Title: PSI 2017/2018 - Scattering Amplitudes in QFT & String Theory - Lecture 4

Date: Apr 12, 2018 11:30 AM

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Abstract:

Lecture IV

Parke-Taylor Generating Function

$$A_n^{(12\cdots n)}\left(\left\{\lambda_\alpha^{(a)}, \tilde{\lambda}_{\dot\alpha}^{(a)}, \tilde{\eta}_I^{(a)}\right\}\right) = \frac{\delta^{2\times 2}\left(\sum_{a=1}^{n} \lambda_\alpha^{(a)} \tilde{\lambda}_{\dot\alpha}^{(a)}\right) \delta^{2\times 4}\left(\sum_{a=1}^{n} \lambda_\alpha^{(a)} \tilde{\eta}_I^{(a)}\right)}{\langle 12\rangle\langle 23\rangle \cdots \langle n-1\, n\rangle\langle n1\rangle}$$

$\alpha = 1,2$ $\qquad I = 1,2,3,4$

Lecture IV

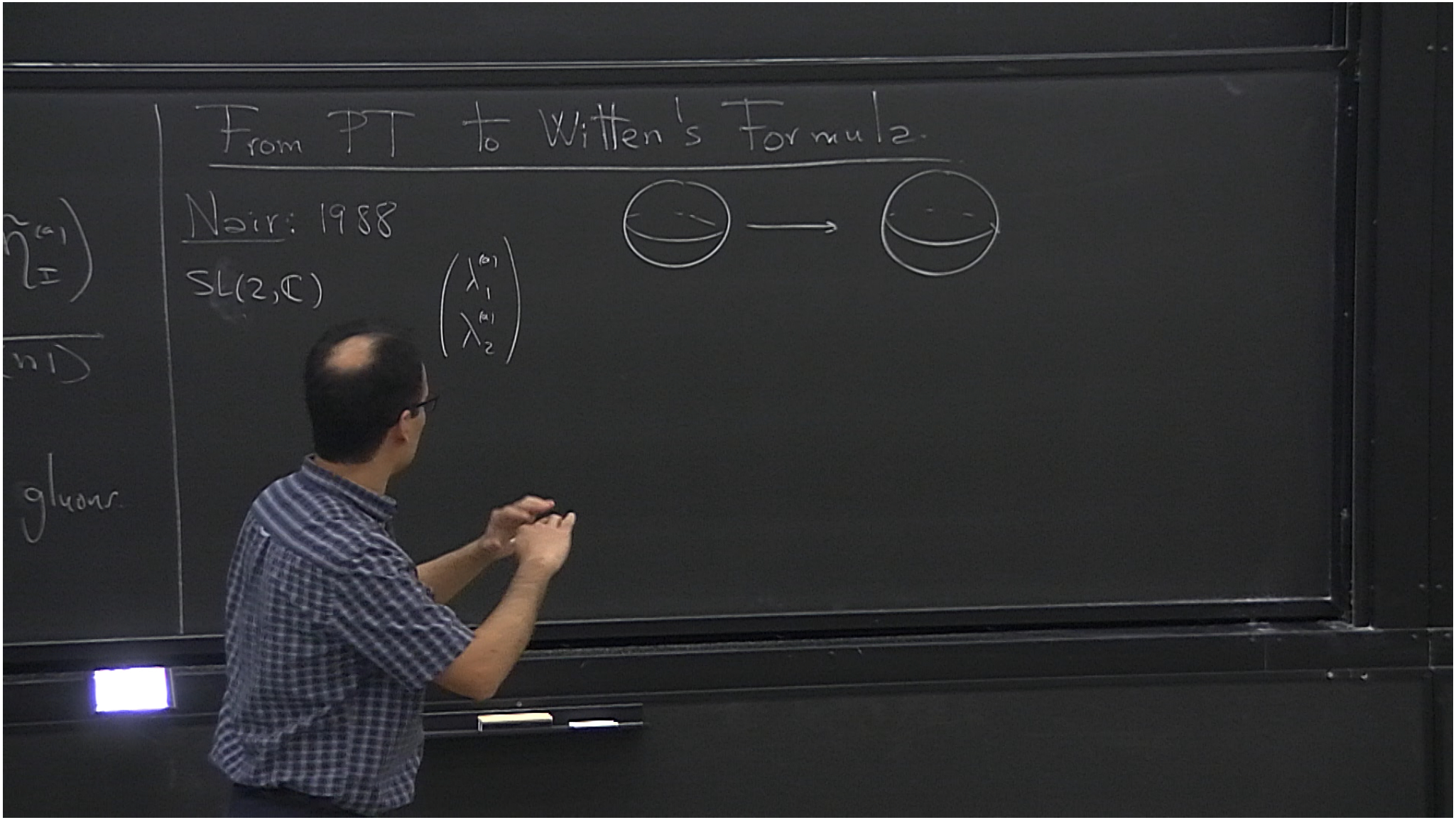
Parke-Taylor Generating Function

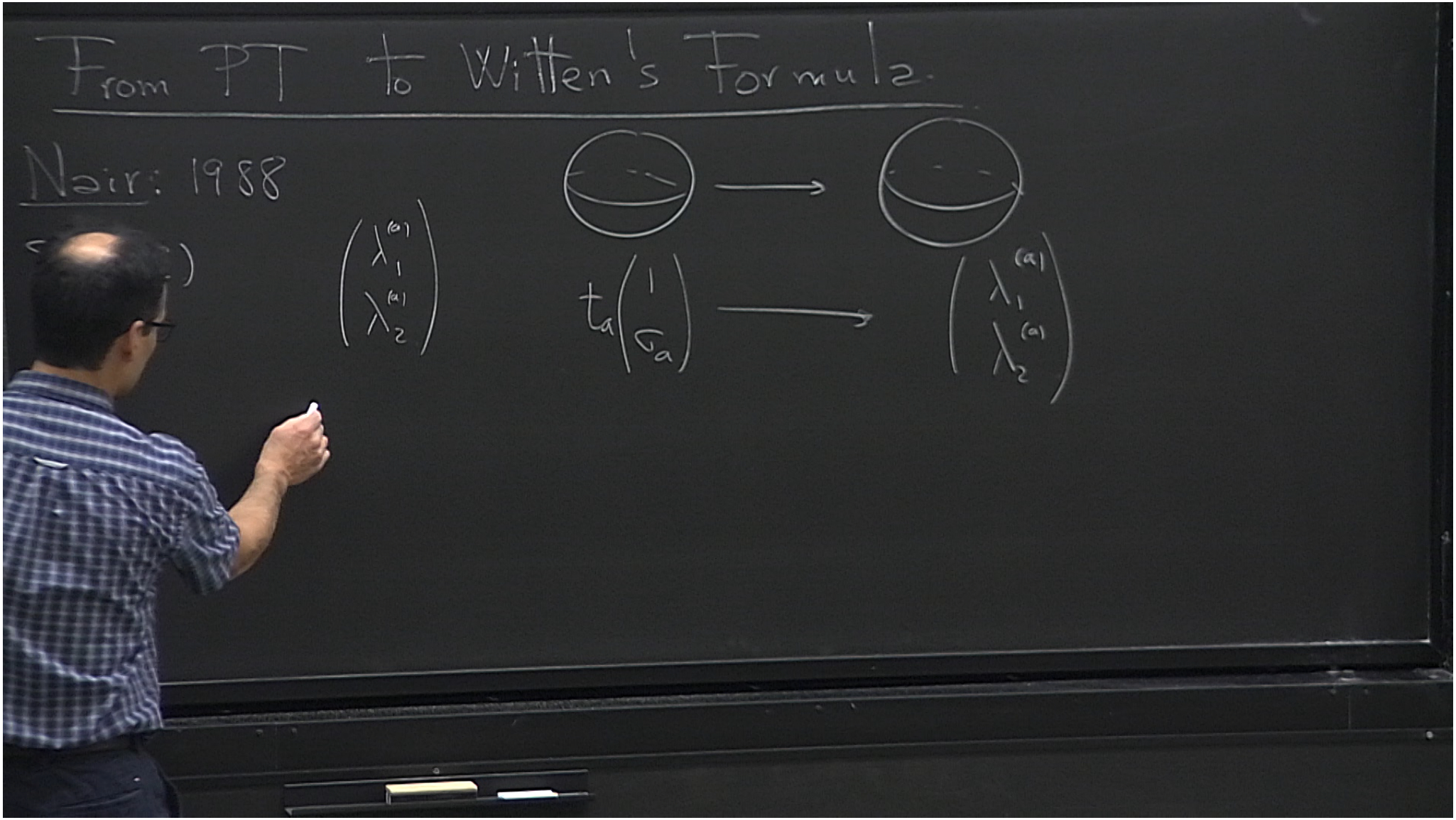
$$A_n^{(12\cdots n)}\left(\left\{\lambda_\alpha^{(a)}, \tilde{\lambda}_{\dot\alpha}^{(a)}, \tilde{\eta}_I^{(a)}\right\}\right) = \frac{\delta^{2\times 2}\left(\sum_{a=1}^{n} \lambda_\alpha^{(a)} \tilde{\lambda}_{\dot\alpha}^{(a)}\right)\delta^{2\times 4}\left(\sum_{a=1}^{n} \lambda_\alpha^{(a)} \tilde{\eta}_I^{(a)}\right)}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1\, n\rangle\langle n1\rangle}$$

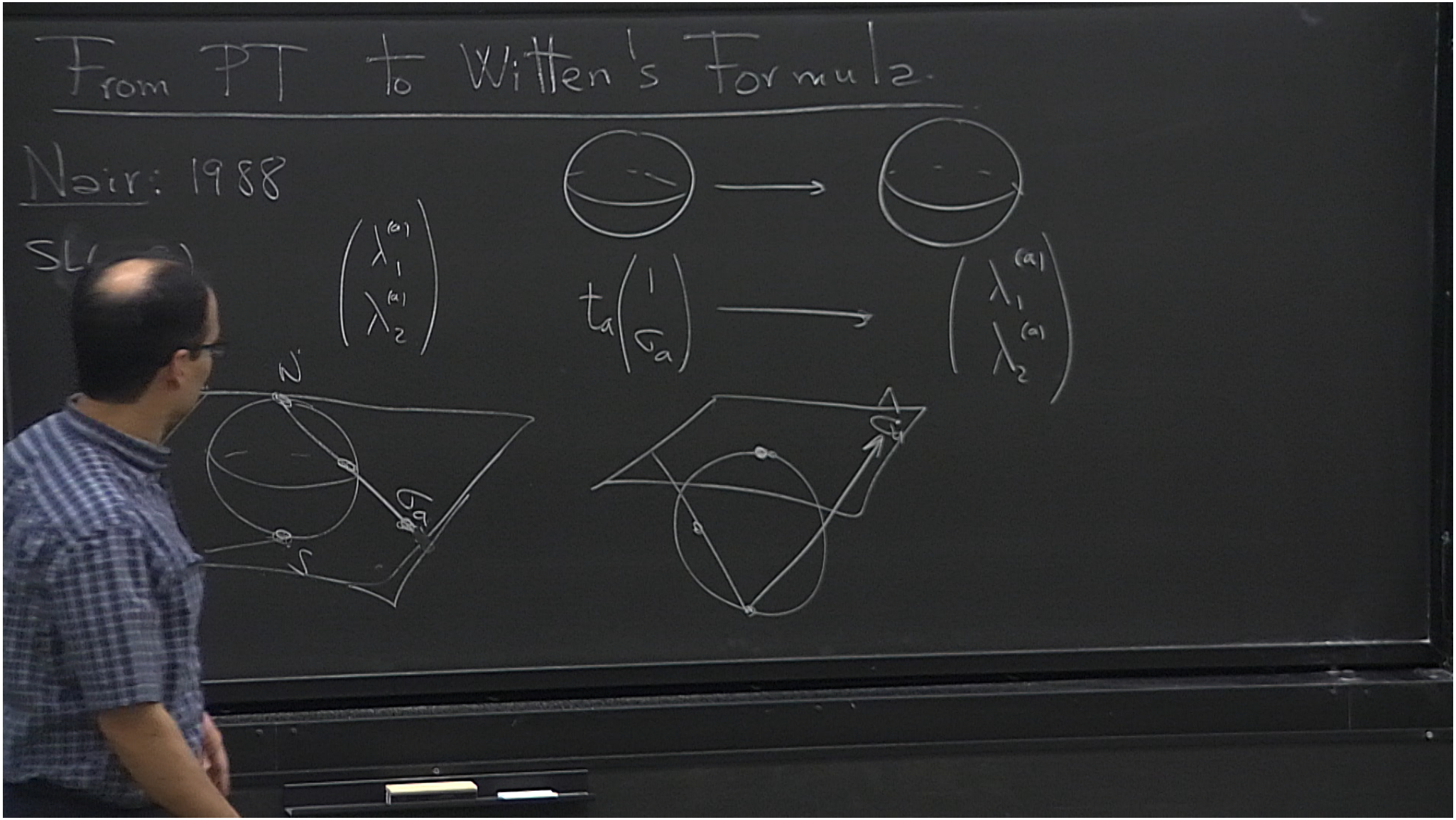
$\{1,2,3,4\}$

$K = \#$ of negative helicity gluons

$K = 2$


From PT to Witten's Formula.
Nair: 1988
SL(2,C)
gluons


From PT to Witten's Formula.
Nair: 1988


From PT to Witten's Formula
Nair: 1988
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From PT to Witten's Formula

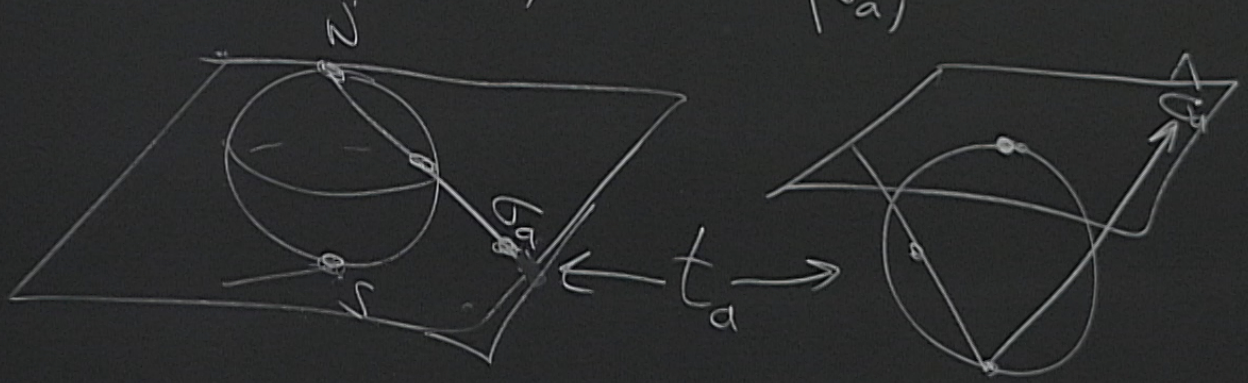
Nair: 1988

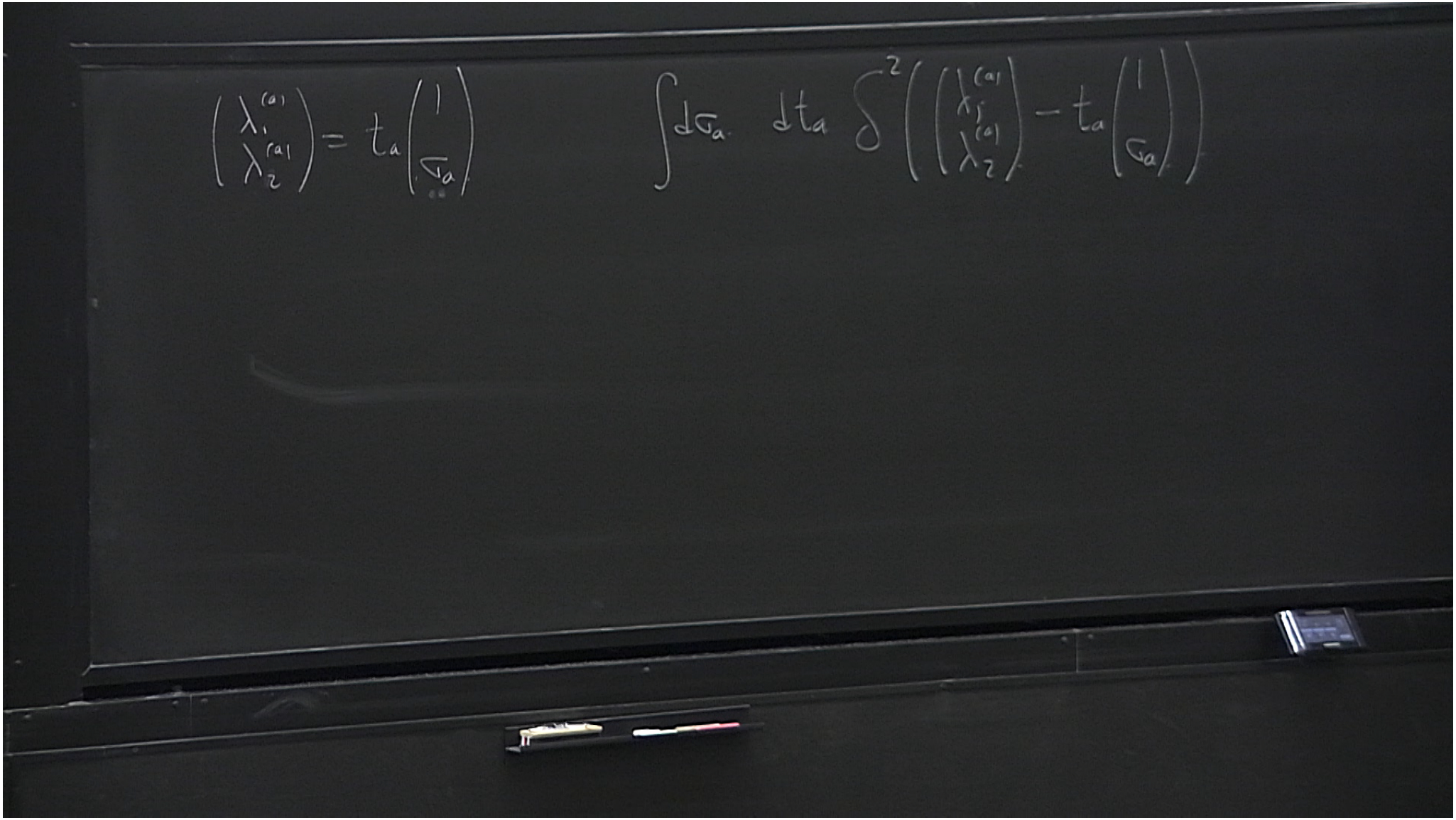
$SL(2,\mathbb{C})$

$\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix}$

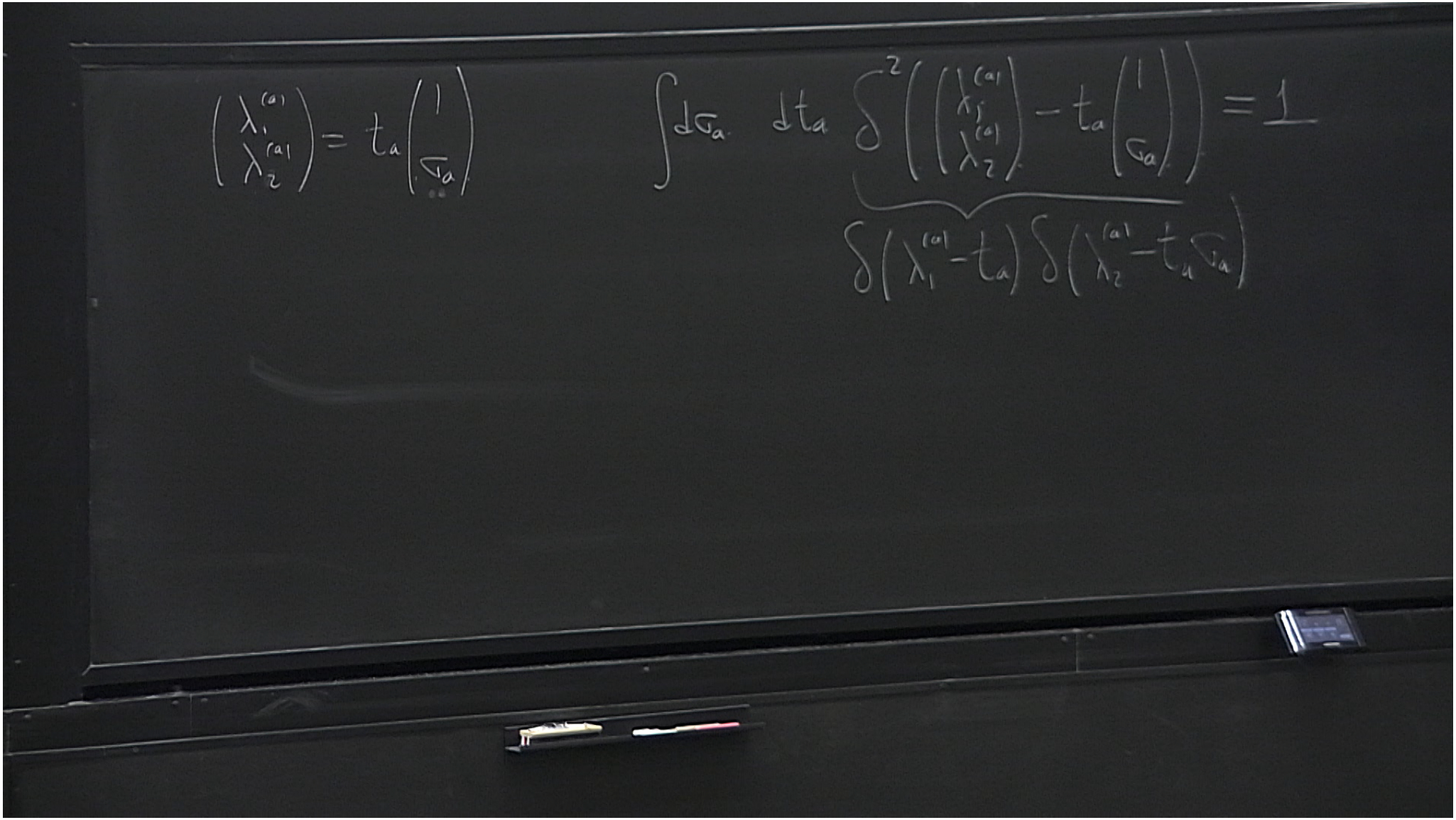
$t_a\begin{pmatrix} 1 \\ G_a \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix}$

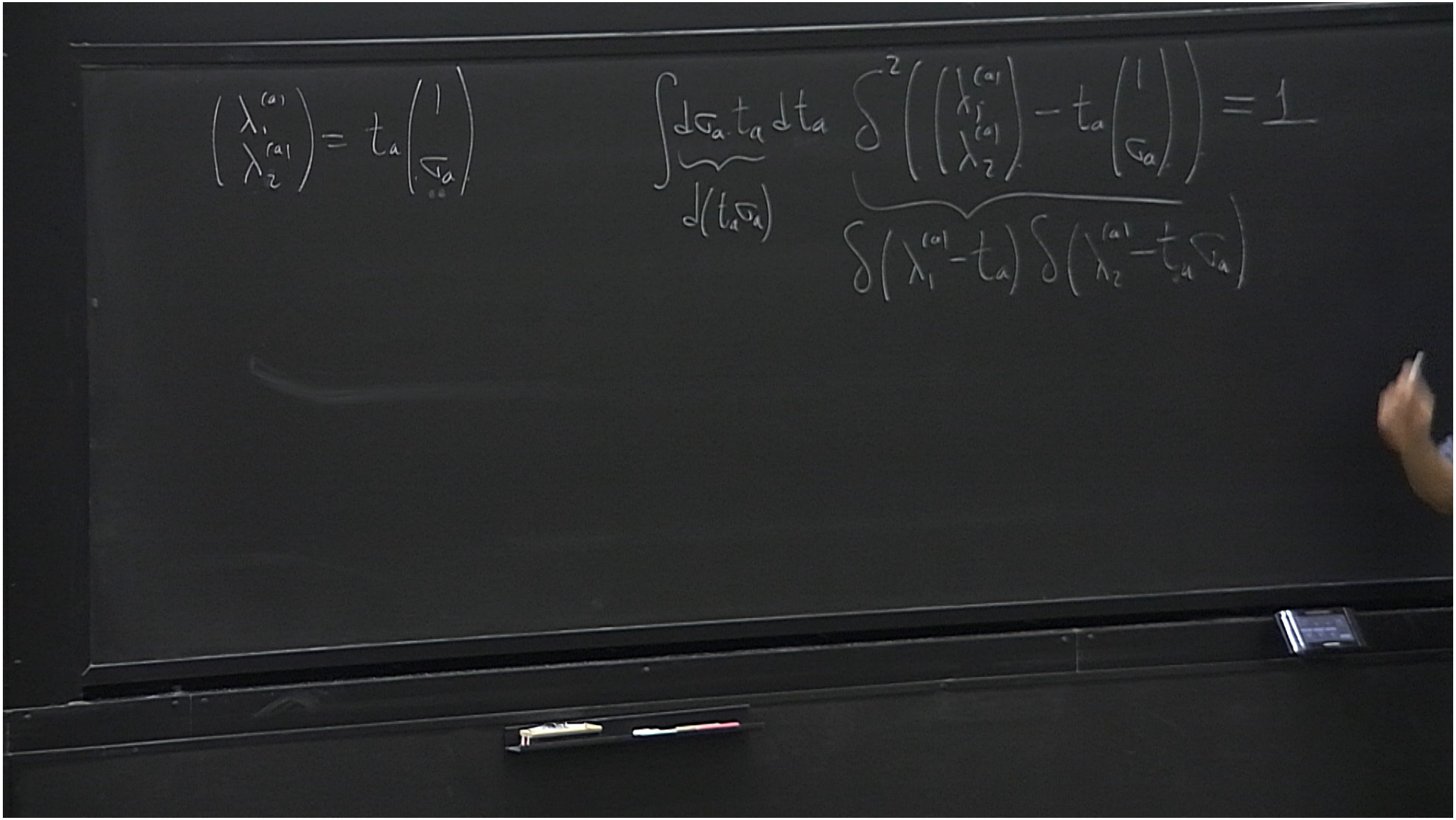
$\mathbb{CP}^1$




$\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} = t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}$
$\int d\sigma_a \, dt_a \, \delta^2\left(\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix} \right)$

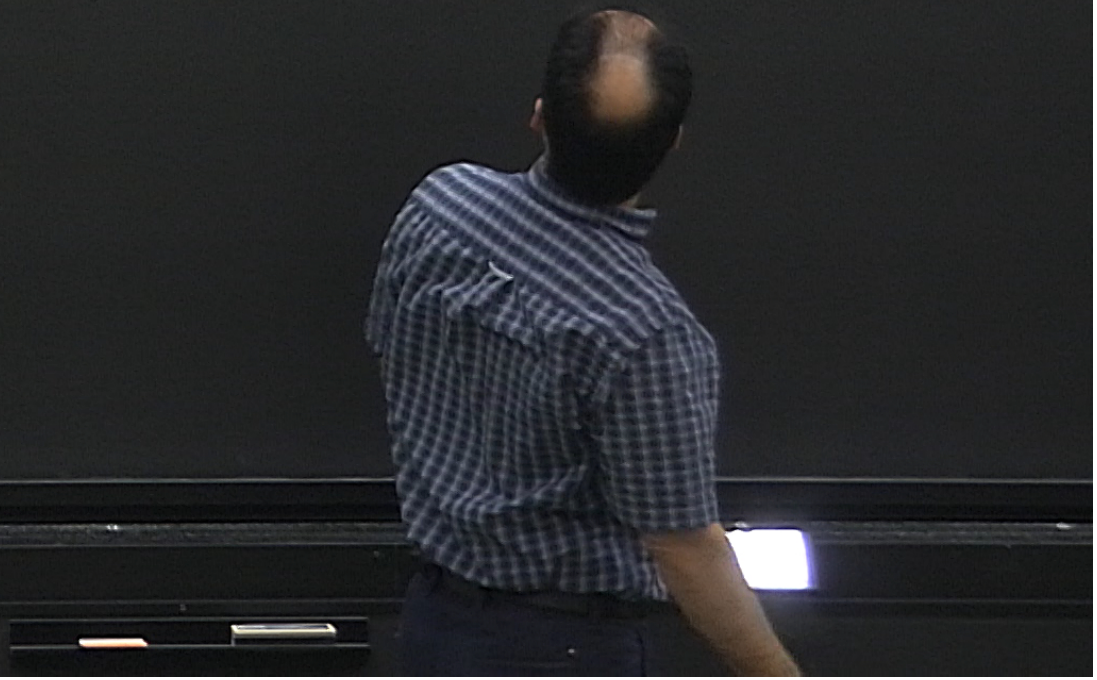
$$\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} = t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix} \qquad \int d\sigma_a \, dt_a \, \delta^2\left(\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix} \right) = 1$$


$\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} = t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}$
$\int d\sigma_a \, dt_a \, \underbrace{\delta^2\left(\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}\right)}_{\delta(\lambda_1^{(a)} - t_a)\,\delta(\lambda_2^{(a)} - t_a \sigma_a)} = 1$


$\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} = t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}$
$\int \underbrace{d\sigma_a \, t_a}_{d(t_a \sigma_a)} dt_a \; \delta^2\left(\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix} \right) = 1$
$\delta(\lambda_1^{(a)} - t_a)\, \delta(\lambda_2^{(a)} - t_a \sigma_a)$

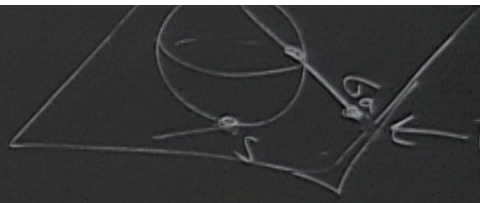
$k=2$

$$A_n = \int \prod_{a=1}^{n} d\sigma_a \, t_a \, d\tilde{t}_a \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a\left(\begin{matrix}1\\ \sigma_a\end{matrix}\right)\right) \delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)$$



$k = \#$ of negative helicity gluons

$k = 2$

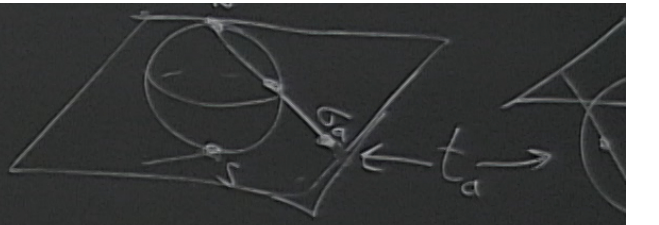


$$A_n = \int \prod_{a=1}^{n} d\sigma_a \, t_a \, dt_a \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}\right) \delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right) \quad (\text{SUSY})$$

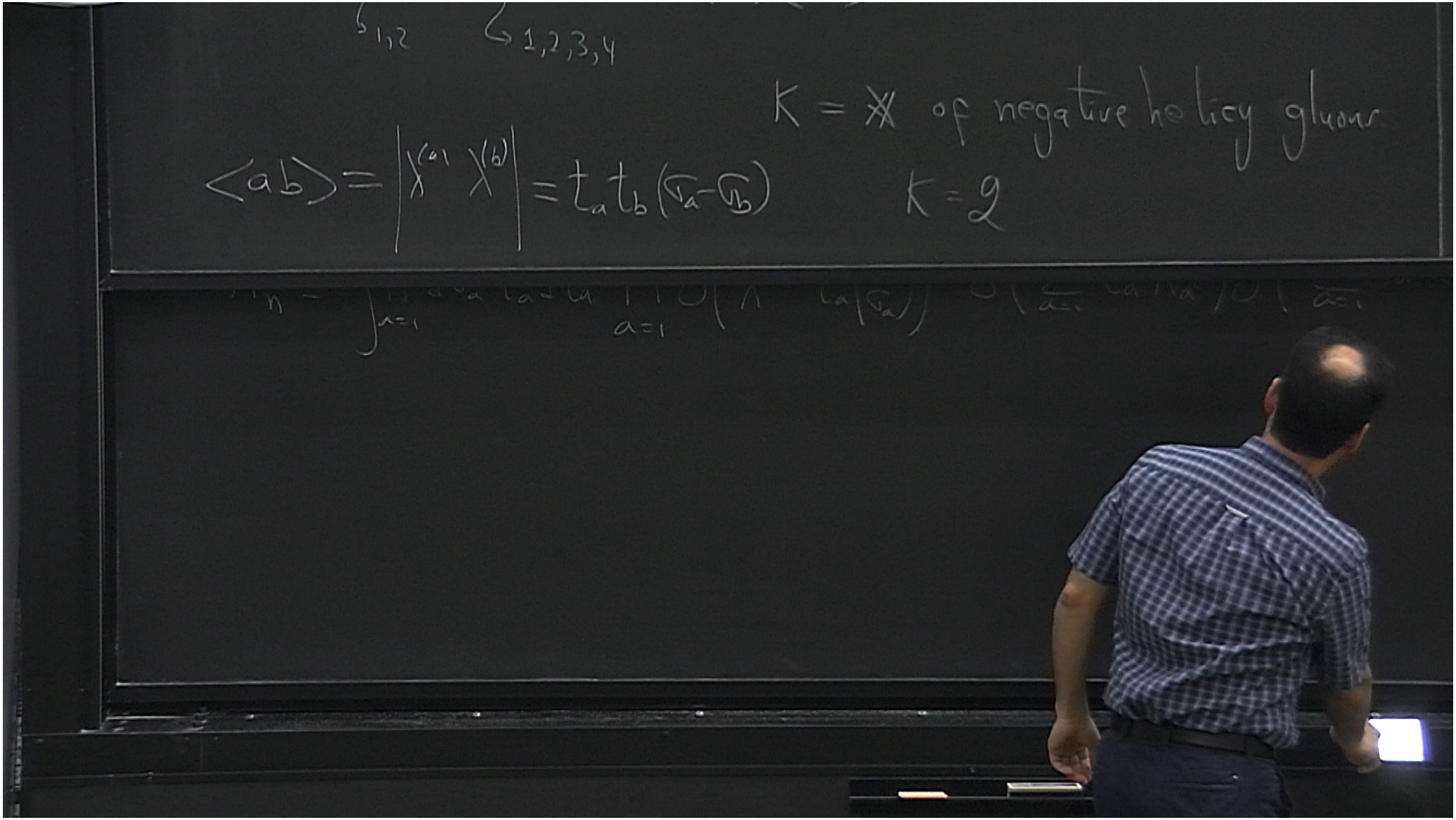
$S_{1,2}$ $\hookrightarrow 1,2,3,4$

$K = \#$ of negative helicy gluons

$K = 2$



$$A_n = \int \prod_{a=1}^{n} d\sigma_a \, t_a \, t_a \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a\left(\frac{1}{\sigma_a}\right)\right) \delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right) \quad \text{(SUSY)} \quad \tilde{\eta} \text{ part}$$


1,2
1,2,3,4
K = # of negative helicity gluons
<ab> = |λ^(a) λ^(b)| = t_a t_b (σ_a − σ_b)
K = 2

$k = \#$ of negative helicity gluons

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$ $\quad k = 2$



$$A_n = \int \prod_{a=1}^{n} d\sigma_a \, t_a \, dt_a \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)}{\left(\prod_{a=1}^{n} t_a^2\right) (\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)} \quad (\text{susy}) \quad \tilde{\eta} \; p \to 1$$

$k = \#$ of negative helicity gluons

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$ $\quad k = 2$



$$A_n = \int \prod_{a=1}^{n} d\sigma_a \, t_a \, dt_a \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)} \quad \text{(SUSY)} \quad \tilde{\eta} \text{ part}$$

$k = \#$ of negative helicity gluons

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$ $\quad k = 2$



$$A_n = \int \prod_{a=1}^{n} d\sigma_a \, d\mu \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a\left(\begin{matrix}1\\ \sigma_a\end{matrix}\right)\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)} \quad \text{(susy)} \quad \tilde{\eta} \text{ part.}$$

$k = \#$ of negative helicity gluons

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$ $\quad k = 2$

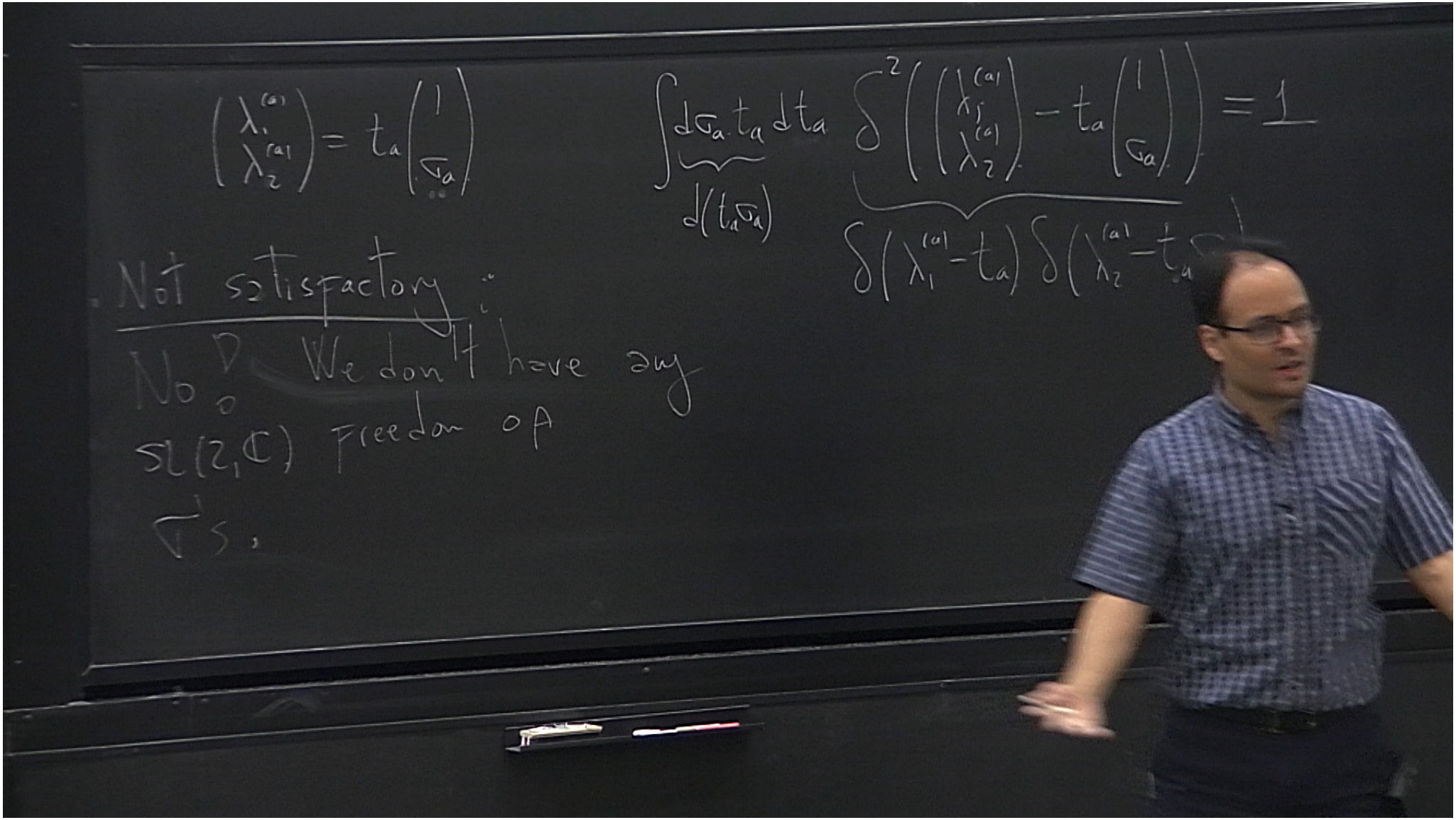


$$A_n = \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)} \quad \text{(susy)} \quad \tilde{\eta} \text{ part.}$$

$$\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} = t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}$$

$$\int \underbrace{d\sigma_a \, t_a}_{d(t_a \sigma_a)} dt_a \; \underbrace{\delta^2\left(\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix} \right)}_{\delta(\lambda_1^{(a)} - t_a)\,\delta(\lambda_2^{(a)} - t_a \sigma_a)} = 1$$

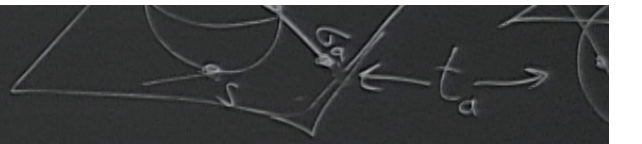
Not satisfactory :


$\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} = t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}$
$\int \underbrace{d\sigma_a \, t_a}_{d(t_a \sigma_a)} dt_a$
$\delta^2\left(\begin{pmatrix} \lambda_1^{(a)} \\ \lambda_2^{(a)} \end{pmatrix} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}\right) = 1$
$\delta(\lambda_1^{(a)} - t_a)\,\delta(\lambda_2^{(a)} - t_a$
Not satisfactory :
No D We don't have any
$SL(2,\mathbb{C})$ Freedom of
$\vec{\nabla}$ s.

