Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 24

Date: Apr 04, 2018 03:00 PM

URL: http://pirsa.org/18040029

Abstract:

The Hawking effect

In 1915, Schwarzschild found this black hole solution to the Einstein equation:

Mass of black hole

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + d\varphi^2 \sin^2\theta\right)$$

Singularity: $r = 0$

Horizon: $r = 2M$

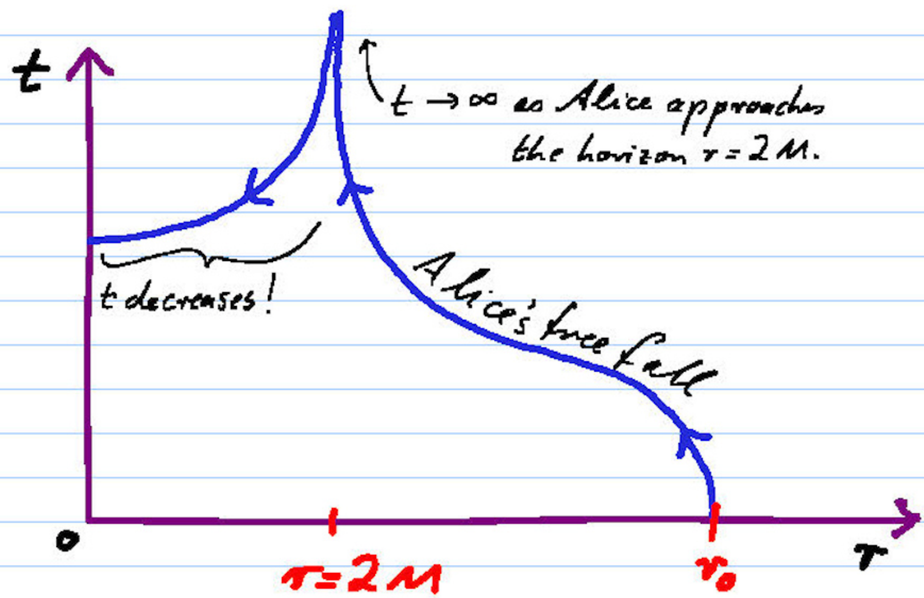
Here, $x = (t, r, \varphi, \theta)$ are called the Schwarzschild coordinates.

Schwarzschild coordinates long misled intuition:

- The singularity at $r = 2M$ is not real: it disappears in other coordinate systems. The curvature is smooth across $r = 2M$.

□ Due to the sign changes across $r = 2M$, for $r < 2M$
dt is spacelike and dr is timelike for $r < 2M$.

□ Consider, for example, a traveler, Alice, who is freely falling from $r = r_0$ to $r = 0$:



$$r(\alpha) = \frac{r_0}{2}\left(1 + \cos(\alpha)\right)$$

$$t(\alpha) = \left(\frac{r_0}{2} + 2M\right) w\alpha + \frac{r_0}{2} w \sin(\alpha) + 2M \log\left|\frac{w + \tan(\alpha/2)}{w - \tan(\alpha/2)}\right|$$

$$\tau(\alpha) = \frac{r_0}{2}\left(\frac{r_0}{2M}\right)^{1/2}\left(\alpha + \sin(\alpha)\right)$$

Here: $0 < \alpha < \pi$ and $w := \left(\frac{r_0}{2M} - 1\right)^{1/2}$

□ For quantization, need better choices of coordinate systems!

Simplification: For now, we drop the φ and θ coordinates.

First design of a new cds (T,R) - Alice's choice (for $r_0=2M$):

- Require $g_{\mu\nu}(T,R)$ to be regular across $r=2M$.
- Require $g_{\mu\nu}(0,0)=\eta_{\mu\nu}$ at $r=2M$. If there's really no singularity at $r=2M$ this must be possible.
- Extend this cds so that $g_{\mu\nu}(T,R)=f(T,R)\,\eta_{\mu\nu}$ because then we know:
  - the action
  - the Klein Gordon equation
  - the solution space of the K.G. equation.
  - which is the mode fctn of the vacuum in this cds.

