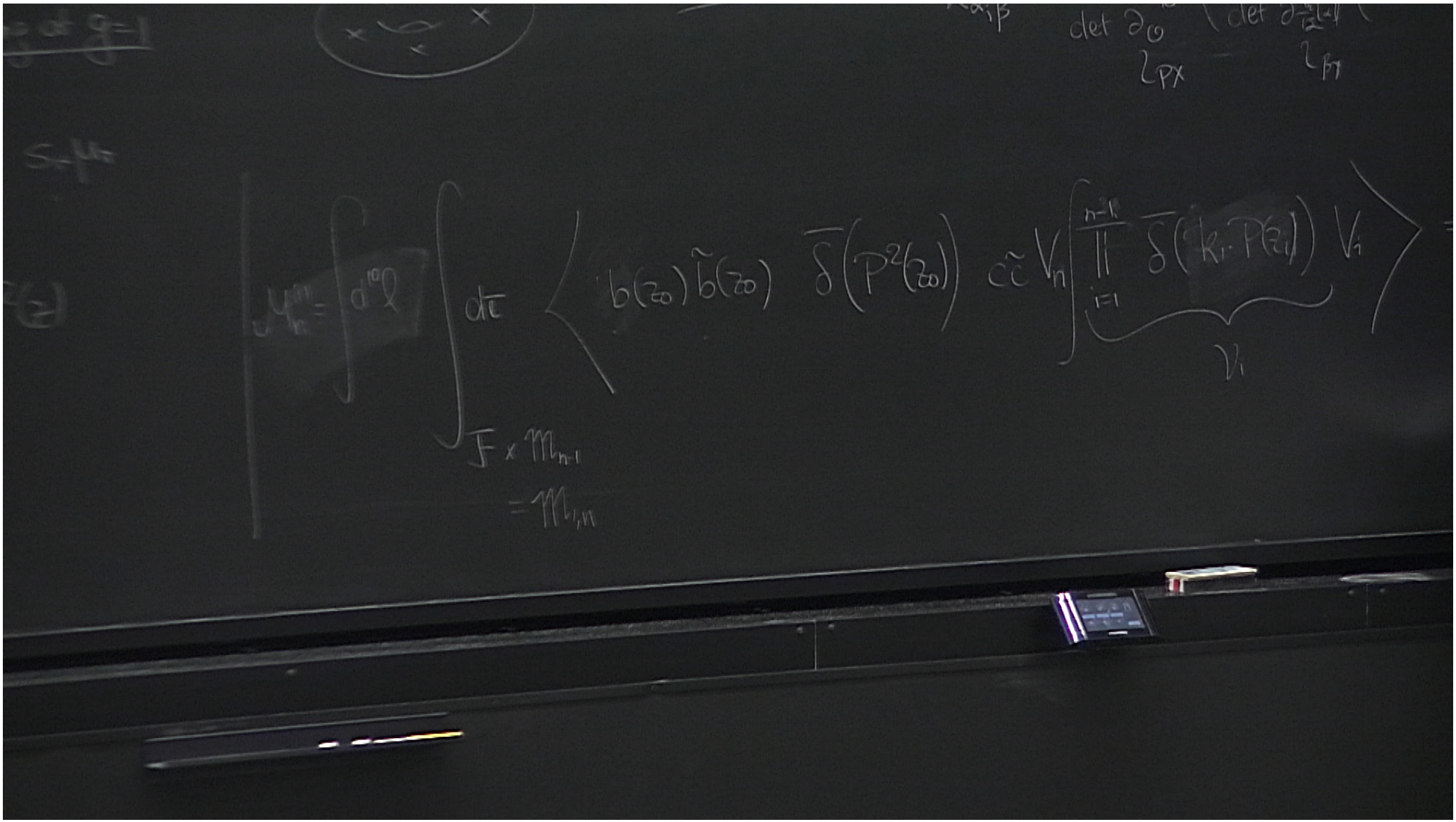

Partition functions : $Z_{\alpha;\beta} = \frac{\det \bar{\partial}_{\Sigma}^{2}}{\det \bar{\partial}_{\mathcal{O}}^{10}}$
$\int_{\mathcal{F}\times\mathcal{M}_{n-1}} d\tau \left\langle b(z_0)\tilde{b}(z_0)\,\bar{\delta}(P^2(z_0))\, c\tilde{c}\, V_n \int \underbrace{\prod_{i=1}^{n-1} \bar{\delta}(k_i\cdot P(z_i))\, V_i}_{\mathcal{V}_i} \right\rangle$

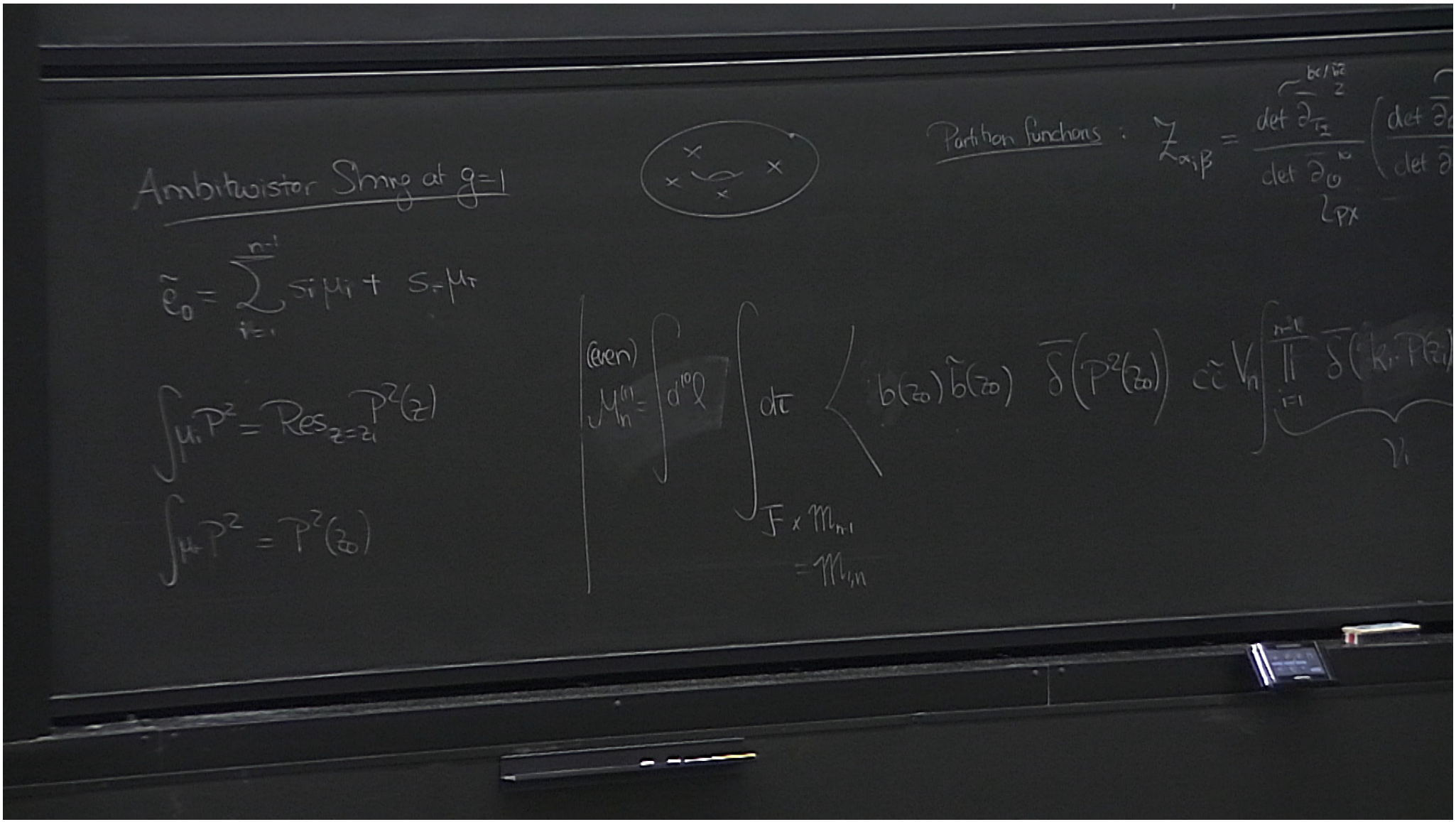
$$Z_{\alpha;\beta} = \frac{\overbrace{\det \bar\partial_{T_\Sigma}}^{bc/\bar b\bar c\ 2}}{\underbrace{\det \bar\partial_{\mathcal{O}}}_{PX}{}^{10}} \left( \frac{\overbrace{\det \bar\partial_{K_\Sigma^{1/2}[\alpha]}}^{\psi\ 5}}{\underbrace{\det \bar\partial_{T_\Sigma^{1/2}[\alpha]}}_{\beta\gamma}} \right) \left( \underbrace{\overbrace{\det \partial \ldots [\beta]}^{\tilde\psi}}_{\tilde\beta\tilde\gamma} \right) = \frac{\vartheta_\alpha^4(0|\tau)\,\vartheta_\beta^4(0,\tau)}{\eta^{24}(\tau)}$$