$\langle ab \rangle = |\lambda^{(a)}\ \lambda^{(b)}| = t_a t_b (\sigma_a - \sigma_b)$

$k =$ # of negative helicity gluons

$k = 2$



$$A_n = \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \int \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a \begin{pmatrix} 1 \\ \sigma_a \end{pmatrix}\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)} \quad \text{(susy)} \quad \tilde{\eta} \text{ part}$$

$$\lambda^{(a)} - t_a \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma_a \right)$$

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$

k = # of negative helicity gluons

$k = 2$



$$A_n = \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \prod_{a=1}^{n} \int^2 \left(\lambda^{(a)} - t_a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \frac{\delta^2 \left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a \right) \delta^2 \left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a \right)}{(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)} \quad \text{(SUSY)} \quad \tilde{\eta} \text{ part}$$

$$\lambda^{(a)} - t_a \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma_a \right)$$

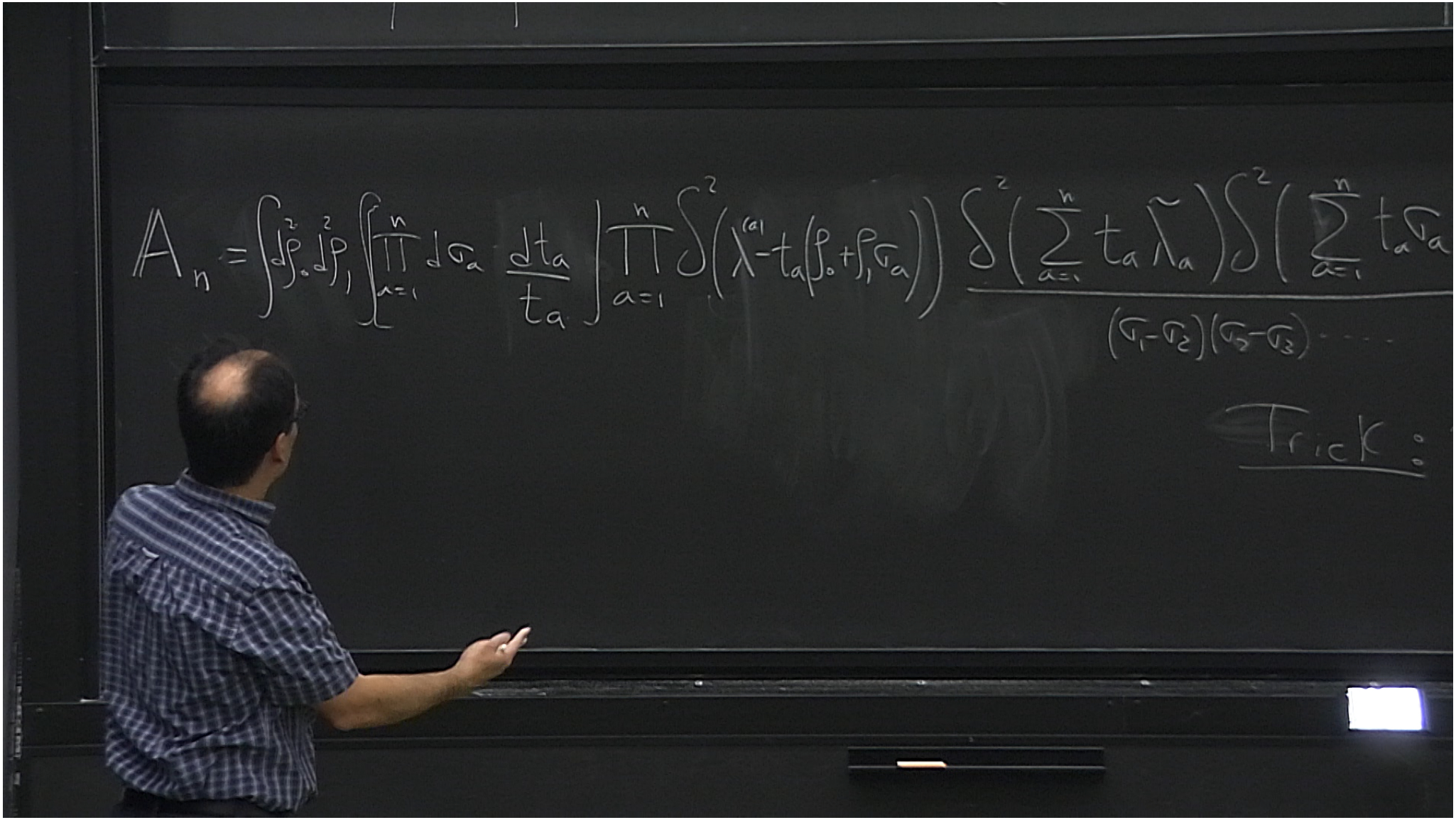
Trick: Bring back $SL(2,\mathbb{C})$

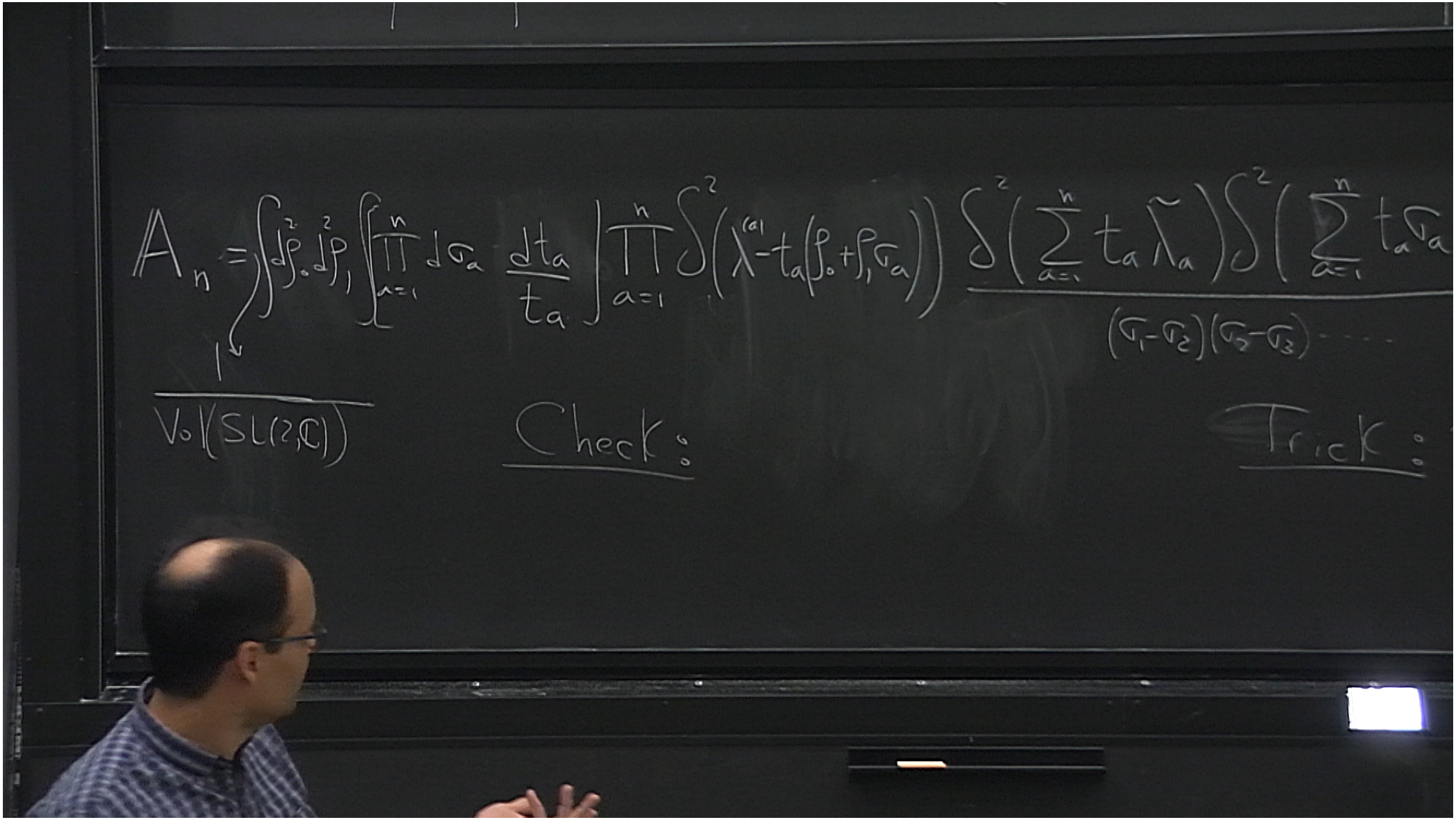
$\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)$ (susy) $\tilde{\eta}$ part.

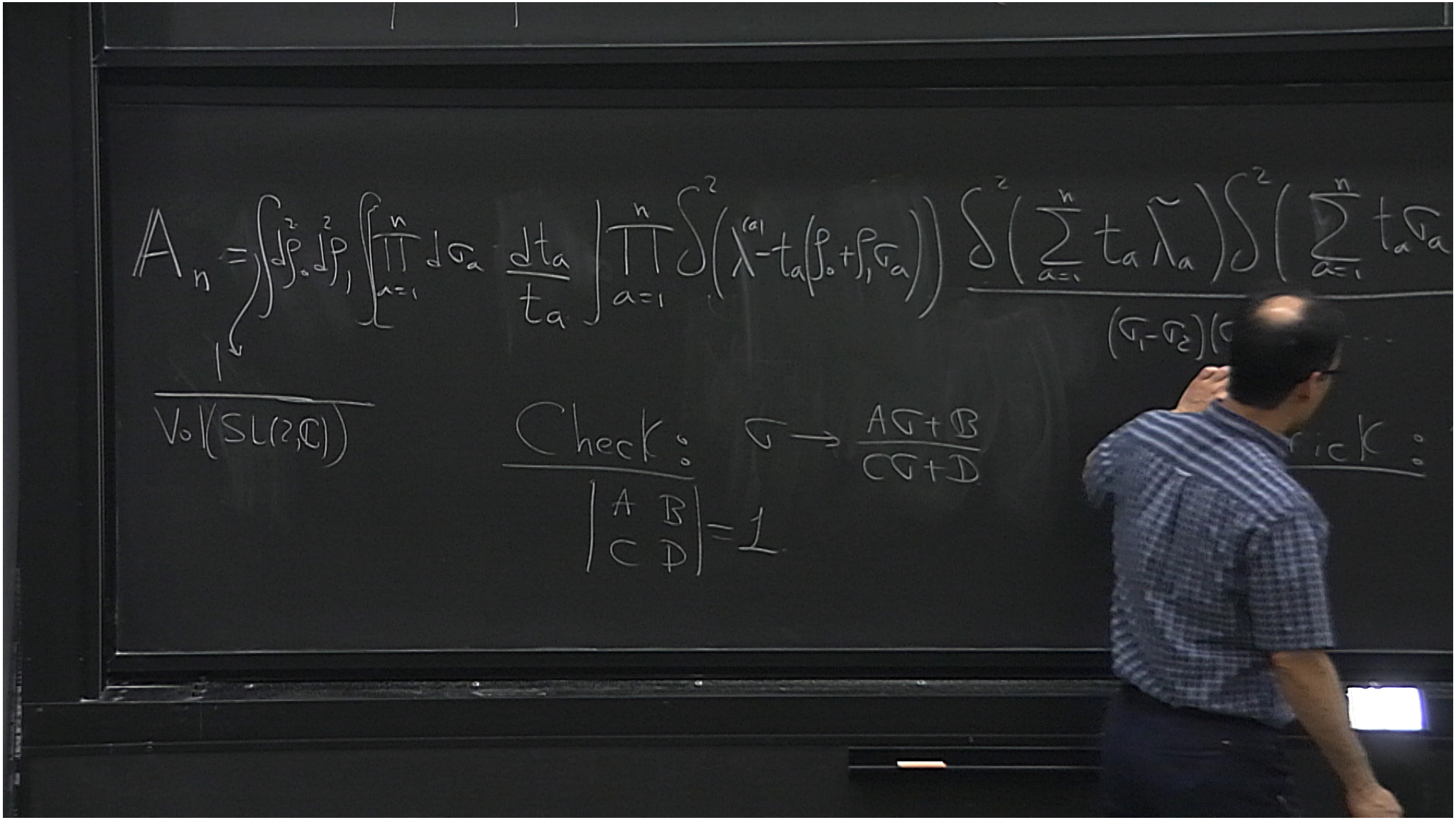
$\sigma_3) \cdots (\sigma_{n-1}-\sigma_n)(\sigma_1-\sigma_n)$

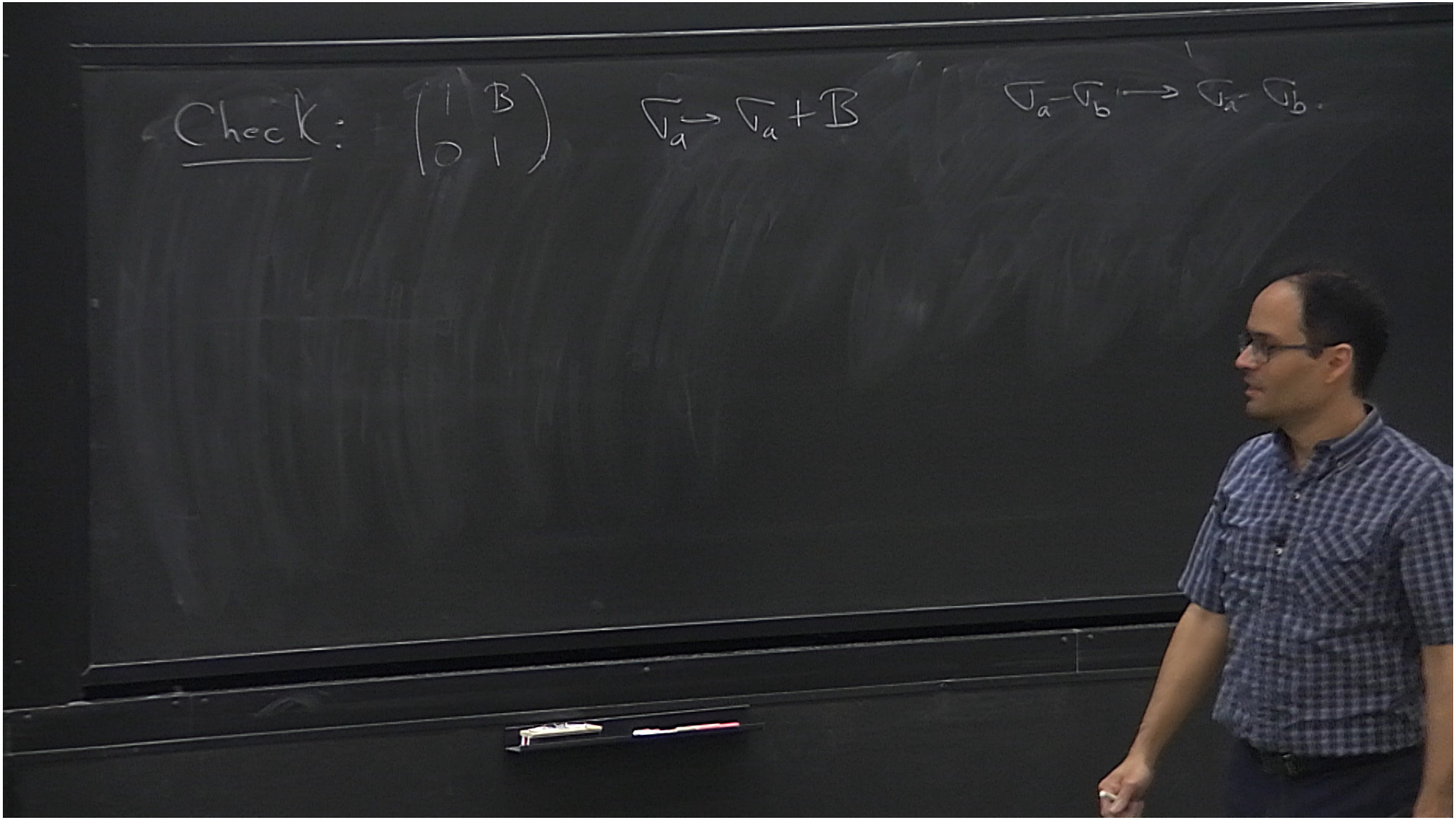
Trick: Bring back $SL(2,\mathbb{C})$ by $\int d^2\rho_0 d^2\rho_1 \, \delta^2\left(\lambda_-^{(a)} - t_a(\rho_0 + \rho_1 \sigma_a)\right)$

$\rho_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\rho_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$


$A_n = \int d^2\rho_0 d^2\rho_1 \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a(\rho_0 + \rho_1\sigma_a)\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a\tilde{\lambda}_a\right)\delta^2\left(\sum_{a=1}^{n} t_a\sigma_a\right.}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots}$
Trick:


$A_n = \int d^2\beta_0 d^2\beta_1 \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a(\beta_0 + \beta_1\sigma_a)\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a\tilde{\lambda}_a\right)\delta^2\left(\sum_{a=1}^{n} t_a\sigma_a\right.}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots}$
$\frac{1}{\mathrm{Vol}(SL(2,\mathbb{C}))}$
Check:
Trick:


$A_n = \int d^2\rho_0\, d^2\rho_1 \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a(\rho_0 + \rho_1\sigma_a)\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a}{(\sigma_1 - \sigma_2)(}$
$\frac{1}{\mathrm{Vol}(SL(2,\mathbb{C}))}$
Check: $\sigma \to \frac{A\sigma + B}{C\sigma + D}$
$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$
ick:


Check: $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ $\nabla_a \to \nabla_a + B$ $\nabla_a - \nabla_b \to \nabla_a - \nabla_b$.

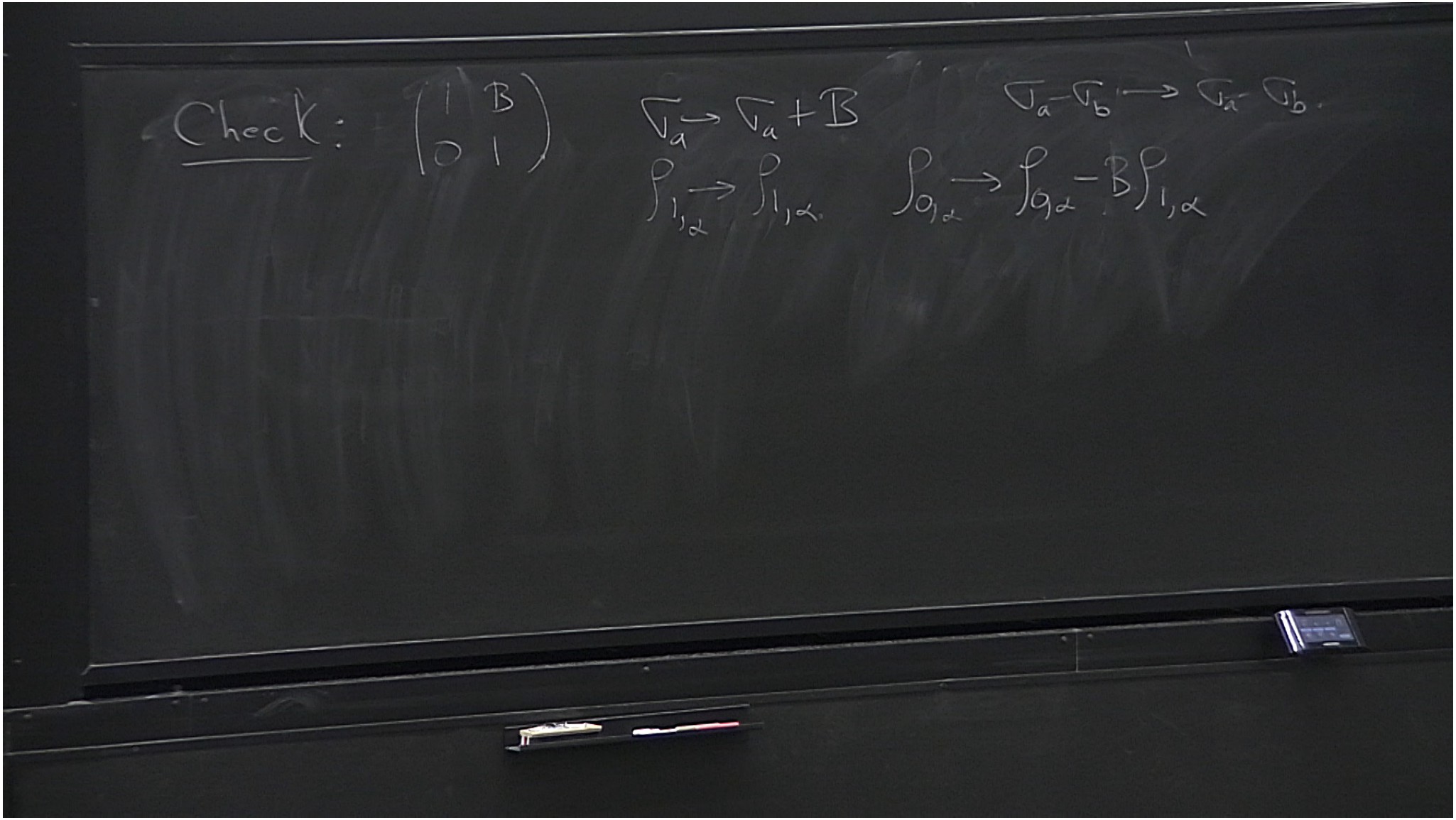
$$A_n = \int d^2\rho_0\, d^2\rho_1 \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \prod_{a=1}^{n} \delta^2\left(\lambda_a^{(a)} - t_a(\rho_0 + \rho_1\sigma_a)\right) \frac{\delta^2\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a\right)}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots}$$

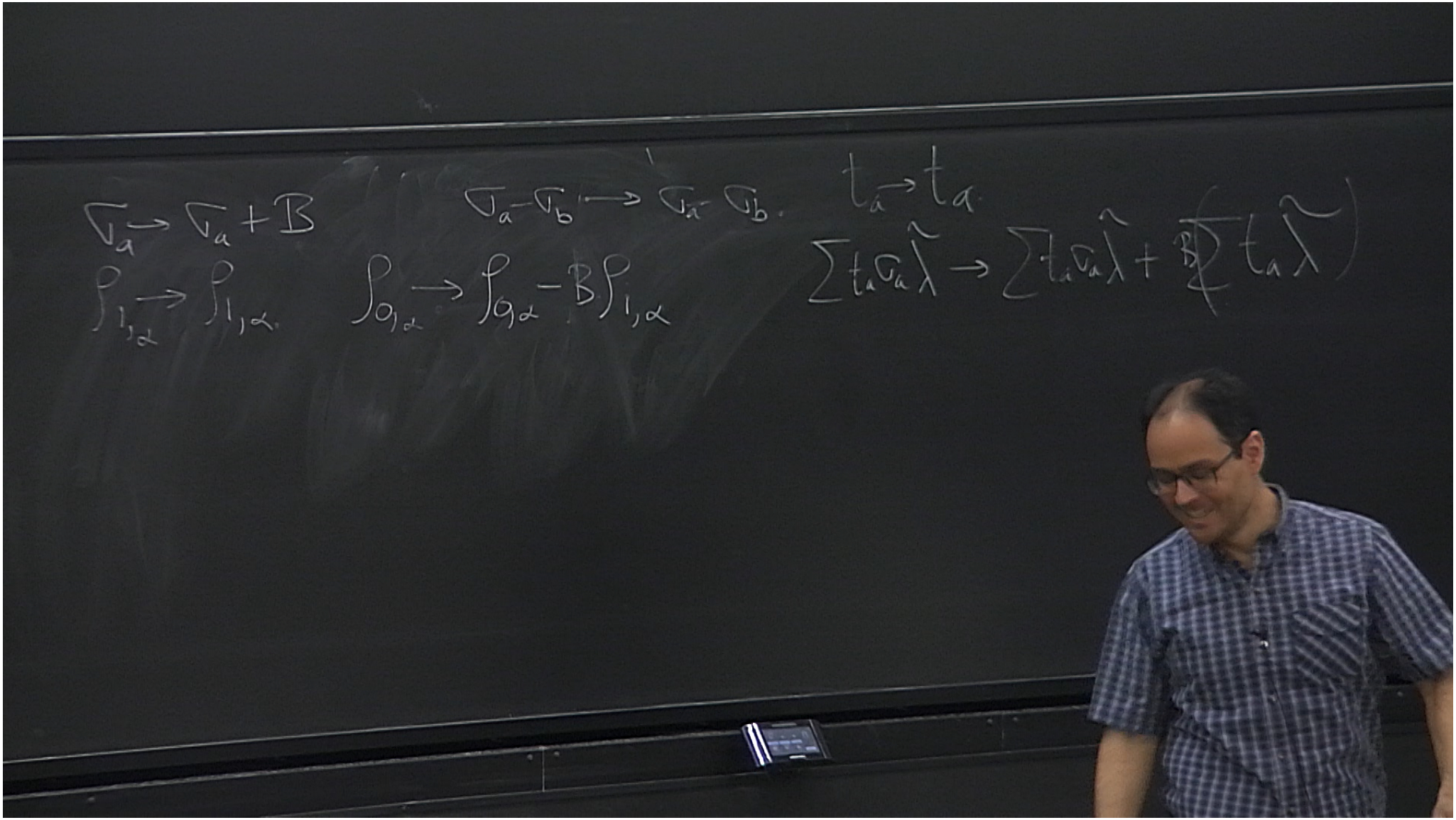
$$\frac{1}{\mathrm{Vol}(SL(2,\mathbb{C}))}$$

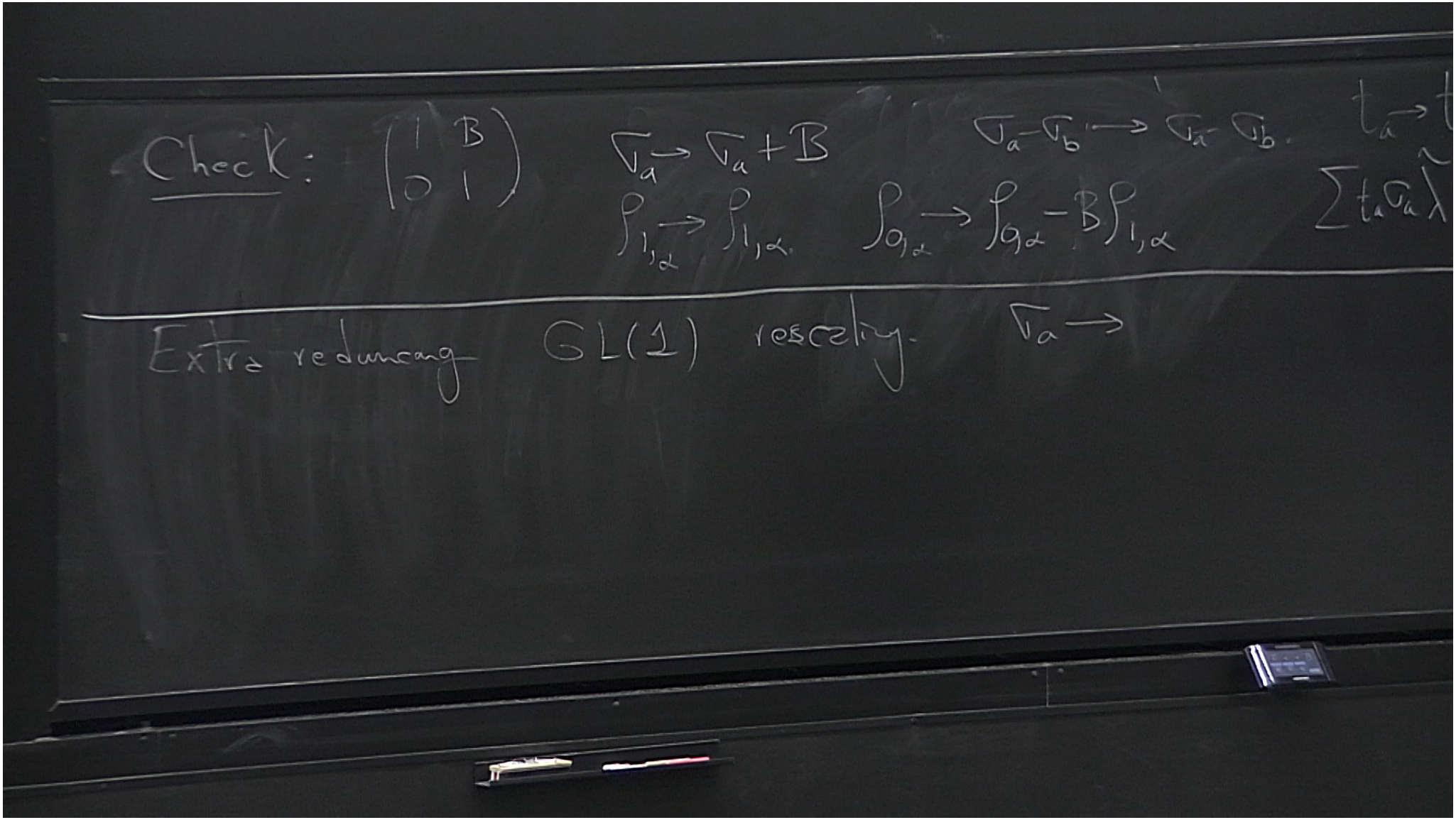
Check: $\sigma \to \frac{A\sigma + B}{C\sigma + D}$

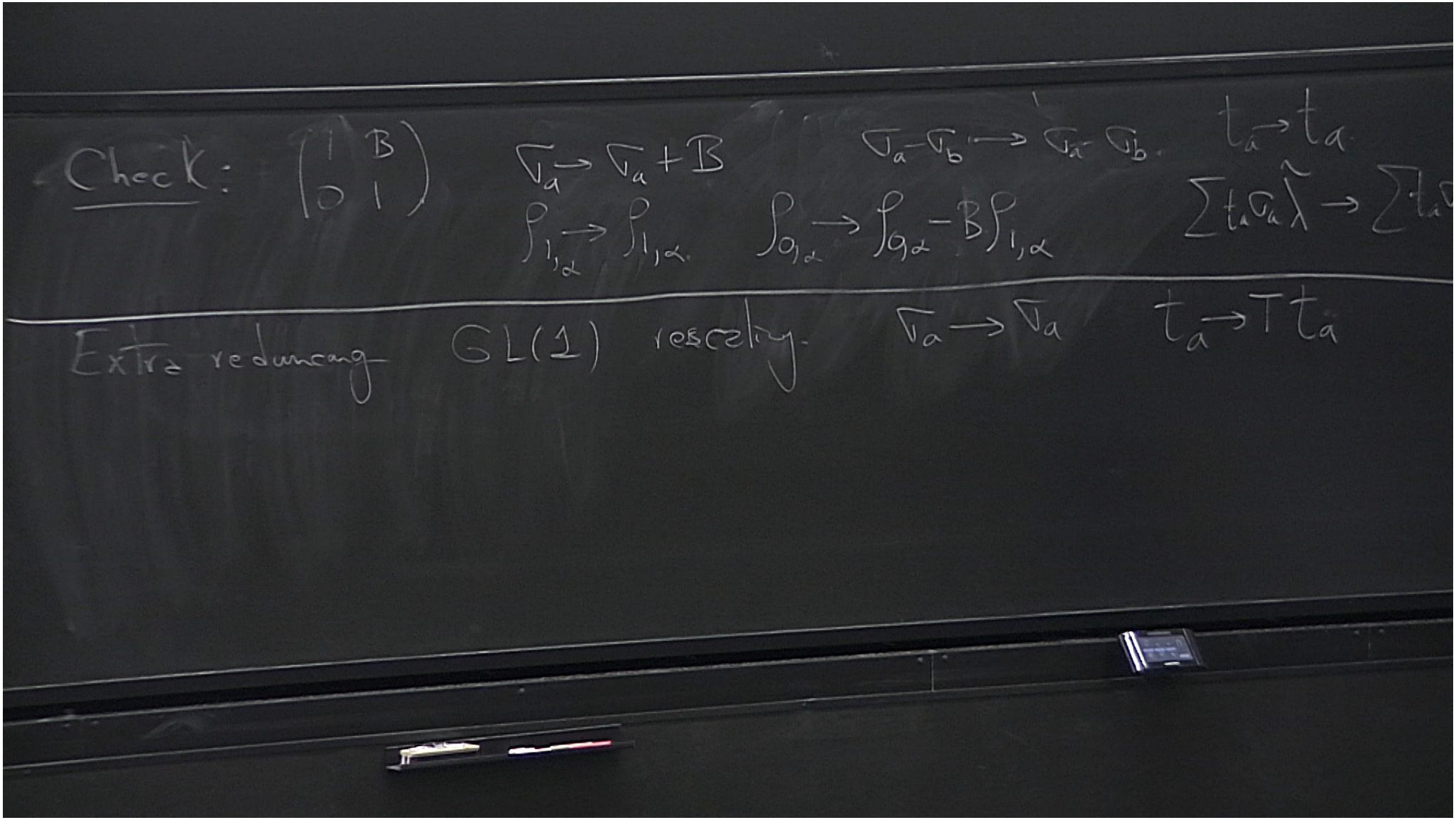
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$$

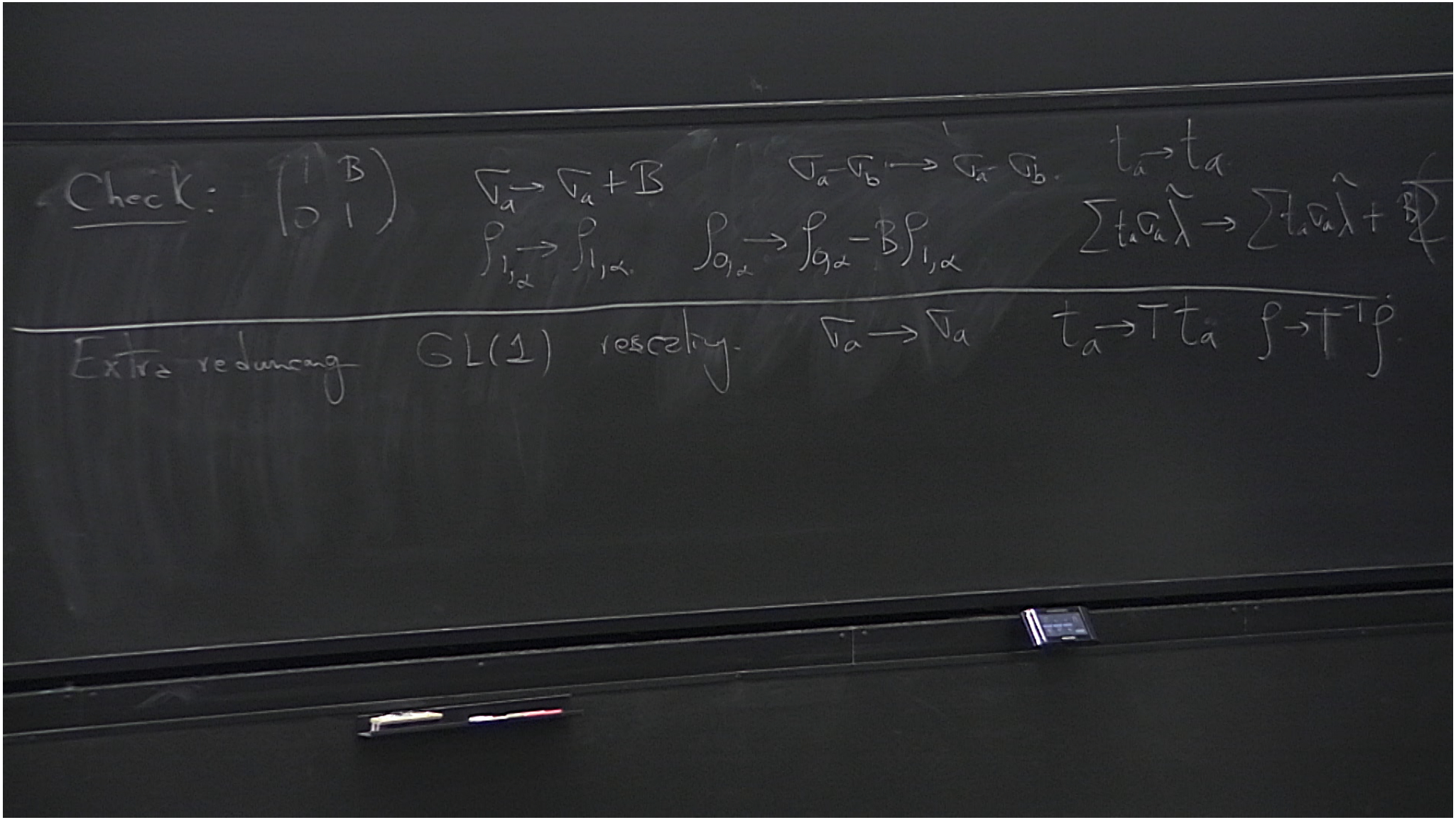
Trick:

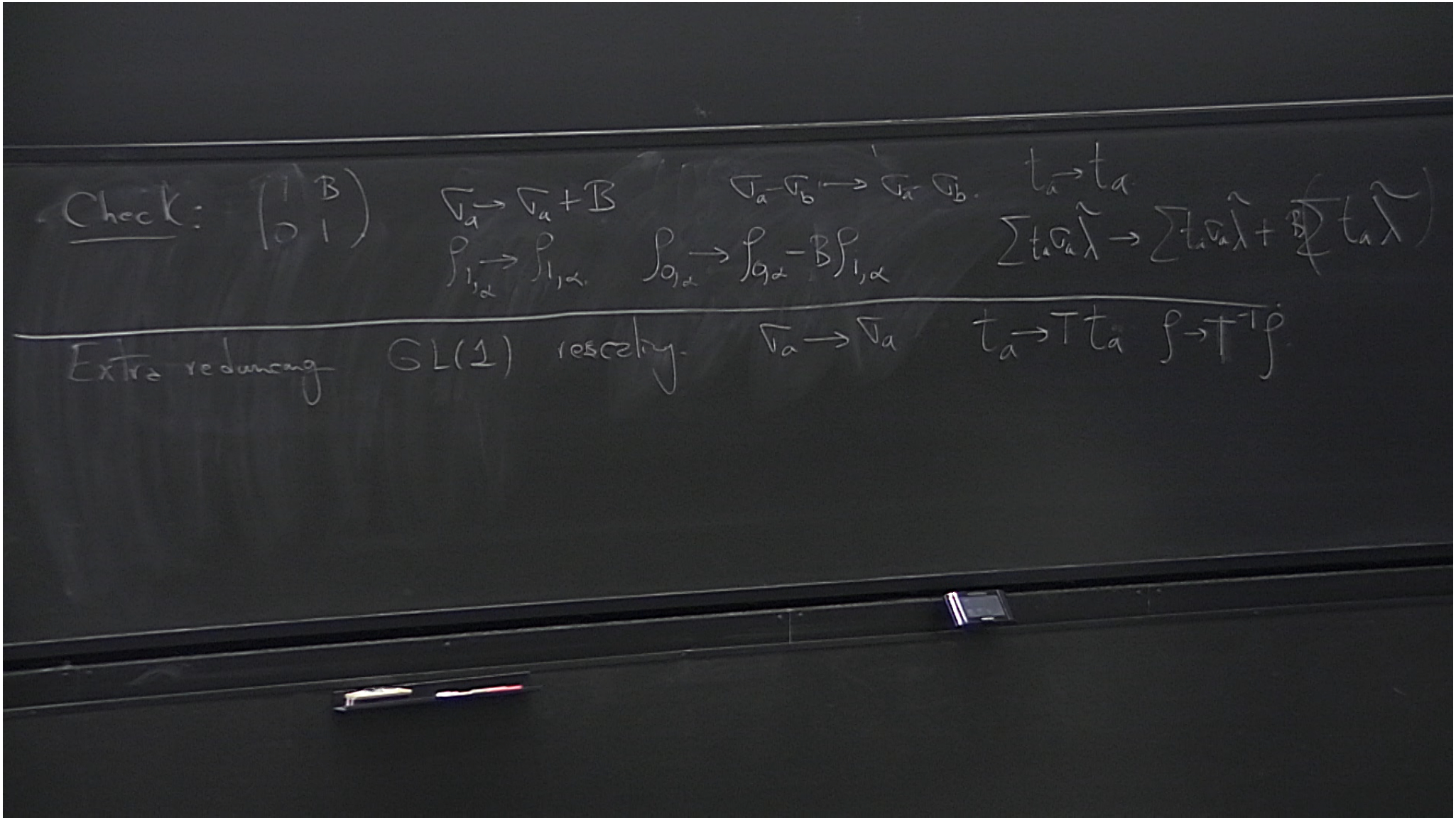

Check: $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$
$\sigma_a \to \sigma_a + B$
$\sigma_a - \sigma_b \to \sigma_a - \sigma_b$
$\rho_{1,\alpha} \to \rho_{1,\alpha}$
$\rho_{0,\alpha} \to \rho_{0,\alpha} - B\rho_{1,\alpha}$