⇒ Alice's choice are the Kruskal-Szekeres coordinates (T, R):

$$T(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\sinh\left(\frac{t}{4M}\right)\theta(r-2M)+\cosh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

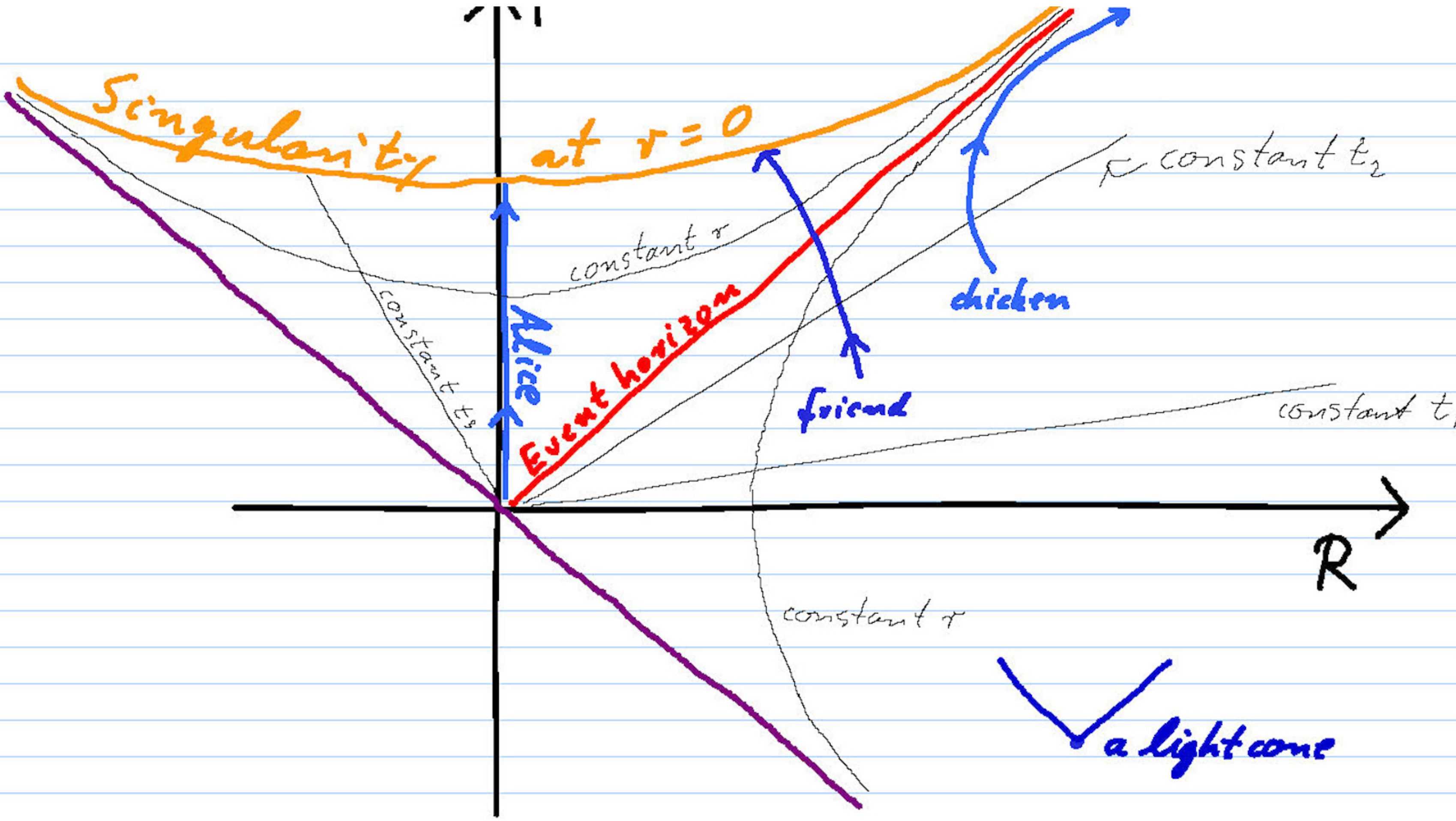
$$R(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\cosh\left(\frac{t}{4M}\right)\theta(r-2M)+\sinh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

This map is, in principle, invertible, to obtain $t(T,R)$, $r(T,R)$.