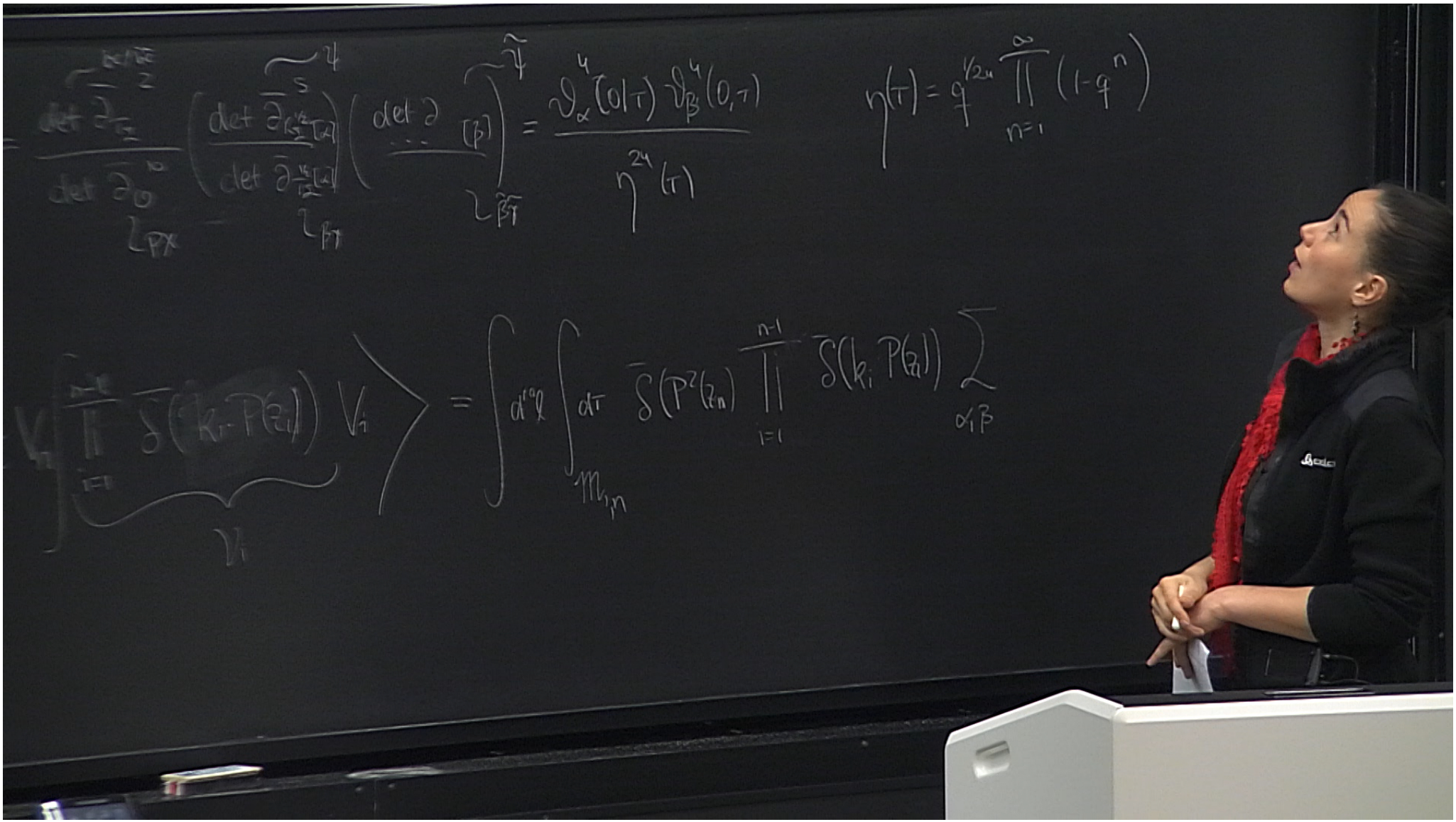
$$\eta(\tau) =$$

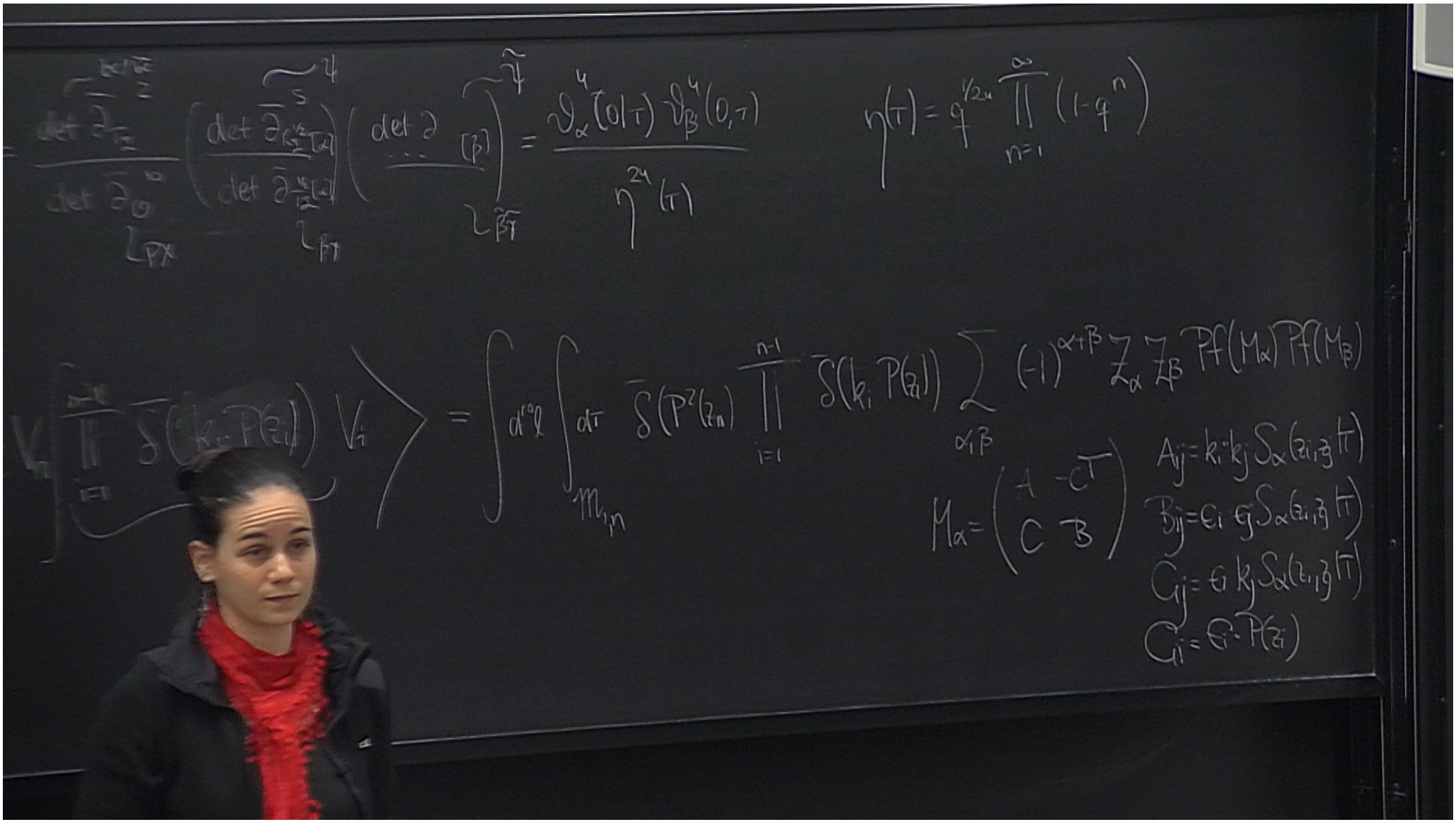
$$\ldots)) \quad c\tilde c\, V_n \int \underbrace{\prod_{i=1}^{n-1} \bar\delta(k_i \cdot P(z_i))\, V_i}_{\mathcal{V}_i} \Big\rangle$$


$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$
$= \frac{\vartheta_\alpha^4(0|\tau)\,\vartheta_\beta^4(0,\tau)}{\eta^{24}(\tau)}$


det ∂_O
ℓ_PX
ℓ_BY
M_n^(1) = ∫ d^10 ℓ ∫_{F × M_{n-1} = M_{1,n}} dτ ⟨ b(z_0) b̃(z_0) δ̄(P^2(z_0)) cc̃ V_n ∫ ∏_{i=1}^{n-1} δ̄(k_i · P(z_i)) V_i ⟩
ν_i


Ambitwistor String at g=1
Partition functions :
(even)


$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$
$= \frac{\vartheta_\alpha^4(0|\tau)\,\vartheta_\beta^4(0,\tau)}{\eta^{24}(\tau)}$
$= \int d^{10}\ell \int_{\mathcal{M}_{1,n}} d\tau\, \delta(P^2(z_n)) \prod_{i=1}^{n-1} \delta(k_i \cdot P(z_i)) \sum_{\alpha,\beta}$