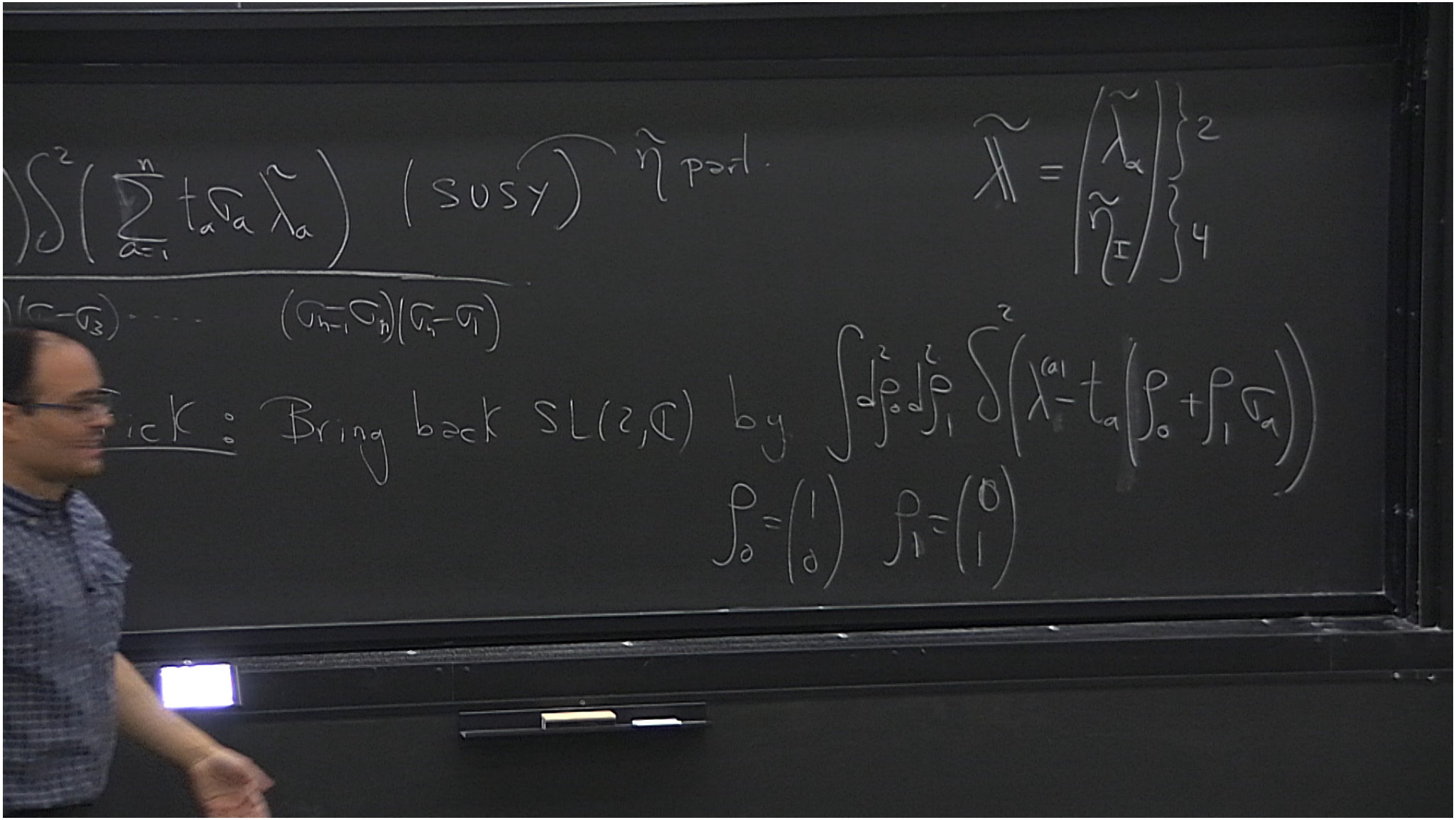

$\sigma_a \to \sigma_a + B$
$\rho_{1,\alpha} \to \rho_{1,\alpha}$
$\sigma_a - \sigma_b \to \sigma_a - \sigma_b$
$\rho_{0,\alpha} \to \rho_{0\alpha} - B\rho_{1,\alpha}$
$t_a \to t_a$
$\sum t_a \sigma_a \tilde{\lambda} \to \sum t_a \sigma_a \tilde{\lambda} + B \sum t_a \tilde{\lambda}$

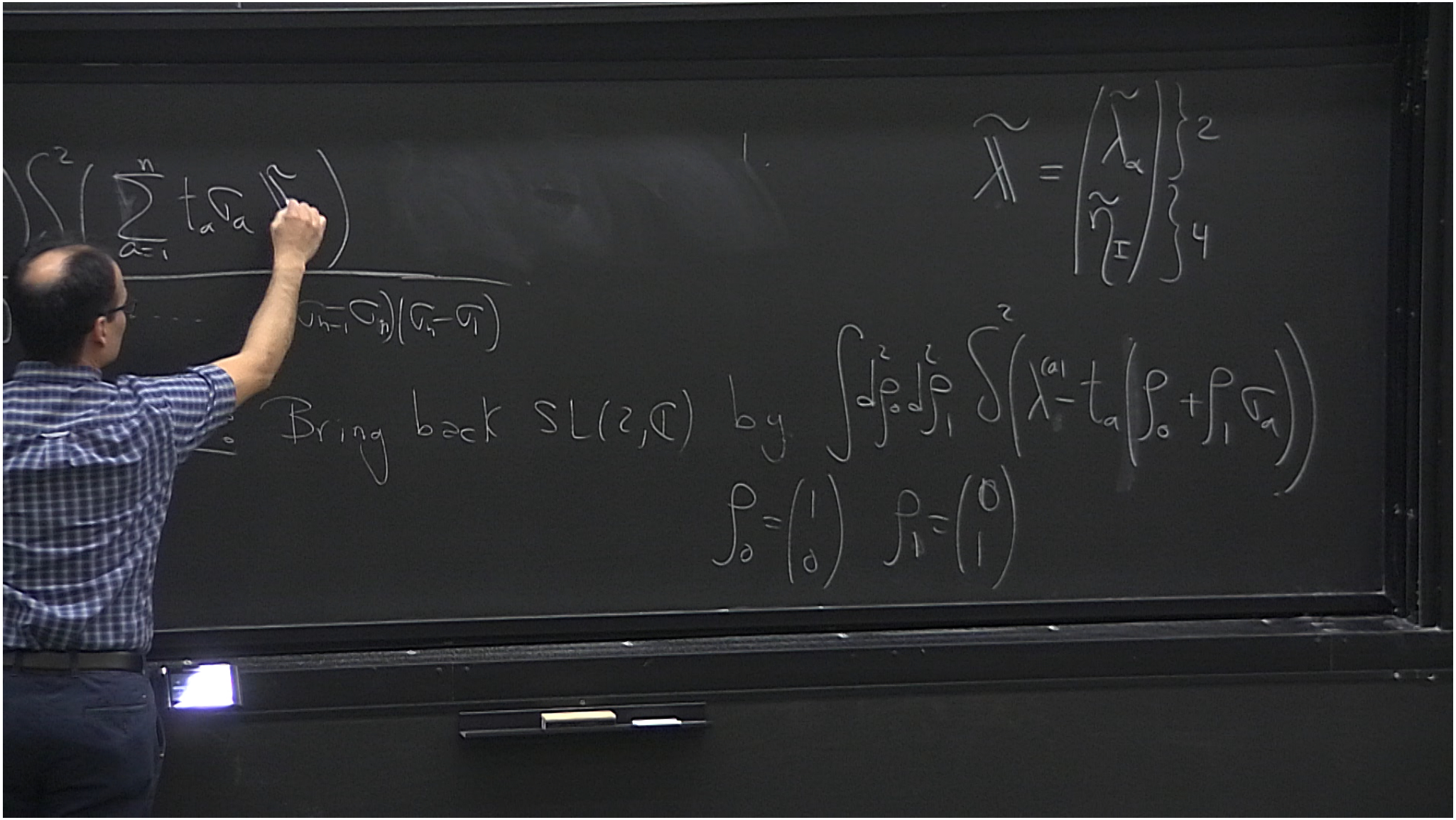

Check: $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ $\Gamma_a \to \Gamma_a + B$ $\Gamma_a - \Gamma_b \to \Gamma_a - \Gamma_b$. $t_a \to t$
$\rho_{1,\alpha} \to \rho_{1,\alpha}$ $\rho_{a,\alpha} \to \rho_{a\alpha} - B\rho_{1,\alpha}$ $\sum t_a \Gamma_a \tilde{\lambda}$
Extra redundancy GL(1) rescaling. $\Gamma_a \to$


Check: $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ $\sigma_a \to \sigma_a + B$ $\sigma_a - \sigma_b \to \sigma_a - \sigma_b$ $t_a \to t_a$
$\rho_{1,\alpha} \to \rho_{1,\alpha}$ $\rho_{0,\alpha} \to \rho_{0,\alpha} - B\rho_{1,\alpha}$ $\sum t_a \sigma_a \tilde{\lambda} \to \sum t_a$
Extra redundancy GL(1) rescaling. $\sigma_a \to \sigma_a$ $t_a \to T t_a$


Check: $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$
$\sigma_a \to \sigma_a + B$
$\sigma_a - \sigma_b \to \sigma_a - \sigma_b$
$t_a \to t_a$
$\rho_{1,\alpha} \to \rho_{1,\alpha}$
$\rho_{0,\alpha} \to \rho_{0,\alpha} - B\rho_{1,\alpha}$
$\sum t_a \sigma_a \tilde{\lambda} \to \sum t_a \sigma_a \tilde{\lambda} + B$
Extra redundancy GL(1) rescaling. $\sigma_a \to \sigma_a$ $t_a \to T t_a$ $\rho \to T^{-1}\rho$


Check: $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$
$\sigma_a \to \sigma_a + B$
$\sigma_a - \sigma_b \to \sigma_a - \sigma_b$
$t_a \to t_a$
$\rho_{1,\alpha} \to \rho_{1,\alpha}$
$\rho_{0,\alpha} \to \rho_{0,\alpha} - B\rho_{1,\alpha}$
$\sum t_a \sigma_a \tilde{\lambda} \to \sum t_a \sigma_a \tilde{\lambda} + B \sum t_a \tilde{\lambda}$
Extra redundancy GL(1) rescaling. $\sigma_a \to \sigma_a$ $t_a \to T t_a$ $\rho \to T^{-1}\rho$


$\delta^2\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)$ (susy) $\tilde{\eta}$ part.
$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_a \\ \tilde{\eta}_I \end{pmatrix} \begin{matrix} \} 2 \\ \} 4 \end{matrix}$
$(\sigma - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)$
Trick: Bring back SL(2,C) by $\int d^2\rho_0 d^2\rho_1 \, \delta^2\left(\lambda_-^{(a)} - t_a(\rho_0 + \rho_1 \sigma_a)\right)$
$\rho_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\rho_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$


$\int d^2\rho_0 d^2\rho_1 \delta^2\left(\lambda_a^{(a)} - t_a(\rho_0 + \rho_1\sigma_a)\right)$
Bring back $SL(2,\mathbb{C})$ by
$\rho_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\rho_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$

$k = \#$ of negative helicity gluons

$k = 2$



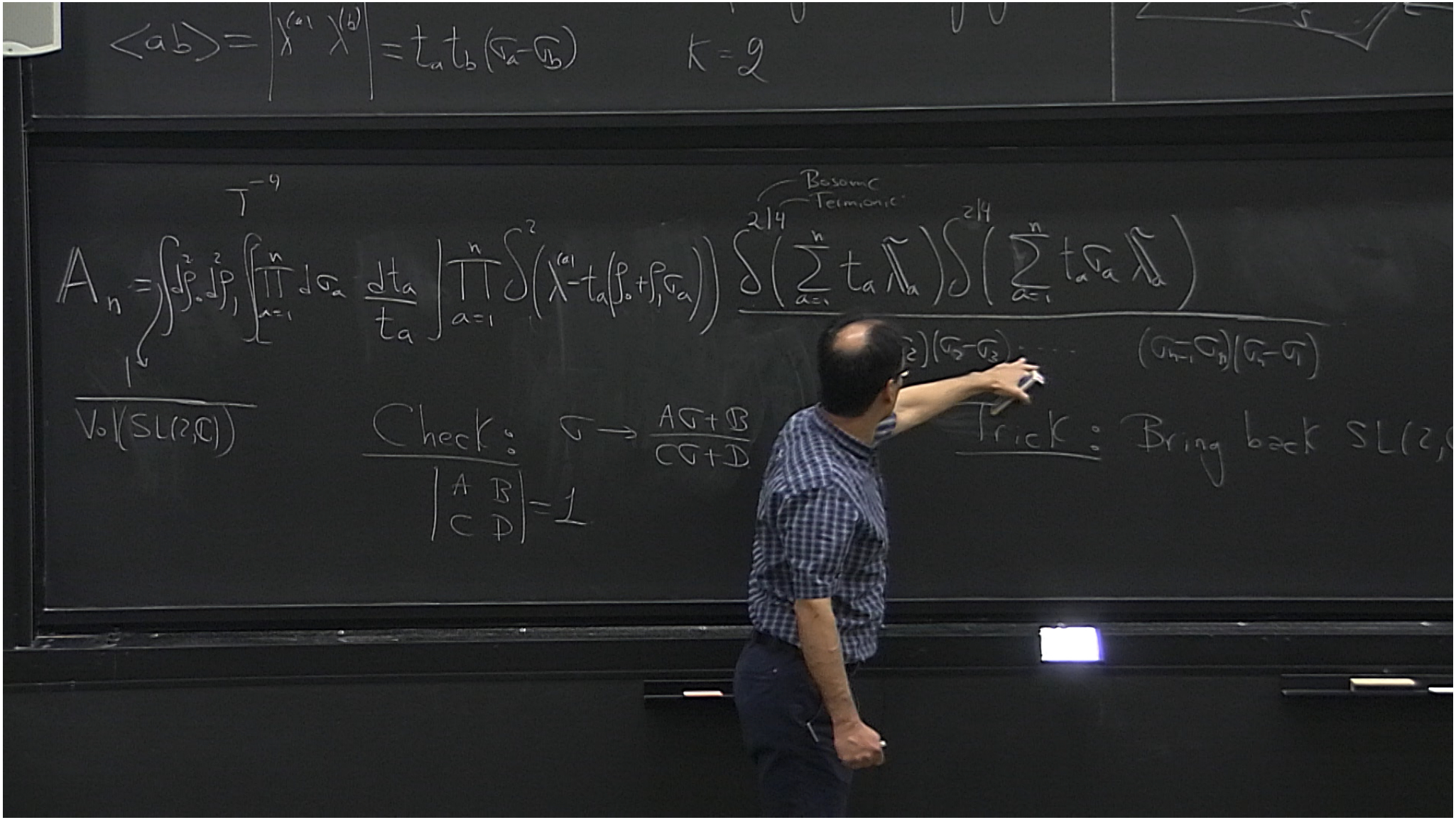
$$A_n = \int d^2\rho_0 \, d^2\rho_1 \left[\prod_{a=1}^{n} d\sigma_a \, \frac{dt_a}{t_a} \right] \prod_{a=1}^{n} \delta^{2}\left(\lambda^{(a)} - t_a(\rho_0 + \rho_1 \sigma_a) \right) \frac{\overbrace{\delta^{2|4}}^{\text{Bosonic, Fermionic}}\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a \right) \delta^{2|4}\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a \right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)}$$

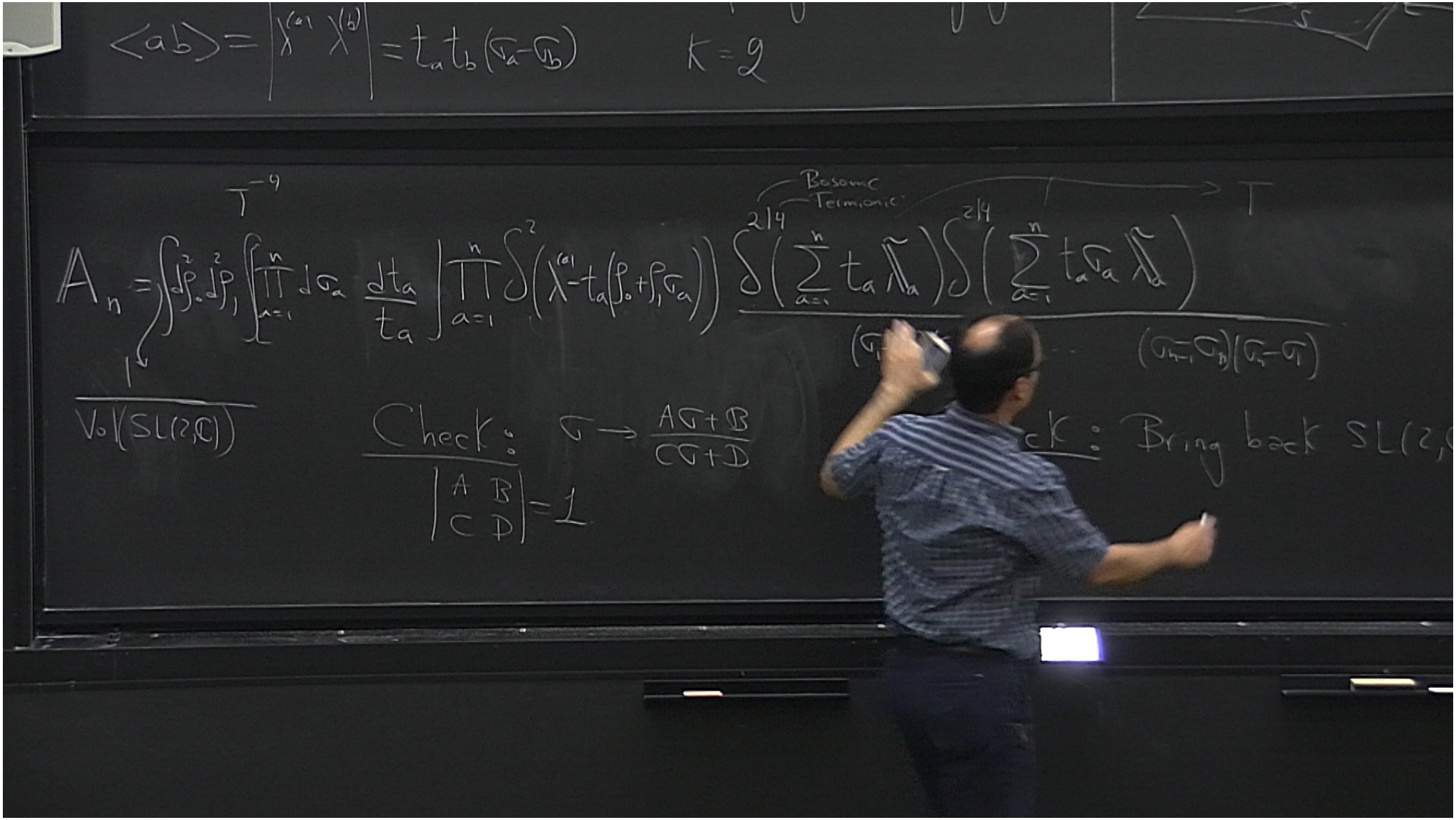
$$\frac{1}{\mathrm{Vol}(SL(2,\mathbb{C}))}$$

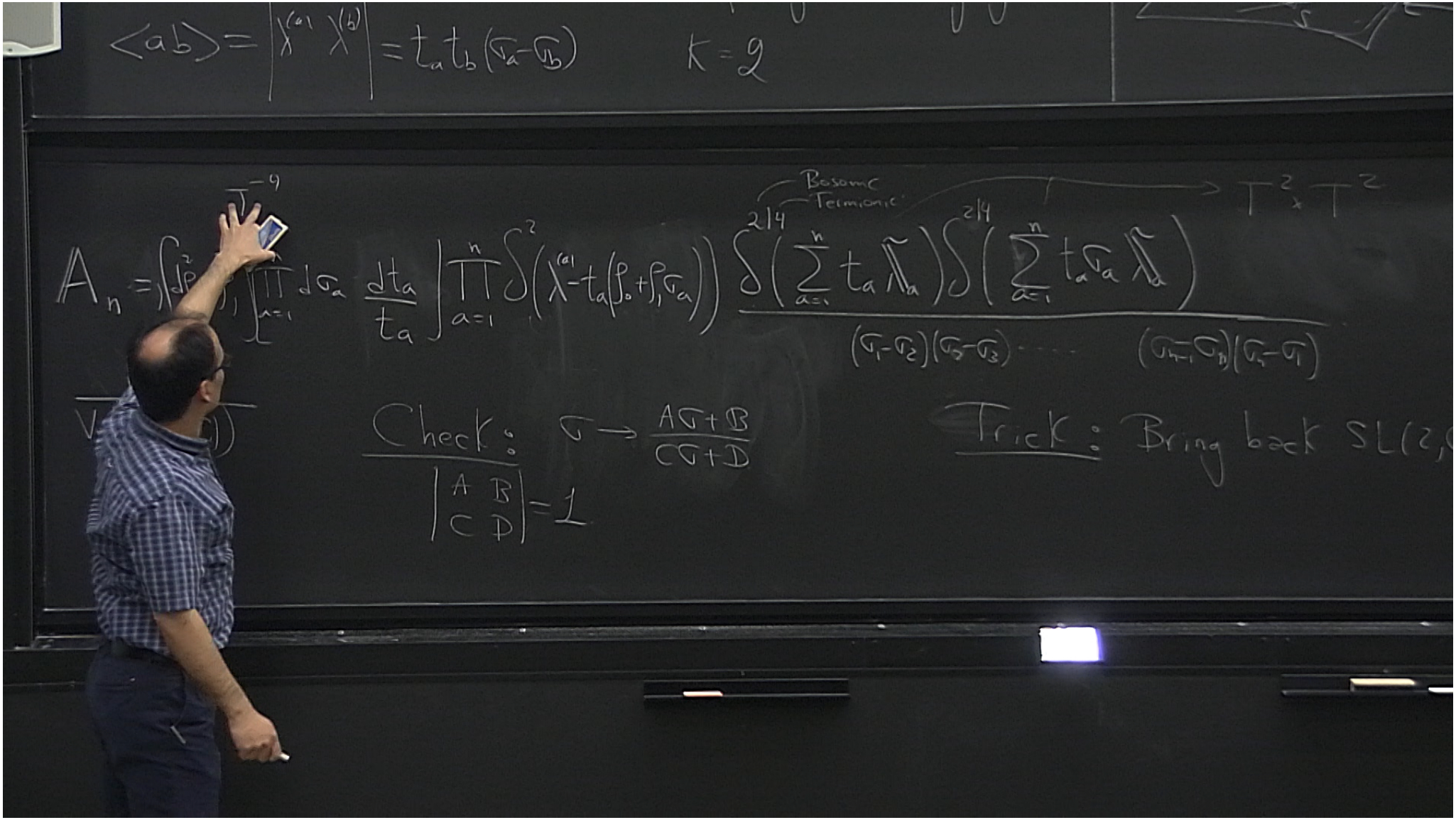
$$\sigma \to \frac{A\sigma + B}{C\sigma + D}$$

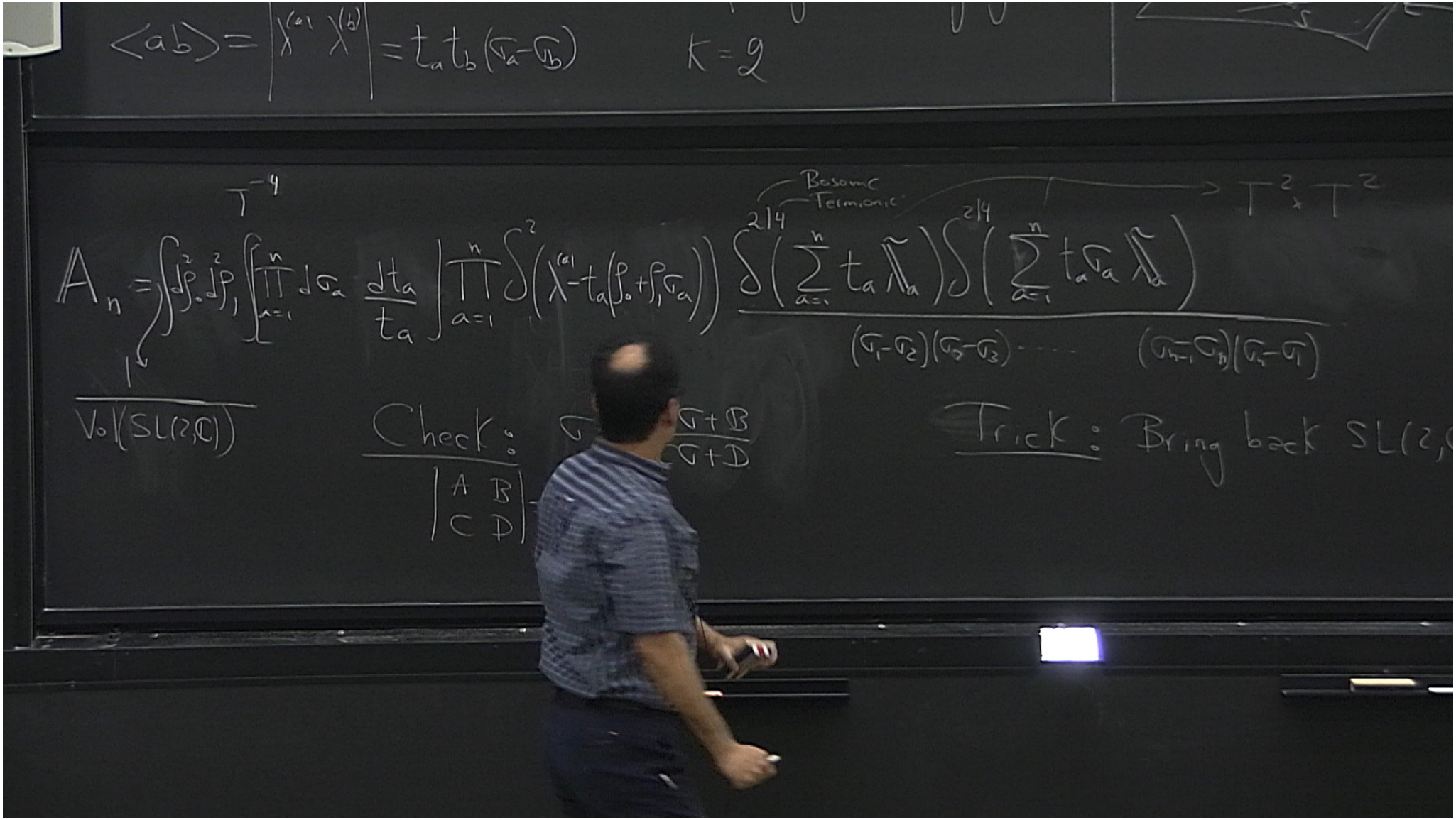
Trick: Bring back $SL(2,\mathbb{C})$ by $\int d^2\rho_0 \, d^2\rho_1$

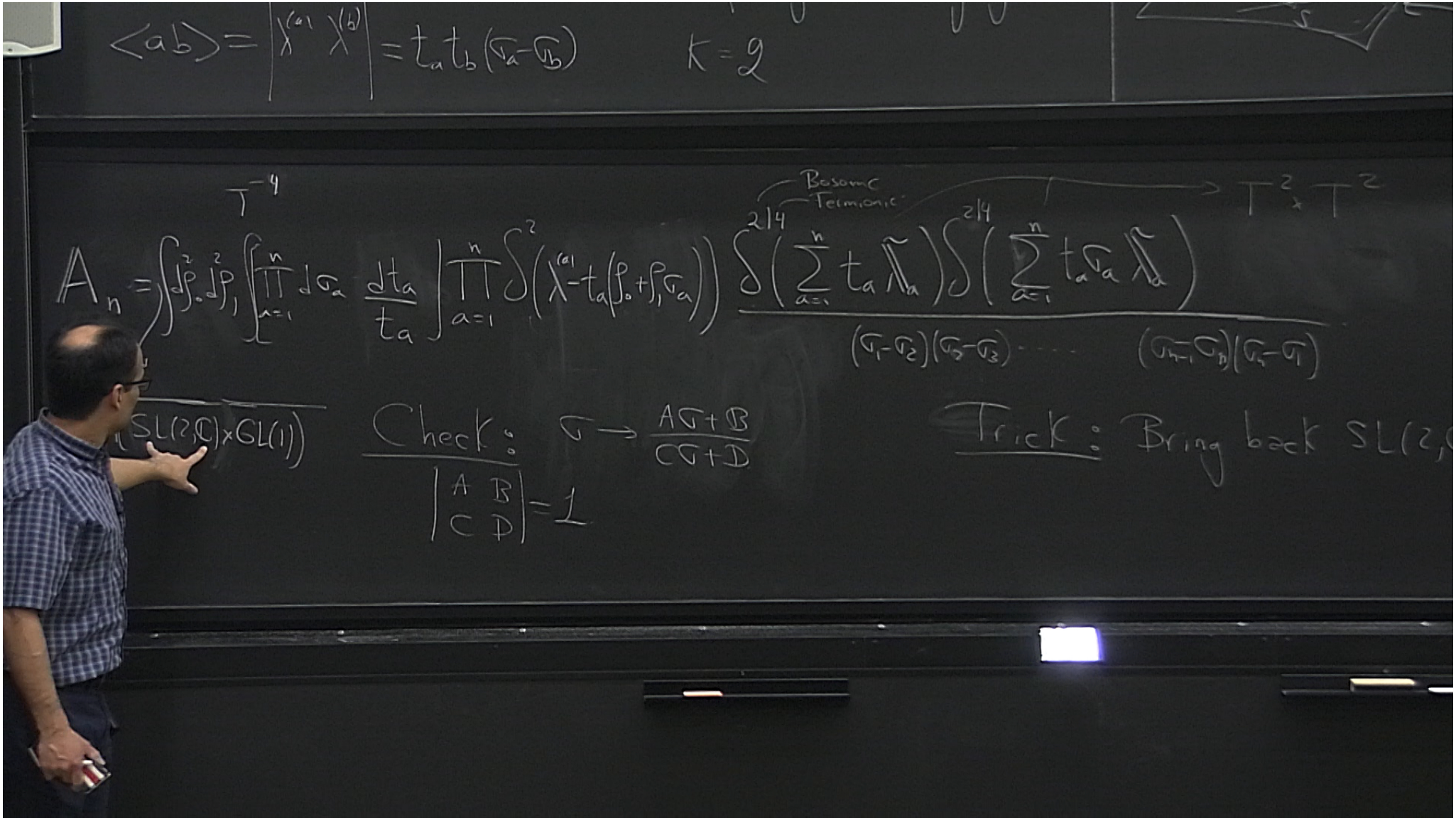
$\rho_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ρ_1

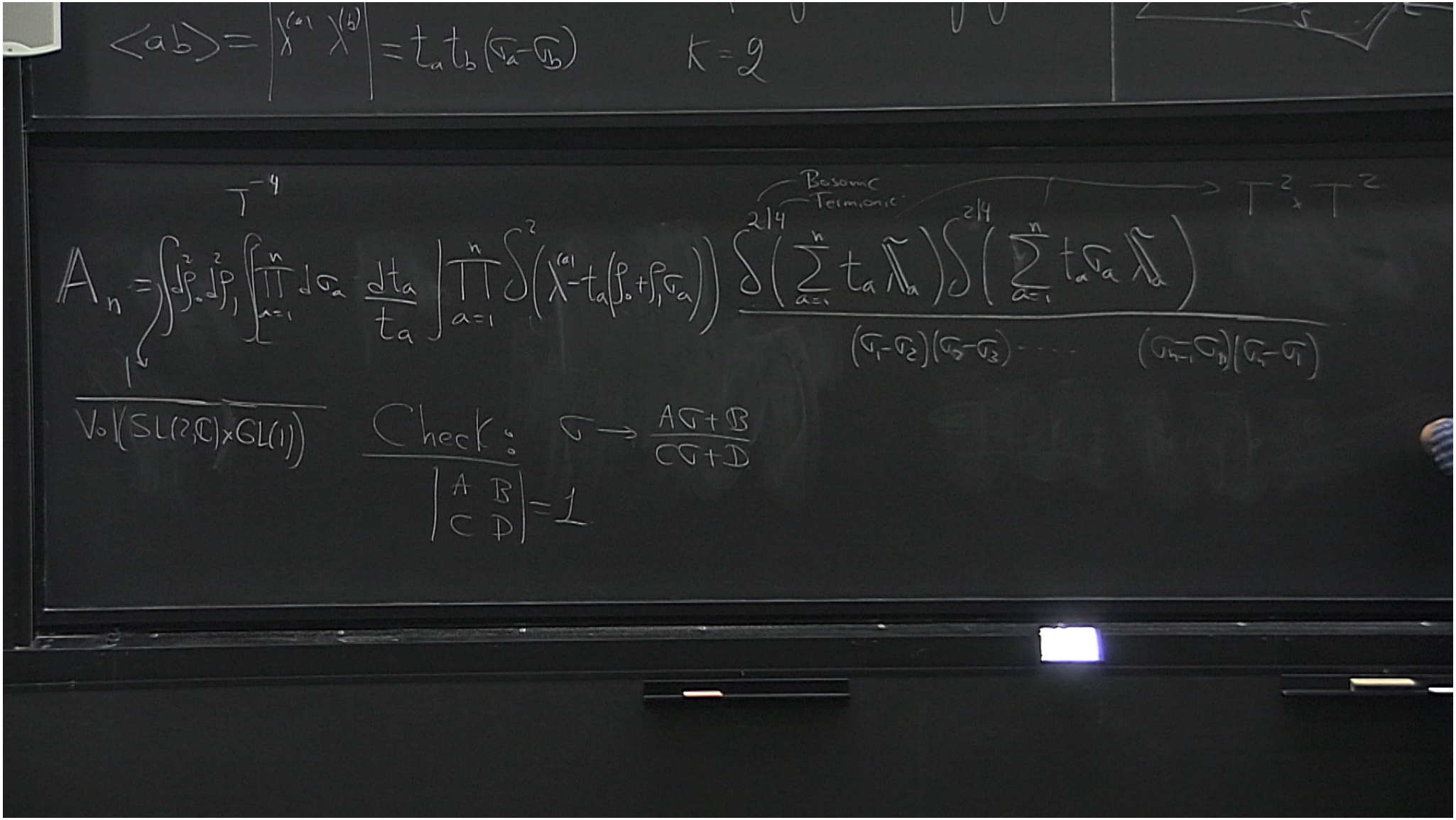

$\langle ab \rangle = |\lambda^{(a)} \lambda^{(b)}| = t_a t_b (\sigma_a - \sigma_b)$
$k = 2$
Bosonic
Fermionic
$A_n = \int d\beta_0 d\beta_1 \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \int \prod_{a=1}^{n} \delta(\lambda^{(a)} - t_a(\beta_0 + \beta_1 \sigma_a)) \delta(\sum_{a=1}^{n} t_a \tilde{\lambda}_a) \delta(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a)$
T^{-4}
$\frac{1}{Vol(SL(2,\mathbb{C}))}$
$(\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)$
Check: $\sigma \to \frac{A\sigma + B}{C\sigma + D}$
$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$
Trick: Bring back SL(2,

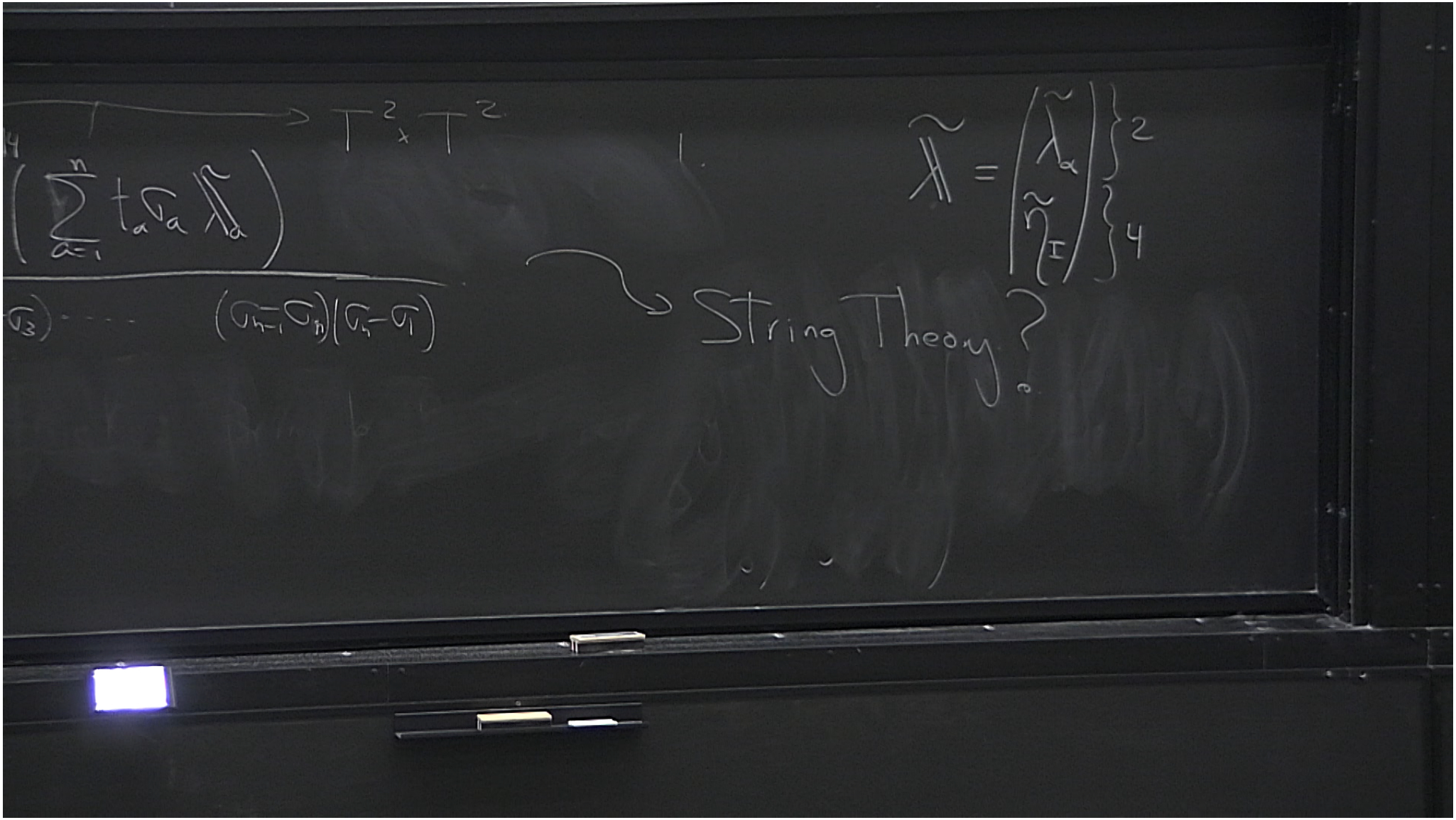

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$
$k = 2$
Bosonic
Fermionic
T^{-4}
T
$A_n = \int \frac{d\beta_0^2 d\beta_1^2}{\mathrm{Vol}(SL(2,\mathbb{C}))} \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a(\beta_0 + \beta_1 \sigma_a)\right) \frac{\delta^{2|4}\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \delta^{2|4}\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)}{(\sigma_1 - \sigma_2)\cdots(\sigma_{n-1}\sigma_n)(\sigma_n - \sigma_1)}$
Check: $\sigma \to \frac{A\sigma + B}{C\sigma + D}$
$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$
ck: Bring back SL(2,