The Schwarzschild metric now takes this form:

$$ds^2 = \underbrace{\frac{2M}{r(T,R)}\, e^{1-\frac{r(T,R)}{2M}}}_{\text{Conformal prefactor } =1 \text{ as } r=2M} \underbrace{\left(dT^2 - dR^2\right)}_{=\eta_{\mu\nu}}$$

Obeys all conditions!



- Alice was at rest at the event horizon.
- The singularity is at $T(R) = \left(R^2 + \frac{16M^2}{e}\right)^{1/2}$ and is spacelike.

Alice's light cone coordinates:

$$u := T - R, \quad v := T + R$$

Metric: $$ds^2 = \underbrace{\frac{2M}{r(u,v)}\, e^{1 - \frac{r(u,v)}{2M}}}_{\substack{\text{conformal factor} \\ \text{(which is 1 at horizon)}}} \underbrace{du\, dv}_{\substack{\text{lightcone} \\ \text{Minkowski}}}$$

$\Rightarrow$ The action $S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\, d^2x$ becomes:

$$= \frac{1}{2}\int_{T>-R} \left(\partial_T \phi(T,R)\right)^2 - \left(\partial_R \phi(T,R)\right)^2 \, dT\, dR$$

$$= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\partial_u \phi(u,v)\right)\left(\partial_v \phi(u,v)\right) dv\, du$$

$\leftarrow$ b/c region $T > -R$ means $T + R > 0$, i.e. $v > 0$.

$\Rightarrow$ Eqn of motion: $\partial_u \partial_v \phi(u,v) = 0$

⇒ solution for the QFT found as before:

$$\hat{\phi}(u,v) = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{(2\omega)^{1/2}} \left(e^{-i\omega u} \hat{a}_\omega + e^{i\omega u} \hat{a}^+_\omega + \text{left movers} \right)$$

obeys the 3 conditions: EoM, CCRs and hermiticity.

Alice's notion of vacuum

- For Alice, as she crosses the horizon, $g_{\mu\nu} = \eta_{\mu\nu}$.
- If her detectors are not clicking, the state of the field is $|0_{Alice}\rangle$, obeying $a_\omega |0_{Alice}\rangle = 0 \; \forall \omega$.

One problem though: In this cds, far away, i.e., as $r \to \infty$, the metric doesn't become the Minkowski $\eta_{\mu\nu}$.

Bob is far from the black hole.

He wants a cds in which:

- $g_{\mu\nu}(x) \to \eta_{\mu\nu}$ as $r \to \infty$.
- $g_{\mu\nu}(x) = f(x)\,\eta_{\mu\nu}$ everywhere.

This is so that in his cds too

- photons travel at $45°$
- we know action, K.G. equation and mode functions.

$\rightsquigarrow$ Bob's choice is the Tortoise coordinate system.

$$t^* := t$$

must require $r > 2M$!