$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$
$= \int d^{10}\ell \int_{\mathcal{M}_{1,n}} d\tau \, \bar{\delta}(P^2(z_n)) \prod_{i=1}^{n-1} \bar{\delta}(k_i P(z_i)) \sum_{\alpha,\beta} (-1)^{\alpha+\beta} Z_\alpha Z_\beta \, Pf(M_\alpha) Pf(M_\beta)$
$M_\alpha = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$
$A_{ij} = k_i k_j S_\alpha(z_i, z_j | \tau)$
$B_{ij} = \epsilon_i \epsilon_j S_\alpha(z_i, z_j | \tau)$
$C_{ij} = \epsilon_i k_j S_\alpha(z_i, z_j | \tau)$
$C_i = \epsilon_i \cdot P(z_i)$

$= \mathcal{M}_{k,n}$

Two features: • "Integrand" $\mathcal{M}_n^{(1)} = \int d^{10}\ell \; \underline{\mathcal{I}_n}$

$= \mathcal{M}_{1,n}$

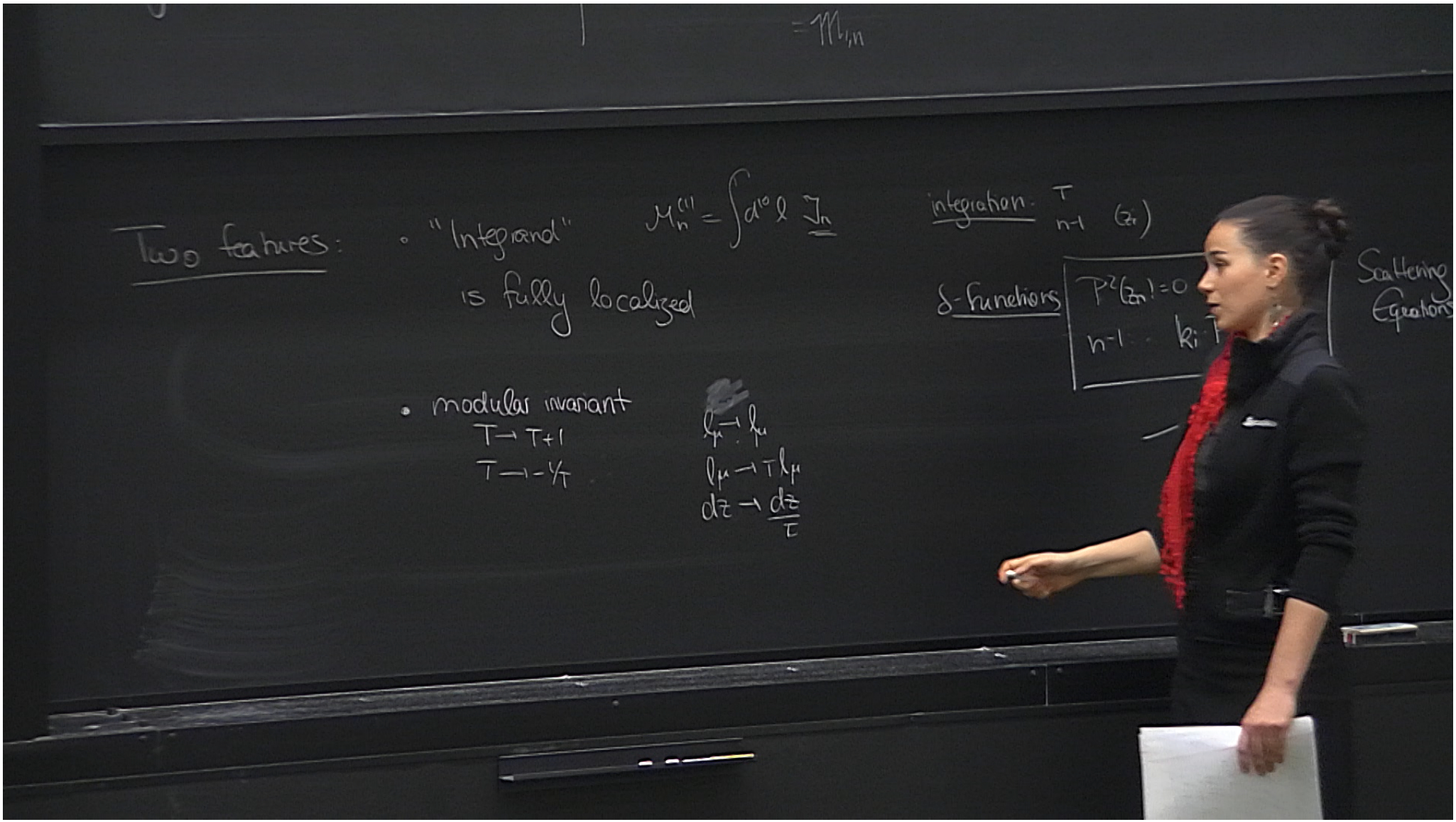
$\mathcal{M}_n^{(1)} = \int d^{10}\ell \; \underline{J_n}$

integration: $\begin{matrix} T \\ n-1 \end{matrix} \; (z_i)$

$\delta$-functions $\boxed{\begin{matrix} P^2(z_n) = 0 \\ n-1 \quad k_i \cdot P(z_i) = 0 \end{matrix}}$ Scattering Equations

$n - n = 0$

Two features:
"Integrand"
is fully localized
integration.
δ-Functions
Scattering Equations


Two features:
"Integrand"
is fully localized
modular invariant
T → T+1
T → -1/T
integration
δ-functions
Scattering Equations

$= \mathcal{M}_{1,n}$

Two features:

- "Integrand" $\mathcal{M}_n^{(1)} = \int d^{10}\ell \, \mathcal{I}_n$ is fully localized

integration: $\int_{n-1}^{T} (z_i)$

$\delta$-functions $\boxed{\begin{array}{l} P^2(z_{2n}) = 0 \\ n-1 \quad k_i \cdot P(z_i) = 0 \end{array}}$ Scattering Equations

$n - n = 0$

- modular invariant

$T \to T+1$ $\quad \ell_\mu \to \ell_\mu$

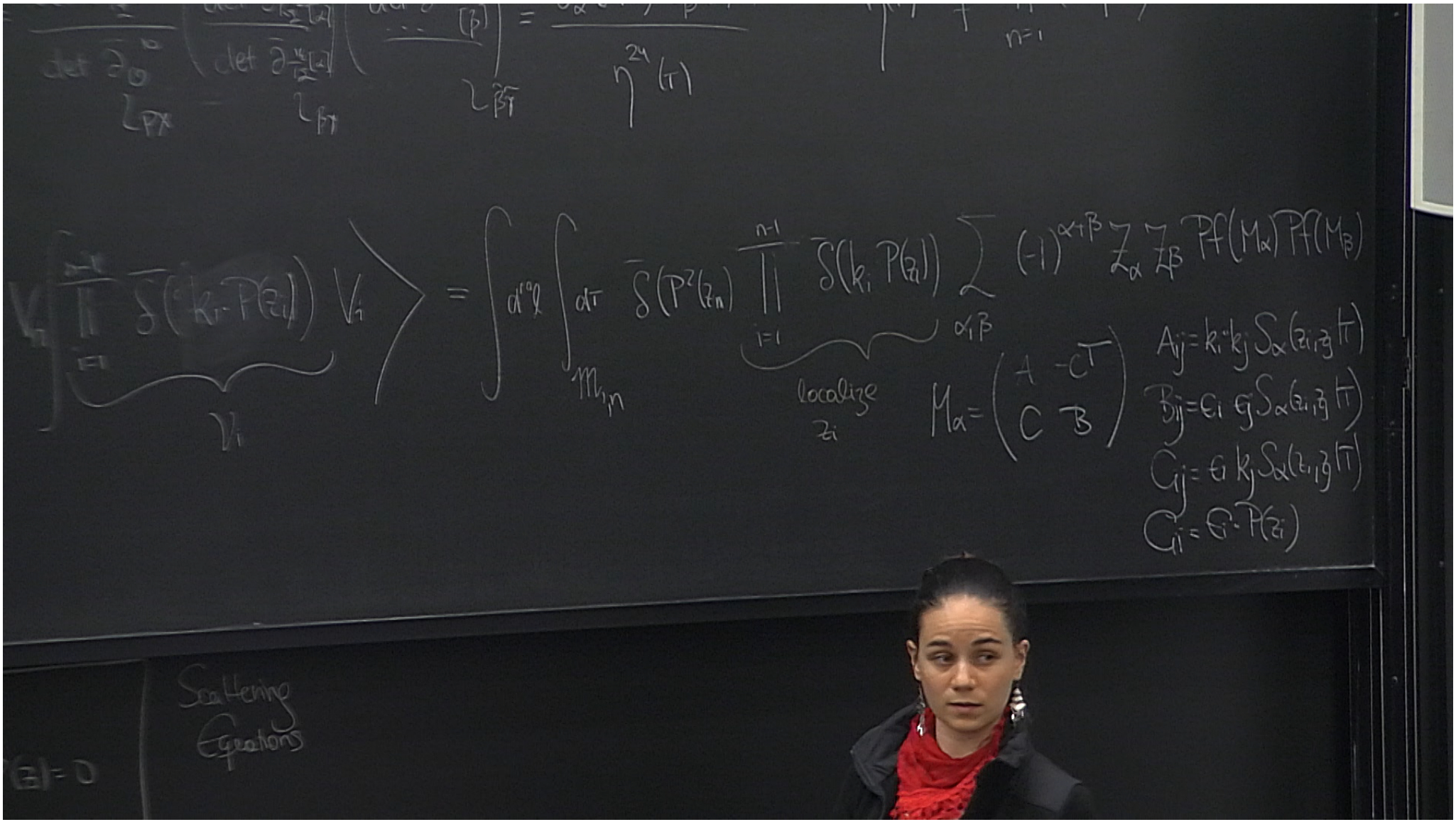
$T \to -1/T$ $\quad \ell_\mu \to T \ell_\mu$