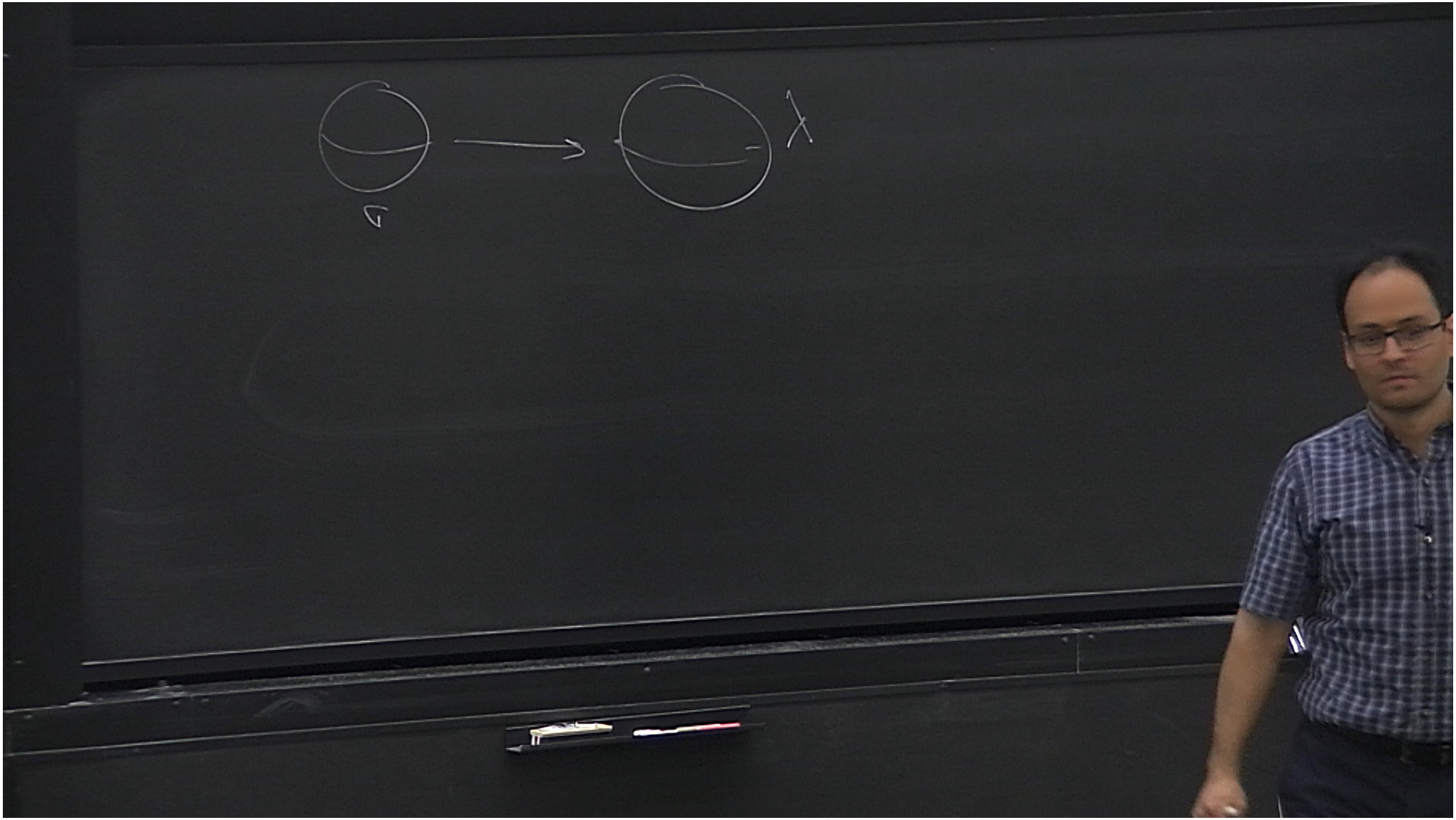

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$
$k = 2$
Bosonic
Fermionic
2|4
2|4
$T^{2}_{x} T^{2}$
$A_n = \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \int \prod_{a=1}^{n} \mathcal{D}\left(\lambda^{(a)} - t_a(\beta_0 + \beta_1 \sigma_a)\right) \frac{\delta\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a\right) \mathcal{D}\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)}$
Check: $\sigma \to \frac{A\sigma + B}{C\sigma + D}$
$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$
Trick: Bring back SL(2,

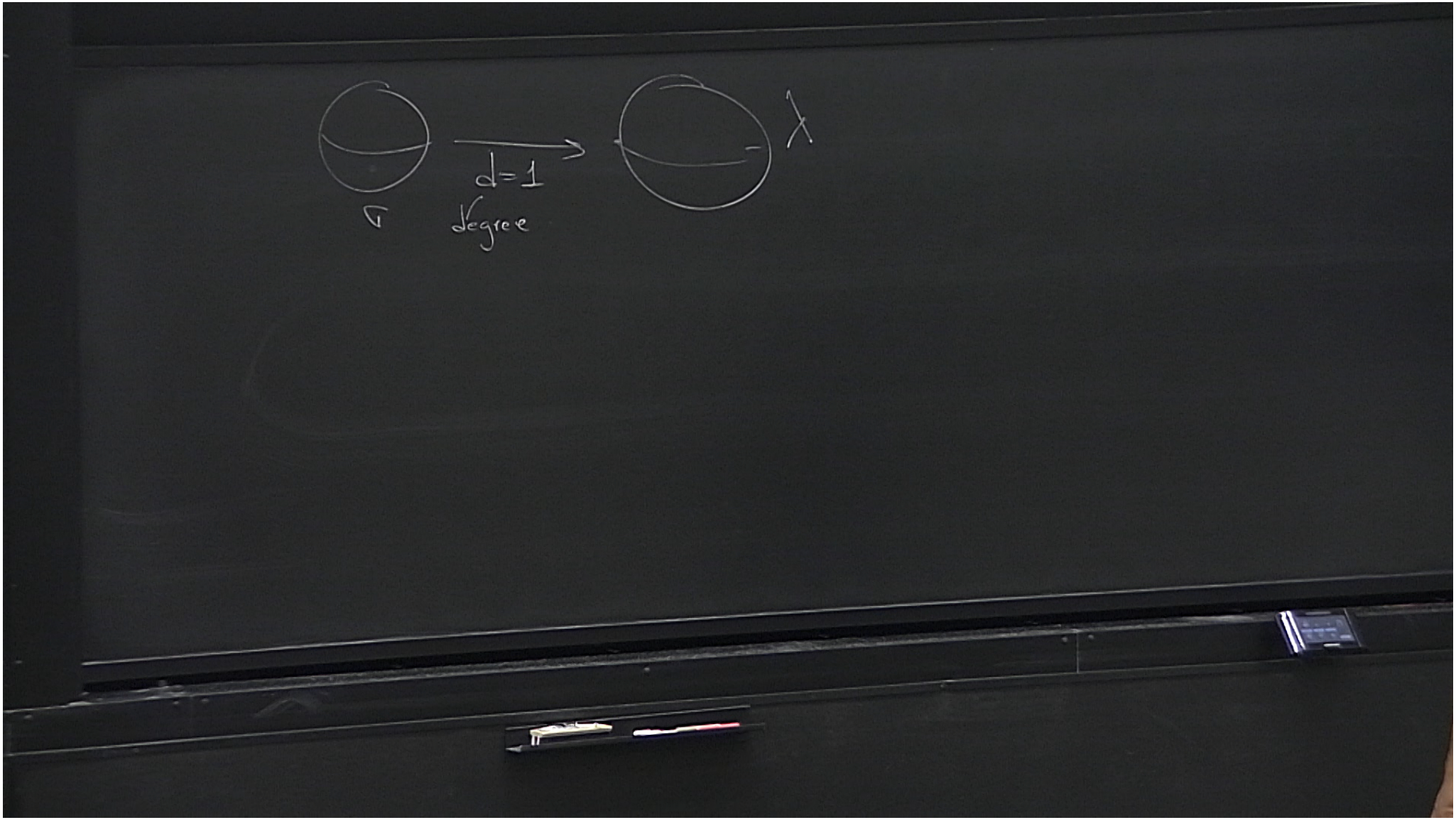

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$
$k = 2$
T^{-4}
Bosonic
Fermionic
$T^2 \times T^2$
$A_n = \frac{1}{\mathrm{Vol}(SL(2,\mathbb{C}))} \int d\beta_0^2 d\beta_1^2 \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \int \prod_{a=1}^{n} \mathcal{D}^2 \left(\lambda^{(a)} - t_a (\beta_0 + \beta_1 \sigma_a) \right) \frac{\delta^{2|4}\left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a \right) \delta^{2|4}\left(\sum_{a=1}^{n} t_a \sigma_a \lambda_a \right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)}$
Check:
$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$
Trick: Bring back $SL(2,$

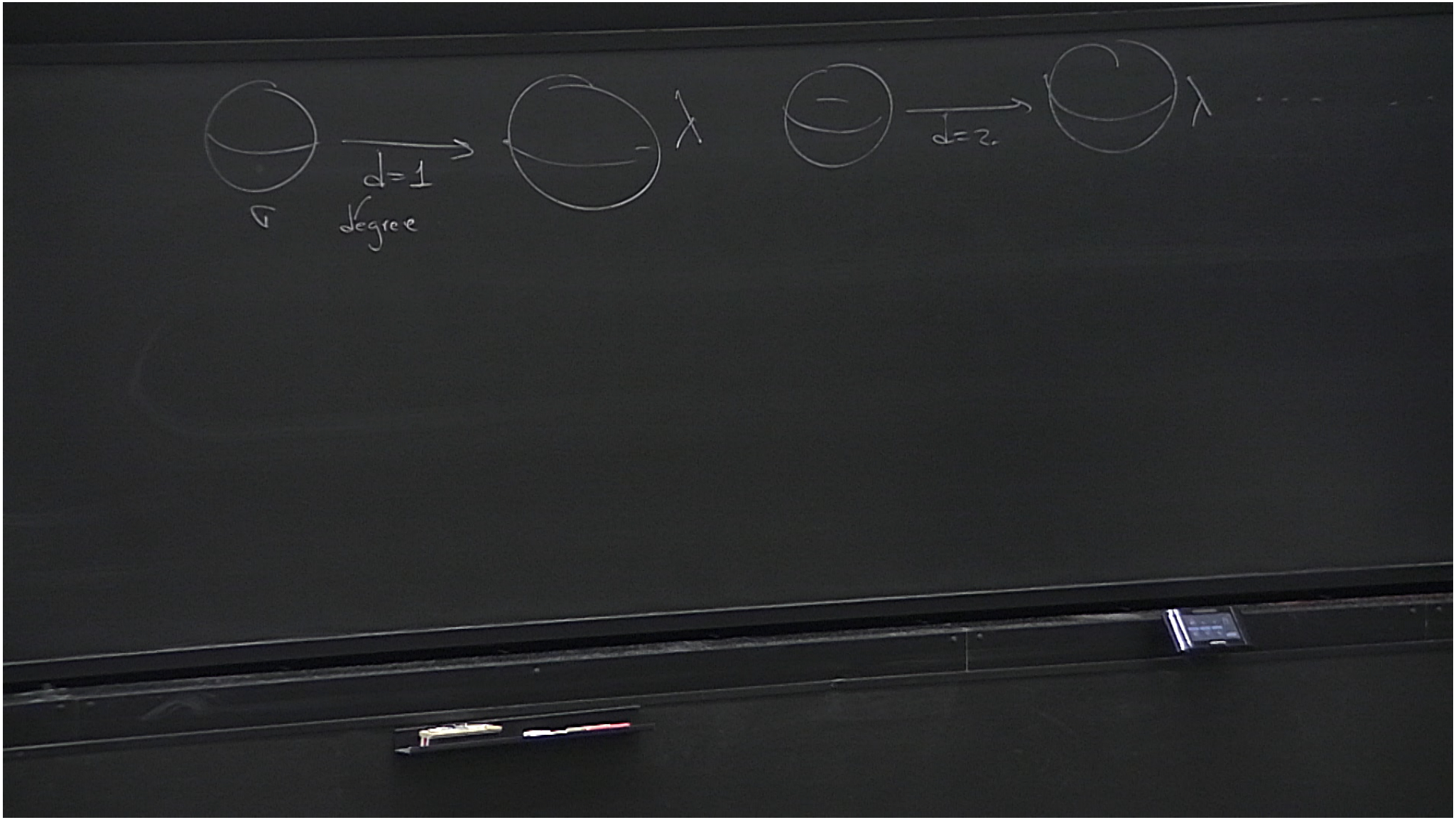

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$
$k = 2$
T^{-4}
Bosonic
Fermionic
$2|4$
$T^2 \times T^2$
$A_n = \int d\beta_0^2 d\beta_1^2 \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \int \prod_{a=1}^{n} \delta^2(\lambda^{(a)} - t_a(\beta_0 + \beta_1 \sigma_a)) \frac{\delta(\sum_{a=1}^{n} t_a \tilde{\lambda}_a) \delta(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)}$
$(SL(2,\mathbb{C}) \times GL(1))$
Check: $\sigma \to \frac{A\sigma + B}{C\sigma + D}$
$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$
Trick: Bring back $SL(2,$

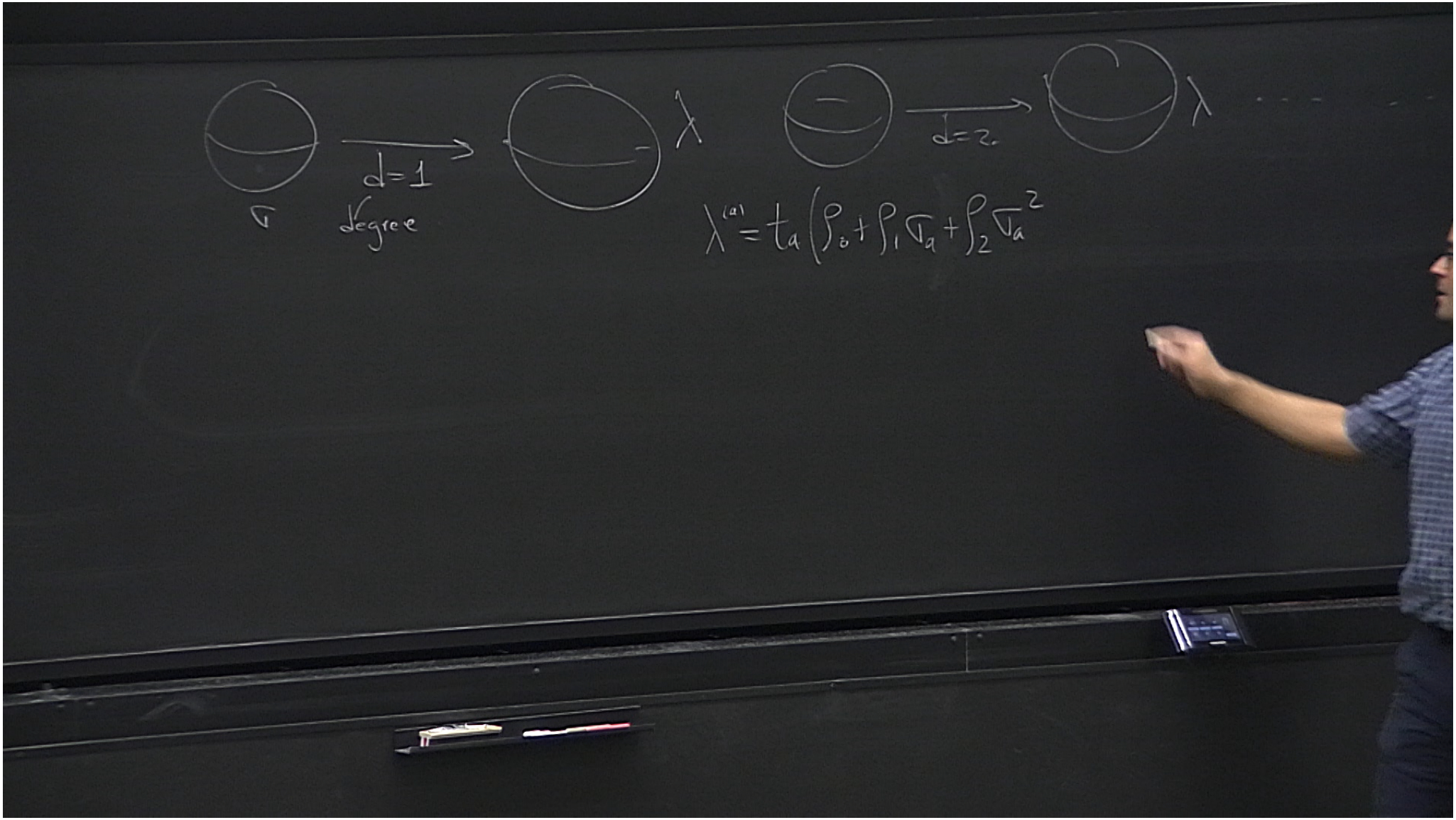

$\langle ab \rangle = \left| \lambda^{(a)} \lambda^{(b)} \right| = t_a t_b (\sigma_a - \sigma_b)$
$k = 2$
T^{-4}
Bosonic
Fermionic
$2|4$
$2|4$
$T^2 \times T^2$
$A_n = \int d\beta_0^2 d\beta_1^2 \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \prod_{a=1}^{n} \delta^2 \left(\lambda^{(a)} - t_a (\beta_0 + \beta_1 \sigma_a) \right) \frac{\delta \left(\sum_{a=1}^{n} t_a \tilde{\lambda}_a \right) \delta \left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a \right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)}$
$\frac{1}{Vol\left(SL(2,\mathbb{C}) \times GL(1)\right)}$
Check: $\sigma \to \frac{A\sigma + B}{C\sigma + D}$
$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$

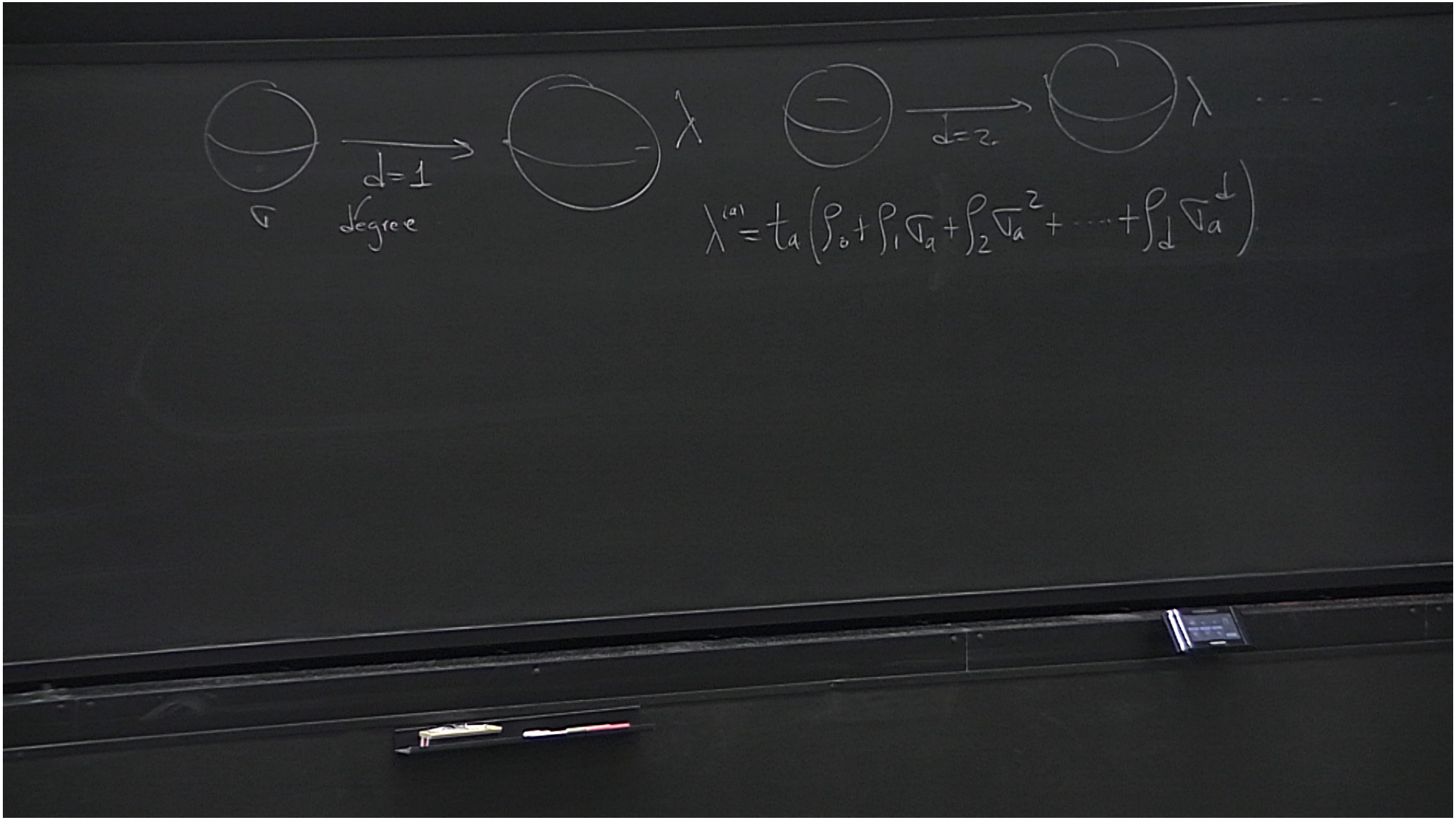

$T^2 \times T^2$
$\left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a\right)$
$(\sigma_3) \cdots (\sigma_{n-1}\sigma_n)(\sigma_1 - \sigma_1)$
$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_\alpha \\ \tilde{\eta}_I \end{pmatrix}$ 2 4
String Theory?

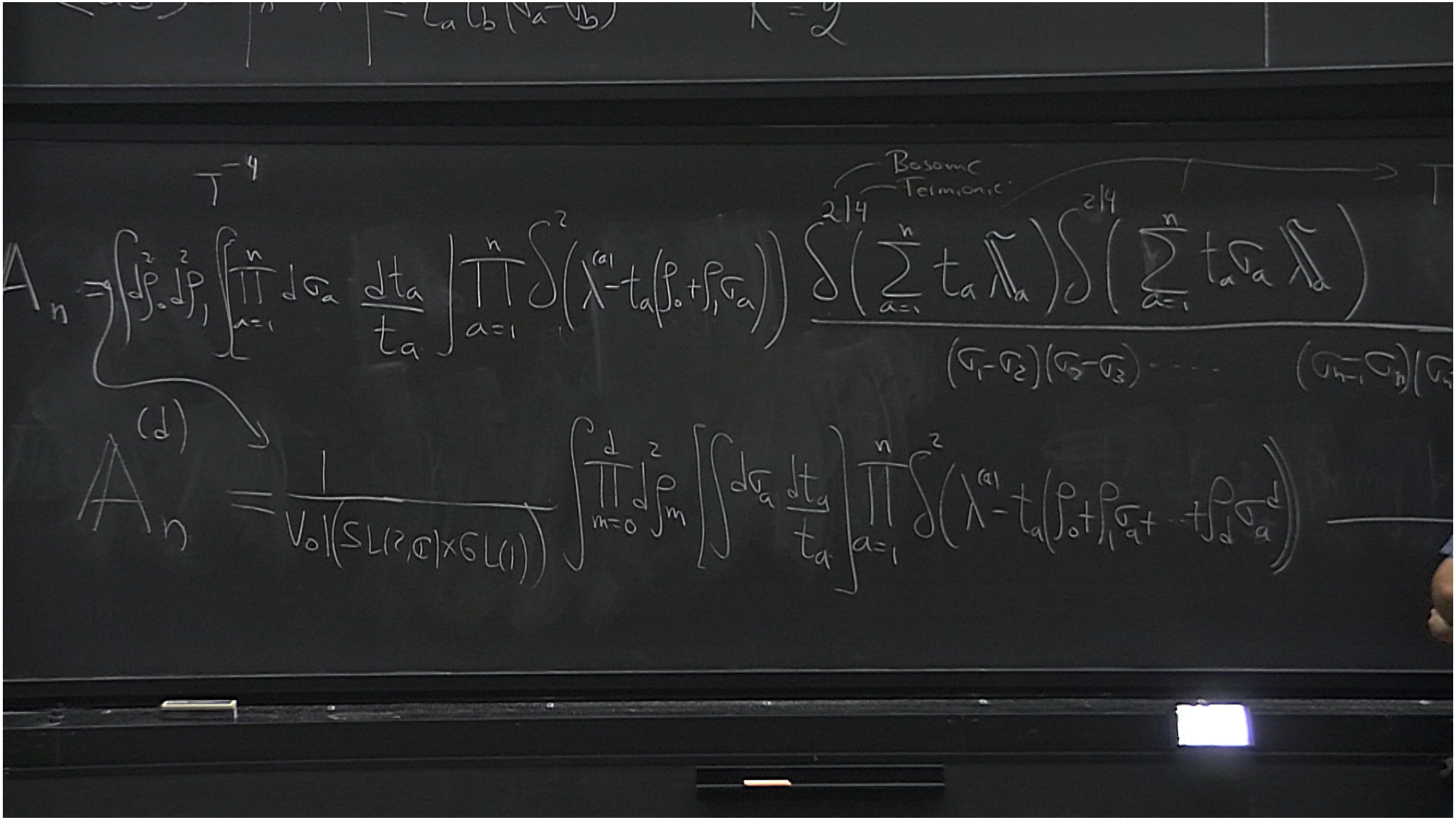


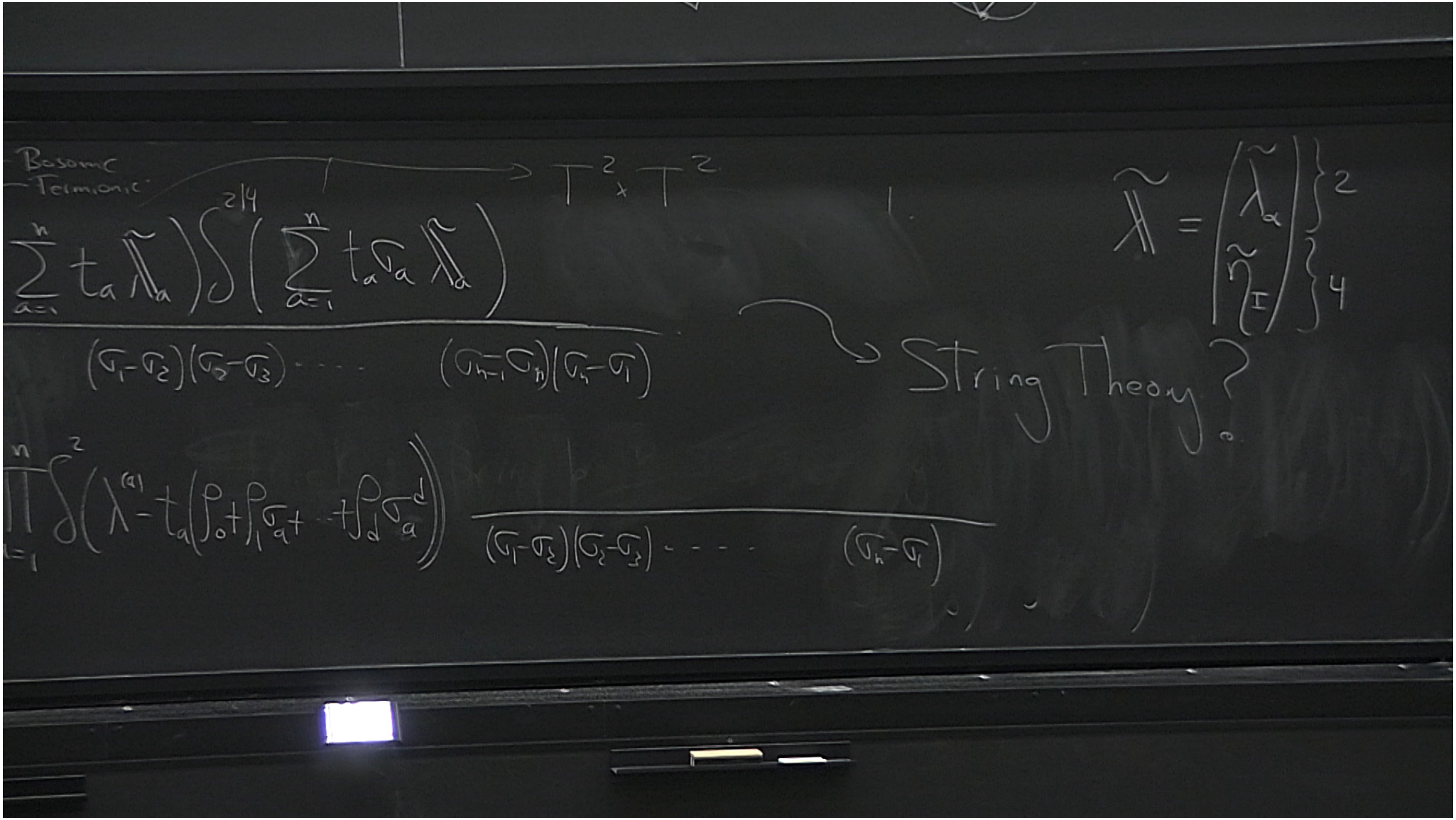

d=1
degree

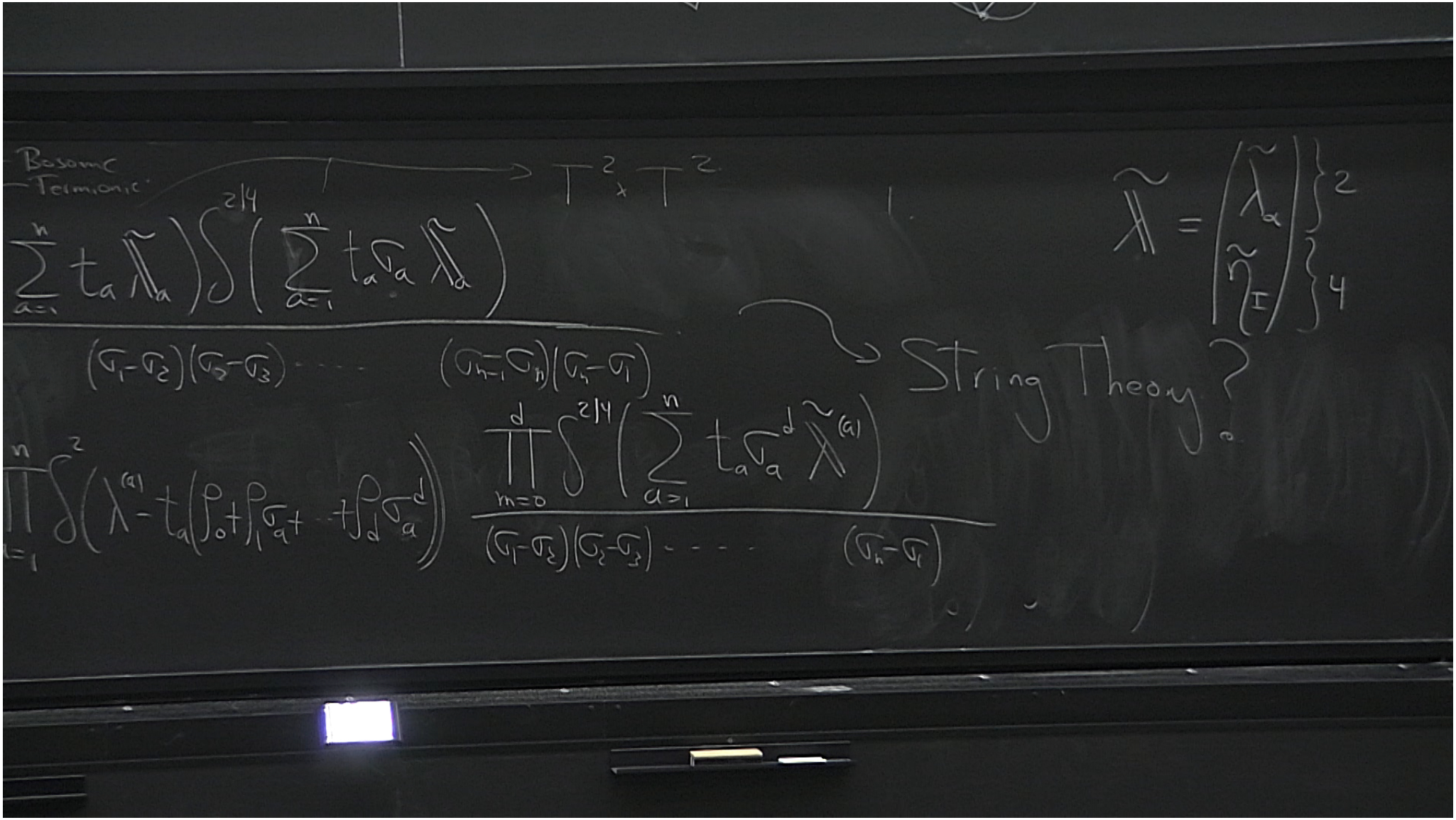

d=1
degree
d=2.

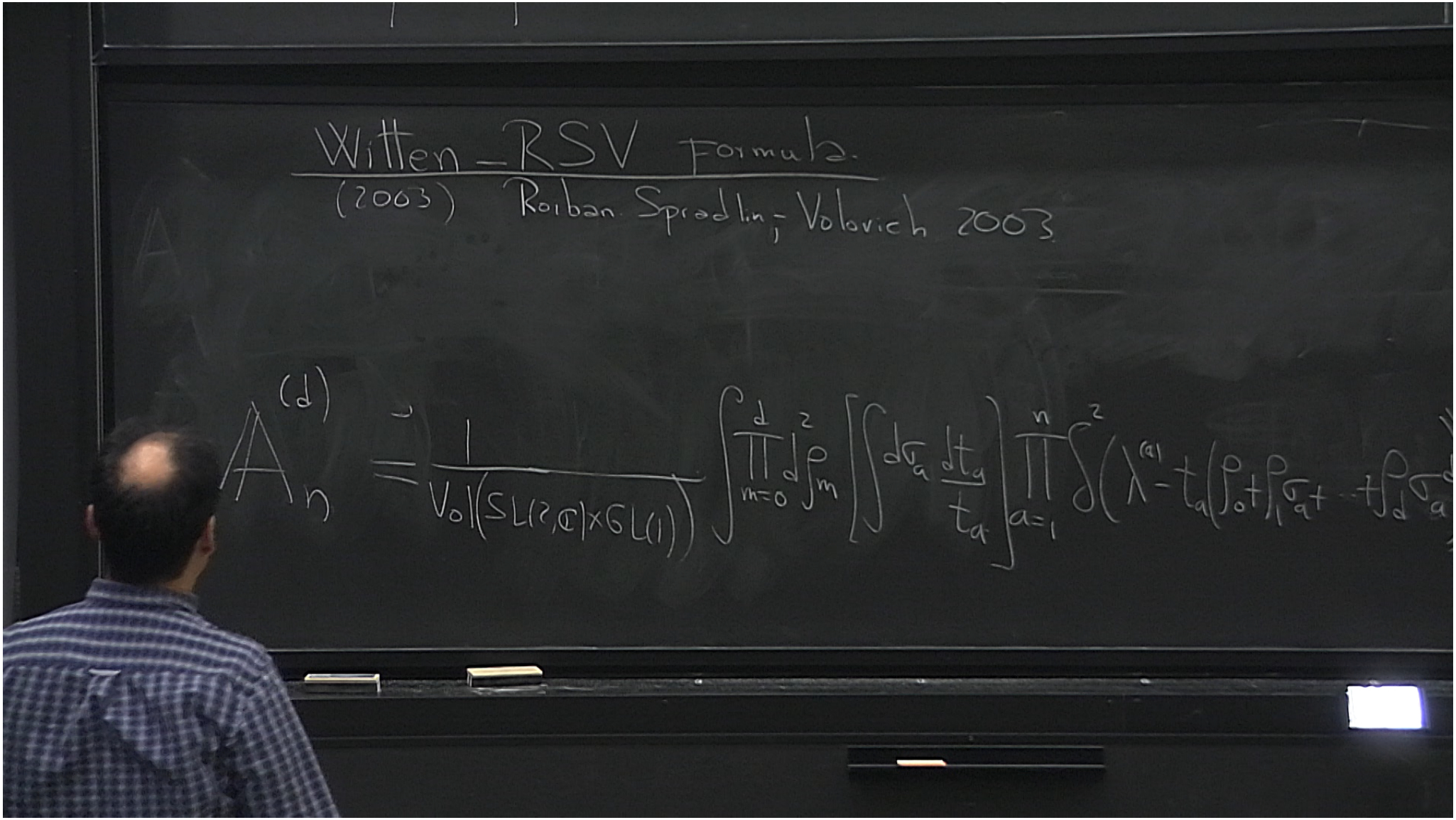

d=1
degree
d=2.
$\lambda^{(a)} = t_a(\beta_0 + \beta_1 \sigma_a + \beta_2 \sigma_a^2$

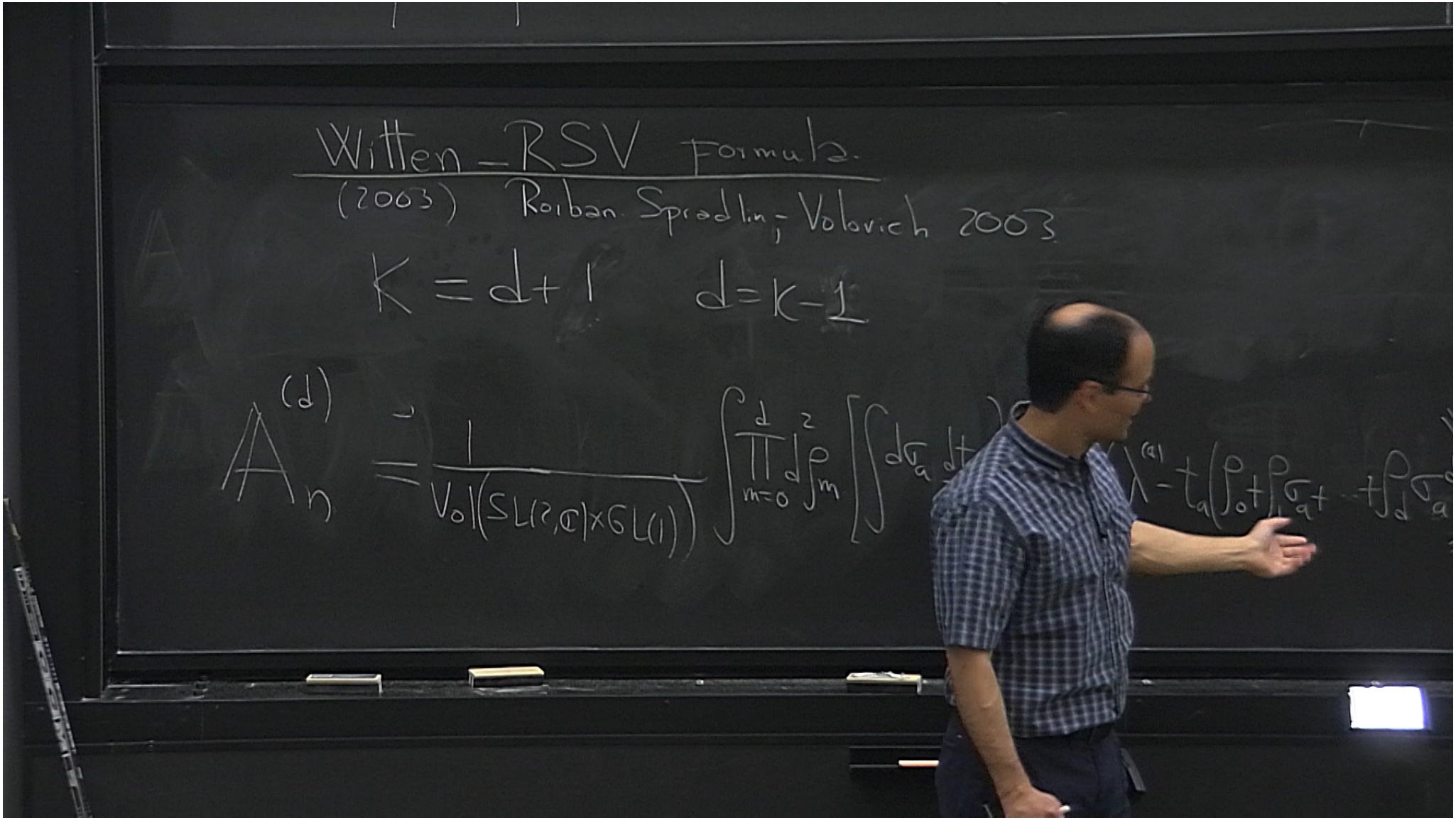

d=1
degree
d=2
$\lambda^{(a)} = t_a\left(\rho_0 + \rho_1 \sigma_a + \rho_2 \sigma_a^2 + \cdots + \rho_d \sigma_a^d\right)$