$$r^* := r - 2M + 2M \log\left(\frac{r}{2M} - 1\right)$$

$\Rightarrow$ Important: This is in principle invertible, to obtain

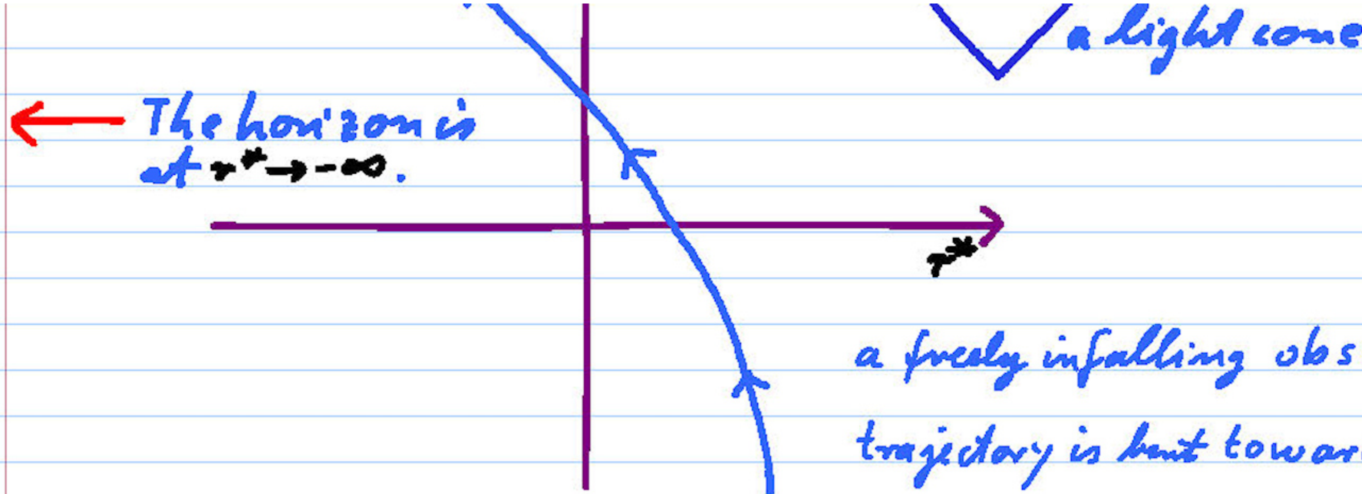
$$r = r(r^*)$$

but only for $r > 2M$!

$\Rightarrow$ The tortoise cds only cover the BH's outside!

Metric: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(r^*)}\right)}_{} \left(dt^{*2} - dr^{*2}\right)$$

Conformal factor $\rightarrow 1$ as $r \rightarrow \infty$, as planned but $\rightarrow 0$ at horizon.



a freely infalling observer's trajectory is bent towards 45° as he speeds up.

Bob's light cone coordinates: $\bar{u} := t^* - r^*$, $\bar{v} := t^* + r^*$

The metric is then: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(\bar{u},\bar{v})}\right)}_{\to 1 \text{ as } r\to\infty \text{ and } \to 0 \text{ as } r \to 2M} d\bar{u}\, d\bar{v}$$

Important later: $u = -4M e^{-\frac{\bar{u}}{4M}}$, $v = 4M e^{\frac{\bar{v}}{4M}}$

→ The action:

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\,d^2x$$ becomes:

$$= \frac{1}{2}\int_{\mathbb{R}^2} \left(\partial_{t^*}\phi(t^*,r^*)\right)^2 - \left(\partial_{r^*}\phi(t^*,r^*)\right)^2 dt^* dr^*$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\partial_{\bar{u}}\phi(\bar{u},\bar{v})\right)\left(\partial_{\bar{v}}\phi(\bar{u},\bar{v})\right) d\bar{v}\, d\bar{u}$$

⇒ Eqn of motion: $\partial_{\bar{u}}\partial_{\bar{v}}\phi(\bar{u},\bar{v}) = 0$

$\Rightarrow$ Solution for the QFT found as before:

$$\hat{\phi}(\bar{u},\bar{v}) = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{(2\omega)^{1/2}} \left(e^{-i\omega\bar{u}} \hat{b}_\omega + e^{i\omega\bar{u}} \hat{b}^+_\omega + \text{left movers} \right)$$

obeys the 3 conditions: EoM, CCRs and hermiticity.