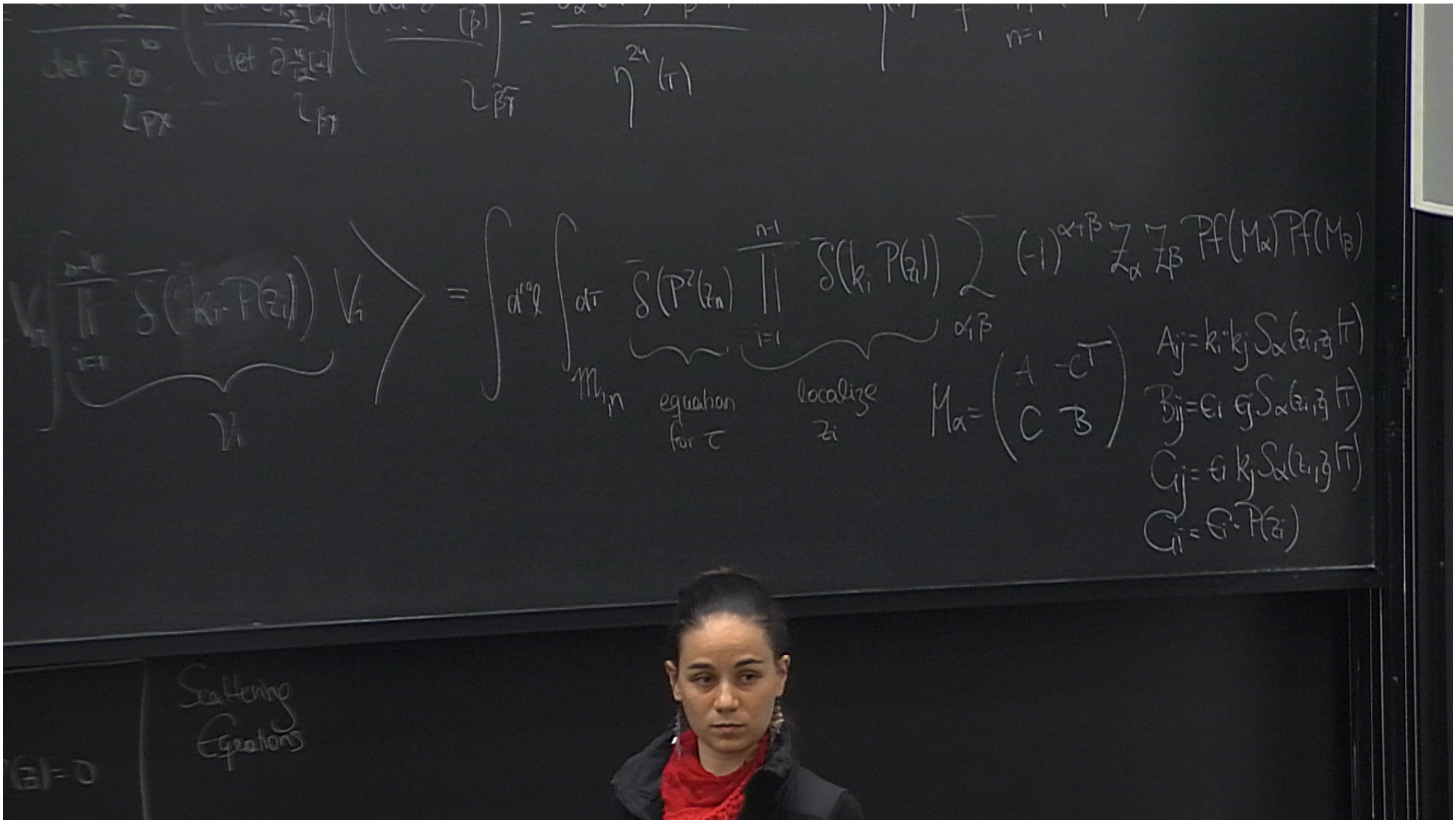
$dz \to \frac{dz}{T}$

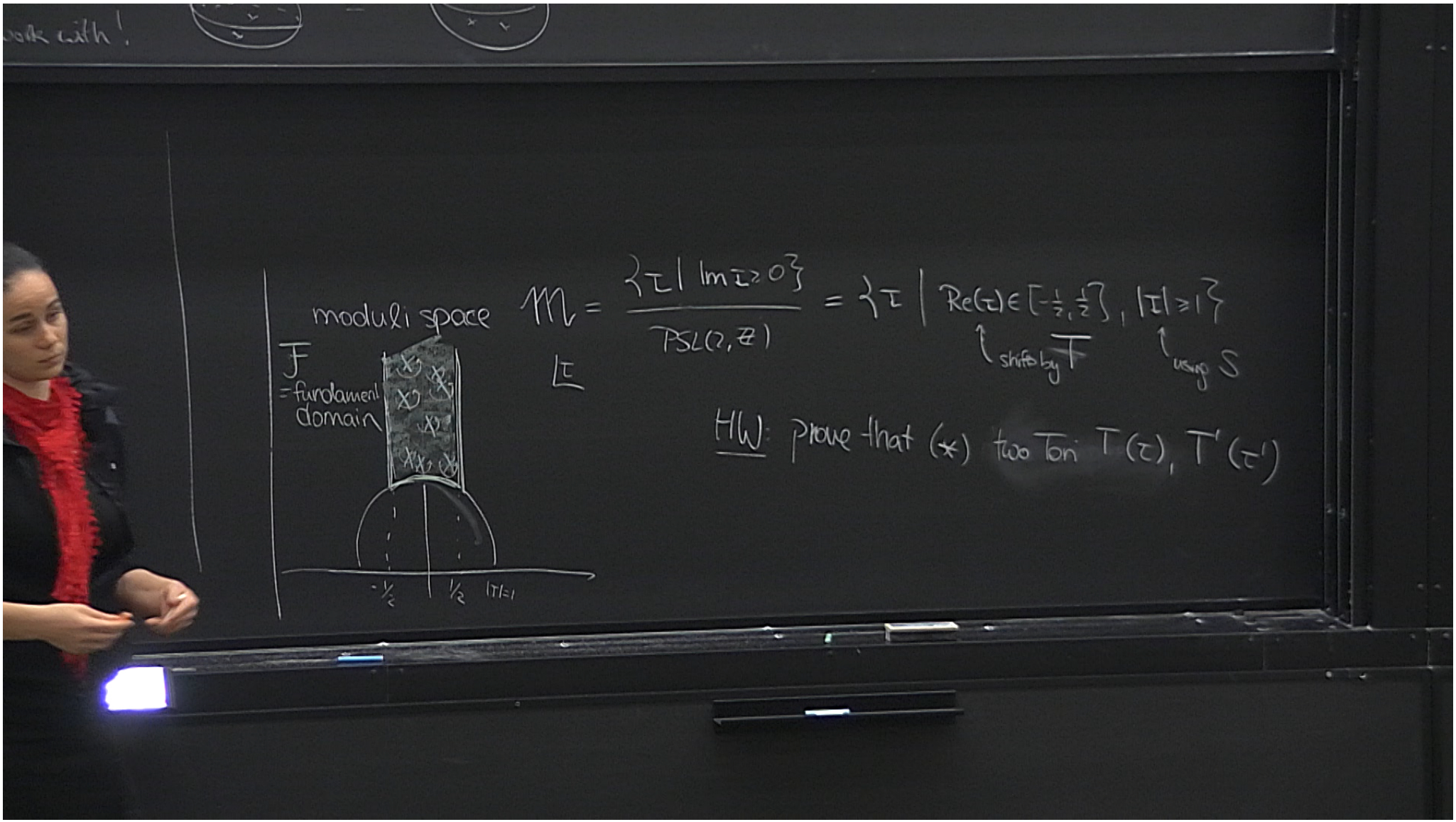
HW: Polchinski I p. 215/216

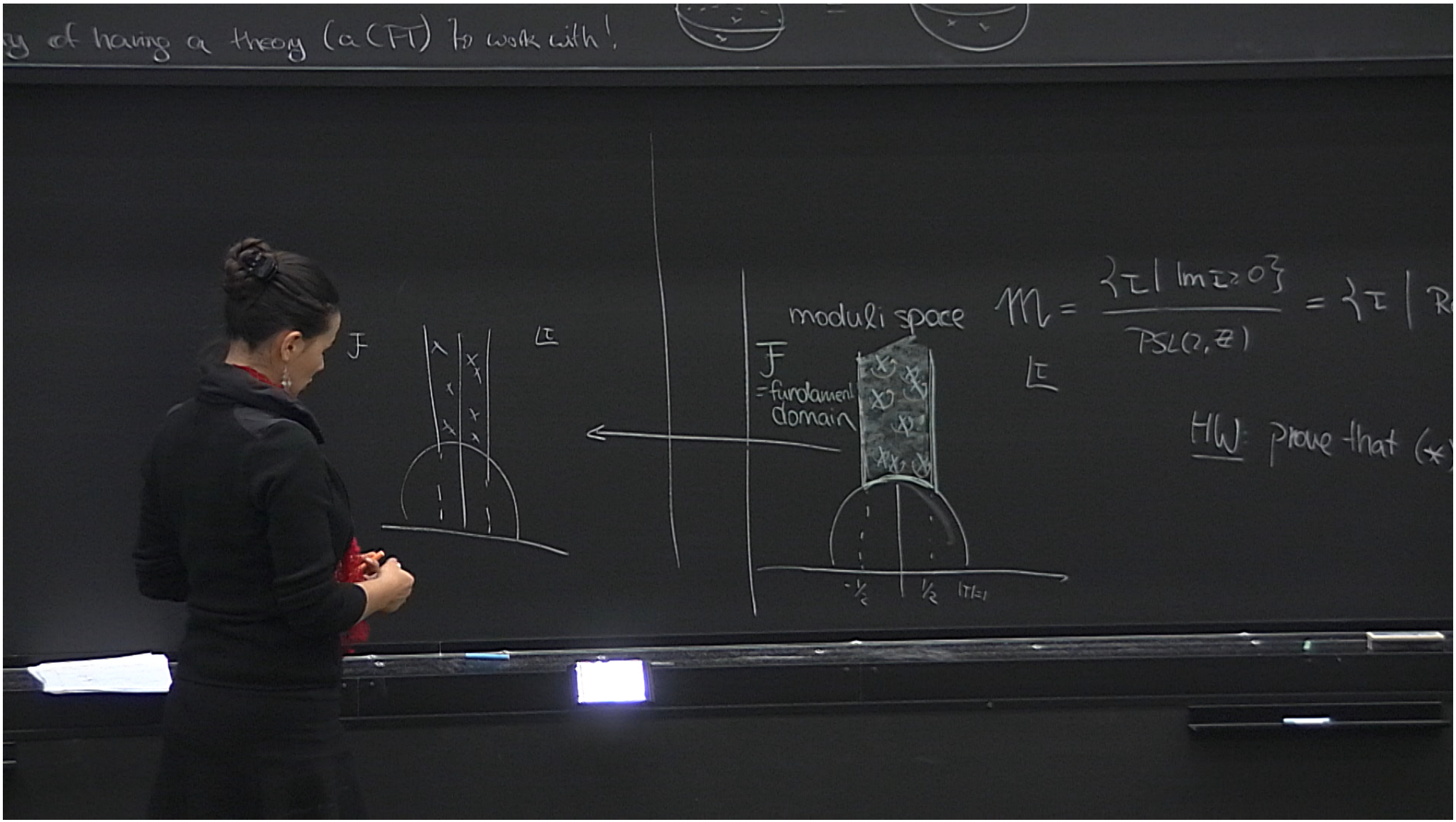
The beauty of having a theory (a CFT) to work with!

TRICK: Residue Theorem

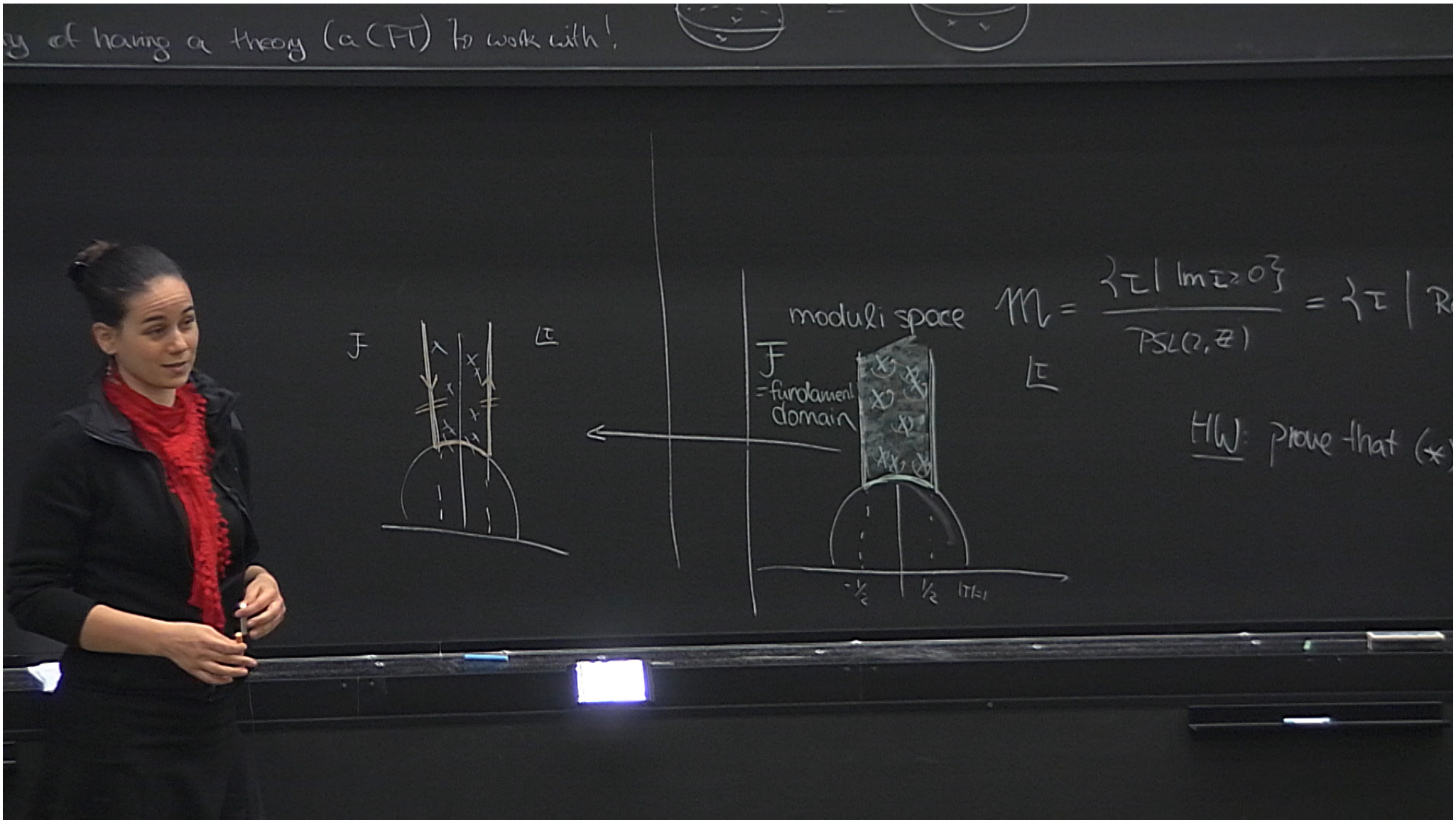

localize
Scattering
Equations


equation for τ
localize $z_i$
$A_{ij} = k_i \cdot k_j S_\alpha(z_i, z_j | \tau)$
$B_{ij} = \epsilon_i \epsilon_j S_\alpha(z_i, z_j | \tau)$
$C_{ij} = \epsilon_i k_j S_\alpha(z_i, z_j | \tau)$
$G_i = \epsilon_i \cdot P(z_i)$
Scattering Equations


moduli space $\mathcal{M} = \frac{\{\tau \mid \mathrm{Im}\,\tau \geq 0\}}{PSL(2,\mathbb{Z})} = \{\tau \mid \mathrm{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}], |\tau| \geq 1\}$
shift by T
using S
$\mathcal{F}$
= fundamental domain
$\tau$
$-1/2$
$1/2$
$|\tau|=1$
HW: prove that (*) two Tori $T(\tau)$, $T'(\tau')$


y of having a theory (a CFT) to work with!
F
τ
moduli space
$\mathcal{M} = \frac{\{\tau \mid \mathrm{Im}\,\tau > 0\}}{PSL(2,\mathbb{Z})} = \{\tau \mid$
F
= fundamental domain
τ
-1/2
1/2
|τ|=1
HW: prove that (*)