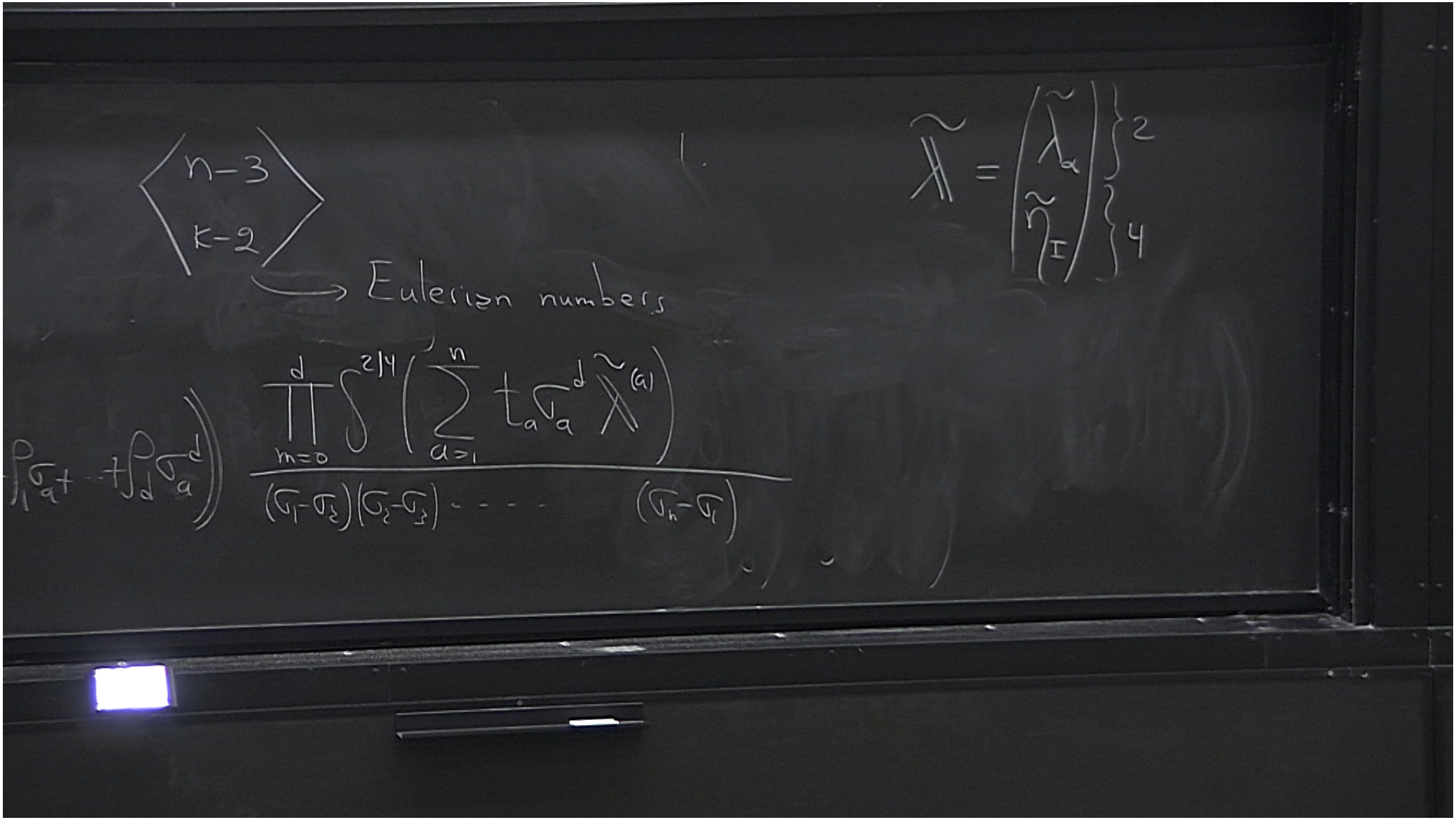

Bosonic
Fermionic

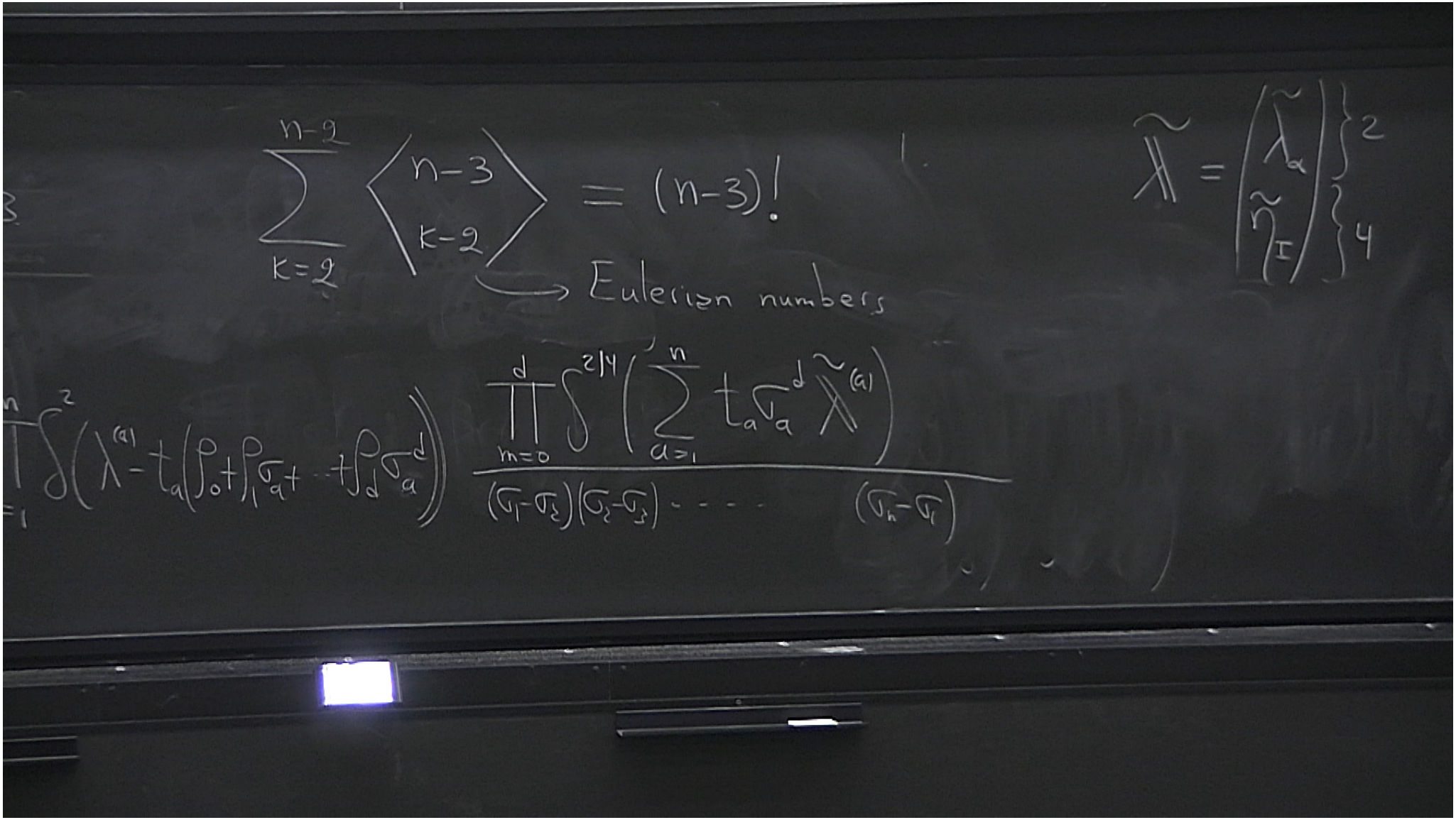

Bosonic
Fermionic
$T^2 \times T^2$
$\sum_{a=1}^{n} t_a \tilde{\lambda}_a \int^{2|4} \left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a \right)$
$(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_{n-1}-\sigma_n)(\sigma_n-\sigma_1)$
$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_\alpha \\ \tilde{\eta}_I \end{pmatrix}$ 2 4
String Theory?
$\prod_{a=1}^{n} \int^{2} \left(\lambda^{(a)} - t_a (\rho_0 + \rho_1 \sigma_a + \cdots + \rho_d \sigma_a^d) \right)$
$(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_n-\sigma_1)$


Bosonic
Fermionic
$T^2 \times T^2$
$\sum_{a=1}^{n} t_a \tilde{\lambda}_a \int^{2|4} \left(\sum_{a=1}^{n} t_a \sigma_a \tilde{\lambda}_a \right)$
$(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_{n-1}-\sigma_n)(\sigma_n-\sigma_1)$
$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_\alpha \\ \tilde{\eta}_I \end{pmatrix} \begin{matrix} \} 2 \\ \} 4 \end{matrix}$
String Theory?
$\prod_{a=1}^{n} \int^{2} \left(\lambda^{(a)} - t_a (\rho_0 + \rho_1 \sigma_a + \cdots + \rho_d \sigma_a^d) \right)$
$\prod_{m=0}^{d} \int^{2|4} \left(\sum_{a=1}^{n} t_a \sigma_a^d \tilde{\lambda}^{(a)} \right)$
$(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_n-\sigma_1)$


Witten-RSV Formula.
(2003) Roiban Spradlin; Volovich 2003
$A_n^{(d)} = \frac{1}{Vol(SL(2,\mathbb{C})\times GL(1))} \int \prod_{m=0}^{d} d^2\rho_m \left[\int d\sigma_a \frac{dt_a}{t_a} \right] \prod_{a=1}^{n} \delta^2(\lambda^{(a)} - t_a(\rho_0 + \rho_1\sigma_a + \cdots + \rho_d\sigma_a^d$


Witten - RSV formula.
(2003) Roiban Spradlin; Volovich 2003
K = d + 1 d = K - 1
$A_n^{(d)} = \frac{1}{Vol(SL(2,\mathbb{C}) \times GL(1))} \int \prod_{m=0}^{d} d^2\rho_m \left[\int d\sigma_a dt \right]$


n-3
k-2
Eulerian numbers
$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_\alpha \\ \tilde{\eta}_I \end{pmatrix}$ 2 4
$\int \rho_1 \sigma_a + \cdots + \rho_d \sigma_a^d \Big)$
$\prod_{m=0}^{d} \int^{2|4} \left(\sum_{a=1}^{n} t_a \sigma_a^d \tilde{\lambda}^{(a)} \right)$
$(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_n - \sigma_1)$


$\sum_{k=2}^{n-2} \left\langle {n-3 \atop k-2} \right\rangle = (n-3)!$
Eulerian numbers
$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_a \\ \tilde{\eta}_I \end{pmatrix}$ 2 4
$\int \left(\lambda^{(a)} - t_a (\rho_0 + \rho_1 \sigma_a + \cdots + \rho_d \sigma_a^d) \right)$
$\prod_{m=0}^{d} \int^{2|4} \left(\sum_{a=1}^{n} t_a \sigma_a^d \tilde{\lambda}^{(a)} \right)$
$(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_n - \sigma_1)$

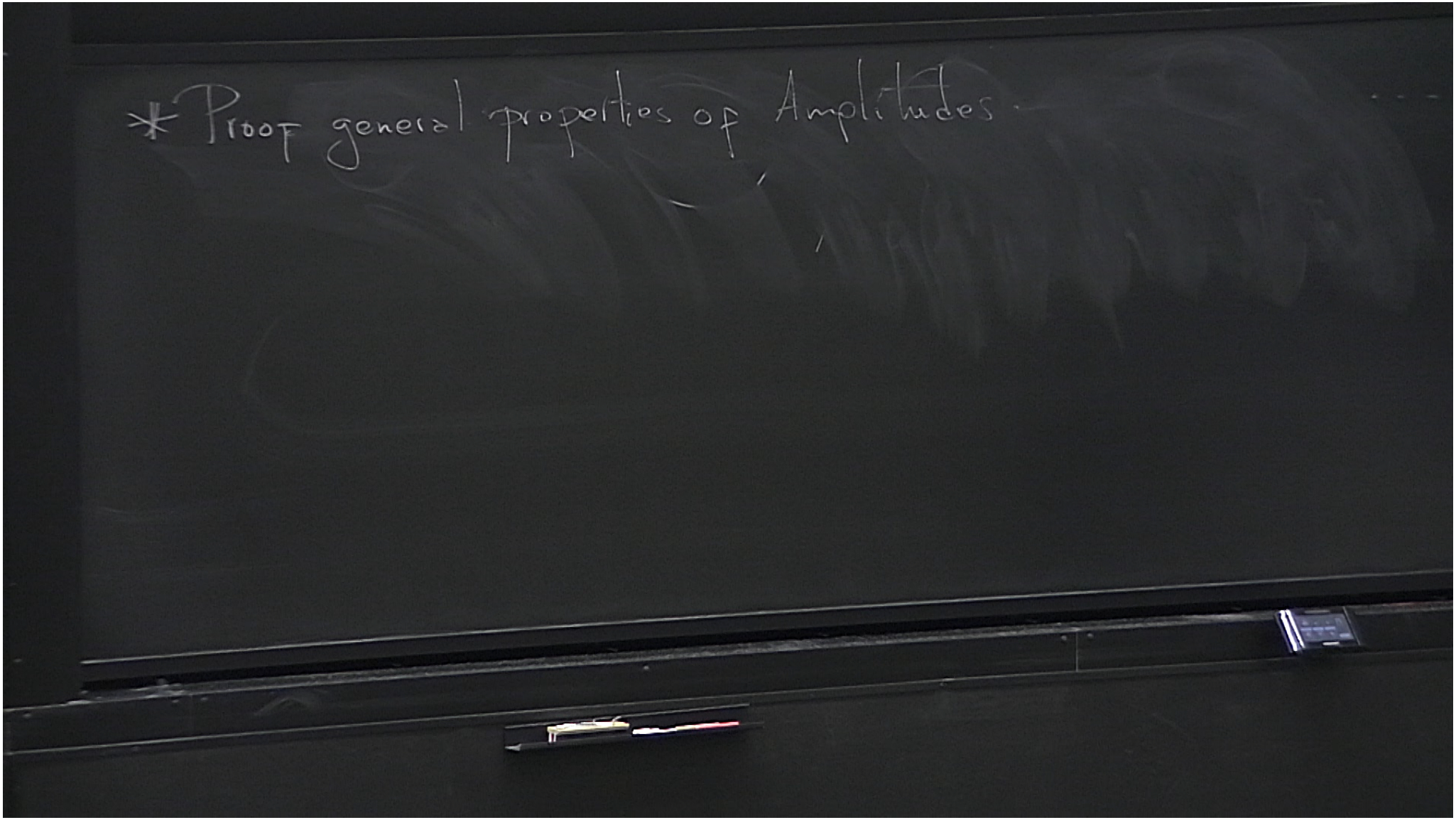
$$\sum_{k=2}^{n-2} \left\langle \begin{matrix} n-3 \\ k-2 \end{matrix} \right\rangle = (n-3)!$$

Eulerian numbers

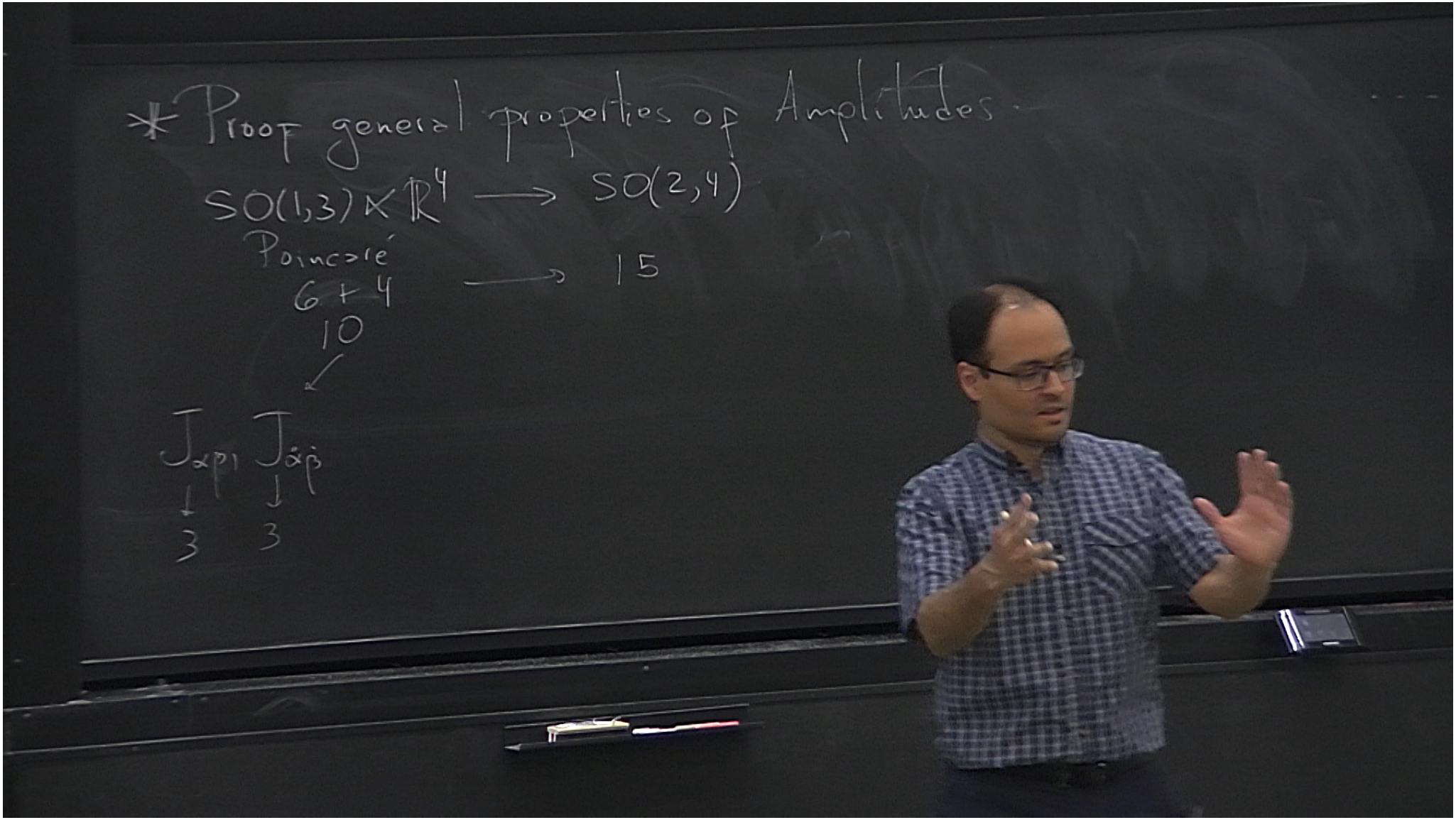
$$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_a \\ \tilde{\eta}_I \end{pmatrix} \begin{matrix} \} 2 \\ \} 4 \end{matrix}$$

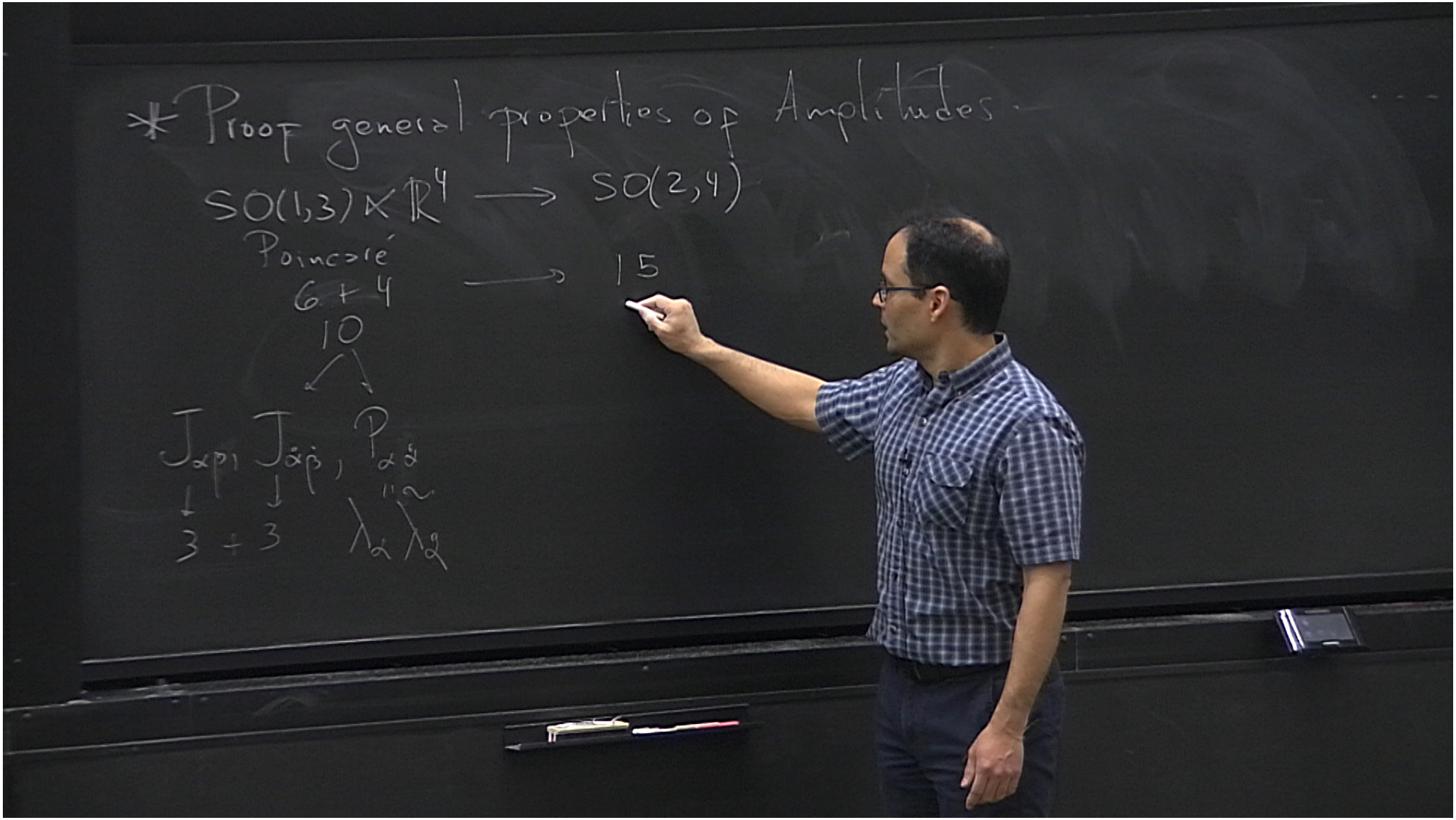
Practical computation
- use BCFW recurs relations.

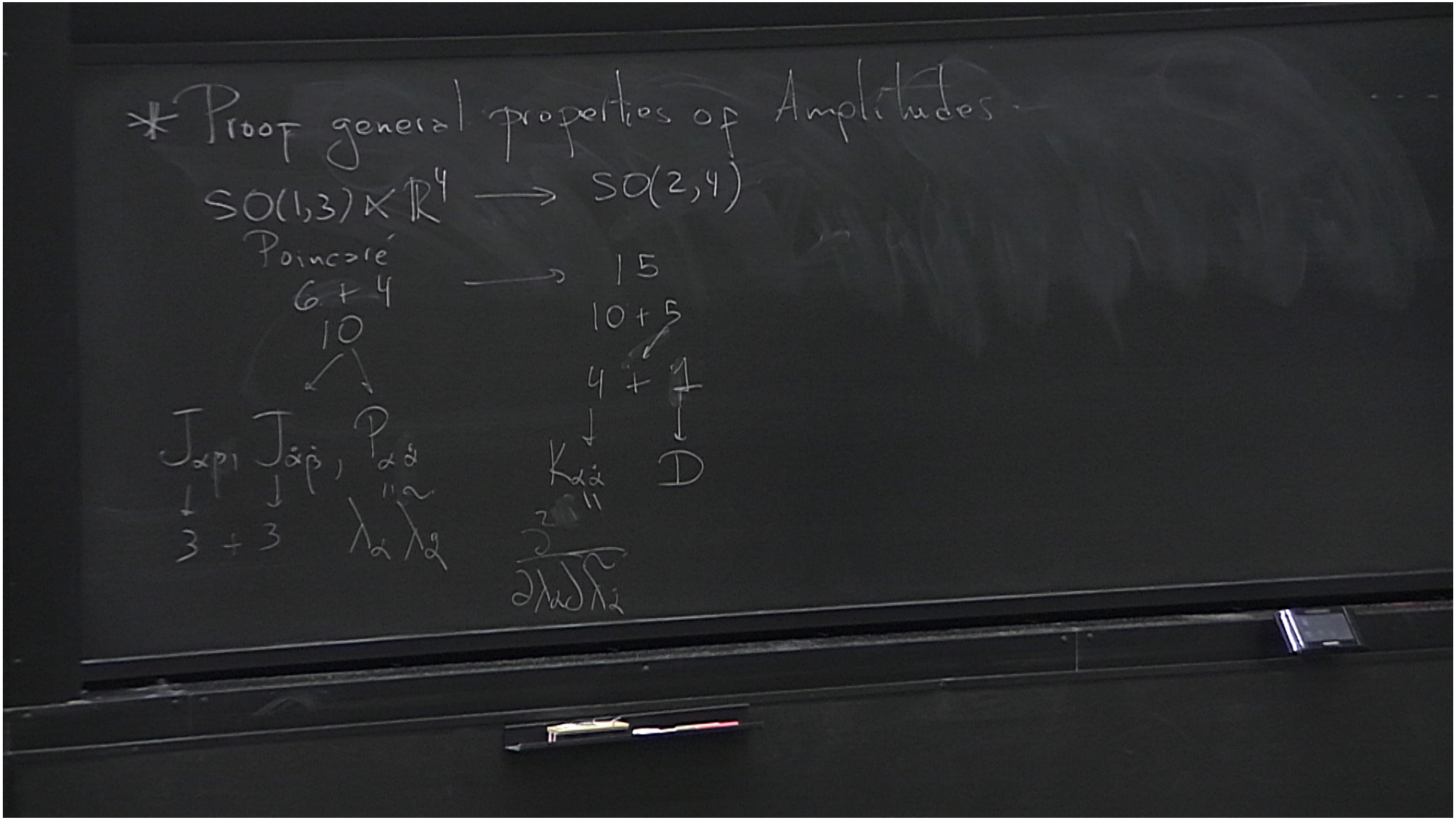
$$\int \prod_{m=0}^{d} \left(\rho_0 + \rho_1 \sigma_a + \cdots + \rho_d \sigma_a^d \right) \frac{\int^{2|4} \left(\sum_{a=1}^{n} t_a \sigma_a^d \tilde{\lambda}^{(a)} \right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_n - \sigma_1)}$$

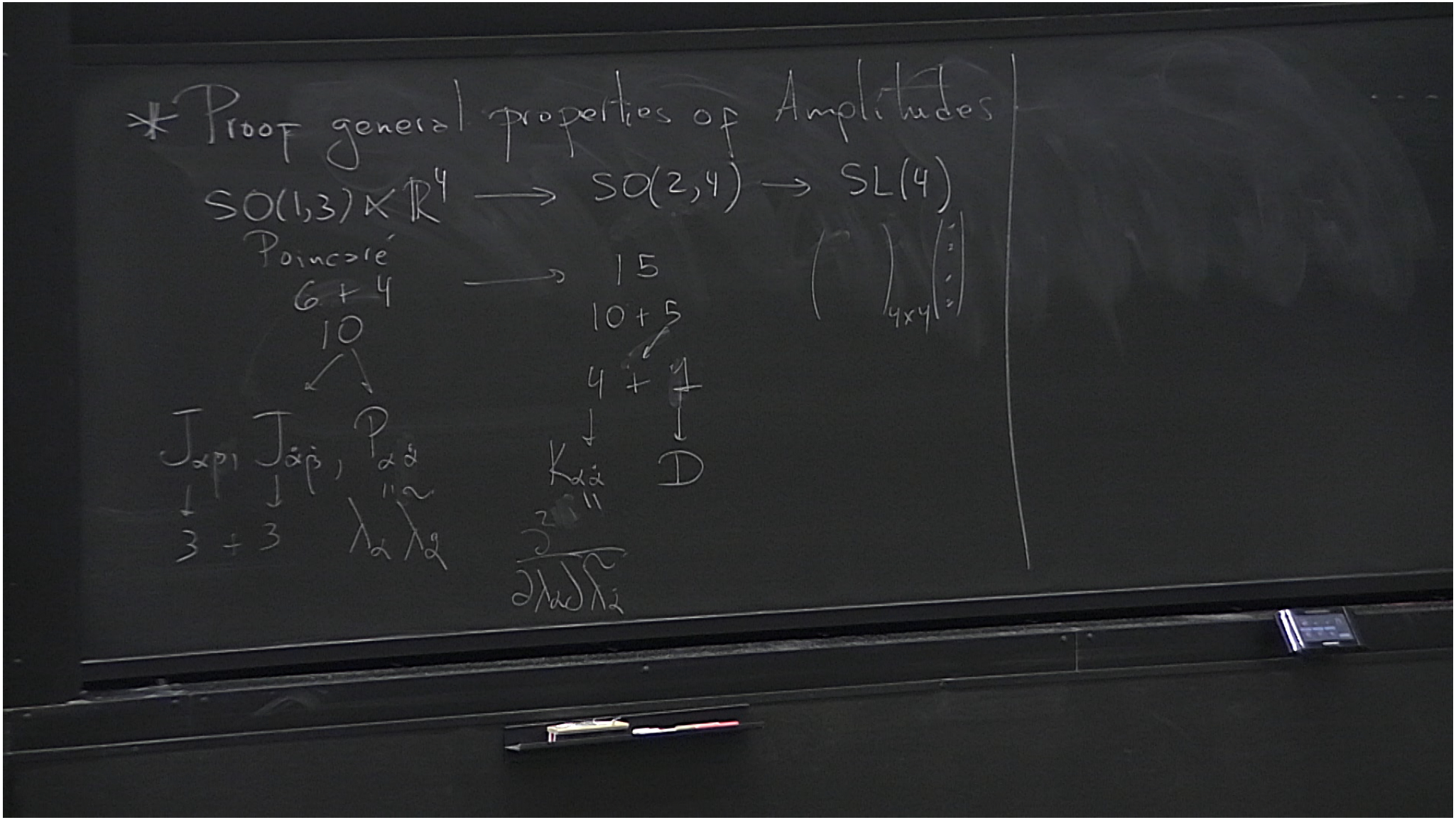

* Proof general properties of Amplitudes

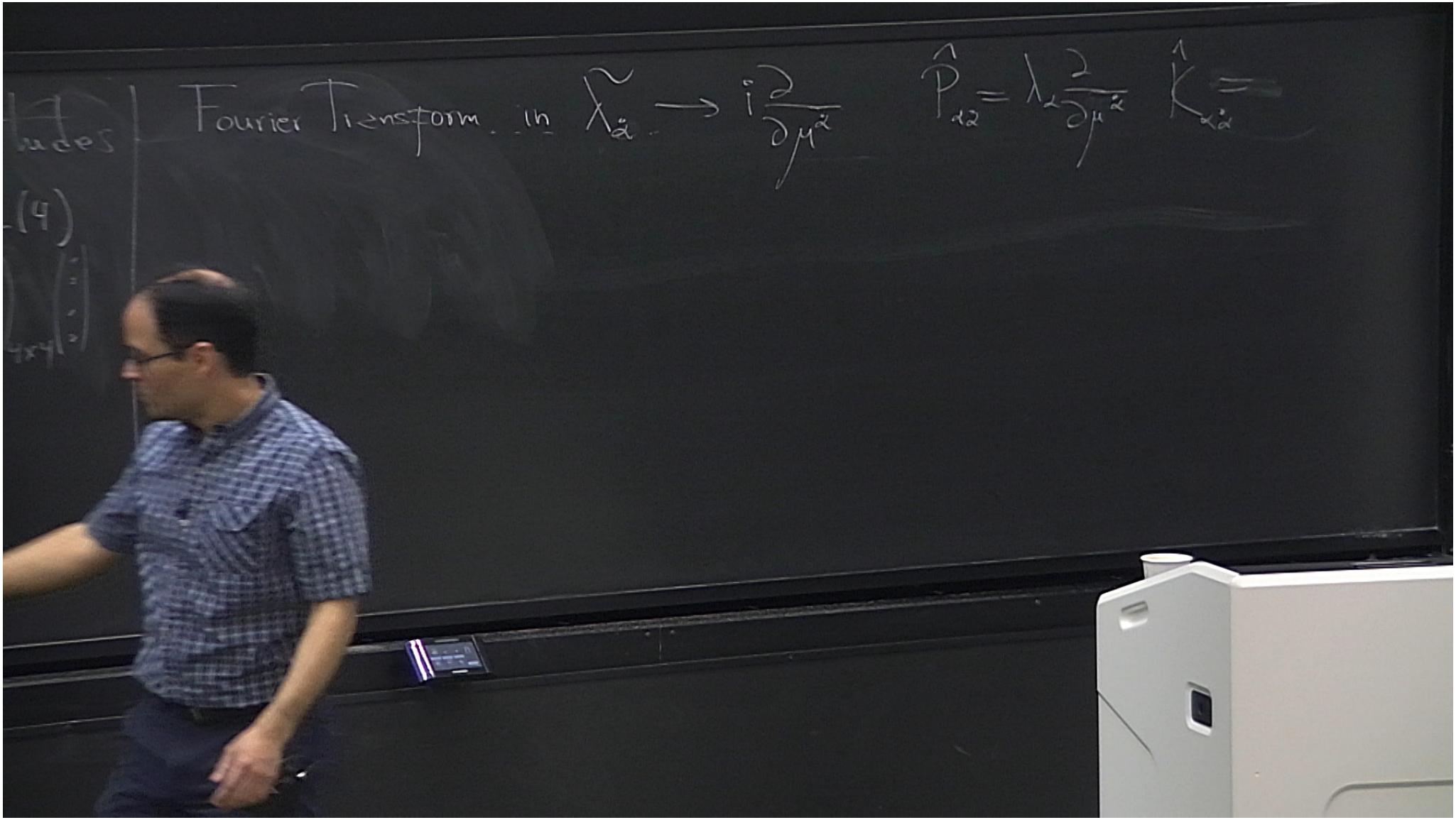

* Proof general properties of Amplitudes
SO(1,3) ⋉ ℝ⁴
Poincaré

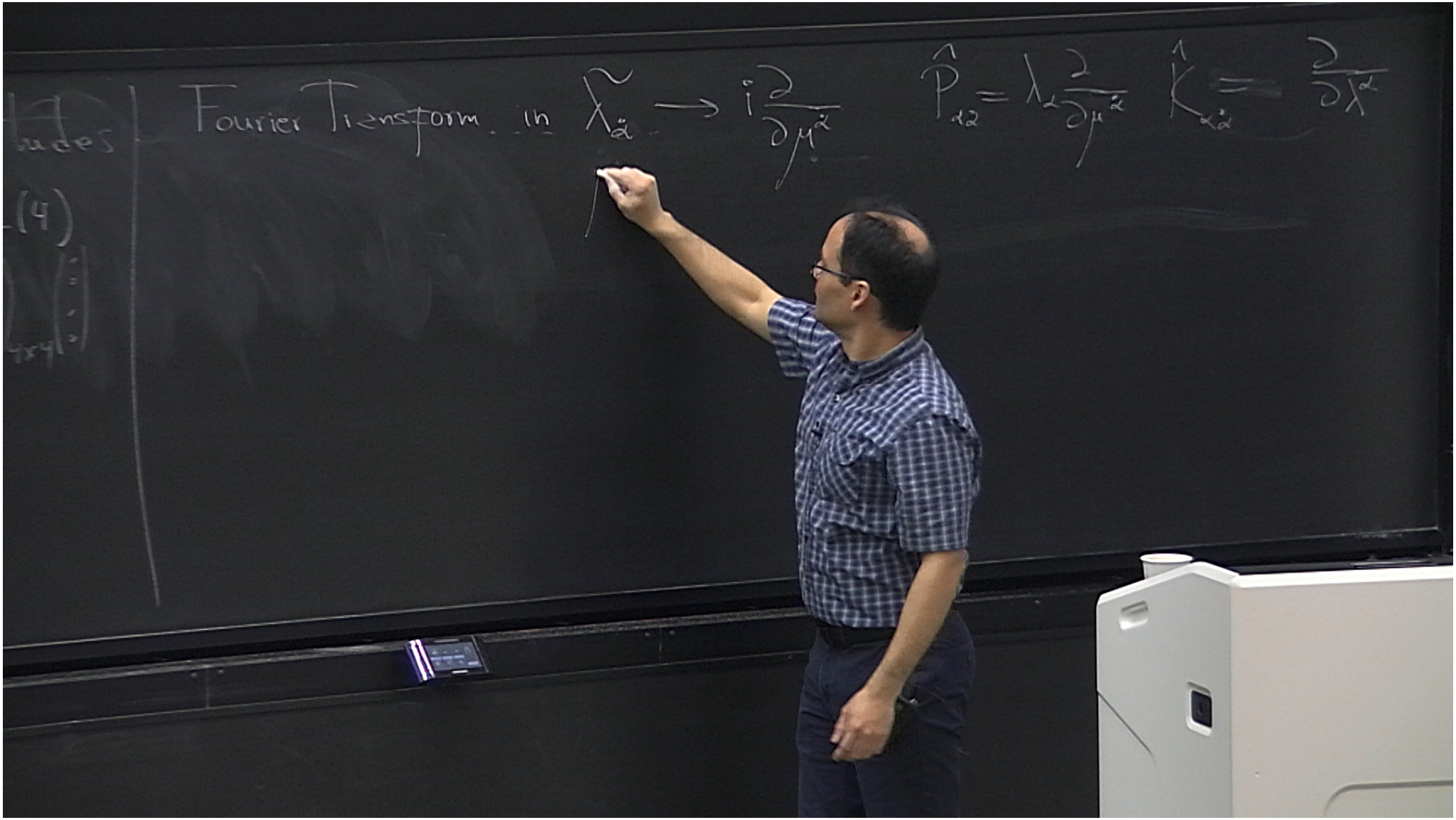

* Proof general properties of Amplitudes
SO(1,3) ⋉ ℝ⁴ ⟶ SO(2,4)
Poincaré
6 + 4
10
15
J_αβ1 J_α̇β̇
3 3

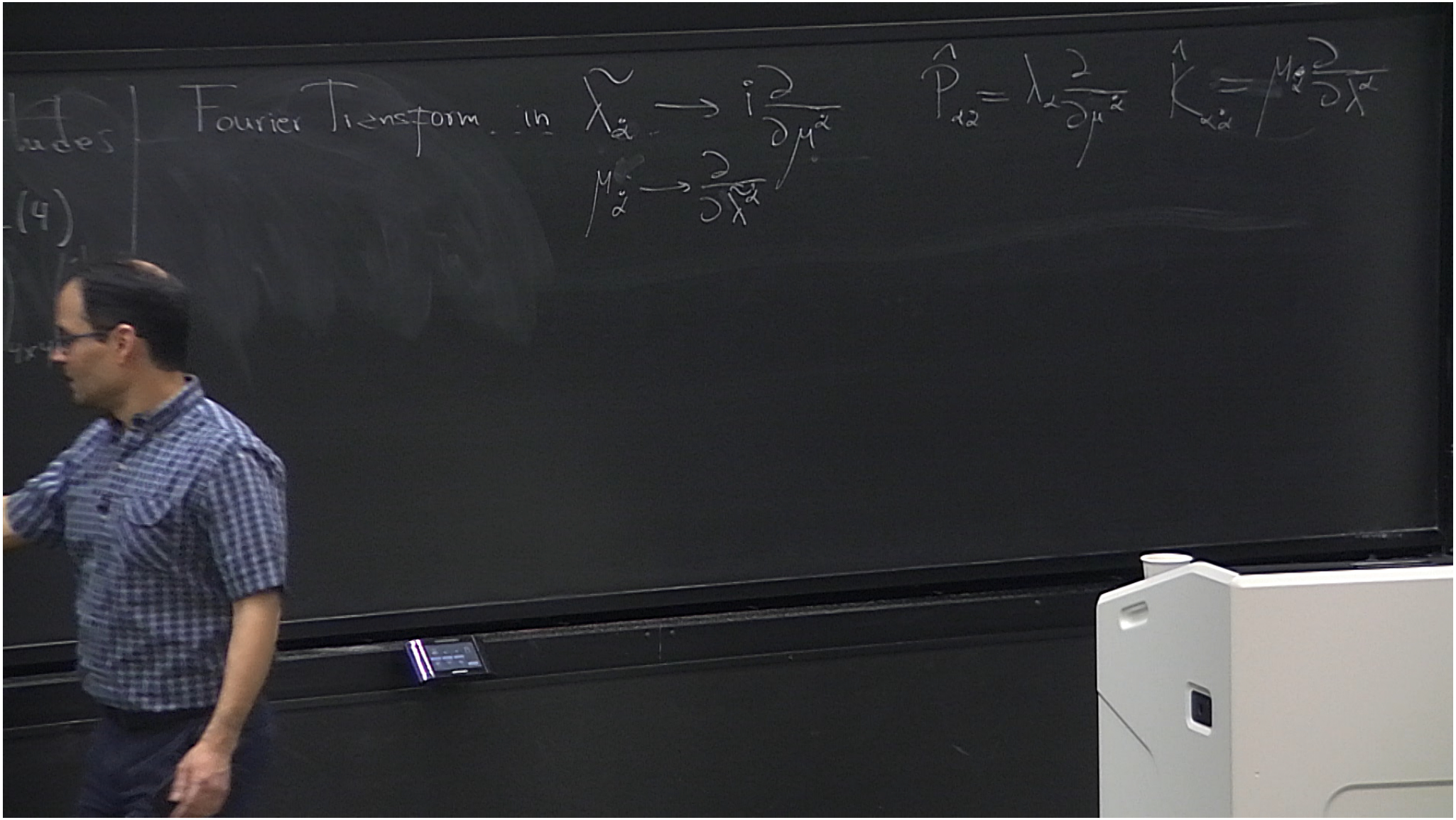

* Proof general properties of Amplitudes
SO(1,3) ⋉ ℝ⁴ → SO(2,4)
Poincaré
6 + 4
10
15
$J_{\alpha\beta}$ $J_{\dot\alpha\dot\beta}$ $P_{\alpha\dot\alpha}$
3 + 3
λ_α $\lambda_{\dot\alpha}$