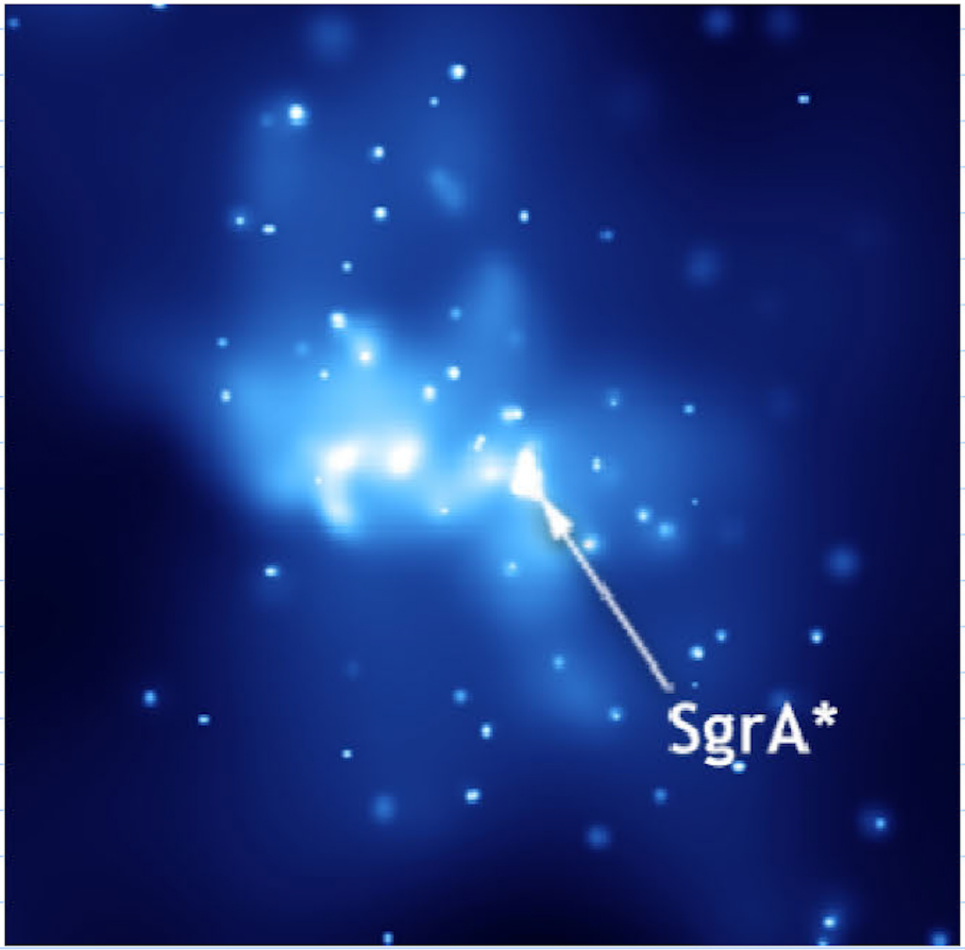
Bob's notion of vacuum

- For Bob, out at $r \to \infty$, the metric is $g_{\mu\nu} = \eta_{\mu\nu}$.
- If Bob's detectors are not clicking, the state of the field is $|0_{Bob}\rangle$, obeying $\hat{b}_\omega |0_{Alice}\rangle = 0 \ \forall \omega$.

E.g.: Sagittarius A*

- 4 Mio stellar masses
- Diameter 44 Mio km
- 26000 light years away at centre of Milky Way.

→ Observations coming up 2018 by Event Horizon Telescope Array (in mm band) with enough resolution to see the event horizon?



Real black holes:

- they have complicating properties, such as
  - a ring down
  - peculiar velocity
  - angular momentum
  - charges
  - and even mass changes (through absorption or emission).

Simple model:

- Let us neglect all these.
- Also assume that the rest of the universe is empty.

$\Rightarrow$ In good approximation the spacetime should be

Q: Is $|\psi\rangle = |0_{Alice}\rangle$ or perhaps $|\psi\rangle = |0_{Bob}\rangle$?

A: Probably both: $|\psi\rangle = |0_{Alice,right}\rangle \otimes |0_{Bob,left}\rangle$

Here: $a_{\omega,right}|0_{Alice,right}\rangle = 0 \ \forall \omega$

$b_{\omega,left}|0_{Bob,left}\rangle = 0 \ \forall \omega$

Why?

We cannot reliably calculate through the collapse process, because it involves tracking waves being infinitely blue-shifted at the horizon ($\rightarrow$ see the Transplanckian problem for BHs).