y of having a theory (a CFT) to work with!
moduli space
$\mathcal{M} = \frac{\{\tau \mid \mathrm{Im}\,\tau > 0\}}{PSL(2,\mathbb{Z})} = \{\tau \mid$
$\mathcal{F}$
= fundamental domain
$-\frac{1}{2}$
$\frac{1}{2}$
$|\tau|=1$
HW: prove that (*)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moduli space
$\mathcal{M} = \frac{\{\tau \mid \mathrm{Im}\,\tau > 0\}}{PSL(2,\mathbb{Z})} = \{\tau \mid$
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$-\frac{1}{2}$
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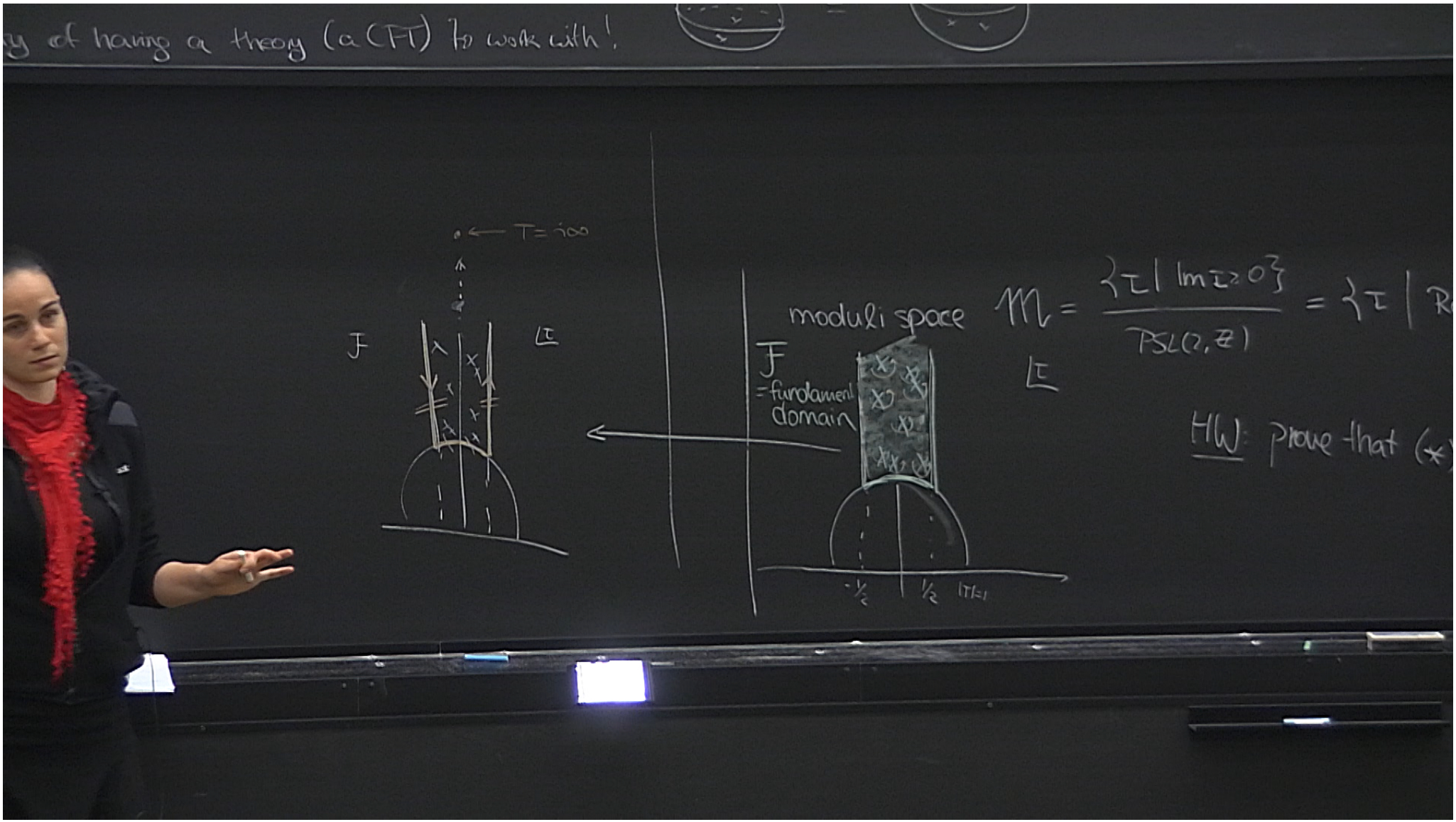
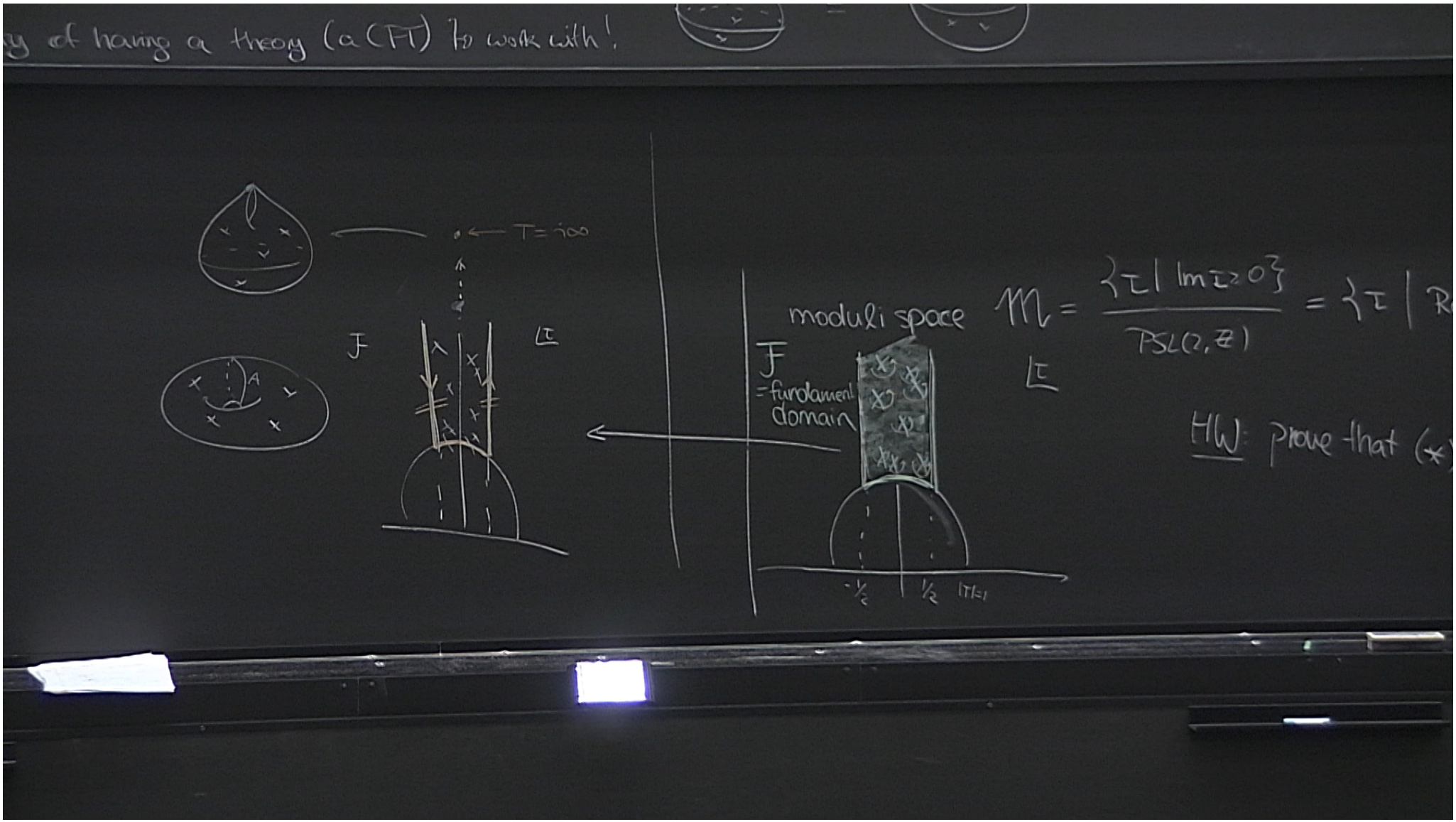
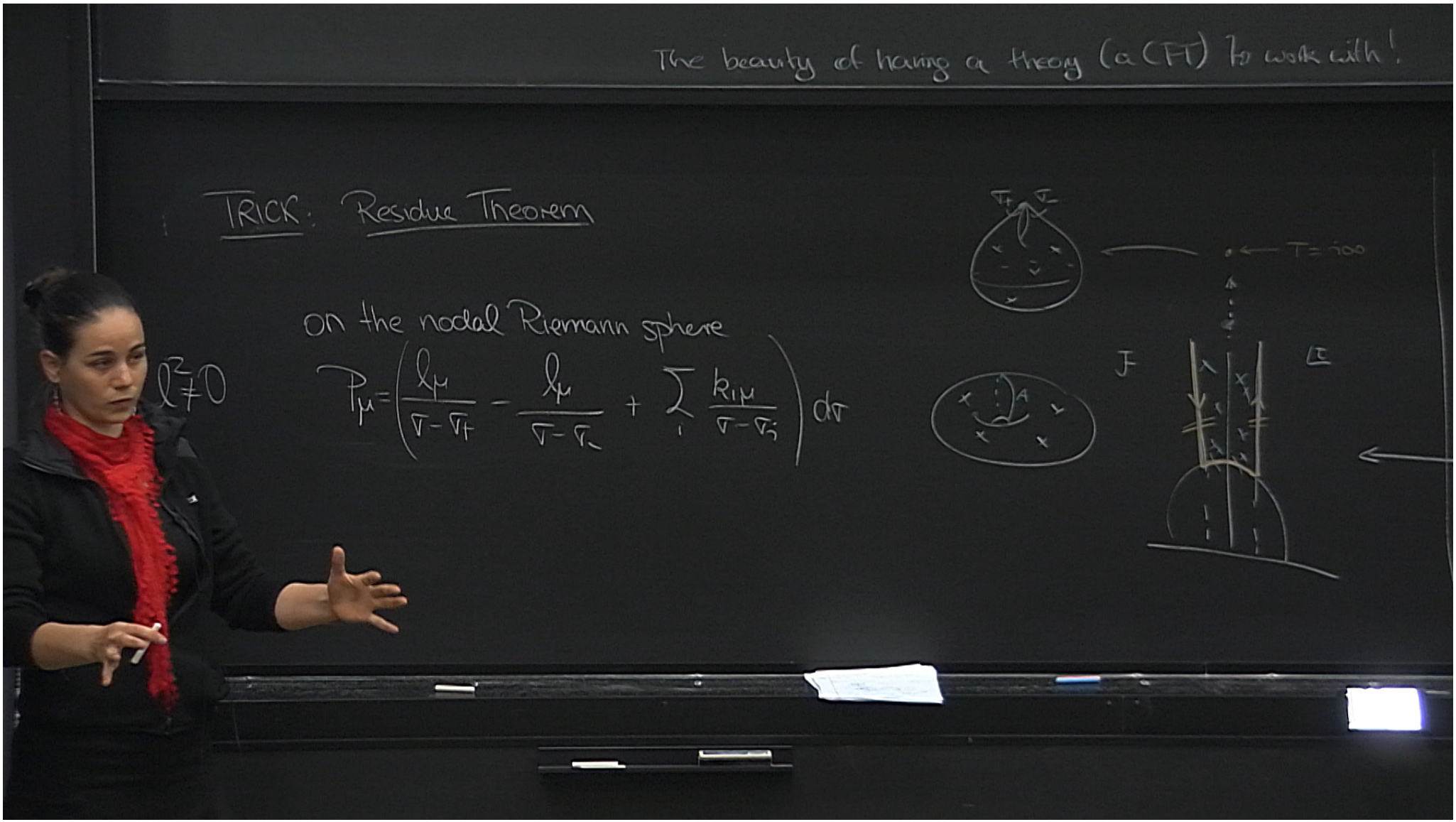
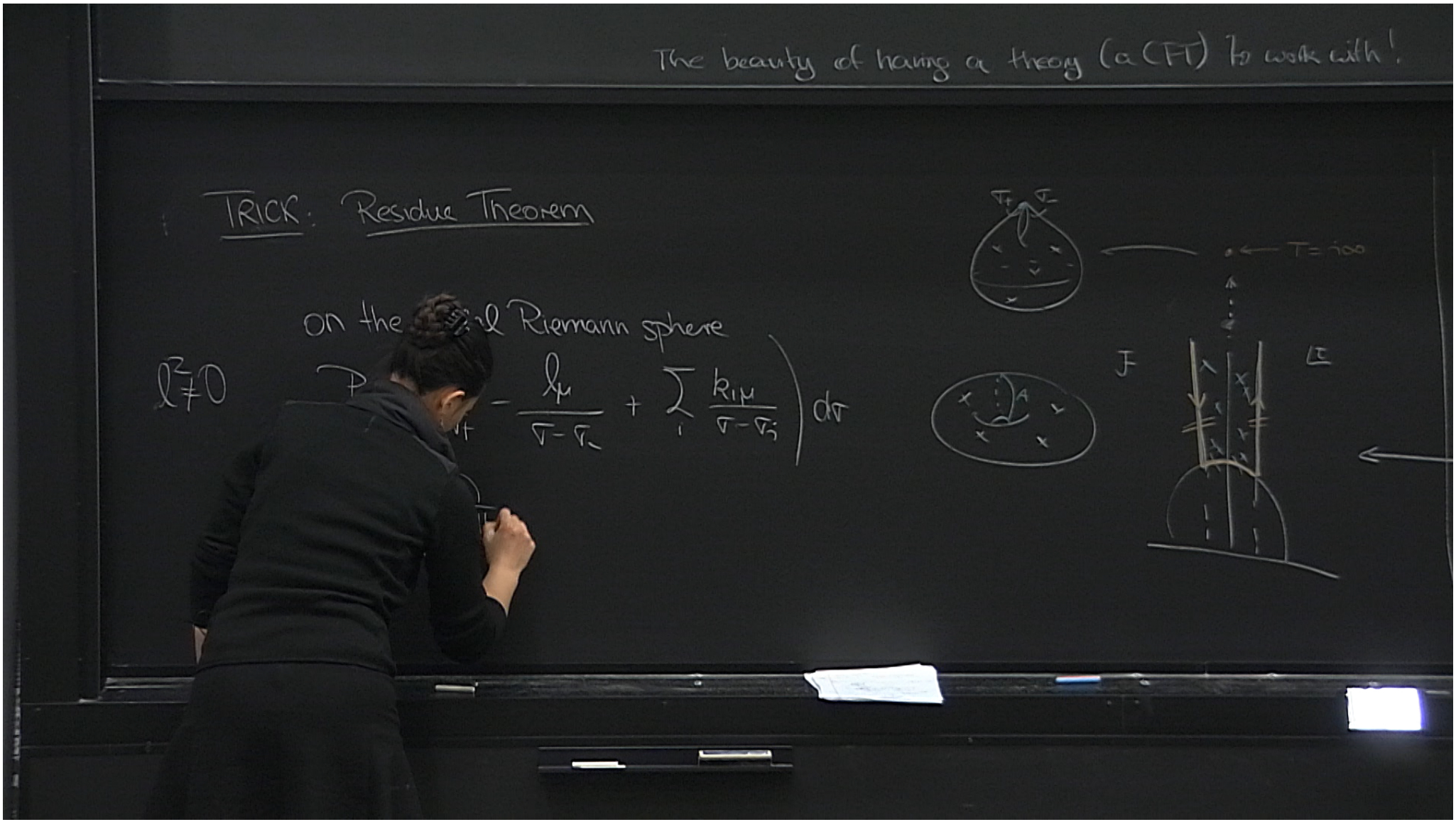
HW: prove that (*)