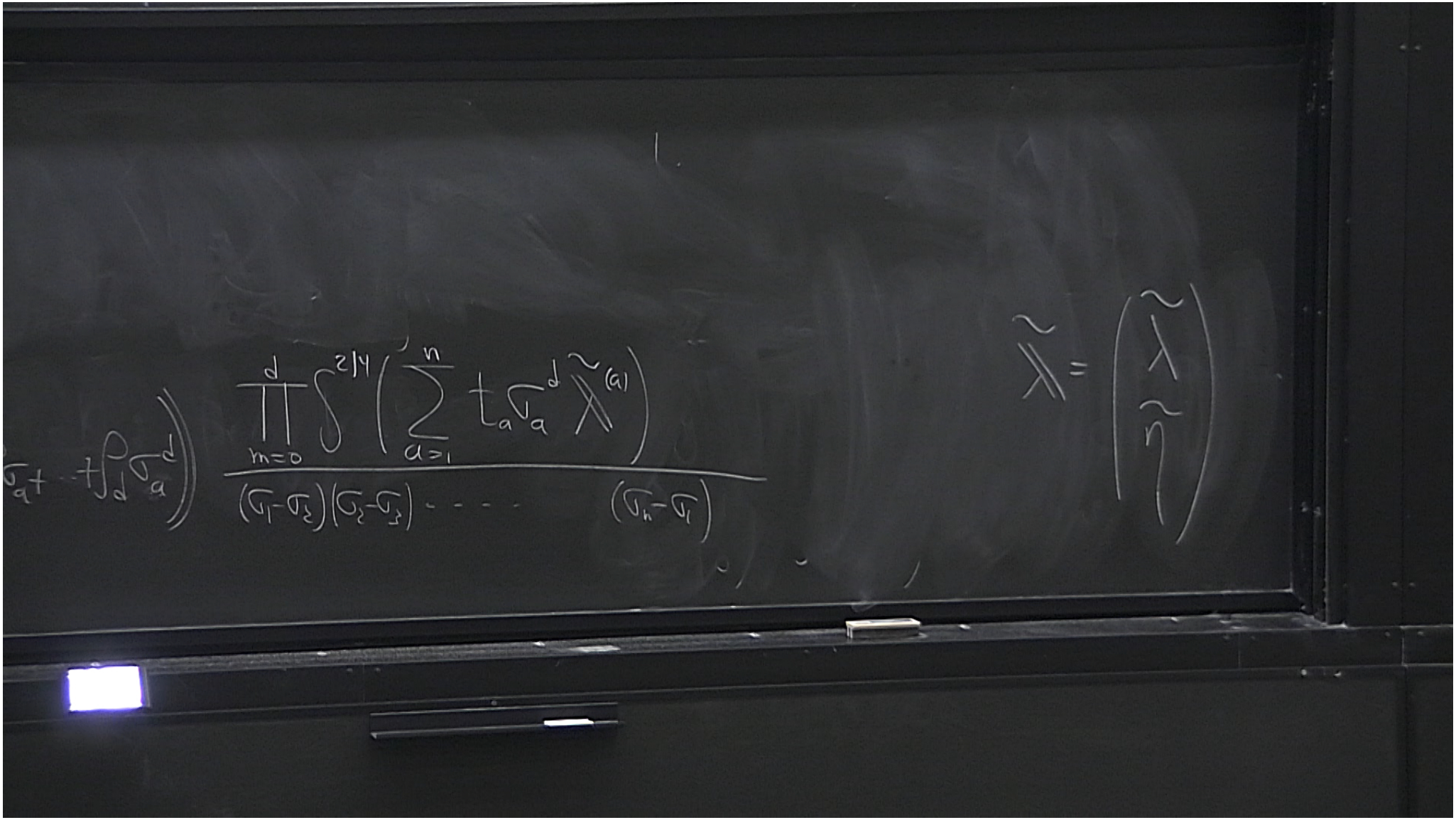

* Proof general properties of Amplitudes
SO(1,3) ⋉ ℝ⁴ ⟶ SO(2,4)
Poincaré
6 + 4
10
15
10 + 5
4 + 1
$J_{\alpha\beta}$ $J_{\dot\alpha\dot\beta}$ $P_{\alpha\dot\alpha}$
$K_{\alpha\dot\alpha}$ D
3 + 3
$\lambda_\alpha \lambda_\beta$
$\partial/\partial\lambda\partial\lambda_2$

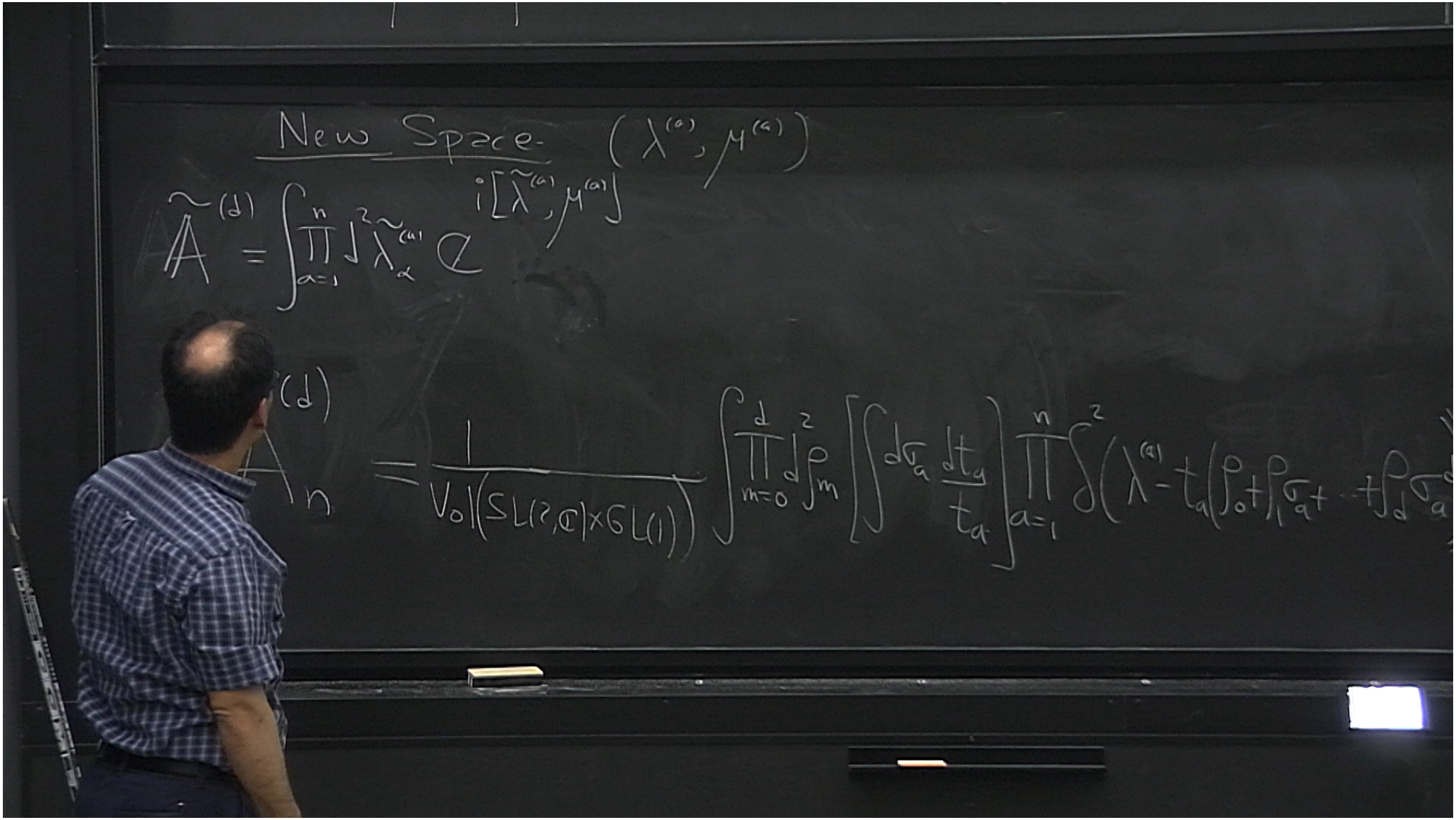

* Proof general properties of Amplitudes
SO(1,3) ⋉ ℝ⁴ → SO(2,4) → SL(4)
Poincaré
6 + 4
10
15
10 + 5
4 + 1
4×4
$J_{\alpha\beta}$ $J_{\dot\alpha\dot\beta}$ $P_{\alpha\dot\alpha}$
$K_{\alpha\dot\alpha}$ D
3 + 3
λ_α λ_β
$\frac{\partial^2}{\partial\lambda_\alpha\partial\lambda_\beta}$

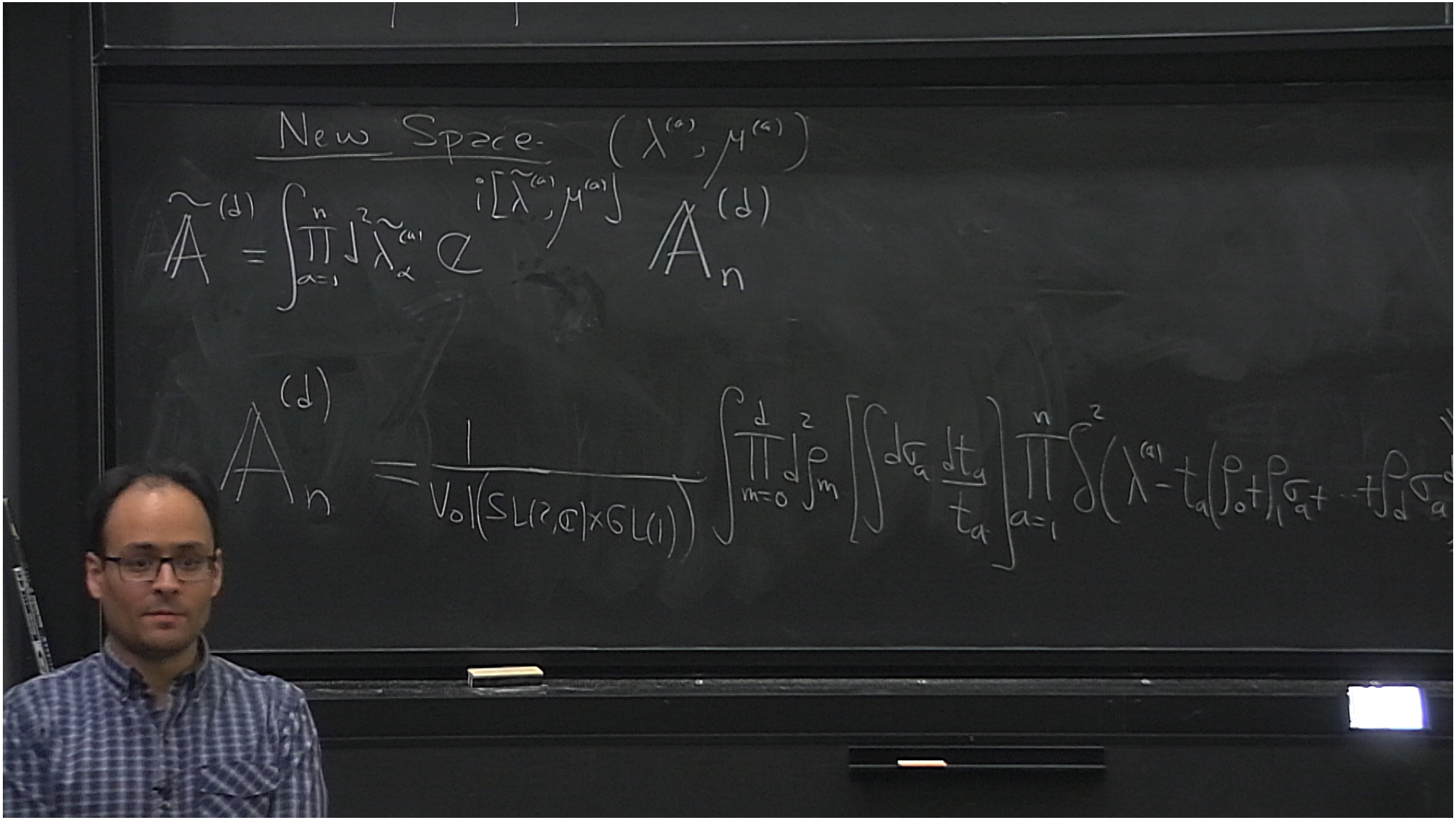

Fourier Transform in $\tilde{\lambda}_{\dot{\alpha}} \to i\frac{\partial}{\partial \mu^2}$
$\hat{P}_{\alpha\dot{\alpha}} = \lambda_\alpha \frac{\partial}{\partial \mu^{\dot{\alpha}}}$
$\hat{K}_{\alpha\dot{\alpha}} =$

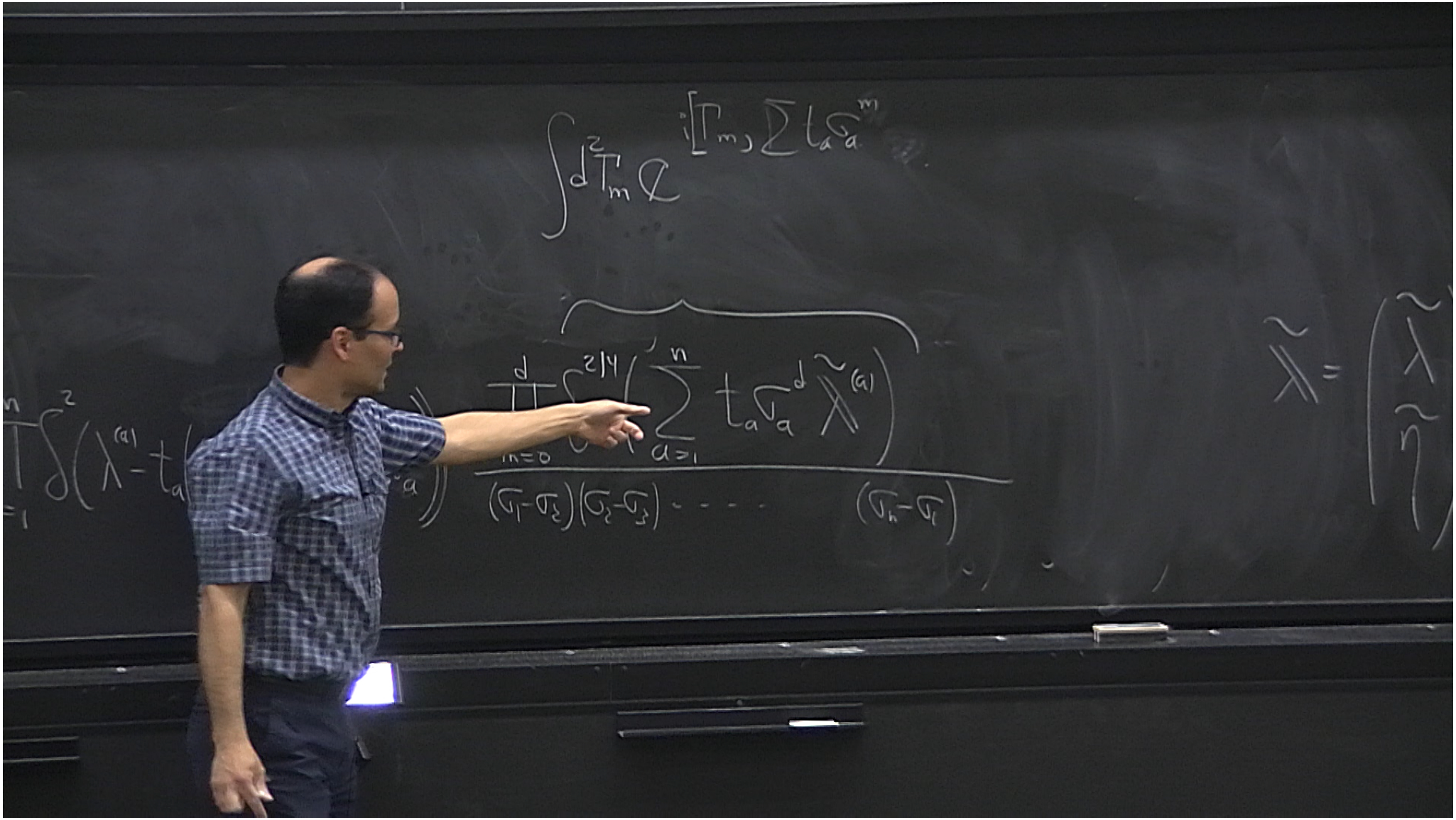

Fourier Transform in $\tilde{\lambda}_{\dot{\alpha}} \rightarrow i\frac{\partial}{\partial \mu^2}$
$\hat{P}_{\mu 2} = \lambda_2 \frac{\partial}{\partial \mu^2}$
$\hat{K}_{\mu\dot{\alpha}} = \frac{\partial}{\partial x^\mu}$

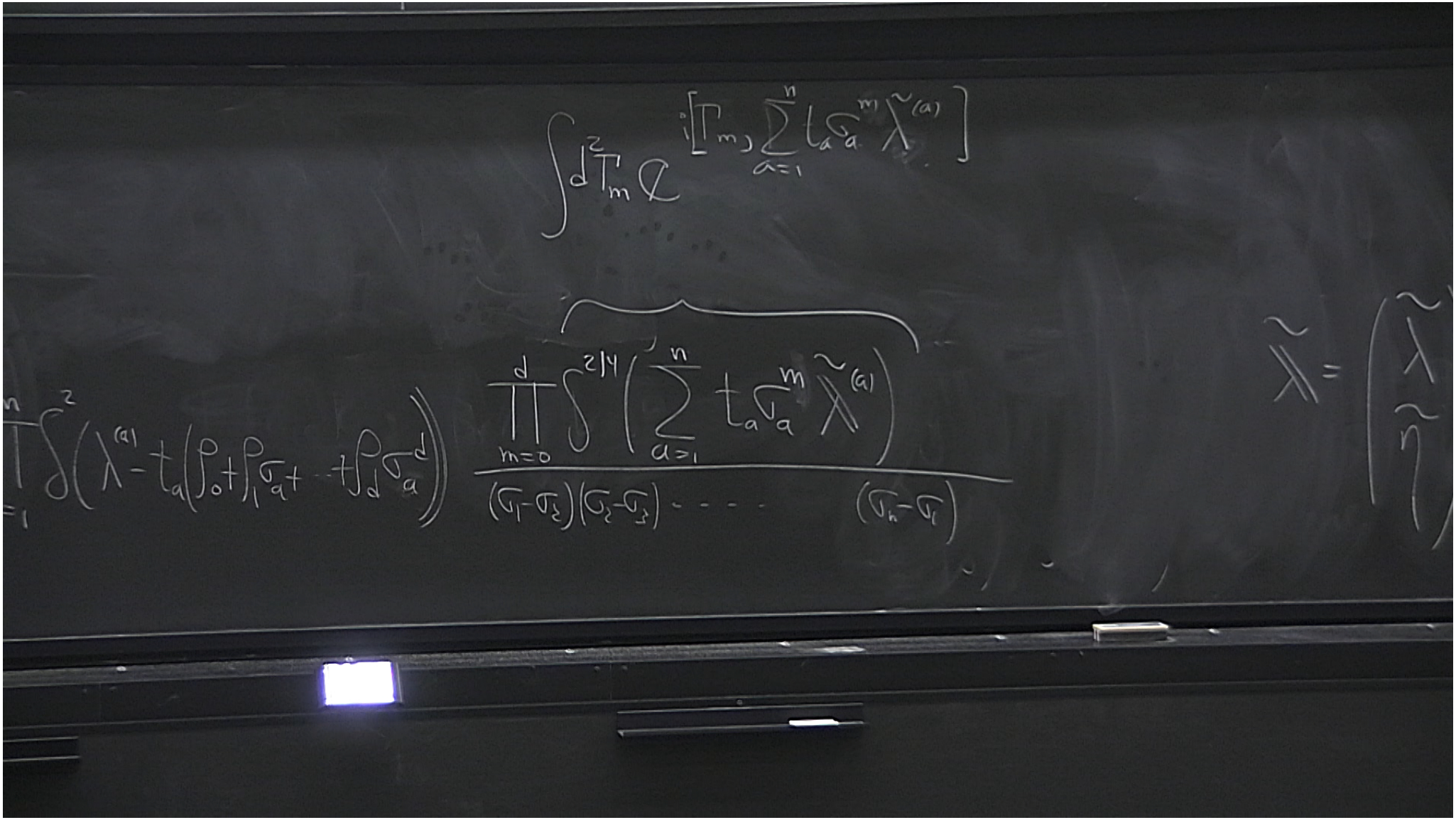

Fourier Transform in

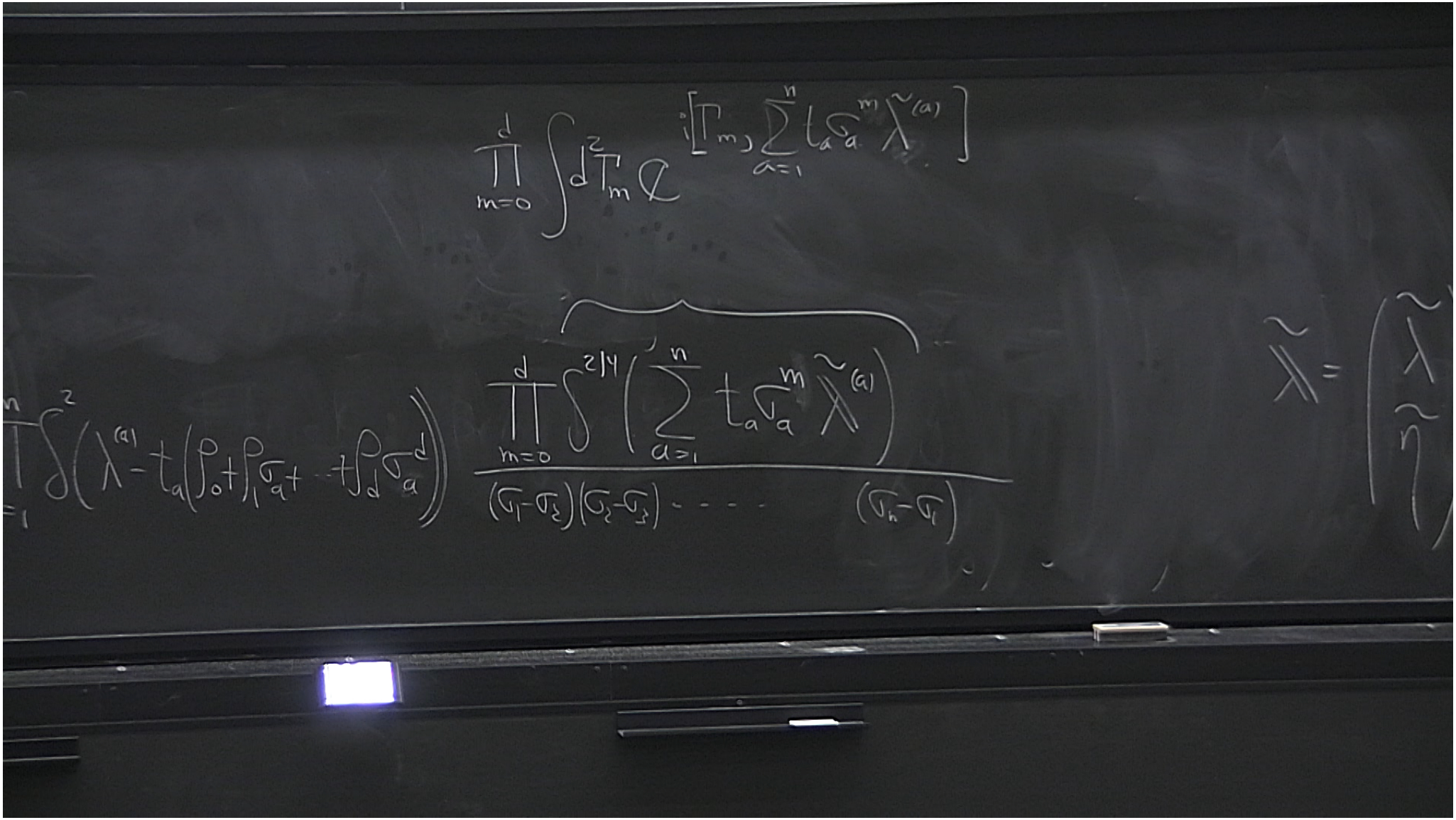

$\sigma_a t + \cdots + t_d \sigma_a^d \Big)\Big)$
$\prod_{m=0}^{d} \int^{z/4} \Big(\sum_{a=1}^{n} t_a \sigma_a^d \tilde{\chi}^{(a)} \Big)$
$(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots\cdots(\sigma_n-\sigma_1)$
$\tilde{\chi} = \begin{pmatrix} \tilde{\lambda} \\ \tilde{\eta} \end{pmatrix}$

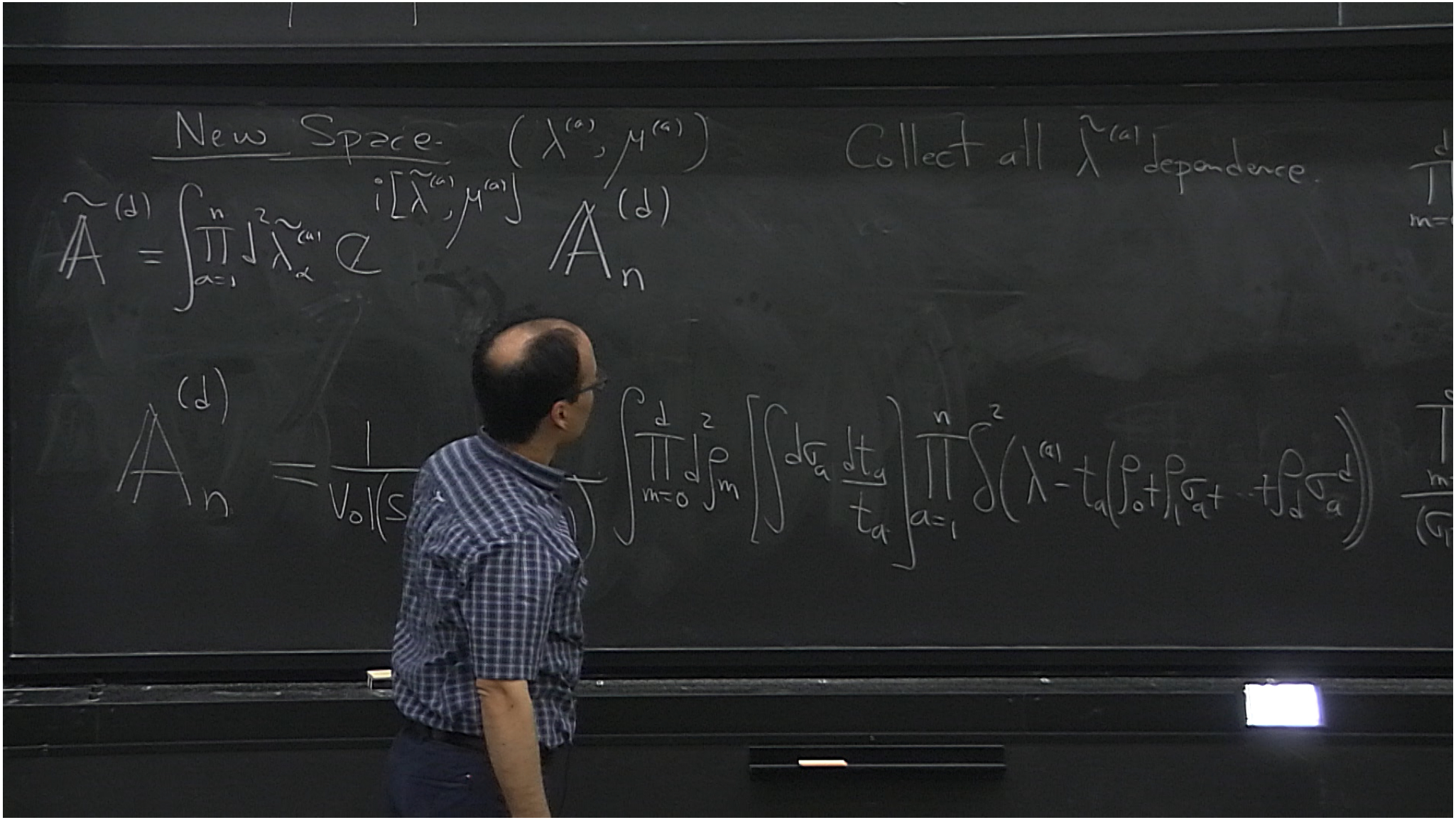

New Space-
$(\lambda^{(a)}, \mu^{(a)})$
$\tilde{A}^{(d)} = \int \prod_{a=1}^{n} d^2\tilde{\lambda}^{(a)}_{\alpha}\, e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}]}$
$A^{(d)}_n = \frac{1}{Vol(SL(2,\mathbb{C})\times GL(1))} \int \prod_{m=0}^{d} d^2\rho_m \left[\int d\sigma_a \frac{dt_a}{t_a}\right] \prod_{a=1}^{n} \delta^2(\lambda^{(a)} - t_a(\rho_0 + \rho_1\sigma_a + \cdots + \rho_d\sigma_a^d$

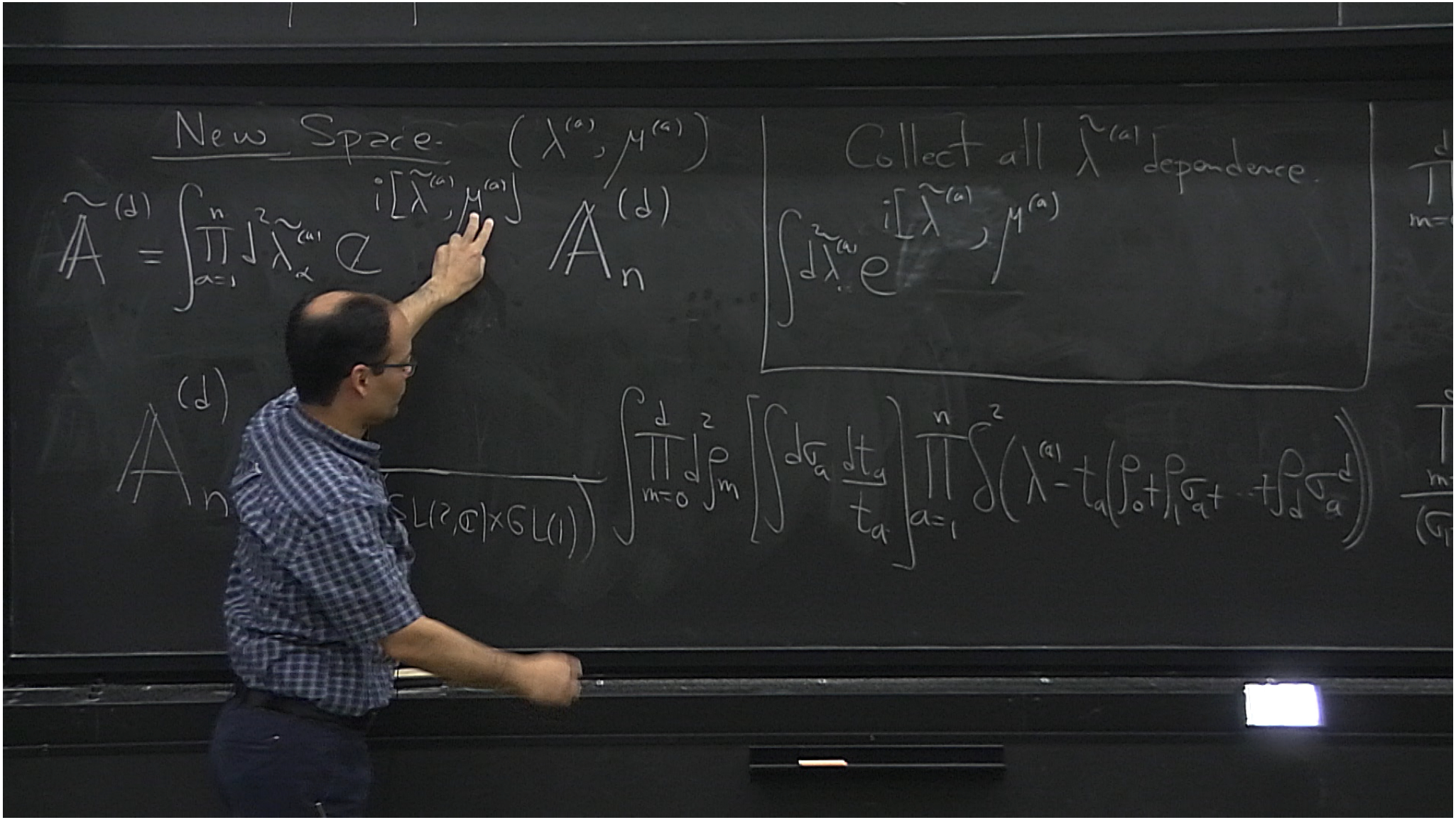

New Space- $(\lambda^{(a)}, \mu^{(a)})$
$\tilde{\mathbb{A}}^{(d)} = \int \prod_{a=1}^{n} d^2\tilde{\lambda}^{(a)}_{\alpha}\, e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}]}\, \mathbb{A}^{(d)}_n$
$\mathbb{A}^{(d)}_n = \frac{1}{Vol(SL(2,\mathbb{C})\times GL(1))} \int \prod_{m=0}^{d} d^2\rho_m \left[\int d\sigma_a \frac{dt_a}{t_a}\right] \prod_{a=1}^{n} \delta^2(\lambda^{(a)} - t_a(\rho_0 + \rho_1\sigma_a + \cdots + \rho_d\sigma_a^d$

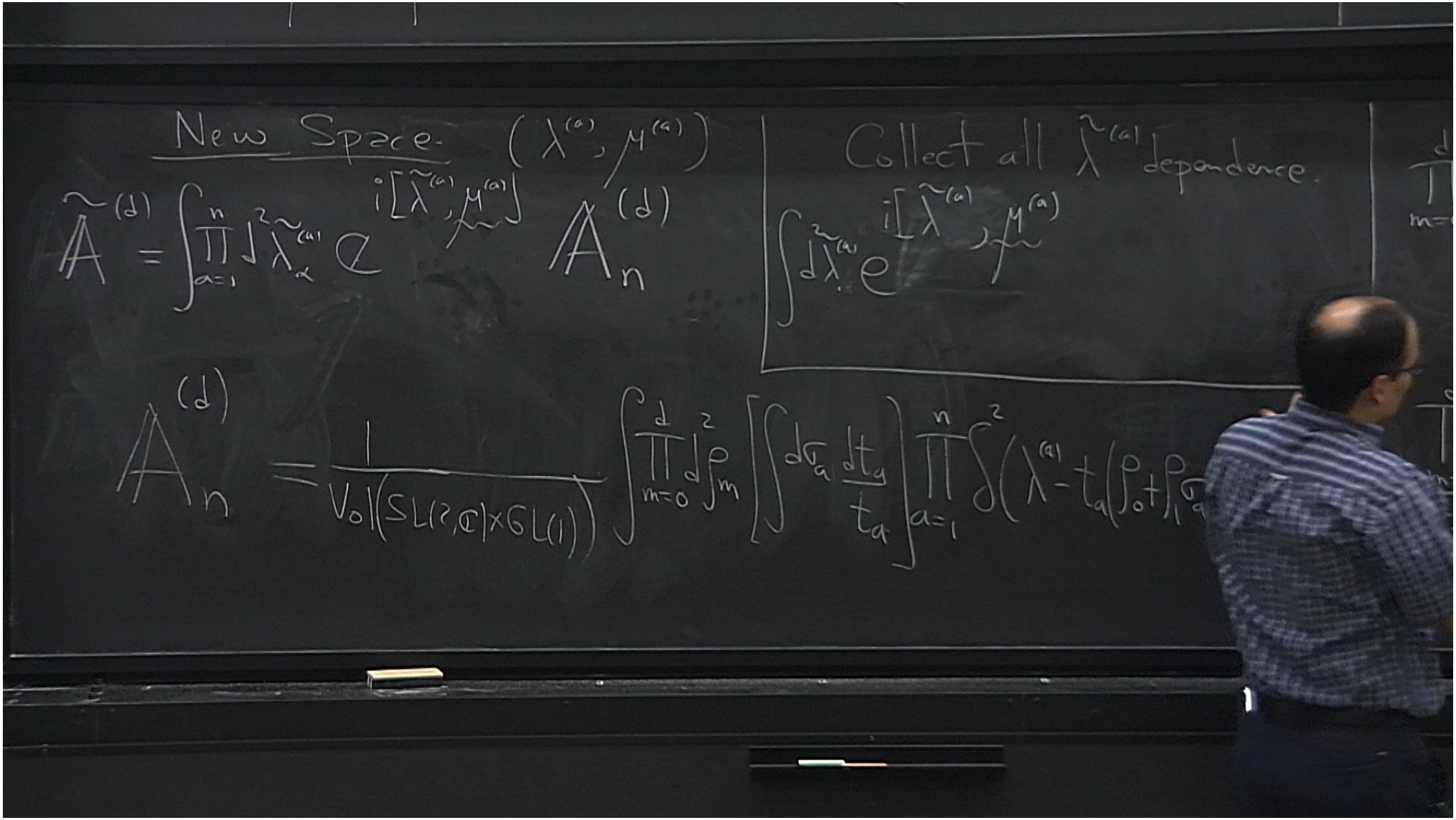

$\int d^2\Gamma_m \, e^{i[\Gamma_m, \sum t_a \sigma_a^m]}$
$\prod_{p=1}^{n} \int^{2\pi} \frac{d\lambda_p}{2\pi} \left(\sum_{a=1}^{n} t_a \sigma_a^d \tilde{\chi}^{(a)} \right)$
$(\sigma_1-\sigma_2)(\sigma_2-\sigma_3) \cdots (\sigma_n-\sigma_1)$
$\tilde{\chi} =$

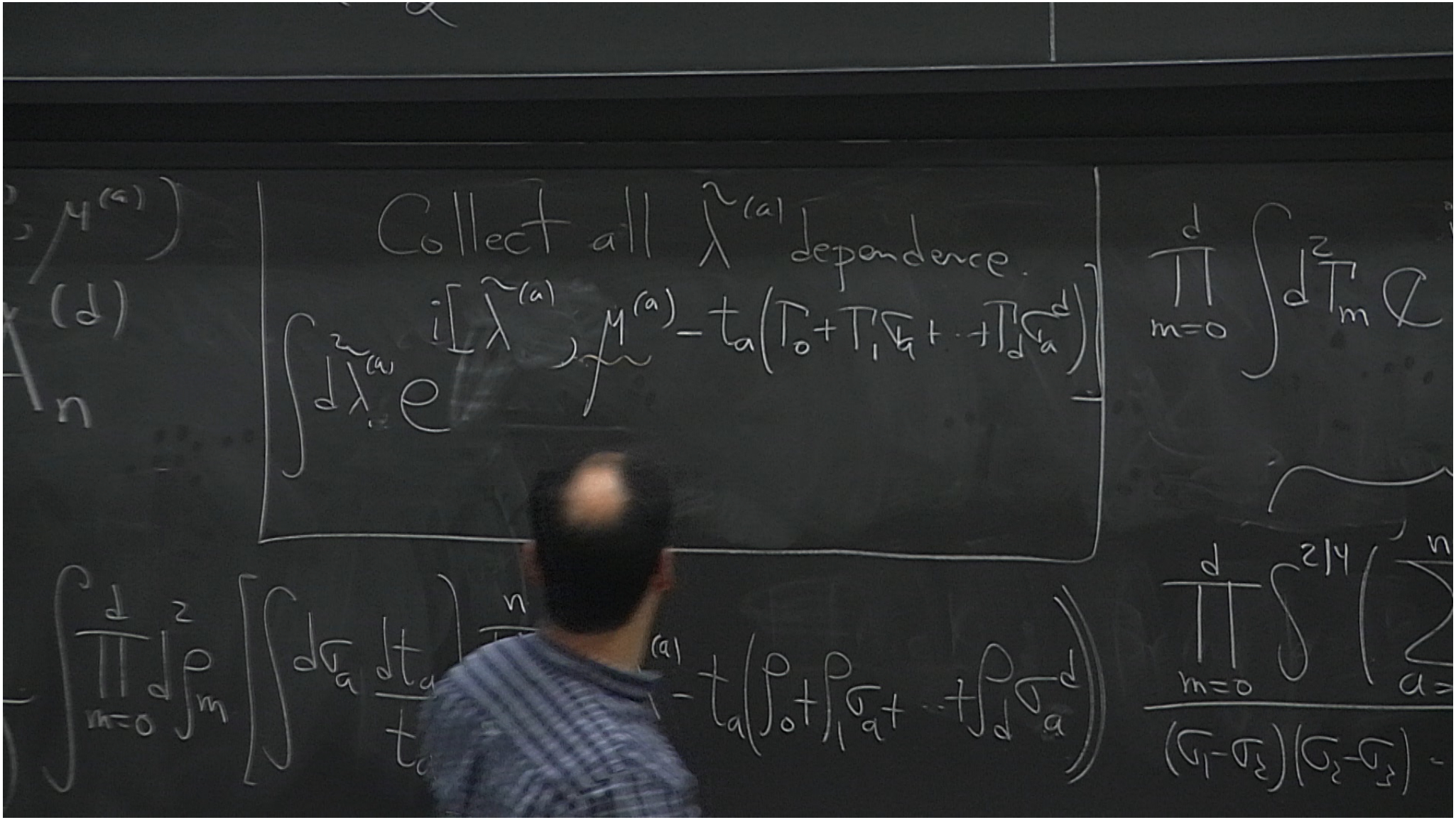

$\int d^2\Gamma_m \, \mathcal{C} \, e^{i\left[\Gamma_m \sum_{a=1}^{n} t_a \sigma_a^m \tilde{\lambda}^{(a)}\right]}$
$\prod_{m=0}^{d} \int^{z/4} \left(\sum_{a=1}^{n} t_a \sigma_a^m \tilde{\lambda}^{(a)} \right)$
$(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots\cdots(\sigma_n-\sigma_1)$
$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda} \\ \tilde{\eta} \end{pmatrix}$