▫ If, as we assume, the rest of the universe has no stars etc, then there should be no flux of quanta into the black hole.

⇒ The left-moving (i.e. ingoing) modes should be in the state

$$|0_{Bob,left}\rangle$$

▫ Can the right moving (i.e. outgoing) modes be in the state $|0_{Bob,right}\rangle$?

No!

$$\boxed{u = -4M e^{-\frac{\bar{u}}{4M}}, \quad v = 4M e^{\frac{\bar{v}}{4M}}}$$

u: Alice's; $\bar{u}$: Bob's

Compare with (from the previous lecture):

$$\boxed{u = -\frac{1}{a} e^{-a\bar{u}}}$$

u: inertial; $\bar{u}$: accelerated

$\Rightarrow$ Alice's and Bob's cds relate in the same way as the inertial and accelerated before,

with $a = \frac{1}{4M}$

$\Rightarrow$ $|0_{Bob, right}\rangle$ has divergent vacuum energy towards the horizon!

⇒ If the QFT is in the state $|0_{Bob,right}\rangle$, then:

- Via the Einstein equation, this would contradict our assumption of spacetime being Schwarzschild, which solves:

$$G_{\mu\nu}(g_{Schwarzschild}) = T_{\mu\nu} \quad \text{with } T_{\mu\nu} = 0.$$

- During the collapse, the quantum state will be energetically prevented to evolve into the state

$$|0_{Bob,right}\rangle$$

- Alice would see a diverging amount of quantum field fluctuations and particles as she crosses the horizon.
  - ⇒ She would be able to tell the location of the horizon by local measurements in a free-falling lab.
  - ⇒ This would contradict the equivalence principle.

Q: What state do the right-moving (outgoing) modes have to be in, so that

- Their contribution to $T_{\mu\nu}$ is smooth across the horizon.
- Alice does not see a burst of particles from the horizon.

⇒ She would be able to tell the location

of the horizon by local measurements in a free-falling lab.

⇒ This would contradict the equivalence principle.

Q: What state do the right-moving (outgoing) modes have to be in, so that

- ☐ Their contribution to $T_{\mu\nu}$ is smooth across the horizon.

- ☐ Alice does not see a burst of particles from the horizon.

A: $|0_{Alice, right}\rangle$ has these properties, (via previous lecture's result(s)).

$$\boxed{|\psi\rangle = |0_{\text{Alice, right}}\rangle \otimes |0_{\text{Bob, left}}\rangle}$$

Q: What, therefore, should we see at rest from far?

A: Our natural cds is Bob's then.

$\Rightarrow$ We see no ingoing (left-moving) radiation.

But we can repeat the calculations of the previous lecture for the outgoing modes, using $a = 1/4M$

$\Rightarrow$ We see an outflux of quanta of temperature:

$$\boxed{T_H = \frac{1}{8\pi M}}$$

Recall: $T_H = \frac{a}{2\pi}$

Minkowski space | Schwarzschild spacetime

Accelerated observer's vacuum: "Rindler vacuum" | Bob's vacuum: "Boulware vacuum"

Inertial observer's vacuum: "Minkowski vacuum" | Alice's vacuum: "Kruskal vacuum"

Remarks: □ The state $|0_{Bob,right}\rangle$ (outgoing) was disqualified due to its contribution to $T_{\mu\nu}$ which would diverge towards the horizon.

Is $|0_{Bob,left}\rangle$ having the same problem?
No, it would have that problem at the past horizon but a real black hole doesn't have one (unlike an accelerated observer.)

□ We dropped the angles φ, θ. Do they matter?

Yes, it leads to a weakening of Hawking radiation:

The mode decomposition now involves the analog of Fourier for angles: spherical harmonics.

⇝ The Klein Gordon equation becomes:

$$\left(\Box + \underbrace{\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right)}_{=:\, V_\ell(r)}\right)\phi_{\ell m}(t,r) = 0$$

⇒ This effective potential needs to be overcome by