of having a theory (a CFT) to work with!
T = i∞
moduli space
= fundamental domain
HW: prove that (*)


y of having a theory (a CFT) to work with!
T = i∞
F
τ
moduli space
$\mathcal{M} = \frac{\{\tau \mid \mathrm{Im}\,\tau > 0\}}{PSL(2,\mathbb{Z})} = \{\tau \mid$
F
= fundamental domain
τ
HW: prove that (*)
-1/2
1/2
|τ|=1


The beauty of having a theory (a CFT) to work with!
TRICK: Residue Theorem
on the nodal Riemann sphere
$l^2 \neq 0$
$P_\mu = \left( \frac{l_\mu}{\sigma - \sigma_+} - \frac{l_\mu}{\sigma - \sigma_-} + \sum_i \frac{k_{i\mu}}{\sigma - \sigma_i} \right) d\sigma$
$\sigma_+$
$\sigma_-$
$\tau = i\infty$
J
E


The beauty of having a theory (a CFT) to work with!
TRICK: Residue Theorem
on the Riemann sphere
$l^2 \neq 0$
$\sigma_+$
$\sigma_-$
$T = i\infty$
J
E

The beauty of having a theory (a CFT) to work with!

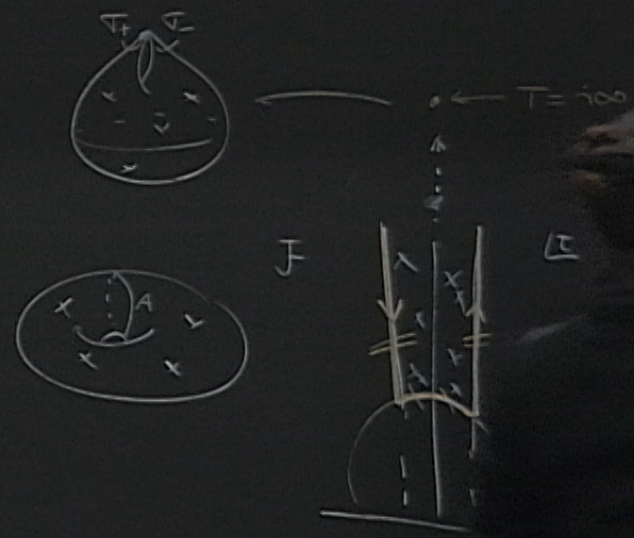
TRICK: Residue Theorem

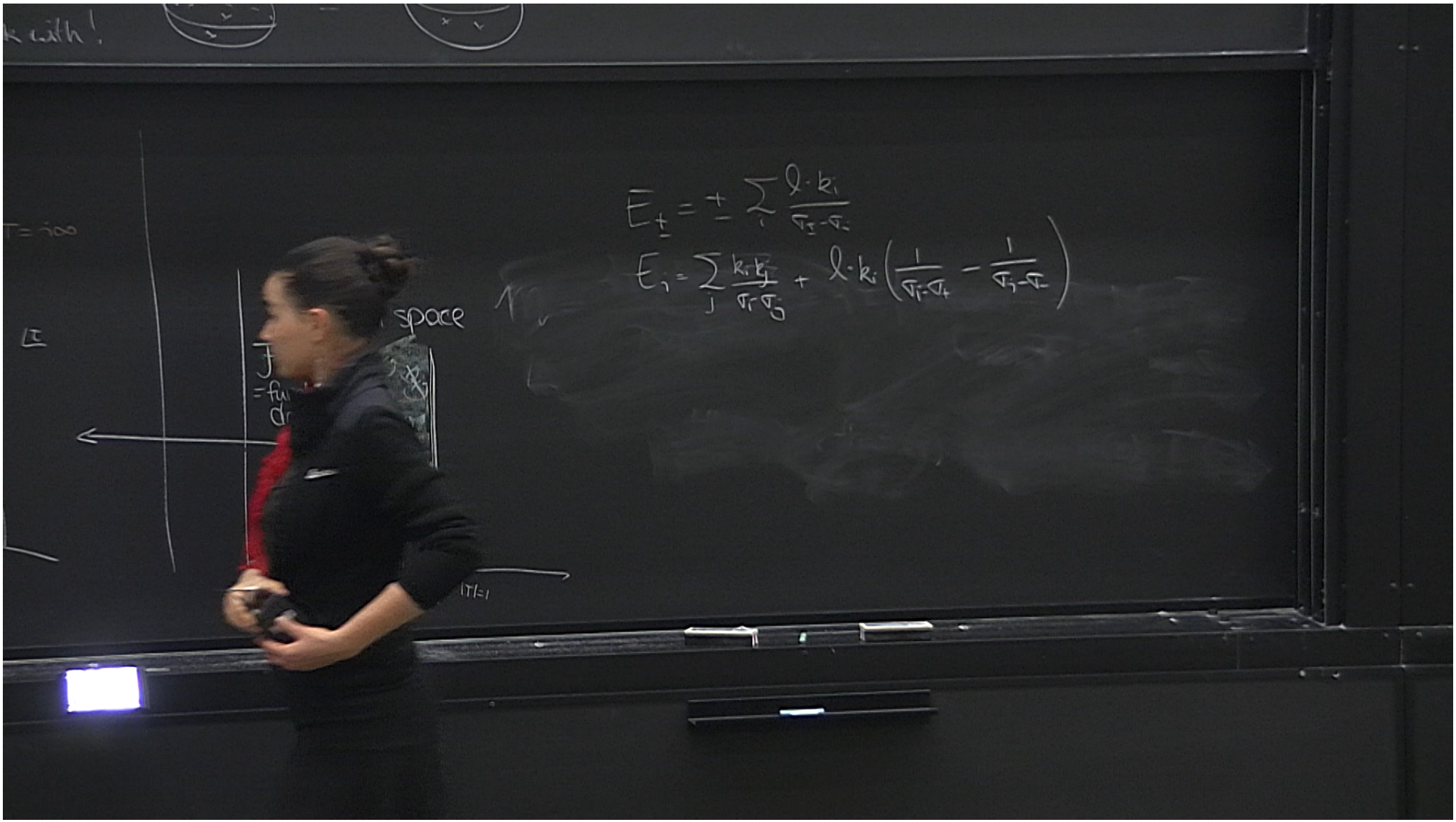
on the nodal Riemann sphere

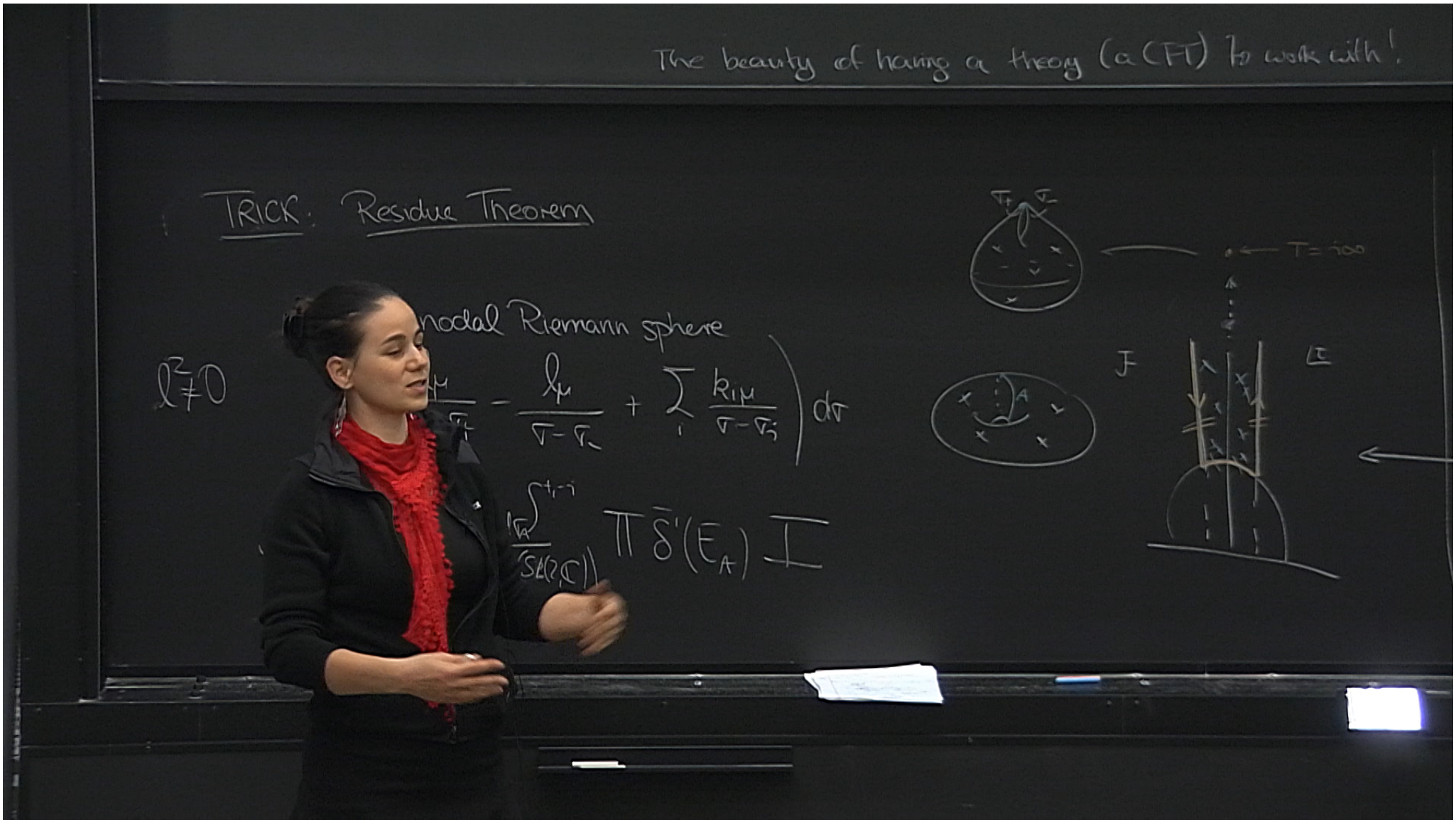
$\ell^2 \neq 0$

$$P_\mu = \left( \frac{\ell_\mu}{\sigma - \sigma_+} - \frac{\ell_\mu}{\sigma - \sigma_-} + \sum_i \frac{k_{i\mu}}{\sigma - \sigma_i} \right) d\sigma$$

$$\mathcal{M}_n = \int d^{10}\ell \int \frac{\prod d\sigma_A}{\mathrm{vol}(SL(2,\mathbb{C}))} \prod \delta'(E_A)$$




$E_\pm = \pm \sum_i \frac{l \cdot k_i}{\sigma_\pm - \sigma_i}$
$E_i = \sum_j \frac{k_i k_j}{\sigma_i - \sigma_j} + l \cdot k_i \left( \frac{1}{\sigma_i - \sigma_+} - \frac{1}{\sigma_i - \sigma_-} \right)$
space


The beauty of having a theory (a CFT) to work with!
TRICK: Residue Theorem
nodal Riemann sphere
$l^2 \neq 0$
$\prod \bar{\delta}'(E_A)\, I$
$T = i\infty$