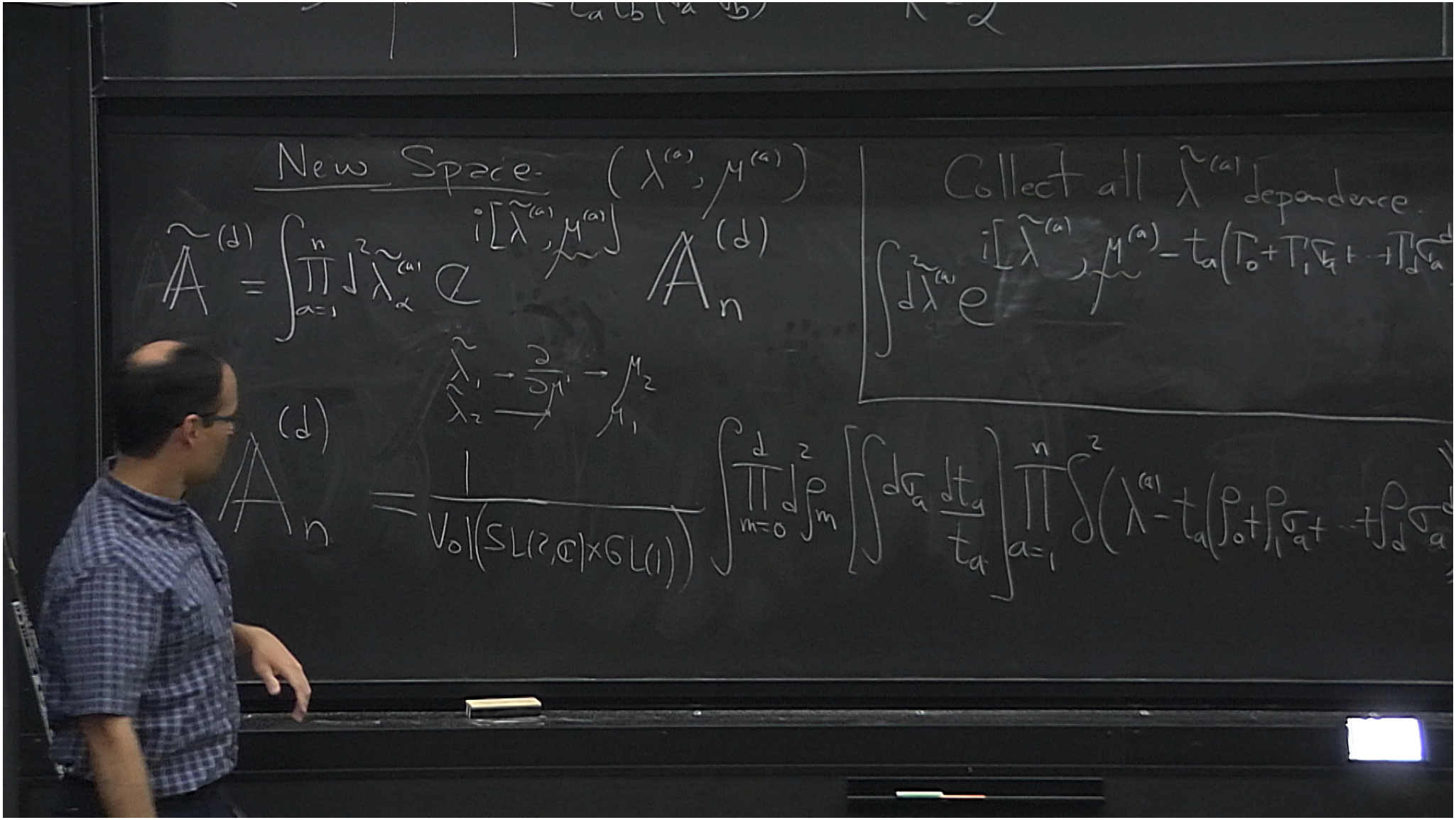

$\prod_{m=0}^{d} \int d^2\Gamma_m \, \mathcal{C}$
$\tilde{\lambda} =$

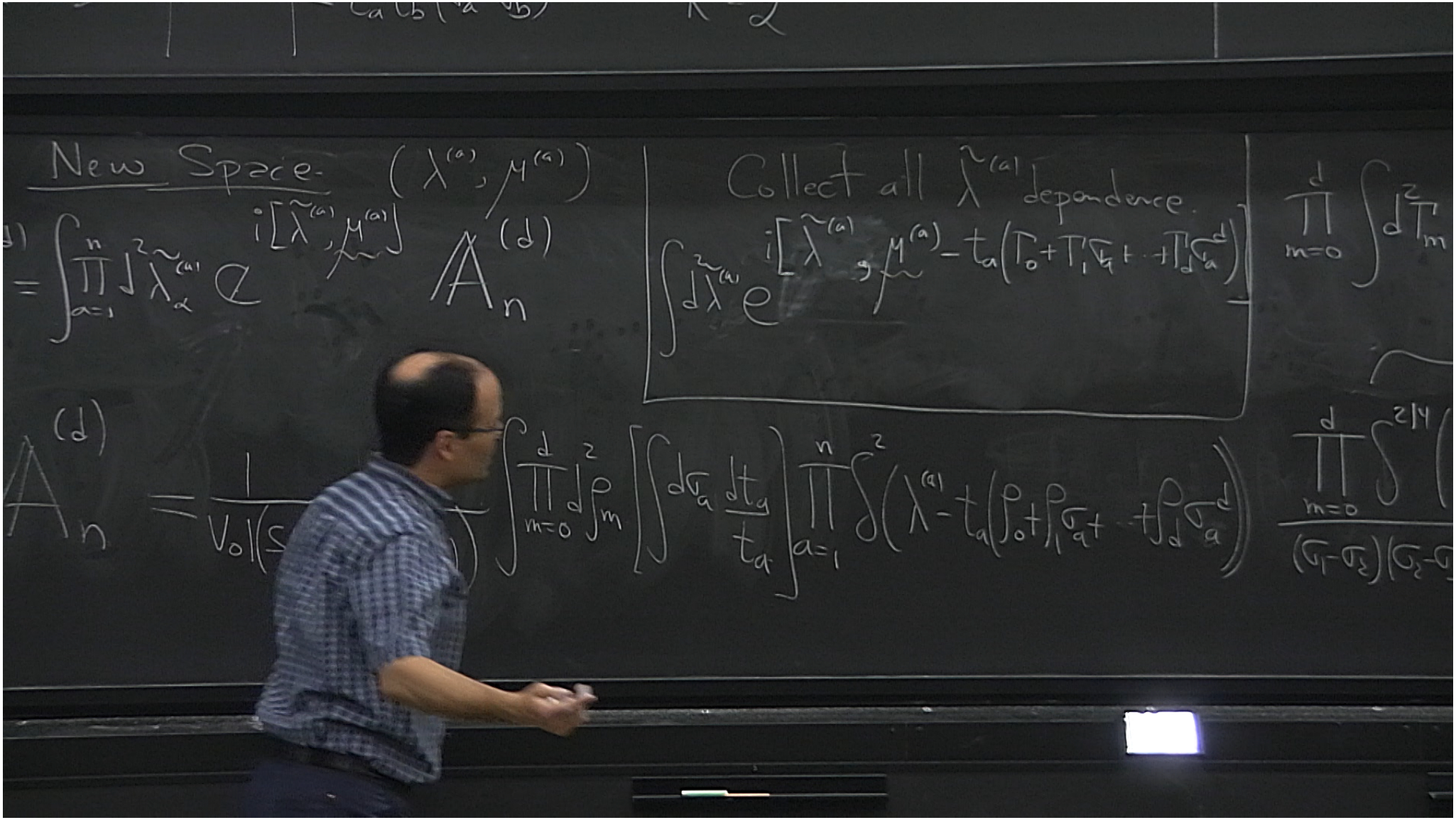

New Space-
$(\lambda^{(a)}, \mu^{(a)})$
Collect all $\tilde{\lambda}^{(a)}$ dependence.
$\tilde{\mathbb{A}}^{(d)} = \int \prod_{a=1}^{n} d^2\tilde{\lambda}^{(a)}_{\alpha}\, e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}]}\, \mathbb{A}^{(d)}_n$
$\mathbb{A}^{(d)}_n = \frac{1}{Vol(S$
$\int \prod_{m=0}^{d} d^2\rho_m \left[\int d\sigma_a \frac{dt_a}{t_a}\right] \prod_{a=1}^{n} \delta^2\left(\lambda^{(a)} - t_a(\rho_0 + \rho_1\sigma_a + \cdots + \rho_d\sigma_a^d)\right)$

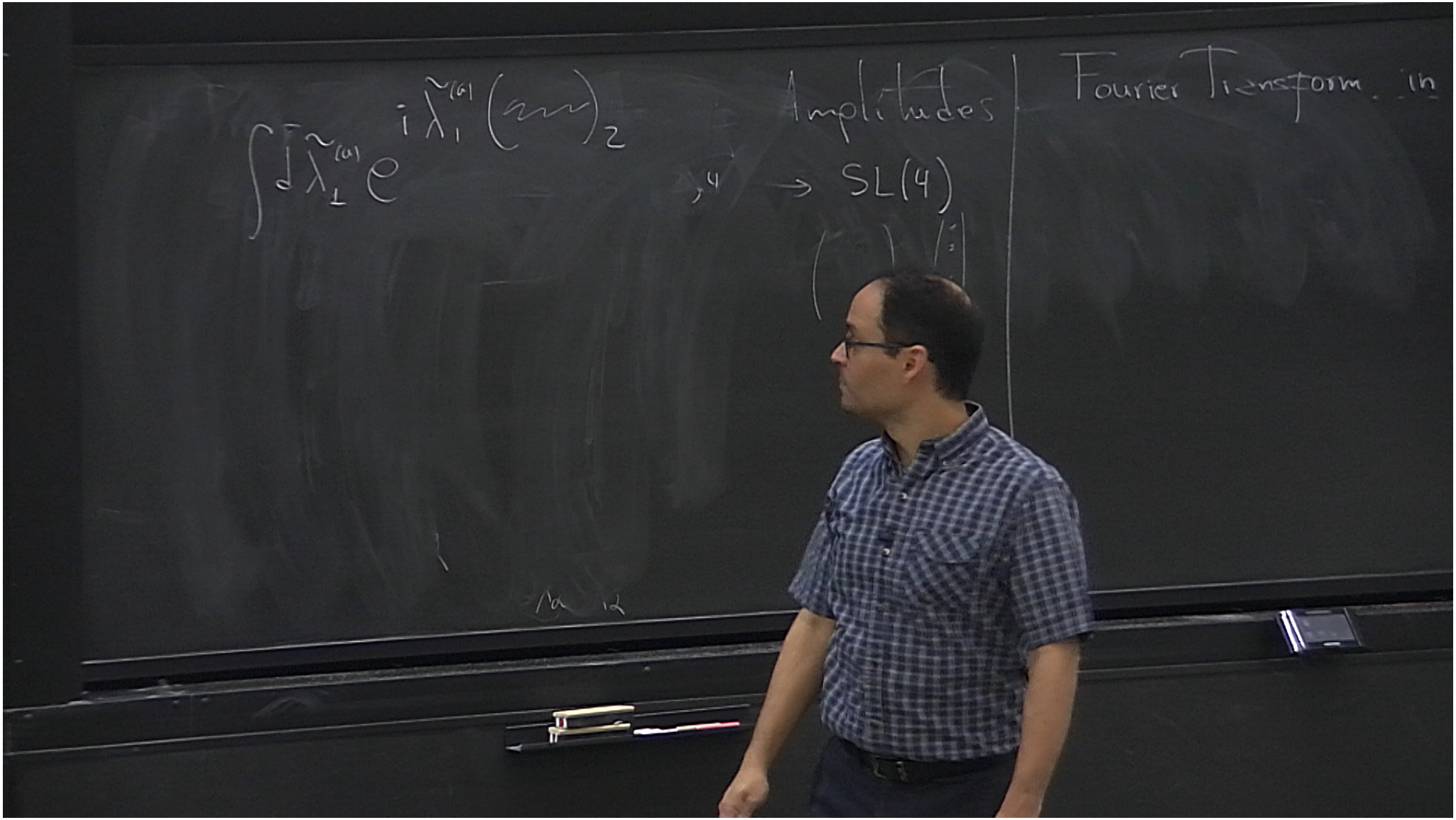

New Space-
Collect all $\tilde{\lambda}^{(a)}$ dependence.

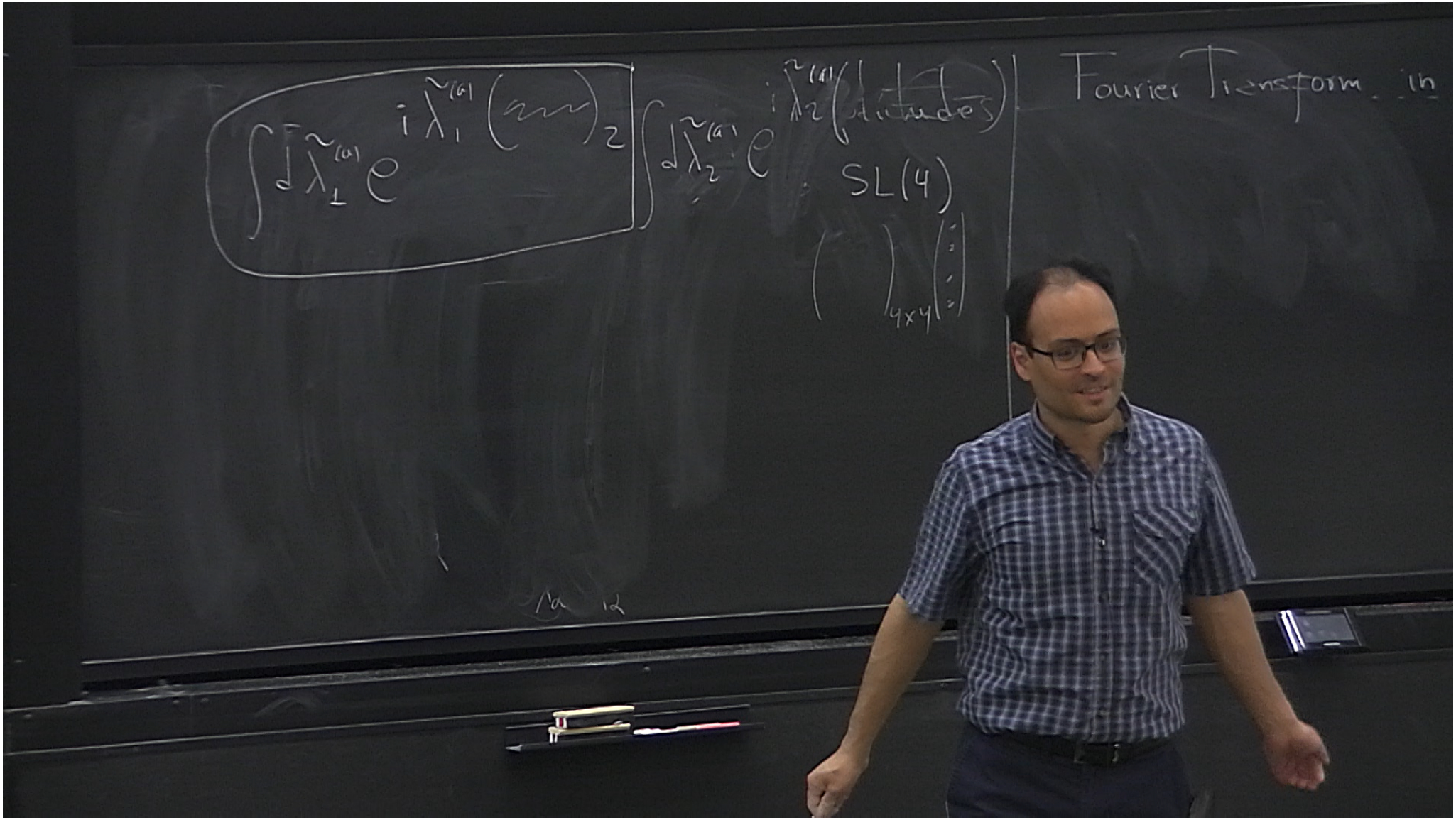

New Space-
$(\lambda^{(a)}, \mu^{(a)})$
$\tilde{A}^{(d)} = \int \prod_{a=1}^{n} d^2\tilde{\lambda}^{(a)}_\alpha \, e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}]} A_n^{(d)}$
Collect all $\tilde{\lambda}^{(a)}$ dependence.
$\int d^2\tilde{\lambda}^{(a)} e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}]}$
$A_n^{(d)} = \frac{1}{Vol(SL(2,\mathbb{C})\times GL(1))} \int \prod_{m=0}^{d} d\rho_m \left[\int d\sigma_a \frac{dt_a}{t_a} \right] \prod_{a=1}^{n} \delta^2(\lambda^{(a)} - t_a(\rho_0 + \rho_1 \sigma_a$

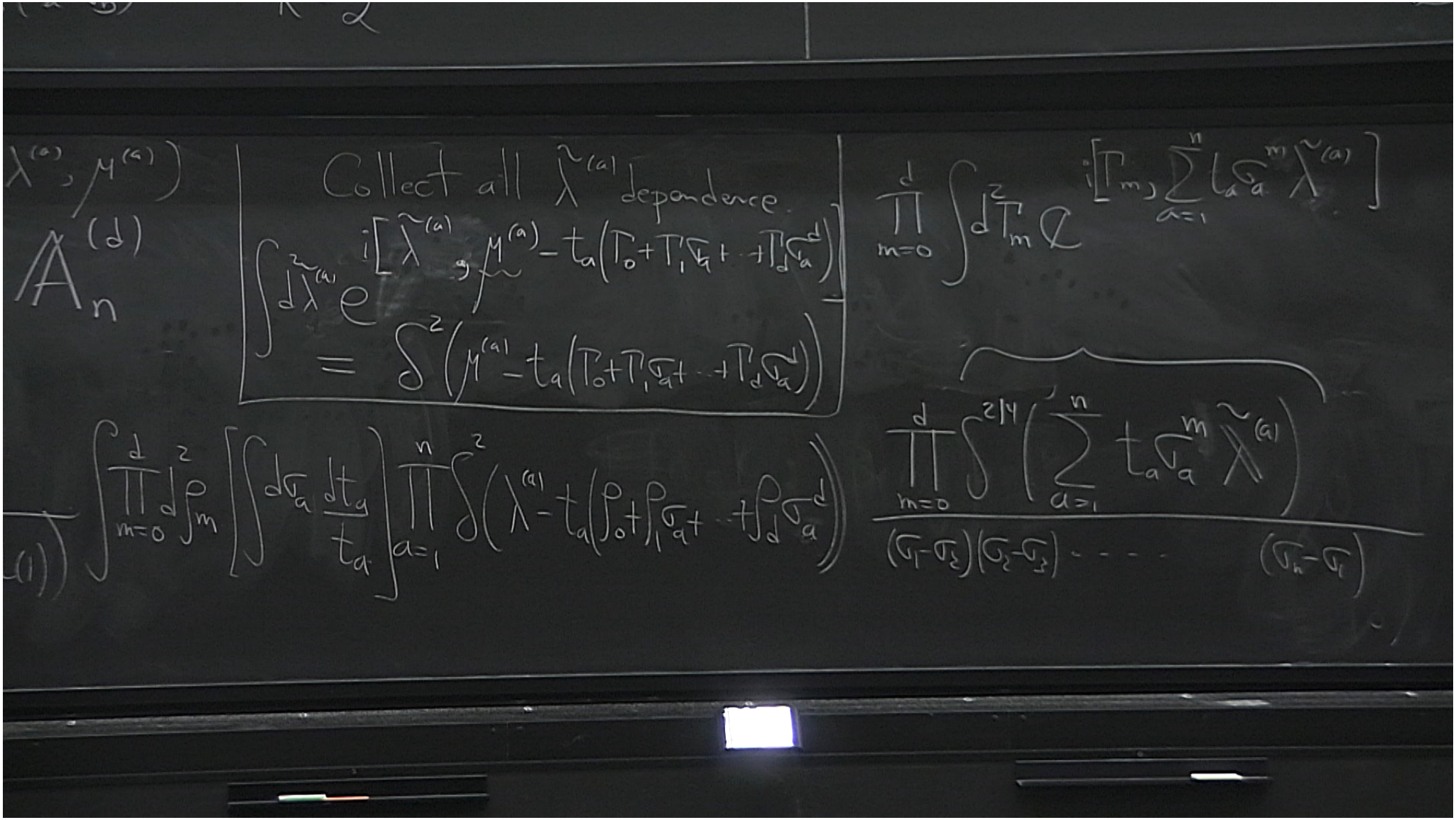

Collect all $\tilde{\lambda}^{(a)}$ dependence.
$\int d\tilde{\lambda}^{(a)} e^{i[\tilde{\lambda}^{(a)}, M^{(a)} - t_a(\Gamma_0 + \Gamma_1 \sigma_a + \cdots + \Gamma_d \sigma_a^d)]}$
$\prod_{m=0}^{d} \int d^2\Gamma_m$
$\prod_{m=0}^{d} \int d^2\rho_m$
$\int d\sigma_a \frac{dt_a}{t_a}$
$\frac{n}{t_a}$
$\tilde{\lambda}^{(a)} - t_a(\rho_0 + \rho_1 \sigma_a + \cdots + \rho_d \sigma_a^d)$
$\prod_{m=0}^{d} \int^{2/4}$
$(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)$

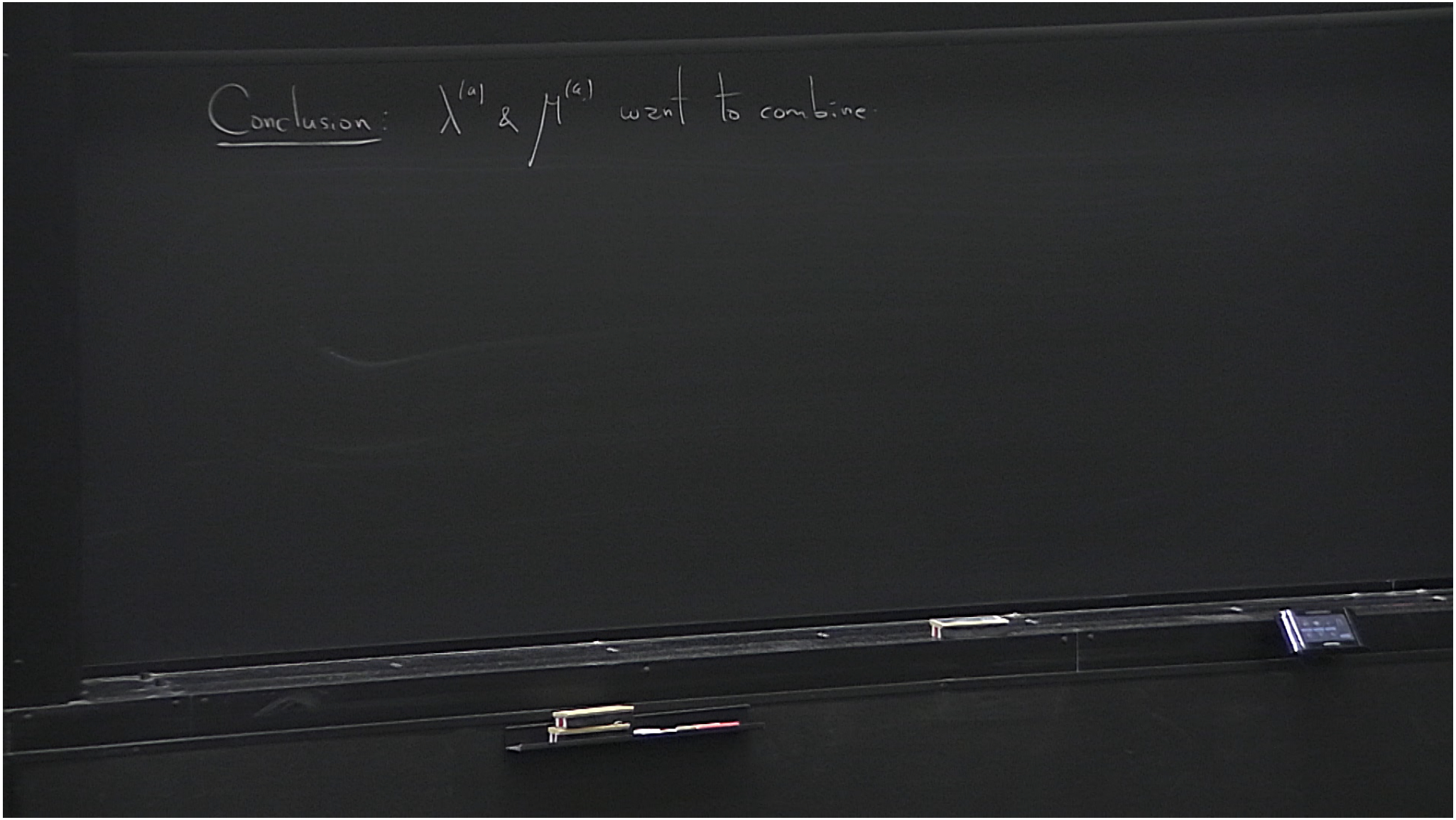

New Space
$(\lambda^{(a)}, \mu^{(a)})$
$\tilde{\mathbb{A}}^{(d)} = \int \prod_{a=1}^{n} d^2\tilde{\lambda}^{(a)}_\alpha \, e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}]} \, \mathbb{A}^{(d)}_n$
$\tilde{\lambda}_1 \to \frac{\partial}{\partial \mu_1} \to \mu_2$
$\tilde{\lambda}_2 \to \mu_1$
$\mathbb{A}^{(d)}_n = \frac{1}{Vol(SL(2,\mathbb{C})\times GL(1))} \int \prod_{m=0}^{d} d^2\rho_m \left[\int d\sigma_a \frac{dt_a}{t_a} \right] \prod_{a=1}^{n} \delta^2(\lambda^{(a)} - t_a(\rho_0 + \rho_1\sigma_a + \ldots + \rho_d\sigma_a^d$
Collect all $\tilde{\lambda}^{(a)}$ dependence.
$\int d^2\tilde{\lambda}^{(a)} e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}] - t_a(\Gamma_0 + \Gamma_1\sigma_a + \ldots + \Gamma_d\sigma_a^d}$

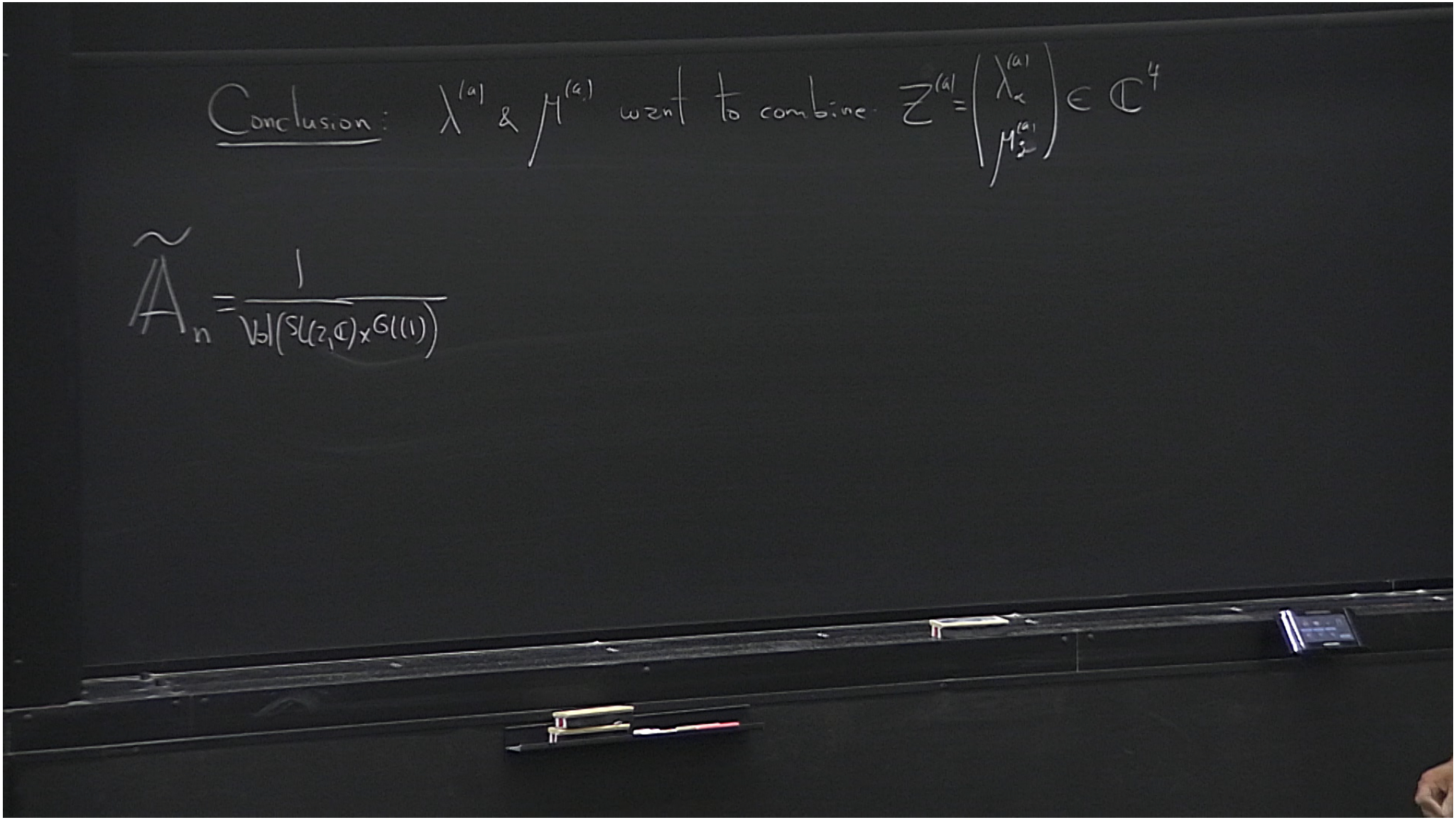

New Space-
Collect all $\tilde{\lambda}^{(a)}$ dependence.

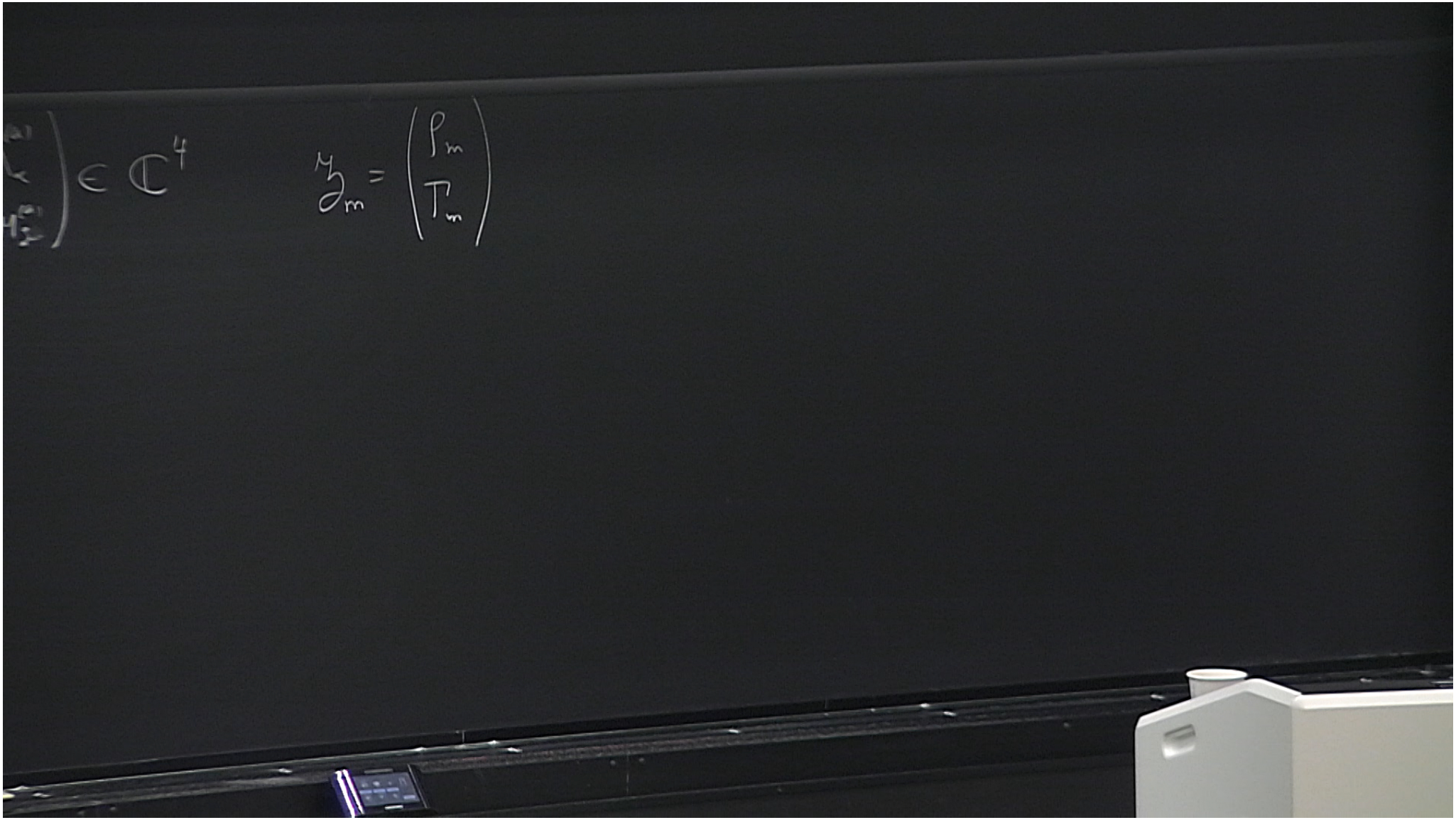

Amplitudes
SL(4)
Fourier Transform in

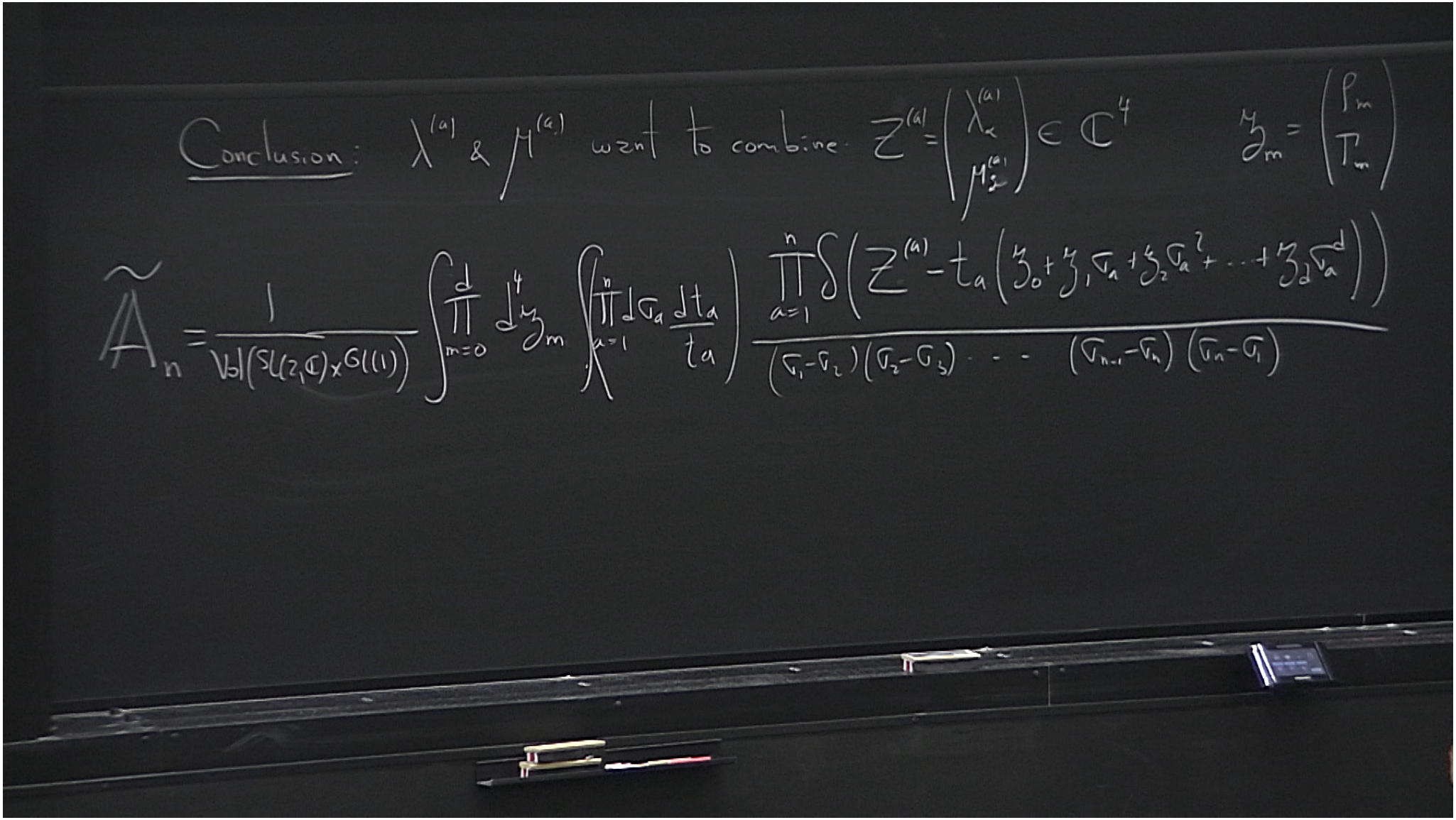

Fourier Transform in
SL(4)
4x4

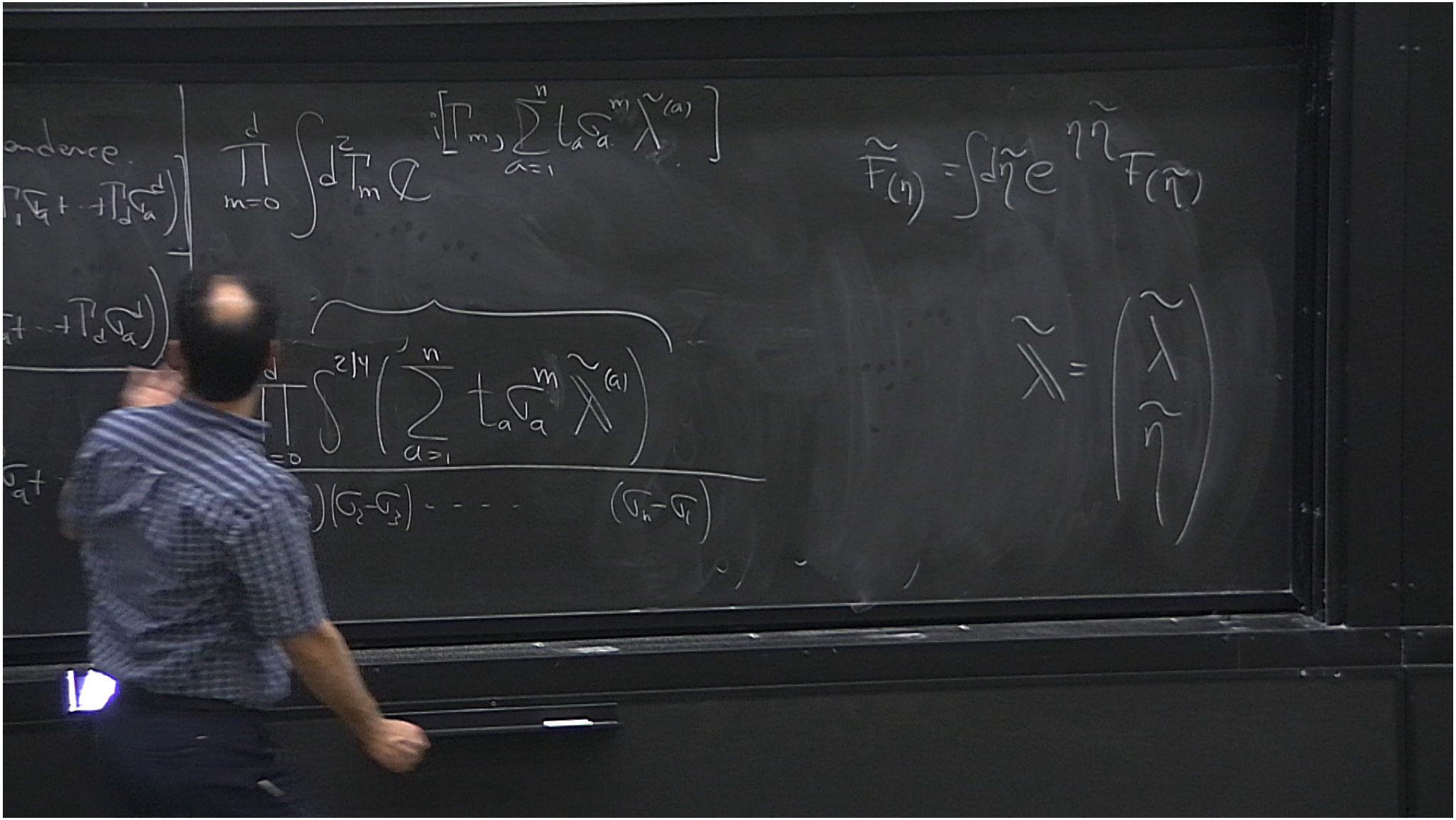

Collect all $\tilde{\lambda}^{(a)}$ dependence.
$\int d\tilde{\lambda}^{(a)} e^{i\left[\tilde{\lambda}^{(a)}\,\mu^{(a)} - t_a\left(\Gamma_0+\Gamma_1\sigma_a+\cdots+\Gamma_d\sigma_a^d\right)\right]}$
$= \delta^2\left(\mu^{(a)} - t_a\left(\Gamma_0+\Gamma_1\sigma_a+\cdots+\Gamma_d\sigma_a^d\right)\right)$
$\prod_{m=0}^{d}\int d^2\Gamma_m\, e^{i\left[\Gamma_m \sum_{a=1}^{n} t_a\sigma_a^m \tilde{\lambda}^{(a)}\right]}$
$\int \prod_{m=0}^{d} d^2\rho_m \left[\int d\sigma_a \frac{dt_a}{t_a}\right] \prod_{a=1}^{n}\delta^2\left(\lambda^{(a)} - t_a\left(\rho_0+\rho_1\sigma_a+\cdots+\rho_d\sigma_a^d\right)\right)$
$\prod_{m=0}^{d}\int^{z|y}\left(\sum_{a=1}^{n} t_a\sigma_a^m\tilde{\lambda}^{(a)}\right)$
$(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots\cdots(\sigma_n-\sigma_1)$

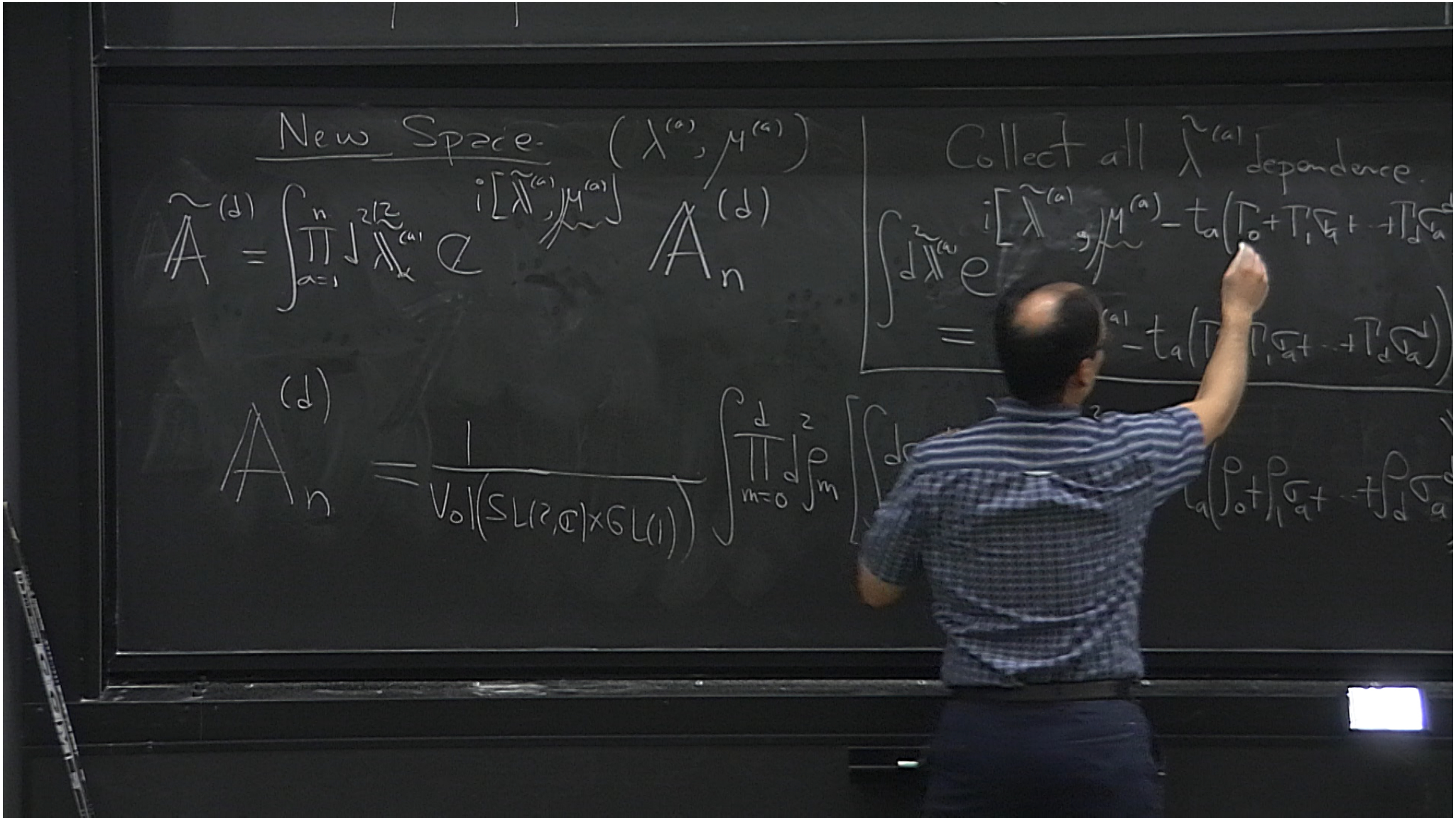

Conclusion: $\lambda^{(a)}$ & $\mu^{(a)}$ want to combine.

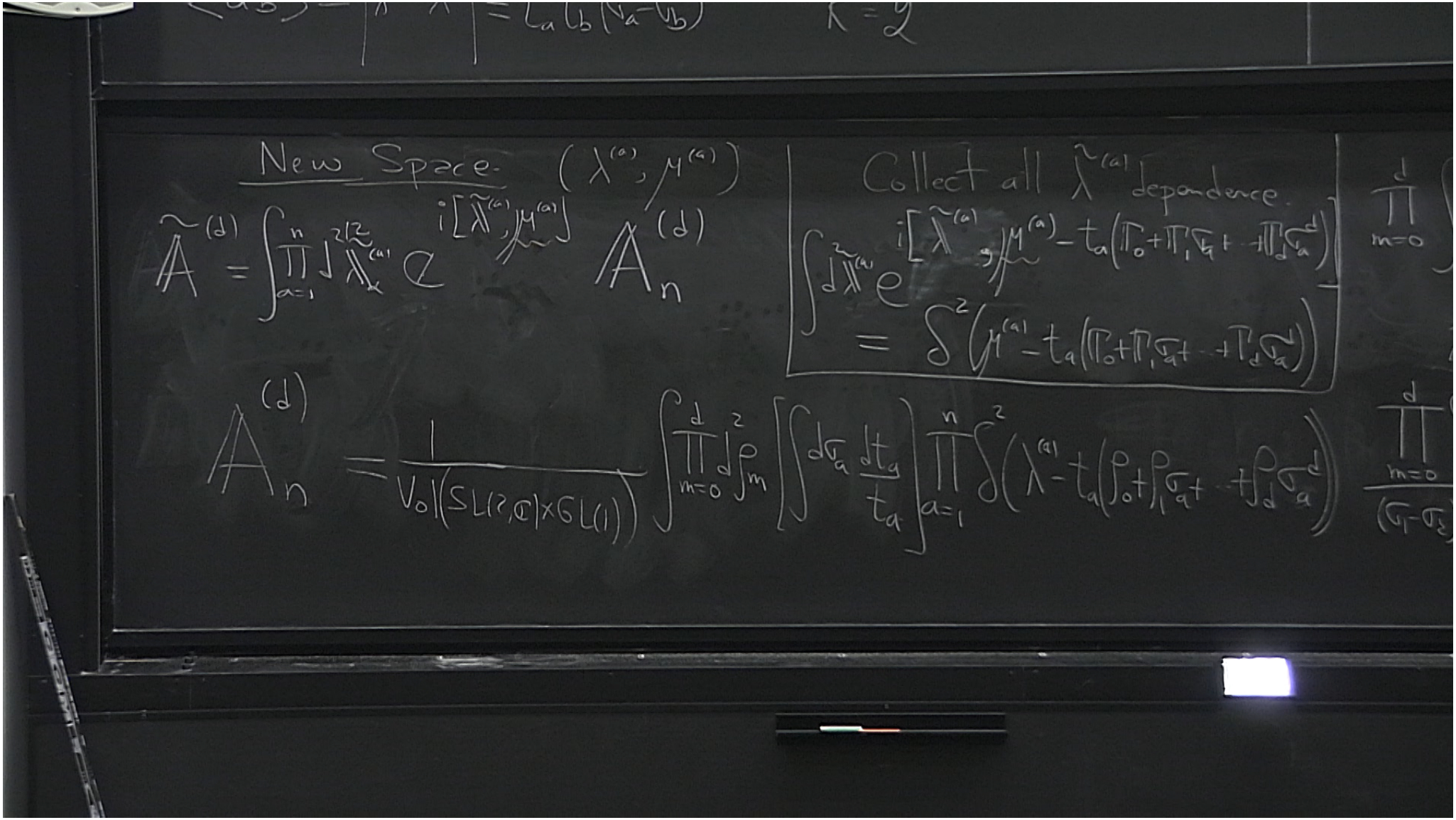

Conclusion: $\lambda^{(a)}$ & $\mu^{(a)}$ want to combine. $Z^{(a)} = \begin{pmatrix} \lambda_\alpha^{(a)} \\ \mu_{\dot\alpha}^{(a)} \end{pmatrix} \in \mathbb{C}^4$
$\tilde{A}_n = \frac{1}{\mathrm{Vol}(SL(2,\mathbb{C}) \times G(1))}$

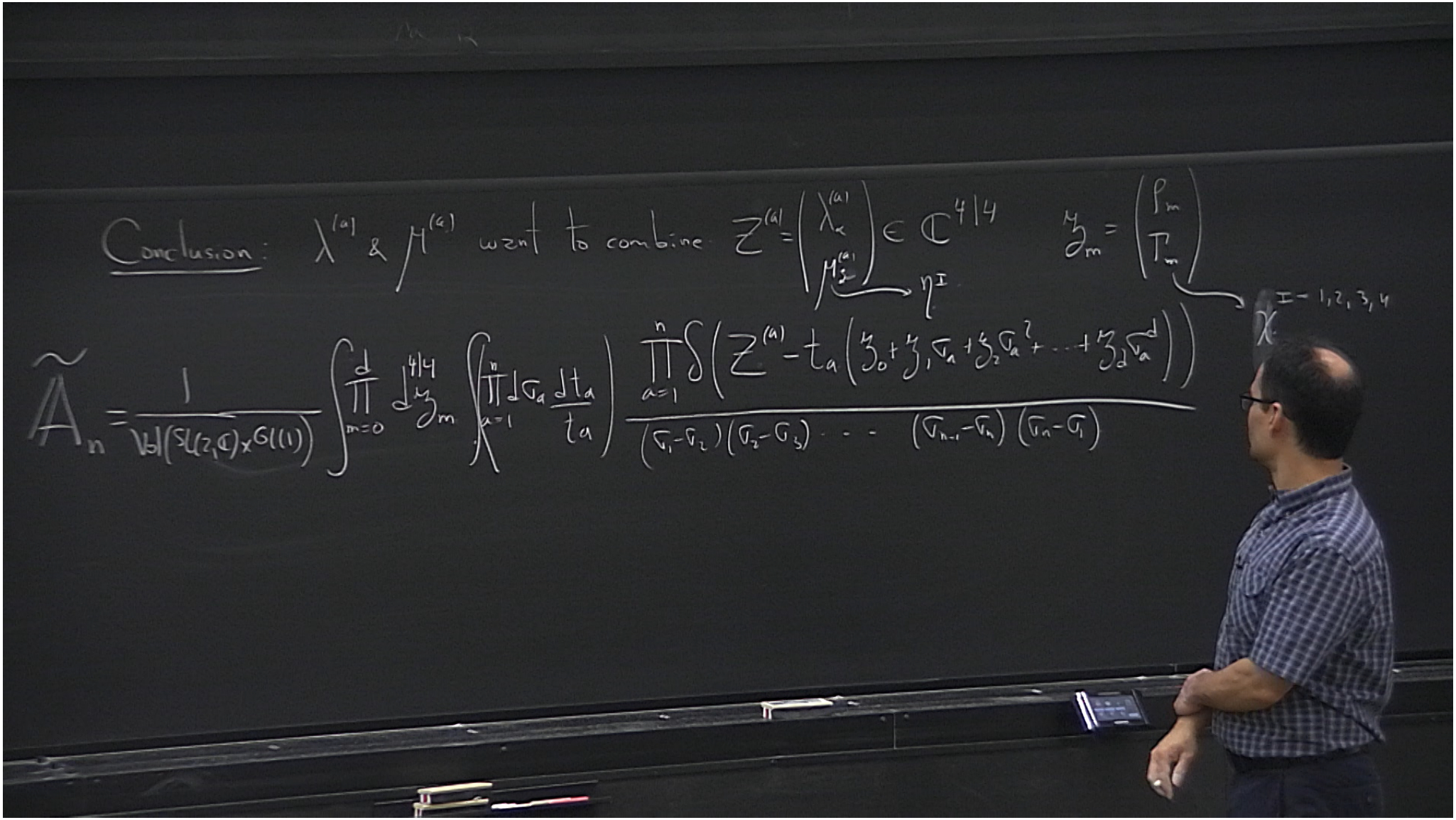

$\in \mathbb{C}^4$
$y_m = \begin{pmatrix} \rho_m \\ T_m \end{pmatrix}$

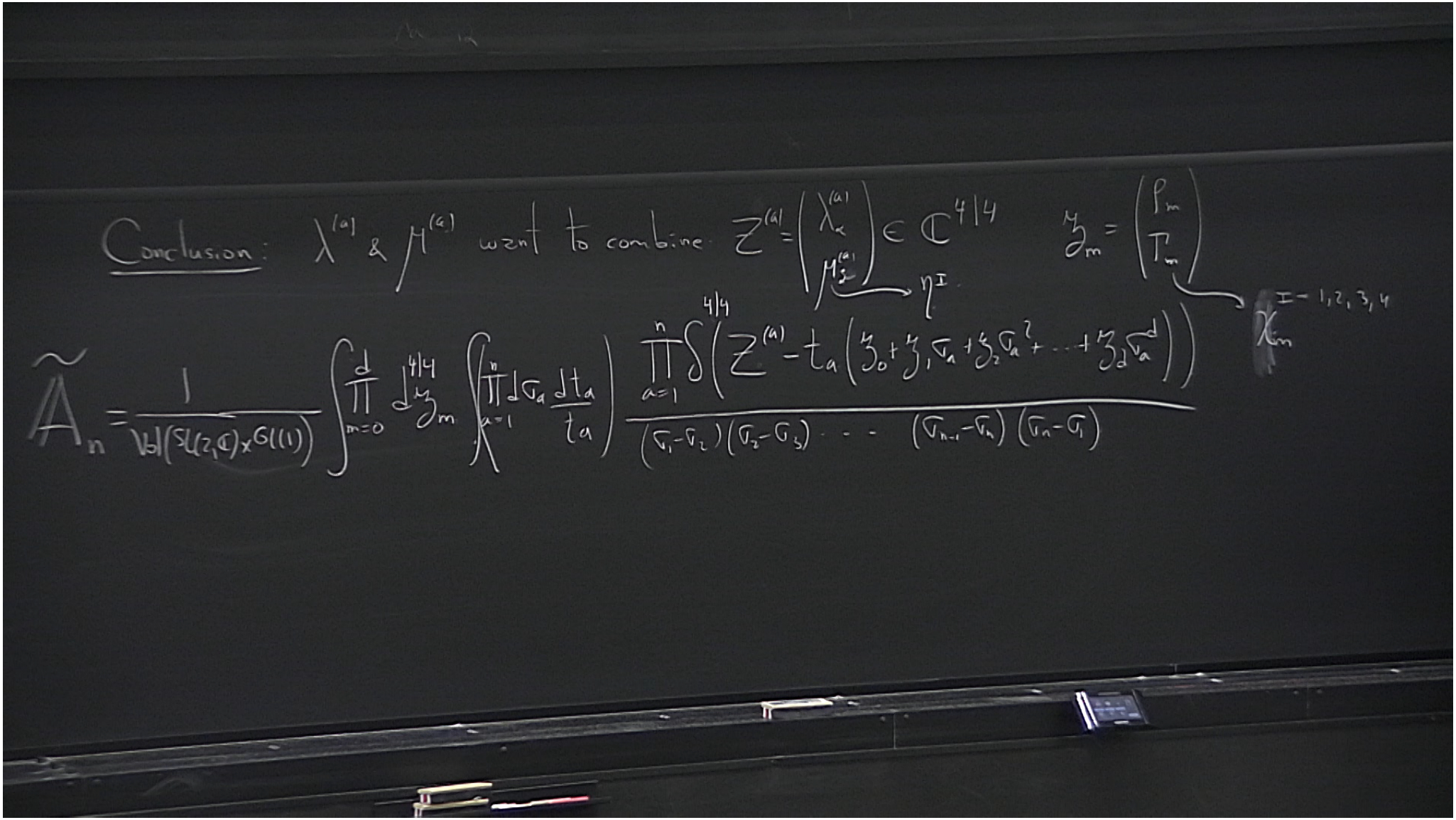

Conclusion: $\lambda^{(a)}$ & $\mu^{(a)}$ want to combine $Z^{(a)} = \begin{pmatrix} \lambda_\alpha^{(a)} \\ \mu_{\dot\alpha}^{(a)} \end{pmatrix} \in \mathbb{C}^4$ $y_m = \begin{pmatrix} p_m \\ r_m \end{pmatrix}$
$\tilde{A}_n = \frac{1}{\mathrm{Vol}(SL(2,\mathbb{C})\times G(1))} \int \prod_{m=0}^{d} d^4 y_m \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \frac{\prod_{a=1}^{n} \delta\left(Z^{(a)} - t_a\left(y_0 + y_1\sigma_a + y_2\sigma_a^2 + \cdots + y_d\sigma_a^d\right)\right)}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_{n-1}-\sigma_n)(\sigma_n-\sigma_1)}$

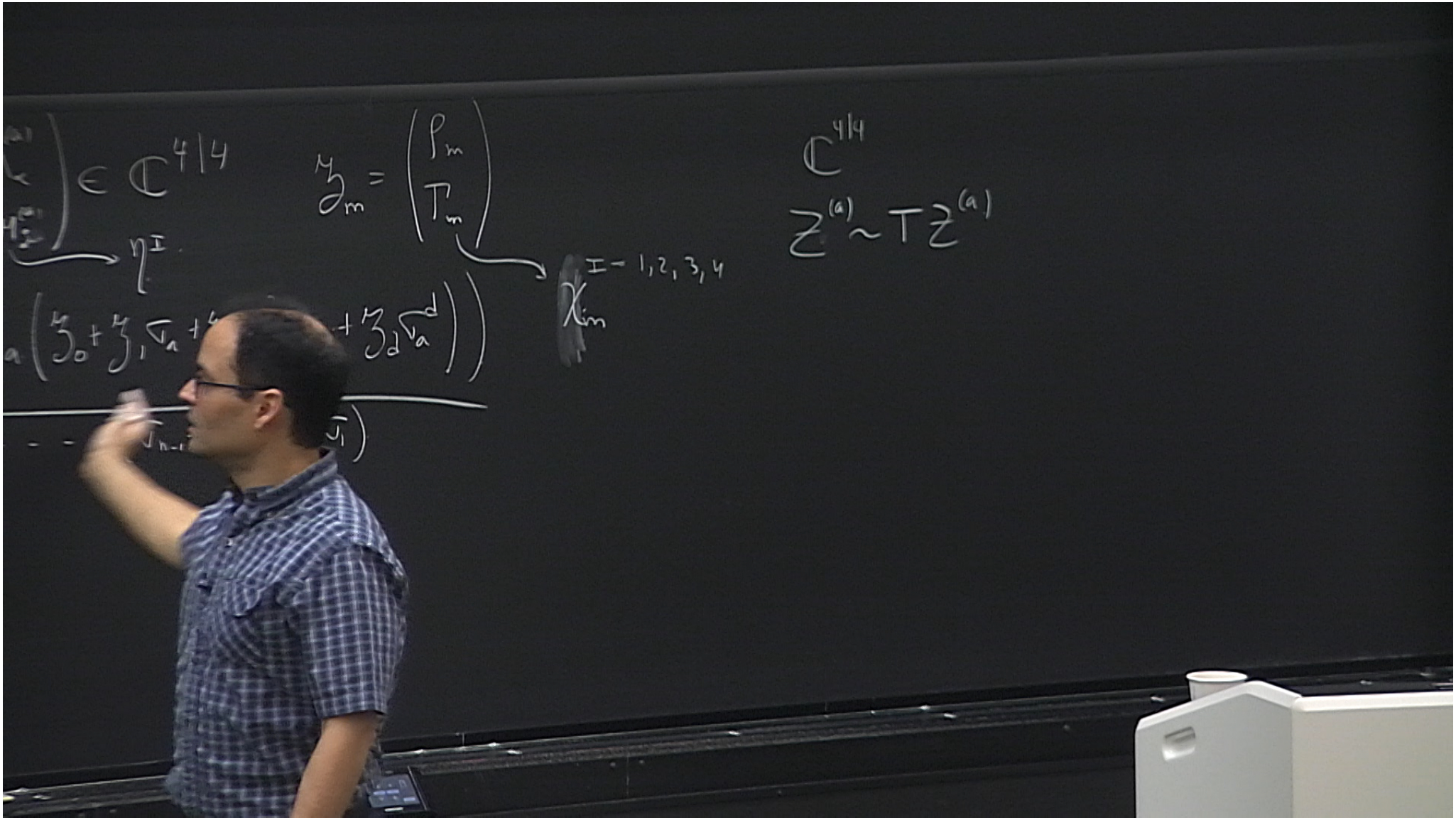

$\prod_{m=0}^{d} \int d\Gamma_m \, e^{i\left[\Gamma_m \sum_{a=1}^{n} t_a \sigma_a^m \tilde{\chi}^{(a)}\right]}$
$\tilde{F}(\eta) = \int d\tilde{\eta}\, e^{\eta\tilde{\eta}} F(\tilde{\eta})$
$\tilde{\chi} = \begin{pmatrix} \tilde{\chi} \\ \tilde{\eta} \end{pmatrix}$
$\prod_{m=0}^{d} \int^{z/4} \left(\sum_{a=1}^{n} t_a \sigma_a^m \tilde{\chi}^{(a)} \right)$
$(\sigma_2 - \sigma_1) \cdots (\sigma_n - \sigma_1)$

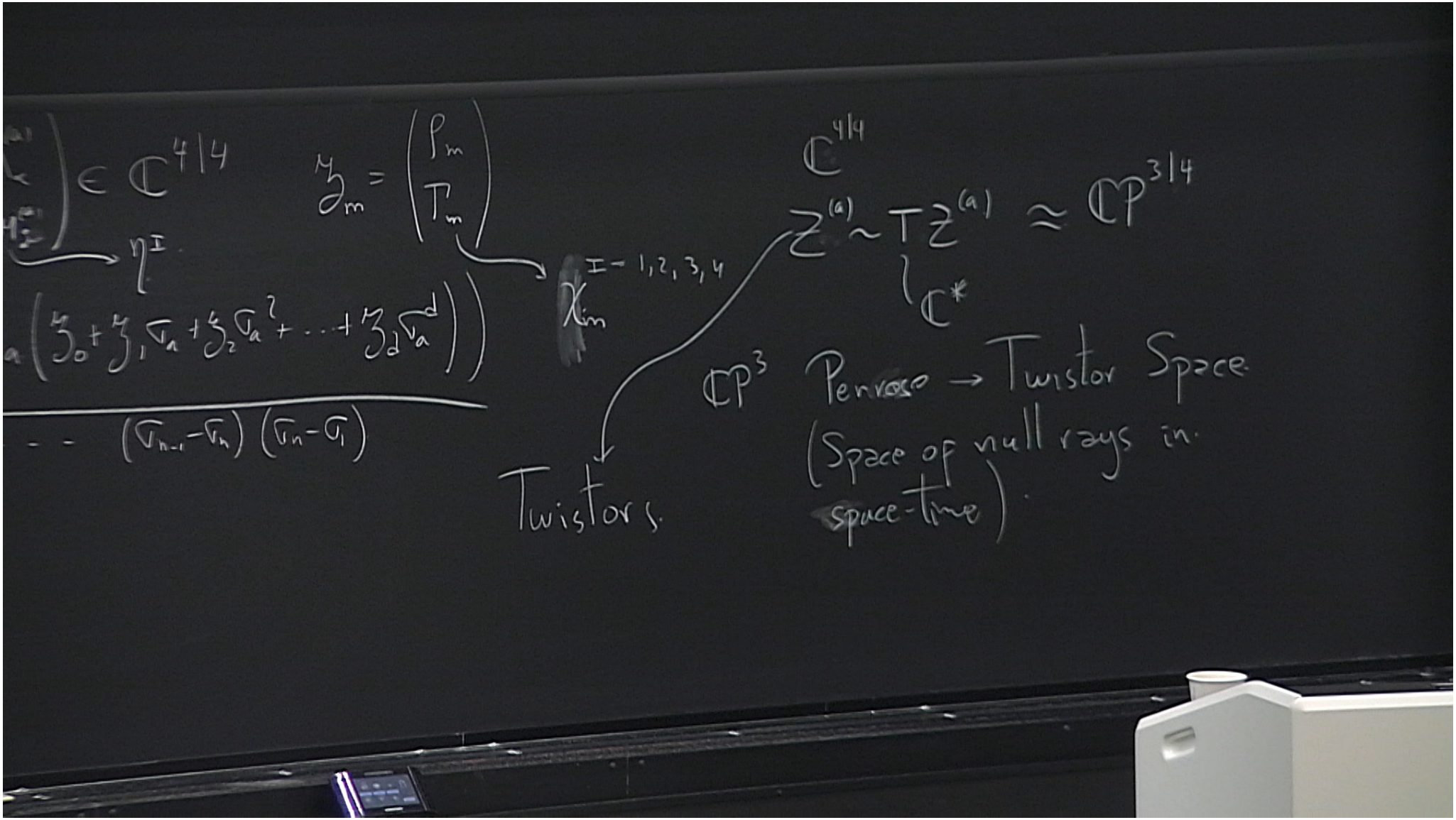

New Space-
Collect all $\lambda^{(a)}$ dependence.


$\langle a_a a_b \rangle = \lambda^{(a)} \lambda^{(b)} = t_a t_b (\sigma_a - \sigma_b)$
$k = 2$
New Space $(\lambda^{(a)}, \mu^{(a)})$
$\tilde{A}^{(d)} = \int \prod_{a=1}^{n} d^2\tilde{\lambda}^{(a)} e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}]} A_n^{(d)}$
Collect all $\tilde{\lambda}^{(a)}$ dependence.
$\int d^2\tilde{\lambda}^{(a)} e^{i[\tilde{\lambda}^{(a)}, \mu^{(a)}] - t_a(\Pi_0 + \Pi_1 \sigma_a + \cdots + \Pi_d \sigma_a^d)} = \delta^2(\mu^{(a)} - t_a(\Pi_0 + \Pi_1 \sigma_a + \cdots + \Pi_d \sigma_a^d))$
$\prod_{m=0}^{d}$
$A_n^{(d)} = \frac{1}{\mathrm{Vol}(SL(2,\mathbb{C}) \times GL(1))} \int \prod_{m=0}^{d} d^2\rho_m \left[\int d\sigma_a \frac{dt_a}{t_a} \right] \prod_{a=1}^{n} \delta^2(\lambda^{(a)} - t_a(\rho_0 + \rho_1 \sigma_a + \cdots + \rho_d \sigma_a^d))$
$\prod_{m=0}^{d}$
$(\sigma_1 - \sigma_2)$


Conclusion: $\lambda^{(a)}$ & $\mu^{(a)}$ want to combine $Z^{(a)} = \begin{pmatrix} \lambda_\alpha^{(a)} \\ \mu_{\dot\alpha}^{(a)} \end{pmatrix} \in \mathbb{C}^{4|4}$ η^I
$y_m = \begin{pmatrix} p_m \\ T_m \end{pmatrix}$
$I = 1,2,3,4$
$\tilde{A}_n = \frac{1}{\mathrm{Vol}(SL(2,\mathbb{C})\times GL(1))} \int_{m=0}^{d} d^{4|4} y_m \left(\int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \right) \frac{\prod_{a=1}^{n} \delta\left(Z^{(a)} - t_a \left(y_0 + y_1 \sigma_a + y_2 \sigma_a^2 + \cdots + y_d \sigma_a^d \right) \right)}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_{n-1}-\sigma_n)(\sigma_n-\sigma_1)}$


Conclusion: $\lambda^{(a)}$ & $\mu^{(a)}$ want to combine $Z^{(a)} = \begin{pmatrix} \lambda_\alpha^{(a)} \\ \mu_{\dot\alpha}^{(a)} \end{pmatrix} \in \mathbb{C}^{4|4}$ η^I
$y_m = \begin{pmatrix} p_m \\ T_m \end{pmatrix}$ $I = 1,2,3,4$ χ_{Im}
$\tilde{A}_n = \frac{1}{\mathrm{Vol}(SL(2,\mathbb{C})\times G(1|1))} \int_{m=0}^{d} d^{4|4} y_m \int \left(\prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a}\right) \frac{\prod_{a=1}^{n} \delta^{4|4}\left(Z^{(a)} - t_a\left(y_0 + y_1\sigma_a + y_2\sigma_a^2 + \cdots + y_d\sigma_a^d\right)\right)}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_{n-1}-\sigma_n)(\sigma_n-\sigma_1)}$


$\in \mathbb{C}^{4|4}$
η^I
$y_m = \begin{pmatrix} \rho_m \\ T_m \end{pmatrix}$
χ_{Im}
$I = 1,2,3,4$
$\mathbb{C}^{4|4}$
$z^{(a)} \sim T z^{(a)}$

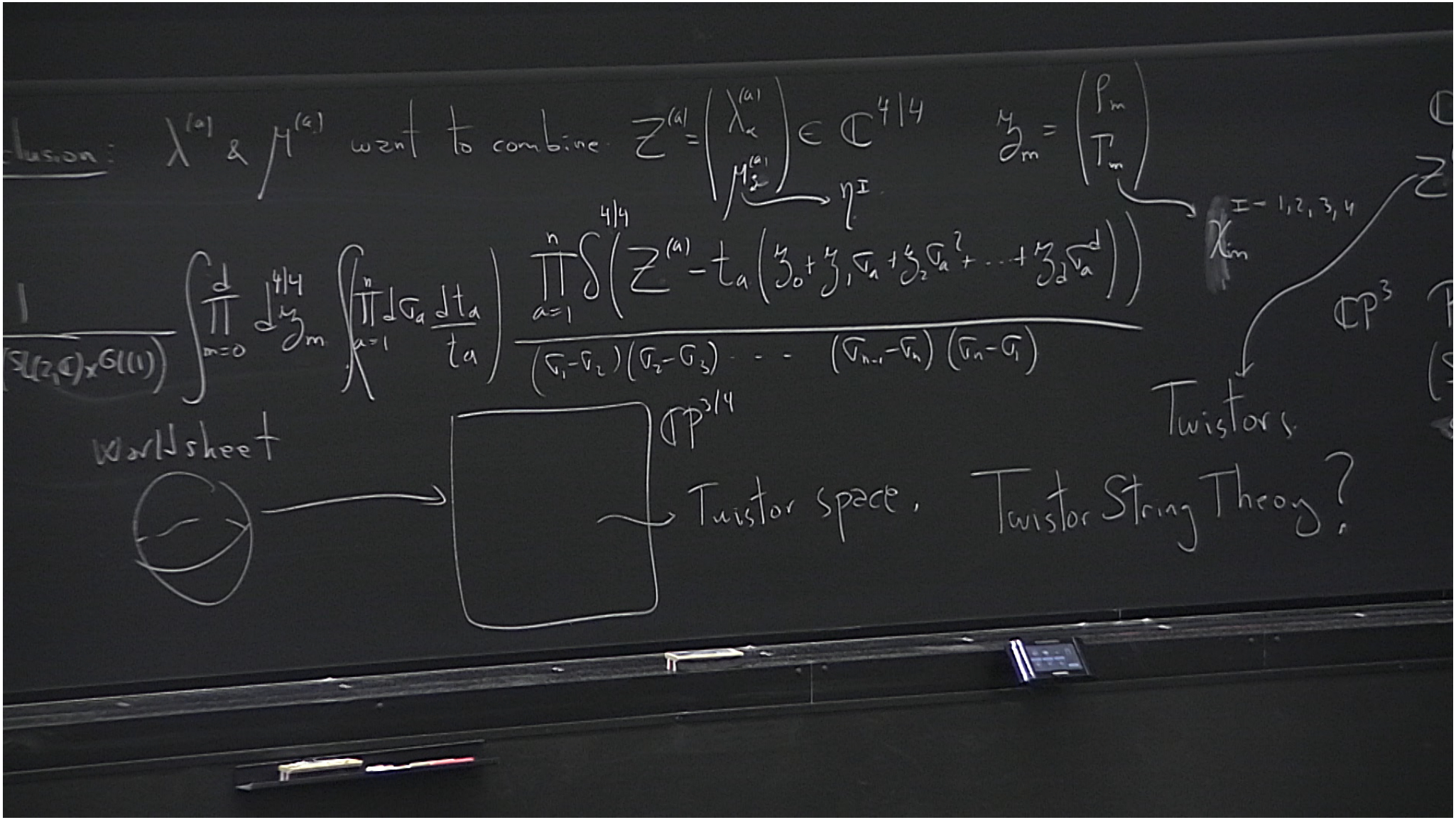

$\in \mathbb{C}^{4|4}$
η^I
$\mu_{\dot{a}}\left(\lambda_{0\dot{a}} + \lambda_{1\dot{a}}\sigma_a + \lambda_{2\dot{a}}\sigma_a^2 + \cdots + \lambda_{d\dot{a}}\sigma_a^d\right)$
$(\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)$
$\mu_{\dot{a}} = \begin{pmatrix} \rho_m \\ \tau_m \end{pmatrix}$
$I = 1,2,3,4$
χ_{im}
Twistors.
$\mathbb{C}^{4|4}$
$Z^{(a)} \sim T Z^{(a)} \approx \mathbb{CP}^{3|4}$
$\mathbb{C}^*$
$\mathbb{CP}^3$ Penrose → Twistor Space
(Space of null rays in space-time)

Conclusion: $\lambda^{(a)}$ & $\mu^{(a)}$ want to combine $Z^{(a)} = \begin{pmatrix} \lambda_\alpha^{(a)} \\ \mu_{\dot\alpha}^{(a)} \end{pmatrix} \in \mathbb{C}^{4|4}$ $\longrightarrow \eta^I$ $\qquad y_m = \begin{pmatrix} p_m \\ T_m \end{pmatrix}$

$$\tilde{A}_n = \frac{1}{\mathrm{Vol}\left(SL(2,\mathbb{C})\times G(1|1)\right)} \int \prod_{m=0}^{d} d^{4|4} y_m \int \left(\prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \right) \frac{\prod_{a=1}^{n} \delta^{4|4}\left(Z^{(a)} - t_a\left(y_0 + y_1\sigma_a + y_2\sigma_a^2 + \cdots + y_d\sigma_a^d \right)\right)}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_{n-1}-\sigma_n)(\sigma_n-\sigma_1)}$$

Worldsheet $\longrightarrow$ $\mathbb{CP}^{3|4}$ Twistor space.

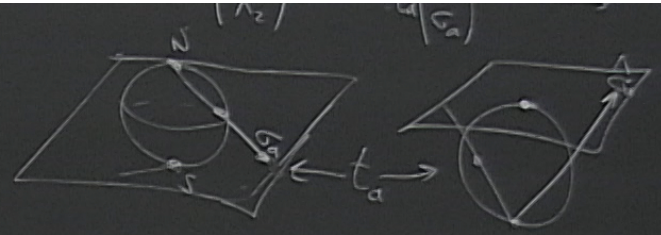
Tw


clusion: $\lambda^{(a)}$ & $\mu^{(a)}$ want to combine $Z^{(a)} = \begin{pmatrix} \lambda_\alpha^{(a)} \\ \mu_{\dot\alpha}^{(a)} \end{pmatrix} \in \mathbb{C}^{4|4}$ η^I
$Y_m = \begin{pmatrix} p_m \\ T_m \end{pmatrix}$
$I = 1,2,3,4$
χ_m
$\mathbb{CP}^3$
$\frac{1}{\mathrm{Vol}(SL(2,\mathbb{C})\times GL(1))} \int \prod_{m=0}^{d} d^{4|4} Y_m \int \prod_{a=1}^{n} d\sigma_a \frac{dt_a}{t_a} \frac{\prod_{a=1}^{n} \delta^{4|4}\left(Z^{(a)} - t_a\left(Y_0 + Y_1\sigma_a + Y_2\sigma_a^2 + \cdots + Y_d\sigma_a^d\right)\right)}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_{n-1}-\sigma_n)(\sigma_n-\sigma_1)}$
Twistors.
Worldsheet
$\mathbb{CP}^{3|4}$
Twistor space, Twistor String Theory?

$\langle 12\rangle\langle 23\rangle \cdots \langle n-1\, n\rangle\langle n1\rangle$

$K = \#$ of negative helicity gluons

$\langle ab\rangle = \left|\lambda^{(a)}\lambda^{(b)}\right| = t_a t_b (\sigma_a - \sigma_b)$ $\quad K = 2$



Q: Is there a full String Theory whose correlation function give the full perturbative S-matrix of $\mathcal{N}=4$ SYM?

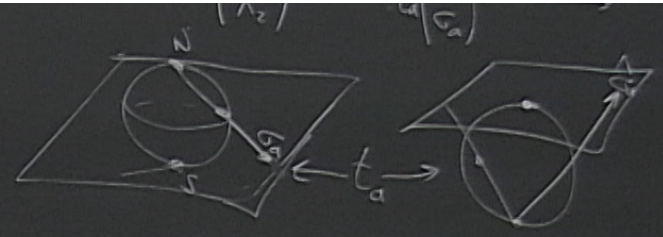
Q: Tree level ✓  One-Loop  → Elliptic Functions ✗ QFT

$\langle 12\rangle\langle 23\rangle \cdots \langle n-1\, n\rangle\langle n1\rangle$

$\{1,2\}$ $\{1,2,3,4\}$

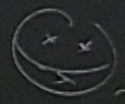
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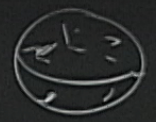
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Q: Tree level ✓  One-Loop  → Elliptic Functions ✗ QFT

Q: What is the origin of all this connection between  & K_n^k?

(Is all this simply a consequence of $\mathcal{N}=4$ SYM?

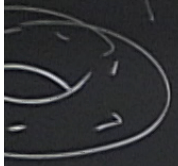
Q: Is there a full String Theory whose correlation func

Q: Tree Level ✓  One-Loop 

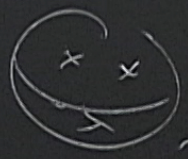
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$\longrightarrow$ Elliptic Functions ~~X~~ QFT.

tion between  & k_a^μ?

of $\mathcal{N}=4$ SYM?